Convergence Rates for Relaxation Schemes Approximating Conservation Laws

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Abstract

In this paper, we prove a global error estimate for a relaxation scheme approximating scalar conservation laws. To this end, we decompose the error into a relaxation error and a discretization error. Including an initial error $\omega(\epsilon)$ we obtain the rate of convergence of $\sqrt{\epsilon}$ in L^1 for the relaxation step. The estimate here is independent of the type of nonlinearity. In the discretization step a convergence rate of $\sqrt{\Delta x}$ in L^1 is obtained. These rates are independent of the choice of initial error $\omega(\epsilon)$. Thereby, we obtain the order 1/2 for the total error.

Key words: Relaxation scheme, relaxation model, convergence rate.
AMS subject classification: 35L65, 65M06, 65M15.

1 Introduction

Let $u : \mathrm{IR} \times \mathrm{IR}^+ \to \mathrm{IR}$ be the unique global entropy solution in the sense of Kružkov [11] to the Cauchy problem for the conservation law

$$u_t + f(u)_x = 0 \tag{1.1}$$

with initial data

$$u(x,0) = u_0(x), \tag{1.2}$$

where $f \in C^1(\mathrm{IR})$ and $u_0 \in L^\infty(\mathrm{IR}) \cap L^1(\mathrm{IR}) \cap BV(\mathrm{IR})$. The solution u satisfies Kružkov's entropy conditions

$$\partial_t|u-k| + \partial_x[\mathrm{sgn}(u-k)(f(u)-f(k))] \leq 0, \quad \text{in } \mathcal{D}', \quad \text{for all } k \in \mathrm{IR}. \tag{1.3}$$

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We are considering here a relaxation scheme, proposed by Jin and Xin [10] to compute the entropy solution of (1.1) using a small relaxation rate ϵ. Our main purpose is to study the convergence rate of the relaxation scheme to the conservation laws as both the relaxation rate ϵ and the mesh length Δx tend to zero.

The relaxation model takes the form

$$\begin{cases} \partial_t u^\epsilon + \partial_x v^\epsilon = 0, \\ \\ \partial_t v^\epsilon + a\partial_x u^\epsilon = -\frac{1}{\epsilon}[v^\epsilon - f(u^\epsilon)]. \end{cases} \tag{1.4}$$

The variables u^ϵ and v^ϵ are the unknowns, $\epsilon > 0$ is referred to as the relaxation rate and a is a positive constant. The system (1.4) was introduced by Jin and Xin [10] as a new way of regularizing hyperbolic systems of the same kind as the scalar equation (1.1). It is also the basis for the construction of relaxation schemes.

In fact, for small ϵ, using the Chapman-Enskog expansion [4], one may deduce from (1.4) the following convection-diffusion equation

$$u_t^\epsilon + f(u^\epsilon)_x = \epsilon\Big([a - f'(u^\epsilon)^2]u_x^\epsilon\Big)_x,$$

see [10], which gives a viscosity solution to the conservation law (1.1) if the well known subcharacteristic condition (cf. Liu [16]) holds:

$$a > f'(u)^2. \tag{1.5}$$

Natalini [20] proved that the solutions to (1.4) converge strongly to the unique entropy solution of (1.1) as $\epsilon \to 0$. Thus the system (1.4) provides a natural way to regularize the scalar equation (1.1). This is in analogy to the regularization of the Euler equations by the Boltzmann equation, see Cercignani [5].

Consider the grid sizes Δx, Δt in space and time as well as for $n \in \mathbb{N}$, $j \in \mathbb{Z}$ a numerical approximate solution $(u_j^n, v_j^n) = (u, v)(j\Delta x, n\Delta t)$. The relaxation scheme associated with the relaxation system (1.4) is given as

$$\begin{cases} u_j^{n+1} - u_j^n + \frac{\lambda}{2}(v_{j+1}^n - v_{j-1}^n) - \frac{\mu}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) &= 0, \\ \\ v_j^{n+1} - v_j^n + \frac{a\lambda}{2}(u_{j+1}^n - u_{j-1}^n) - \frac{\mu}{2}(v_{j+1}^n - 2v_j^n + v_{j-1}^n) &= -k(v_j^{n+1} - f(u_j^{n+1})), \end{cases} \tag{1.6}$$

where $\lambda = \frac{\Delta t}{\Delta x}$ is the mesh ratio, $\mu = \sqrt{a}\lambda$ and $k = \frac{\Delta t}{\epsilon}$. We refer to Aregba-Driollet and Natalini [1] for a class of fractional-step relaxation schemes for the relaxation system (1.4). The first order scheme proposed there can be easily rewritten in the form of (1.6) with $\kappa = \exp(\Delta t/\epsilon) - 1$, where the homogeneous (linear) part is treated by some monotone scheme and then the source term can be solved exactly due to its particular structure. Throught this paper we assume the usual CFL condition

$$0 < \mu = \sqrt{a}\lambda < 1. \tag{1.7}$$

Convergence theory for this kind of relaxation schemes can be found in Aregba-Driollet and Natalini [1], Wang and Warnecke [31] and Yong [32]. Based on proper total variation bounds on the approximate solutions, independent of ϵ, convergence of a subsequence of $(u_j^n, v_j^n)_{j\in\mathbb{Z}, n\in\mathbb{N}}$ to the unique week solution of (1.1) was established by standard compactness arguments.

Currently, there are only very few computational results for relaxation schemes available in the literature, see e.g. Jin and Xin [10], who introduced these schemes, as well as Aregba-Driollet and Natalini [2]. Therefore, it is very hard to tell how useful they may be for practical computations in the future. The main advantages of these schemes are that they neither require the use of Riemann solvers nor the computation of nonlinear flux Jacobians. This seems to become an important advantage when considering fluids with non-standard equations of state, e.g. in multiphase mixtures. Note also that an extension of our results to second order schemes seems easily feasible since we are only making use of the TVD property which can be achieved using flux limiters, see Jin and Xin [10].

The relaxation approximation to conservation laws is in spirit close to the description of the hydrodynamic equations by the detailed microscopic evolution of gases in kinetic theory. The rigorous theory of kinetic approximation for solutions with shocks is well developed when the limit equation is scalar. For works using the continuous velocity kinetic approximation, see Giga and Miyakawa [9], Lions, Perthame and Tadmor [17] and Perthame and Tadmor [19], for discrete velocity approximation of entropy solutions to multidimensional scalar conservation laws see Natalini [21], also Katsoulakis and Tzavaras [13]. Based on a discrete kinetic approximation for multidimensional systems of conservation laws [21], the authors in [2] constructed a class of relaxation schemes approximating the scalar conservation laws.

It was pointed out by Natalini [21] that the relaxation system (1.4) can be rewritten into the two velocities "kinetic" formulation by just setting $\lambda_2 = -\lambda_1 = \sqrt{a} > 0$ and defining the Riemann invariants

$$R_1^\epsilon = \frac{1}{2}\left(u^\epsilon - \frac{v^\epsilon}{\sqrt{a}}\right), \qquad R_2^\epsilon = \frac{1}{2}\left(u^\epsilon + \frac{v^\epsilon}{\sqrt{a}}\right),$$

and the Maxwellians

$$M_1(u^\epsilon) := \frac{1}{2}\left(u^\epsilon - \frac{f(u^\epsilon)}{\sqrt{a}}\right), \qquad M_2(u^\epsilon) := \frac{1}{2}\left(u^\epsilon + \frac{f(u^\epsilon)}{\sqrt{a}}\right).$$

The relaxation model (1.4) becomes

$$\partial_t R_i^\epsilon + \lambda_i \partial_x R_i^\epsilon = \frac{1}{\epsilon}[M_i(u^\epsilon) - R_i^\epsilon)], \quad i = 1,\ 2. \tag{1.8}$$

The relaxation rate plays the role of the mean free path in kinetic theories. Indeed the system (1.8) provides more insight into the properties of the relaxation system. In our investigation of the convergence rates we will use this formulation for the relaxation model as well as for the corresponding relaxation scheme.

The main goal of this paper is to improve on the previous convergence results, see Aregba-Driollet and Natalini [1], Wang and Warnecke [31] and Yong [32] for the relaxation scheme (1.6) by looking at the accuracy of the relaxation scheme for solving the conservation law (1.1). We do this here by studying the error of approximation $u - u_\Delta^\epsilon$ between the exact solution u and the numerical solution u_Δ^ϵ measured in the L^1 norm. The parameters ϵ and Δx determine the scale of approximation and converge to zero as the scales become finer. We shall call the order of this error in these parameters the convergence rate of the numerical solution generated by the relaxation scheme.

To make this point precise, we choose the initial data for (1.4) as

$(\mathbf{H}_1)$ $\qquad u_0^\epsilon(x) := u_0(x), \qquad v_0^\epsilon = f(u_0(x)) + K(x)\omega(\epsilon)$

where $K \in L^\infty \cap L^1(\mathrm{IR}) \cap BV(\mathrm{IR})$, $\omega : [0, \infty[\to [0, \infty[$ is continuous and $\omega(0) = 0$. Here we allow for an initial error $K(x)\omega(\epsilon)$ instead of $v_0^\epsilon = f(u_0)$ because we want to see the contribution of this initial error to the global error. We mention that it is possible to consider perturbed data in the u-component, then in the final result an initial error $\|u_0^\epsilon - u_0\|_{L^1(\mathrm{IR})}$ would persist in time and may prevent the convergence of u^ϵ to the entropy solution, as shown in Theorem 3.4. However, the initial error in the v-component persists only for a short time of order ϵ, thereby it does not prevent the convergence of u^ϵ.

We initialize the relaxation scheme (1.6) by cell averaging the initial data $(u_0^\epsilon, v_0^\epsilon)$ in the usual way

$$(u_j^0, v_j^0) = \frac{1}{\Delta x}\int (u_0^\epsilon(x), v_0^\epsilon(x))\chi_j(x)dx. \tag{1.9}$$

Here and elsewhere $\int$ without the integral limit denotes the integral on the whole of IR, and $\chi_j(x)$ denotes the indicator function $\chi_j(x) := 1_{\{|x - j\Delta x| \leq \Delta x/2\}}$.

Let us now introduce some notations. The L^1-norm is denoted by $\|\cdot\|_1$ and $TV(\cdot)$ denotes the total variation, defined on a subset $\Omega \subseteq \mathrm{IR}$ by

$$TV(u) := \sup_{h \neq 0} \int_\Omega \frac{|u(x+h) - u(x)|}{|h|} dx.$$

The BV-norm is defined as

$$\|u\|_{BV} = \|u\|_1 + TV(u).$$

For grid functions the total variation is defined by

$$TV(u^n) = \sum_{i\in\mathbb{Z}} |u_i^n - u_{i-1}^n|,$$

and $\|\cdot\|_1$ denotes the discrete l^1-norm

$$\|u^n\|_1 = \Delta x \sum_{i\in\mathbb{Z}} |u_i^n|.$$

Taking initial data (1.9), we summarize our main convergence rate result by stating

Theorem 1.1. *Take any $T > 0$ and let for a suitable $N \in \mathrm{IN}$ and time step Δt the relation $T = N\Delta t$ be satisfied. Further let u be the entropy solution of (1.1)-(1.2) with initial data u_0 in $L^\infty(\mathrm{IR}) \cap BV(\mathrm{IR})$, and let (u^N, v^N) be a piecewise constant representation on $\mathrm{IR}\times[0,T]$ of the approximate solution $(u_i^n, v_i^n)_{i\in\mathbb{Z}, 0\le n\le N}$ generated by the relaxation scheme (1.6) with initial data satisfying (H_1) and (1.9). Then for fixed $\lambda = \frac{\Delta t}{\Delta x}$ satisfying the CFL condition (1.7) there exists a constant C_T, independent of Δx, Δt and ϵ, such that*

$$\|u^N - u(\cdot,T)\|_1 \le C_T\left[\sqrt{\epsilon} + \sqrt{\Delta x}\right]. \tag{1.10}$$

Theorem 1.1 suggests that the accumulation of error comes from two sources: the relaxation error and the discretization error. The theorem will be a consequence of Theorem 2.2, giving a rate of convergence to the unique entropy solution of (1.1) in the relaxation step of the solutions (u^ϵ, v^ϵ) to the relaxed system (1.4), as well as Theorem 2.3, giving a discretization error bound for the relaxation scheme (1.6) as an approximation to the relaxation system (1.4).

It may be helpful, at the outset, to explain the structure of the proof. Consider that the relaxation scheme was designed through two steps: the relaxation step and the discretization step. Our basic idea is to investigate the error bound of the two steps separately and then the total convergence rate by combining the relaxation error and discretization error. The basic assumptions and the error bounds of the two steps will be given in detail in Section 2.

We split the error $e_\Delta^\epsilon = u(\cdot,T) - u_\Delta^\epsilon(\cdot,t_N)$ into a relaxation error e^ϵ with $\|e^\epsilon\|_1 \le C_T\sqrt{\epsilon}$, and a discretization error e_Δ with $\|e_\Delta\|_1 \le C_T\sqrt{\Delta x}$, i.e., we have the decomposition

$$e_\Delta^\epsilon = e^\epsilon + e_\Delta$$

with

$$e^\epsilon = u(\cdot,T) - u^\epsilon(\cdot,T) \;\text{ and }\; e_\Delta = u^\epsilon(\cdot,T) - u_\Delta(\cdot,t_N).$$

In order to get the desired approximate entropy inequality, we work with the reformulated system (1.8), in place of the original system (1.4).

We would like to mention that an analogous result for a class of relaxation systems was already obtained by Kurganov and Tadmor [12] by using the Lip$'$-framework initiated by Nessyahu and Tadmor [22]. But their argument uses the convexity of the flux function. For the case of a possibly nonconvex flux function f, our work uses Kuznetzov-type error estimates, see [15] and [3]. Recently, Teng [29] proved the first order convergence rate for piecewise smooth solutions with finitely many discontinuities with the assumption of convex fluxes $f(u)$. Based on Teng's result, Tadmor and Tang [28] provided the optimal pointwise convergence rate for the relaxation approximation to convex scalar conservation laws with piecewise solutions. They use an innovative idea that they introduced in their paper [27] which enables them to convert a global L^1 error estimate into a local error estimate.

In the discretization step the same error bound was obtained by Schroll, Tveito and Winther [25] for a model that arises in chromatography, their argument is in the spirit of Kuznetsov [15] and Lucier [18]. The results in [25] rely on the assumption that the initial data are close to an equilibrium state of order ϵ, i.e. $\omega(\epsilon) = \epsilon$. Our result shows that the uniform estimate does not depend on $\omega(\epsilon)$, which is more natural since in the discretization step ϵ is kept a constant. Taking $\epsilon = 0$ in discretization step, we immediately recover the optimal convergence rate of order $1/2$ for monotone schemes, see Tang and Teng [26], Sabac [23]. We thank a referee for pointing out the possibility of an extension of the present arguments to multi-speeds kinetic schemes introduced in [2], even in the multidimensional case [21].

The paper is organized as follows. In Section 2 we state the assumptions on the system (1.4) and recall properties of the solutions to (1.4) and the scheme (1.5)-(1.6). The main results on the relaxation error and discretization error are given. Their proofs are presented in Section 3 and Section 4, respectively. The authors thank a referee for bringing the paper [14] to their attention. In that paper, a discrete version of the theorem by Bouchut and Perthame [3], see Theorem 3.3 in Section 3, is established and convergence rates for some relaxation schemes based on the relaxation approximation proposed in [13] were considered.

2 Preliminaries and Main Theorems

In this section we review some assumptions and the analytic results concerning the relaxation model and relaxation schemes. After further preparing the initial data, we state our main theorems.

Let us first recall some results obtained by Natalini [20] concerning the analytical properties of the problem (1.4) with the specific initial data $(u_0^\epsilon, v_0^\epsilon)$, which will be of use in our error analysis. Let us make the following assumptions:

(H$_2$) the flux function f is a C^1 function with $f(0) = f'(0) = 0$;

(H$_3$) the initial data satisfy $(u_0^\epsilon, v_0^\epsilon) \in L^1(\mathrm{IR}) \cap L^\infty(\mathrm{IR}) \cap BV(\mathrm{IR})$ and there exist constants $\rho_0 > 0$, $M > 0$ not depending on ϵ such that

$$\rho_0 = \max\left(\sup_{\epsilon>0}\|v_0^\epsilon\|_\infty,\ \sup_{\epsilon>0}\|u_0^\epsilon\|_\infty\right), \qquad \|(u_0^\epsilon, v_0^\epsilon)\|_{BV} := \|u_0^\epsilon\|_{BV} + \|v_0^\epsilon\|_{BV} \le M.$$

and for the flux function f as well as K given in (H$_1$)

$$\mathrm{Lip}(f) := \sup_{x\neq y}\left|\frac{f(x)-f(y)}{x-y}\right| \le M, \qquad \|K\|_1 \le M.$$

$\square$

Let us define, for any $\rho > 0$, the notations

$$F(\rho) := \sup_{|\xi|\le\rho}|f(\xi)|,$$

$$B(\rho) := 2\rho + F(2\rho)$$

and

$$M(\rho) := \sup_{|\xi|\le B(\rho)}|f'(\xi)|.$$

Theorem 2.1. (Natalini [20]) *Assume* (H$_2$) *and* $(u_0^\epsilon, v_0^\epsilon) \in L^\infty(\mathrm{IR})^2$, *for any* $\rho_0 > 0$ *and* $\epsilon > 0$, *if*

$$\sqrt{a} > M(\rho_0), \tag{2.1}$$

then the system (1.4) admits a unique, global solution (u^ϵ, v^ϵ) *in* $C([0,\infty[; [L^1_{\mathrm{loc}}(\mathrm{IR})]^2)$ *satisfying*

$$\|v^\epsilon \pm \sqrt{a}u^\epsilon\|_{L^\infty(\mathrm{IR}\times]0,\infty[)} \le \sqrt{a}B(\rho_0), \tag{2.2}$$

$$|f'(u^\epsilon(x,t))| < \sqrt{a}, \tag{2.3}$$

for all $\epsilon > 0$ *and for almost every* $(x,t) \in \mathrm{IR}\times]0,\infty[$.

We refer to Natalini [20] for detailed discussions on the existence, uniqueness and convergence of solutions to the relaxation model (1.4).

Equipped with assumptions in (H_1)-(H_3), it has been proved that, as $\epsilon \to 0^+$, the solution sequence to (1.4) converges strongly to the unique entropy solution of (1.1)-(1.2), see Natalini [20]. We will study the convergence rate in Section 3, our main result on the limit $\epsilon \downarrow 0$ is summarized in the following theorem.

Theorem 2.2. *Consider the system (1.4), subject to $L^\infty(\mathrm{IR}) \cap BV(\mathrm{IR})$-perturbed initial data satisfying* (H$_1$)-(H$_3$). *Then the global solution (u^ϵ, v^ϵ) converges to $(u, f(u))$ as $\epsilon \downarrow 0$ and the following error estimates hold,*

$$\|u^\epsilon(\cdot,t) - u(\cdot,t)\|_1 \le \mathrm{C}_T\sqrt{\epsilon}, \tag{2.4}$$

$$\|v^\epsilon(\cdot,t) - f(u^\epsilon(\cdot,t))\|_1 \le C_T[\mathrm{e}^{-\frac{t}{\epsilon}}\omega(\epsilon) + \epsilon(1-e^{-\frac{t}{\epsilon}})], \quad 0 \le t \le T. \tag{2.5}$$

Thus, (2.5) reflects two sources of error: the initial contribution of size $\omega(\epsilon)$ and the relaxation error of order ϵ. It should be mentioned that, however, the effect of initial contribution persists only for a short time of order ϵ, beyond this time the nonequilibrium solution approaches a state close to equilibrium at an exponential rate. Note that the proof of estimate (2.5) is included in the proof of Lemma 3.2 below. The proof of (2.4) will follow from Theorem 3.4.

Now we turn to the formulation of the relaxation schemes.

In order to approximate the solutions of the initial value problem (1.4), we first discretize the initial data $(u_0^\epsilon, v_0^\epsilon)$ satisfying (H$_1$) and (H$_3$). To make this more precise, we denote a family of approximate solutions given as piecewise constant functions, dropping the superscript ϵ for notational convenience, by

$$(u_\Delta^n, v_\Delta^n) := (u_\Delta, v_\Delta)(\cdot, n\Delta t) := \sum_{j\in\mathbb{Z}} (u_j^n, v_j^n)\chi_j(x). \tag{2.6}$$

Here, $n \in \mathrm{IN}$ represents the number of time steps performed. For the initial conditions (u_Δ^0, v_Δ^0) we take the orthogonal projection of the initial data $(u_0^\epsilon, v_0^\epsilon)$ onto the space of piecewise constant functions on the given grid

$$\begin{aligned} u_\Delta^0 &= P_\Delta u^0 = \sum_{j\in\mathbb{Z}} u_j^0 \chi_j(x), \quad u_j^0 := \frac{1}{\Delta x}\int u_0^\epsilon(x)\chi_j(x)dx, \\ v_\Delta^0 &= P_\Delta v^0 = \sum_{j\in\mathbb{Z}} v_j^0 \chi_j(x), \quad v_j^0 := \frac{1}{\Delta x}\int v_0^\epsilon(x)\chi_j(x)dx. \end{aligned} \tag{2.7}$$

Thus it follows from (H$_1$), (H$_3$) and (2.7) that the discrete initial data satisfy

$$(a) \qquad \max\left(\|u_\Delta^0\|_\infty, \|v_\Delta^0\|_\infty\right) \le \rho_0, \tag{2.8}$$

$$(b) \qquad TV(u_\Delta^0) + TV(v_\Delta^0) \le M, \tag{2.9}$$

$$(c) \qquad \left\|v_\Delta^0 - f(u_\Delta^0)\right\|_1 \le M\omega(\epsilon). \tag{2.10}$$

The grid parameters Δx and Δt are assumed to satisfy $\frac{\Delta t}{\Delta x} = \mathrm{Const}$. We note that, since $\lambda = \Delta t/\Delta x$ is assumed constant, $\Delta x \to 0$ implies $\Delta t \to 0$ as well.

It is well known that the projection error is of order Δx. More precisely, we have

$$\begin{aligned}\|u_\Delta^0 - u_0^\epsilon\|_1 \leq & \Delta x TV(u_0^\epsilon),\\ \|v_\Delta^0 - v_0^\epsilon\|_1 \leq & \Delta x TV(v_0^\epsilon).\end{aligned} \tag{2.11}$$

As was shown by Aregba-Driollet and Natalini [1] as well as Wang and Warnecke [31], for a large enough constant a a uniform bound for the numerical approximations given by the scheme (1.6) can be found. Precisely, there exists a positive constant $\overline{M}(\rho_0)$ such that if

$$\sqrt{a} > \overline{M}(\rho_0),$$

then the numerical solution satisfies

$$(u_j^n, v_j^n) \in K_{\rho_0} := \left\{ (u, v) \in \mathrm{IR}^2, \left| u \pm \frac{v}{\sqrt{a}} \right| \leq \overline{B}(\rho_0) \right\} \tag{2.12}$$

where $\overline{B}(\rho_0)$ is a constant depending only on ρ_0.

Starting with the discrete initial data satisfying (2.9)-(2.11), by using the Riemann invariants, the TVD bound of the approximate solutions of (1.6) was proved previously by Aregba-Driollet and Natalini [1], Wang and Warnecke [31] and Yong [32]. By Helly's compactness theorem, the piecewise constant approximate solution $(u_\Delta(x, t_n), v_\Delta(x, t_n)) = \sum_j (u_j^n, v_j^n)\chi_j(x)$ converges strongly to the unique limit solution $(u, v)(x, t_n)$ as we refine the grid taking $\Delta x \downarrow 0$. This, together with equi-continuity in time and the Lax-Wendroff theorem, yields a weak solution (u^ϵ, v^ϵ) of the relaxation system (1.4). We note that the initial bounds (2.9)-(2.11) still hold for the piecewise constant numerical data (u_Δ^0, v_Δ^0) for which

$$TV\left(u_\Delta(\cdot, t)\right) \leq TV(u) \ \text{ and } \ \|u_\Delta(\cdot, t)\|_1 \leq \|u\|_1.$$

We refer to [1], [31] and [32] for the convergence of approximate solutions (u_Δ, v_Δ) to the unique weak solution of (1.4). Our goal here is to improve the previous convergence theory by establishing the following L^1-error bound of the relaxation scheme. This theorem will be proved in Section 4.

Theorem 2.3. *Let (u^ϵ, v^ϵ) be the weak solution of (1.4) with initial data $(u_0^\epsilon, v_0^\epsilon)$, and let (u^N, v^N) be a piecewise constant representation of the data $(u_i^N, v_i^N)_{i\in\mathbb{Z}}$ generated by (1.6) starting with (u_Δ^0, v_Δ^0). Then, for any fixed $T = N\Delta t \geq 0$, there is a finite constant C_T independent of Δx, Δt and ϵ such that*

$$\left\|v^\epsilon(\cdot, T) - v^N\right\|_1 + \left\|u^\epsilon(\cdot, T) - u^N\right\|_1 \leq C_T\sqrt{\Delta x}.$$

Remark 1. Our uniform error bound is independent of the relaxation parameter ϵ and initial error $\omega(\epsilon)$. This is more general than the result obtained by [25] assuming the initial error $\omega(\epsilon)$ to be ϵ. $\square$

As remarked earlier, the error is split into a relaxation error e^ϵ and a discretization error e_Δ, combining the two errors we arrive at the desired total error for the scheme (1.6) as stated in Theorem 1.1.

3 Relaxation Error

In this section we establish Kuznetzov-type error estimates for the approximation of the entropy solution of the scalar conservation law

$$u_t + f(u)_x = 0 \tag{3.1}$$

by solutions (u^ϵ, v^ϵ) of the relaxation system

$$\begin{cases} u^\epsilon_t + v^\epsilon_x = 0, \\ v^\epsilon_t + a u^\epsilon_x = -\frac{1}{\epsilon}\left[v^\epsilon - f(u^\epsilon)\right]. \end{cases} \tag{3.2}$$

For the above purpose we have to show that u^ϵ satisfies an approximate entropy condition. Let us begin with rewriting (3.2) in terms of the Riemann invariants $(R^\epsilon_1, R^\epsilon_2)$ and Maxwellians $(M_1(u^\epsilon), M_2(u^\epsilon))$ defined in the introduction and recall some basic facts that will be used in our analysis. It is easy to see that

$$u^\epsilon = R^\epsilon_1 + R^\epsilon_2, \qquad v^\epsilon = \sqrt{a}\big(R^\epsilon_2 - R^\epsilon_1\big). \tag{3.3}$$

Then the system is rewritten as (1.8), i.e.,

$$\partial_t R^\epsilon_i + \lambda_i \partial_x R^\epsilon_i = \frac{1}{\epsilon}\big[M_i(u^\epsilon) - R^\epsilon_i\big], \quad i = 1, 2$$

where the functions $M_i(u)$ $(i = 1, 2)$ have the following properties

$$\sum_{i=1}^{2} M_i(u) = u, \quad \sum_{i=1}^{2} \lambda_i M_i(u) = f(u). \tag{3.4}$$

We note that the L^∞ estimate obtained by Natalini [20] implies that

$$(u^\epsilon, v^\epsilon) \in K_{\rho_0} := \left\{(u, v) \in \mathrm{IR}^2 | \ \ |f'(u)| \le \sqrt{a}, |u \pm \frac{v}{\sqrt{a}}| \le B(\rho_0)\right\}. \tag{3.5}$$

Further, we set

$$I_{\rho_0} := \{u \in \mathrm{IR}, (u, v) \in K_{\rho_0}\}.$$

The $M_i(u)$ are monotone (non-decreasing) functions of $u \in I_{\rho_0}$ because due to the subcharacteristic condition (1.5)

$$\frac{d}{du}M_i(u) = \frac{1}{2}\left(1 \pm \frac{f'(u)}{\sqrt{a}}\right) \geq 0.$$

Thus we have for any u^ϵ and $\tilde{u}^\epsilon$ in I_{ρ_0}

$$\sum_{i=1}^{2} |M_i(u^\epsilon) - M_i(\tilde{u}^\epsilon)| = \left|\sum_{i=1}^{2} M_i(u^\epsilon) - \sum_{i=1}^{2} M_i(\tilde{u}^\epsilon)\right| = |u^\epsilon - \tilde{u}^\epsilon|. \tag{3.6}$$

Starting with the initial distribution

$$R_1^\epsilon(\cdot,0) = \frac{1}{2}\left(u_0^\epsilon - \frac{v_0^\epsilon}{\sqrt{a}}\right), \quad R_2^\epsilon(\cdot,0) = \frac{1}{2}\left(u_0^\epsilon + \frac{v_0^\epsilon}{\sqrt{a}}\right) \tag{3.7}$$

the model evolves according to the system (1.8), which is well-posed. In fact, we rewrite (1.8) in the form

$$\partial_t R_i^\epsilon + \lambda_i \partial_x R_i^\epsilon + \frac{1}{\epsilon} R_i^\epsilon = \frac{1}{\epsilon} M_i(u^\epsilon), \quad i = 1,2. \tag{3.8}$$

From (3.8) one obtains the L^1-contraction property using a Gronwall inequality, see [21], which is

$$\sum_{i=1}^{2} \|R_i^\epsilon(\cdot,t) - S_i^\epsilon(\cdot,t)\|_1 \leq \sum_{i=1}^{2} \|R_i^\epsilon(\cdot,0) - S_i^\epsilon(\cdot,0)\|_1, \ \ \forall \ t \geq 0. \tag{3.9}$$

As shown in [20], the above nice property for general data of bounded variation yields the following TV estimate

$$\|(u^\epsilon(\cdot,t), v^\epsilon(\cdot,t))\|_{BV} \leq C\|(u_0^\epsilon, v_0^\epsilon)\|_{BV}. \tag{3.10}$$

Lemma 3.1. *For any $\rho_0 > 0$ and $\epsilon > 0$, if*

$$\sqrt{a} > M(\rho_0),$$

then the global solution $(R_1^\epsilon, R_2^\epsilon) \in C([0,\infty[; L^1_{loc}(\mathsf{IR})^2)$ of the problem (1.8), (3.7) for any $k \in \mathsf{IR}$ satisfies the entropy-type inequality

$$\partial_t\left(\sum_{i=1}^{2} |R_i^\epsilon - M_i(k)|\right) + \partial_x\left(\sum_{i=1}^{2} \lambda_i |R_i^\epsilon - M_i(k)|\right) \leq 0, \quad in \ \mathcal{D}'. \tag{3.11}$$

Proof. See Natalini [21, Proposition 3.8]. $\square$

Before establishing the desired convergence rate in Theorem 2.2, we first need a Lemma giving a bound on the distance of $(R_1^\epsilon, R_2^\epsilon)$ from equilibrium. This bound is actually equivalent to the estimate (2.5). The following lemma is a generalization of Natalini [20, Proposition 4.7].

Lemma 3.2. *Suppose that the assumptions $(H_1)-(H_3)$ hold. Let (u^ϵ, v^ϵ) be a solution of (1.4) with initial data $(u_0^\epsilon, v_0^\epsilon)$. Then for every $t>0$, it holds that*

$$A^\epsilon(t) := \sum_{i=1}^{2} \|R_i^\epsilon(\cdot,t) - M_i(u^\epsilon(\cdot,t))\|_1 \le C[e^{-\frac{t}{\epsilon}}\omega(\epsilon) + (1-e^{-\frac{t}{\epsilon}})\epsilon].$$

Proof. In view of the relations between $(R_1^\epsilon, R_2^\epsilon)$ and (u^ϵ, v^ϵ) we have

$$\sqrt{a}\sum_{i=1}^{2} |R_i^\epsilon - M_i(u^\epsilon)| = |v^\epsilon - f(u^\epsilon)|, \tag{3.12}$$

which serves as a measure of the distance from equilibrium. Let $\xi^\epsilon = v^\epsilon - f(u^\epsilon)$; then the function ξ^ϵ satisfies

$$\xi_t^\epsilon + \frac{1}{\epsilon}\xi^\epsilon = -au_x^\epsilon + f'(u^\epsilon)v_x^\epsilon$$

for $(x,t) \in \mathrm{IR}\times]0,\infty[$ and initial data

$$\xi^\epsilon(x,0) = \xi_0^\epsilon := v_0^\epsilon(x) - f(u_0^\epsilon(x)), \quad \text{for } x \in \mathrm{IR}.$$

Then multiplying by $\operatorname{sgn}(\xi^\epsilon)e^{\frac{t}{\epsilon}}$ and integrating over $\mathrm{IR}\times[0,t]$ one gets

$$\int |\xi^\epsilon| dx \le e^{-\frac{t}{\epsilon}} \int |\xi_0^\epsilon| dx + \int_0^t \int e^{\frac{-(t-s)}{\epsilon}} |au_x^\epsilon + f'(u^\epsilon)v_x^\epsilon| dx ds.$$

Note that by (3.10) we have for sufficiently regular initial data

$$\|au_x^\epsilon + f'(u^\epsilon)v_x^\epsilon\| \le C\|(\partial_x u_0^\epsilon, \partial_x v_0^\epsilon)\|_1.$$

For example one could use C^∞ data in combination with an approximation theorem in $BV(\mathrm{IR})$, see the Theorem 1.17 in Giusti [8]. Combining the above facts, for general data of bounded variation, gives

$$||v^\epsilon - f(u^\epsilon)||_1 \le e^{-\frac{t}{\epsilon}}||v_0^\epsilon - f(u_0^\epsilon)||_1 + C\epsilon||(u_0^\epsilon, v_0^\epsilon)||_{BV}(1-e^{-\frac{t}{\epsilon}}).$$

Together with (H_3) and (3.12), this estimate implies the result as asserted. □

Equipped with Lemmas 3.1-3.2, now we turn to the proof of our main result.

Our further analysis uses a result by Bouchut and Perthame [3, Theorem 2.1]. We first state in a less general form their central result.

Theorem 3.3. (**Bouchut and Perthame**) *Let $u, v \in L^\infty_{\mathrm{loc}}([0,\infty[, L^1_{\mathrm{loc}}(\mathrm{IR}))$ be right continuous with values in $L^1_{\mathrm{loc}}(\mathrm{IR})$. Assume that u satisfies the entropy condition (1.3) and v satisfies, for all $k \in \mathrm{IR}$,*

$$\partial_t|v-k| + \partial_x\left[\operatorname{sgn}(v-k)\big(f(v)-f(k)\big)\right] \le \partial_t J_k + \partial_x H_k, \ \text{ in } \mathcal{D}', \tag{3.13}$$

where J_k, H_k are locally finite Radon measures such that for some nonnegative k-independent Radon measures α_J and $\alpha_H \in L^\infty_{\text{loc}}([0,\infty[, L^1_{\text{loc}}(\mathrm{IR}))$, satisfy in the sense of measures

$$\big|J_k(x,t)\big| \le \alpha_J(x,t) \quad \text{and} \quad \big|H_k(x,t)\big| \le \alpha_H(x,t). \tag{3.14}$$

Then for any $T > 0, x_0 \in \mathrm{IR}, \rho > 0, \delta > 0$ and the balls

$$B_t = B(x_0, \rho + M(T-t) + \delta),$$

we have

$$\begin{aligned}\int_{|x-x_0|\le\rho} &|u(x,T) - v(x,T)|\,dx \\ &\le \int_{B_0} |u(x,0) - v(x,0)|\,dx + C\left(E^t + E^x + E^J + E^H\right),\end{aligned}$$

where C is a uniform constant and

$$\begin{aligned}&0 \le E^t \le M\delta TV(v_0), \quad 0 \le E^x \le M\delta TV(u_0), \\ &E^J = \left(1 + \frac{(M+1)T}{\delta}\right) \sup_{t\in]0,2T[} \int_{B_t} \alpha_J(x,t)dx, \\ &E^H = \frac{1}{\delta}\int_0^T \int_{B_t} \alpha_H dxdt.\end{aligned}$$

Based on this general result we will prove

Theorem 3.4. *Under the assumptions of Lemma 3.2. Let $\bar{u}$ be the entropy solution of (1.1) with initial data $\bar{u}_0(x)$. Then, for any fixed $T > 0$ and all $t \le T$,*

$$\|u^\epsilon(\cdot,t) - \bar{u}(\cdot,t)\|_1 \le \|u^\epsilon(\cdot,0) - \bar{u}(\cdot,0)\|_1 + C\sqrt{\epsilon}.$$

Here, C is a positive constant depending on the flux function f and the L^∞-norm of the data $(u_0^\epsilon, v_0^\epsilon)$.

Proof. Since $M_i(u)$ is monotone for $i = 1, 2$, we have by (3.6)

$$\begin{aligned}|u^\epsilon - k| &= \sum_{i=1}^{2} |M_i(u^\epsilon) - M_i(k)| \\ &= \sum_{i=1}^{2} |R_i^\epsilon - M_i(k)| + J^\epsilon\end{aligned} \tag{3.15}$$

where

$$J^\epsilon = \sum_{i=1}^{2} |M_i(u^\epsilon) - M_i(k)| - \sum_{i=1}^{2} |R_i^\epsilon - M_i(k)|.$$

The term J^ϵ is bounded from above by $\sum_{i=1}^2 |R_i^\epsilon - M_i(u^\epsilon)|$ due to the inverse triangle inequality. Let

$$\alpha_J = \sum_{i=1}^{2} |R_i^\epsilon - M_i(u^\epsilon)|$$

then we have $|J^\epsilon| \leq \alpha_J$ and by Lemma 3.2

$$\alpha_J \in L^\infty_{\mathrm{loc}}([0,\infty[, L^1_{\mathrm{loc}}(\mathrm{IR})) \quad \text{with} \quad \|\alpha_J\|_1 = A^\epsilon(t) < \infty. \tag{3.16}$$

Setting

$$H^\epsilon = \sum_{i=1}^{2} \lambda_i \left(\operatorname{sgn}(u^\epsilon - k)(M_i(u^\epsilon) - M_i(k)) - |R_i^\epsilon - M_i(k)|\right)$$

we similarly get using (3.4)

$$\begin{aligned}\operatorname{sgn}(u^\epsilon - k)[f(u^\epsilon) - f(k)] =& \operatorname{sgn}(u^\epsilon - k)\left[\sum_{i=1}^{2} \lambda_i M_i(u^\epsilon) - \sum_{i=1}^{2} \lambda_i M_i(k)\right] \\ =& \sum_{i=1}^{2} \lambda_i |R_i^\epsilon - M_i(k)| + H^\epsilon. \end{aligned} \tag{3.17}$$

Due to the monotonicity property of M_i, we have

$$\operatorname{sgn}(u^\epsilon - k)(M_i(u^\epsilon) - M_i(k)) = |M_i(u^\epsilon) - M_i(k)|.$$

Using this fact and the inverse triangle inequality one obtains

$$|H^\epsilon| \leq \alpha_H := \sqrt{a} \sum_{i=1}^{2} |R_i^\epsilon - M_i(u^\epsilon)| = \sqrt{a}\alpha_J.$$

Therefore, we also have

$$\alpha_H \in L^\infty_{\mathrm{loc}}([0,\infty), L^1_{\mathrm{loc}}(\mathrm{IR})) \quad \text{with} \quad \|\alpha_H\|_1 = \sqrt{a} A^\epsilon(t) < \infty.$$

Combining the previous expressions (3.15) and (3.17) yields

$$\begin{aligned}&\partial_t |u^\epsilon - k| + \partial_x \left[\operatorname{sgn}(u^\epsilon - k)\big(f(u^\epsilon) - f(k)\big)\right] \\ &= \partial_t \left(\sum_{i=1}^{2} |R_i^\epsilon - M_i(k)|\right) + \partial_x \left(\sum_{i=1}^{2} \lambda_i |R_i^\epsilon - M_i(k)|\right) + \partial_t J^\epsilon + \partial_x H^\epsilon.\end{aligned}$$

However by Lemma 3.1 the Riemann invariants $(R_1^\epsilon, R_2^\epsilon)$ satisfy the entropy-type inequalities (3.11); consequently one obtains

$$\partial_t |u^\epsilon - k| + \partial_x \left[\operatorname{sgn}(u^\epsilon - k)\big(f(u^\epsilon) - f(k)\big)\right] \leq \partial_t J^\epsilon + \partial_x H^\epsilon, \text{ in } \mathcal{D}'. \tag{3.18}$$

The functions J^ϵ, H^ϵ are bounded by the L^1-functions α_J, α_H respectively as required in (3.14). Since the solutions u^ϵ, R_i^ϵ are bounded and the flux function f is assumed to be Lipschitz, we may apply Lemma 3.3. Letting $\rho \to \infty$, Lemma 3.3 gives

$$
\begin{aligned}
\|u^\epsilon(\cdot,T) - \bar{u}(\cdot,T)\|_1 &\leq \|u^\epsilon(\cdot,0) - \bar{u}(\cdot,0)\|_1 + C\left(E^t + E^x + E^J + E^H\right) \\
&\leq ||u^\epsilon(\cdot,0) - \bar{u}(\cdot,0)||_1 \\
&\quad + C\left(2M\delta||\bar{u}_0||_{BV} + \left[1 + \frac{(M+1)T}{\delta}\right] \sup_{t\in]0,2T[} \int \alpha_J dx + \frac{1}{\delta}\int_0^T \int \alpha_H dx dt\right) \\
&\leq |u^\epsilon(\cdot,0) - \bar{u}(\cdot,0)||_1 \\
&\quad + C\left(\delta||\bar{u}_0||_{BV} + \left[1 + \frac{(M+1)T}{\delta}\right] \sup_{t\in]0,2T[} A^\epsilon(t) + \frac{\sqrt{a}}{\delta}\int_0^T A^\epsilon(t) dt\right)
\end{aligned} \tag{3.19}
$$

for all $\delta > 0$. Choosing $\delta = \sqrt{\sup_{t\in]0,2T[} A^\epsilon}$ to minimize the right hand side of (3.19), then we have by choosing a suitably large constant C_T, including $\|\bar{u}_0\|_{BV}$, that

$$
||u^\epsilon(\cdot,T) - \bar{u}(\cdot,T)||_1 \leq ||u^\epsilon(\cdot,0) - \bar{u}(\cdot,0)||_1 + C_T\sqrt{\sup_{t\in]0,2T[} A^\epsilon(t)}. \tag{3.20}
$$

Note that the estimate in Lemma 3.2 yields

$$
A^\epsilon(t) \leq C \max\{\epsilon, \omega(\epsilon)\}.
$$

If $\omega(\epsilon) \leq \epsilon$, the above in combination with (3.20) yields Theorem 3.4.

Next we treat the case $\omega(\epsilon) > \epsilon$. To obtain the desired estimate we again apply Lemma 3.3 on the interval $[\tau, T]$ with $\tau > 0$ to be determined, thus we have as above

$$
||u^\epsilon(\cdot,T) - \bar{u}(\cdot,T)||_1 \leq ||u^\epsilon(\cdot,\tau) - \bar{u}(\cdot,\tau)||_1 + C_T\sqrt{\sup_{t\in]\tau,2T[} A^\epsilon(t)}. \tag{3.21}
$$

Using the fact that both u^ϵ and $\bar{u}$ lie in a bounded subset in $\mathrm{Lip}(\mathrm{IR}^+, L^1(\mathrm{IR}))$, see Katsoulakis and Tzavaras [13] and Smoller [24], we have

$$
\begin{aligned}
||u^\epsilon(\cdot,\tau) - \bar{u}(\cdot,\tau)||_1 &\leq \|u^\epsilon(\cdot,0) - \bar{u}(\cdot,0)||_1 + ||u^\epsilon(\cdot,\tau) - u^\epsilon(\cdot,0)||_1 + ||\bar{u}(\cdot,\tau) - \bar{u}(\cdot,0)||_1 \\
&\leq \|u^\epsilon(\cdot,0) - \bar{u}(\cdot,0)||_1 + C\tau.
\end{aligned} \tag{3.22}
$$

It follows from the estimate in Lemma 3.2 that

$$
\sup_{\tau\leq t\leq 2T} A^\epsilon(t) \leq C(\epsilon + e^{-\frac{\tau}{\epsilon}}). \tag{3.23}
$$

Taking $\tau = \epsilon^\alpha$ with $\alpha \in [0,1[$ and applying (3.22) and (3.23) into (3.21) one gets

$$
||u^\epsilon(\cdot,T) - \bar{u}(\cdot,T)||_1 \leq ||u^\epsilon(\cdot,0) - \bar{u}(\cdot,0)||_1 + C_T\left[\epsilon^\alpha + \sqrt{\epsilon + e^{-\epsilon^{\alpha-1}}}\right]. \tag{3.24}
$$

Now, choosing $\alpha = 1/2$, we easily recover the order $1/2$ estimate from (3.24) and

$$
\exp(-\epsilon^{-1/2}) \leq \sqrt{\epsilon}.
$$

This completes the proof. □

4 Discretization Error

The purpose of this section is to derive the error estimate given in Theorem 2.3. Let us define the computational cells as

$$I_j = [x_{j-1/2}, x_{j+1/2}), \quad j \in \mathbb{Z}.$$

We will prove that the error bound for the relaxation scheme (1.6) approximating the relaxation system (1.4) is of order $\sqrt{\Delta x}$ in L^1, without requiring that the initial data be close to equilibrium.

To this end, let us rewrite the relaxation scheme as a splitting or fractional step method in terms of the Riemann invariants $(R_1^\epsilon, R_2^\epsilon)$. Dropping the superscript ϵ and noting that $\kappa = \frac{\Delta t}{\epsilon}$, the scheme takes the form

$$R_{i,j}^{n+1} = R_{i,j}^{n+1/2} + \kappa[M_i(u_j^{n+1}) - R_{i,j}^{n+1}], \quad i = 1, 2, \tag{4.1}$$

for the source terms while the intermediate states $(R_{1,j}^{n+1/2}, R_{2,j}^{n+1/2})$ are generated by the following consistent monotone scheme in conservative form for the convections, namely the upwind scheme,

$$\begin{aligned} R_{1,j}^{n+1/2} &= R_{1,j}^n + \sqrt{a}\lambda(R_{1,j+1}^n - R_{1,j}^n), \\ R_{2,j}^{n+1/2} &= R_{2,j}^n - \sqrt{a}\lambda(R_{2,j}^n - R_{2,j-1}^n). \end{aligned} \tag{4.2}$$

The discrete values $(R_{1,j}^n, R_{2,j}^n)$ computed from (4.1) are considered as approximations of R_i^ϵ in the whole cell I_j at time $t_n = n\Delta t$, which can also be obtained through (u_j^n, v_j^n) generated by (1.6). The discrete variables are related to each other by

$$R_{1,j}^n = \frac{1}{2}\left(u_j^n - \frac{v_j^n}{\sqrt{a}}\right), \qquad R_{2,j}^n = \frac{1}{2}\left(u_j^n + \frac{v_j^n}{\sqrt{a}}\right) \tag{4.3}$$

and

$$M_1(u_j^n) = \frac{1}{2}\left(u_j^n - \frac{f(u_j^n)}{\sqrt{a}}\right), \qquad M_2(u_j^n) = \frac{1}{2}\left(u_j^n + \frac{f(u_j^n)}{\sqrt{a}}\right). \tag{4.4}$$

Conversely (u_j^n, v_j^n) can also be expressed as

$$u_j^n = R_{1,j}^n + R_{2,j}^n, \qquad v_j^n = \sqrt{a}(R_{2,j}^n - R_{1,j}^n). \tag{4.5}$$

For the initial conditions $R_{i,j}^0$ we take the orthogonal projection of the initial data $R_i^\epsilon(x, 0)$ onto the space of piecewise constant functions on the given grid. It is given by the averages

$$R_{i,j}^0 = \frac{1}{\Delta x}\int R_i^\epsilon(x, 0)\chi_j(x)dx, \qquad i = 1, 2, \tag{4.6}$$

for each integer j.

In the previous studies concerning relaxation schemes, some important properties for the numerical scheme were obtained through investigating the reformulated scheme using the Riemann invariants. These properties include: the L^∞ boundedness, the TVD property and the L^1 continuity in time. Here we also rewrite the relaxation scheme in terms of the Riemann invariants because they provide more insight into the convergence behavior of the scheme. The above properties for our scheme are summarized as follows and will be used in the later error analysis.

Lemma 4.1. *Suppose that the initial conditions $(u_0^\epsilon, v_0^\epsilon)$ are bounded and of bounded variation, both uniformly with respect to ϵ. Further, assume that the initial data also satisfy (H_1), i.e. as a consequence*

$$||v_0^\epsilon - f(u_0^\epsilon))||_1 \le M\omega(\epsilon),$$

then there exists a constant C_0, independent of ϵ and Δt such that for $\kappa = \frac{\Delta t}{\epsilon}$ and K_{ρ_0} as defined by (3.5)

(a) $(R_{1,j}^n, R_{2,j}^n) \in K_{\rho_0}$, *for all* $(j,n) \in \mathbb{Z} \times \mathbb{N}$,
(b) $\sum_{i=1}^2 TV(R_i^n) \le C_0, \ \forall n \in \mathbb{N}$,
(c) $\|M_i(u^n) - R_i^n||_1 \le (1+\kappa)^{-n}\|M_i(u^0) - R_i^0\|_1 + C_0\sqrt{a}\frac{\Delta t}{\kappa}[1-(1+\kappa)^{-n}], \ \forall n \in \mathbb{N}$,
(d) $\sum_{i=1}^2 \|R_i^n - R_i^m\||_1 \le C_0 \Delta t|n-m|, \ \forall n,m \in \mathbb{N}$.

Proof. The proof of (a) can be found in Wang and Warnecke [31]. The proof of (b) and (d) is straightforward, see Aregba-Driollet and Natalini [1] and Yong [32]. Since here we choose initial data satisfying (H_1) instead of $v_j^0 = f(u_j^0)$, we prove (c) as follows.

Set

$$Z_j^n := M_1(u_j^n) - R_{1,j}^n.$$

It follows from (4.3) and (4.4) that $Z_j^n := -[M_2(u_j^n) - R_{2,j}^n]$. Summation of the scheme (4.1) over $i = 1,2$ and noting that $u_j^n = \sum_{i=1}^2 R_{i,j}^n$, we obtain

$$u_j^{n+1} = u_j^{n+1/2}.$$

Again let us consider (4.1) for $i = 1$, i.e.

$$R_{1,j}^{n+1} = R_{1,j}^{n+1/2} + \kappa Z_j^{n+1}.$$

Adding $-M_1(u_j^{n+1}) = -M_1(u_j^{n+1/2})$ on both sides gives

$$-Z_j^{n+1} = -Z_j^{n+1/2} + \kappa Z_j^{n+1}$$

by which we obtain

$$(1+\kappa)Z_j^{n+1} = Z_j^{n+1/2}. \tag{4.7}$$

Using the first equation of scheme (4.2) and the definition of Z_j^n, one gets

$$\begin{aligned}
Z_j^{n+1/2} &= M_1(u_j^{n+1/2}) - R_{1,j}^{n+1/2} \\
&= M_1(u_j^{n+1/2}) - R_{1,j}^n - \sqrt{a}\lambda(R_{1,j+1}^n - R_{1,j}^n) \\
&= Z_j^n + M_1(u_j^{n+1/2}) - M_1(u_j^n) - \sqrt{a}\lambda(R_{1,j+1}^n - R_{1,j}^n).
\end{aligned} \tag{4.8}$$

Then by the Mean Value Theorem there exists a value $\tilde{u}_j^n$ between $u_j^{n+1/2} = u_j^{n+1}$ and u_j^n such that for $0 \leq \gamma_j^n = \frac{\partial M_1}{\partial u}(\tilde{u}_j^n)$ we obtain

$$M_1(u_j^{n+1/2}) - M_1(u_j^n) = \gamma_j^n(u_j^{n+1/2} - u_j^n) \tag{4.9}$$

due to the monotonicity property of $M_1(u)$. Substituting (4.9) into (4.8), then using the relation (4.5) and the scheme (4.2) gives

$$Z_j^{n+1/2} = Z_j^n + (1-\gamma_j^n)\sqrt{a}\lambda(R_{1,j}^n - R_{1,j+1}^n) + \gamma_j^n\sqrt{a}\lambda(R_{2,j-1}^n - R_{2,j}^n)$$

By summation over j and multiplying by Δx one obtains from (4.7) and using (b)

$$(1+\kappa)\|Z^{n+1}\|_1 \leq \|Z^n\|_1 + \sqrt{a}\lambda\Delta x\sum_{i=1}^{2} TV(R_i^n) \leq \|Z^n\|_1 + C_0\sqrt{a}\Delta t. \tag{4.10}$$

From (4.10) it is easy to verify by iteration and the geometric sum that

$$\|Z^n\|_1 \leq (1+\kappa)^{-n}\|Z^0\|_1 + C_0\sqrt{a}\frac{\Delta t}{\kappa}[1-(1+\kappa)^{-n}].$$

This completes the proof of Lemma 4.1. □

In the error analysis, we have to consider the finite difference solution as a function defined in the whole upper-half plane, and $t \geq 0$, not only at the mesh points. There are very simple ways of constructing a step function corresponding the grid function. For the analysis we want to construct a different kind of approximate function that automatically satisfies a convenient form of an entropy inequality. For this purpose we use the well known interpretation of the upwind scheme (4.2) as the Godunov scheme. To accomplish this, we construct a family of functions iteratively. Using piecewise constant data for the averages $\overline{R}_i$ this construction is actually equivalent to taking a characteristic scheme since we have linear transport equations, see Childs and Morton [6], followed by an implicit step like (4.1) for the source term. This is like a splitting method using the Godunov operator splitting. We follow the approach used by Schroll et al. [25]. For notational clarity, we will use the variables (y,τ) instead of (x,t) in the context below.

The iteration is initialized by

$$\overline{R}_i(y,0) = R^0_{i,j}, \quad y \in I_j,$$

$$R^0_{i,j} = P_\Delta R_i(y,0) := \frac{1}{\Delta x}\int R_j(x,0)\chi_j(x)dx,$$

where χ_j was defined as the characteristic function for the interval I_j.

(i) We use the notation $t_n^\pm$ to denote limits from above or below. In the interval (t_n^+, t_{n+1}^-), $\overline{R}_i$ is the solution of the linear equation

$$\partial_\tau \overline{R}_i + \lambda_i \partial_y \overline{R}_i = 0, \qquad i = 1,2 \tag{4.11}$$

with initial data $\overline{R}_i(y, t_n^+)$ and $\lambda_1 = -\sqrt{a} = -\lambda_2$.

(ii) At t_{n+1} we project back onto the mesh by taking cell averages

$$\overline{R}_i(y, t_{n+1}) = P_\Delta(\overline{R}_i(\cdot, t_{n+1}^-))(y), \quad i = 1,2. \tag{4.12}$$

(iii) The initial data for the next iteration are defined by the implicit formula treating the source term

$$\overline{R}_i(y, t_{n+1}^+) = \overline{R}_i(y, t_{n+1}) + \kappa[M_i(\bar{u}(y, t_{n+1}^+)) - \overline{R}_i(y, t_{n+1}^+)]. \tag{4.13}$$

□

From the definition of $M_i(u)$ it follows by (4.5) using (4.13) that

$$\bar{u}(y, t_{n+1}^+) = \bar{u}(y, t_{n+1}).$$

Thus the $\overline{R}_i(y, t_{n+1}^+)$ can actually be obtained in the explicit form

$$\overline{R}_i(y, t_{n+1}^+) = \frac{1}{1+\kappa}\left[\,\overline{R}_i(y, t_{n+1}) + \kappa M_i(\bar{u}(y, t_{n+1}))\right]. \tag{4.14}$$

Assuming $\overline{R}_i(y, t_n^+) = R^n_{i,j}$ for $y \in I_j, j \in \mathbb{Z}$, we conclude from the integral form of (4.11) on the rectangle $I_j \times [t_n, t_{n+1}[$

$$\begin{aligned}\overline{R}_1(y, t_{n+1}) &= R^n_{1,j} - \lambda_1\lambda[R^n_{1,j+1} - R^n_{1,j}],\\ \overline{R}_2(y, t_{n+1}) &= R^n_{2,j} - \lambda_2\lambda[R^n_{1,j} - R^n_{1,j-1}].\end{aligned} \tag{4.15}$$

Thus our step functions $\overline{R}_i$, $i = 1,2$, respresenting the grid functions possess the following property, due to $\lambda_{1,2} = \pm\sqrt{a}$,

$$\overline{R}_i(y, t_{n+1}) = R^{n+\frac{1}{2}}_{i,j}, \qquad \overline{R}_i(y, t_{n+1}^+) = R^{n+1}_{i,j}, \qquad i = 1,2. \tag{4.16}$$

The discrete estimates in Lemma 4.1 and the relations (4.16) yield the following properties of $\overline{R}_i$.

Lemma 4.2. *For all t, $\tau \in \mathbb{R}^+$, $y \in \mathbb{R}$, $n, m \in \mathbb{N}$ we have*

(a) $(\overline{R}_1, \overline{R}_2)(y,\tau) \in K_{\rho_0}$,

(b) $\sum_{i=1}^{2} TV(\overline{R}_i(\cdot,\tau)) \le C_0$,

(c) $\|M_i(\bar{u}(\cdot,t_n^+)) - \overline{R}_i(\cdot,t_n^+)\| \le C_0\epsilon$,

(d) $\sum_{i=1}^{2} \|\overline{R}_i(\cdot,t_n^+) - \overline{R}_i(\cdot,t_m^+)\| \le C_0(|t_n - t_m|)$,

(e) $\sum_{i=1}^{2} \|\overline{R}_i(\cdot,t) - \overline{R}_i(\cdot,\tau)\| \le C_0(|t-\tau| + \Delta t)$.

Proof. The relations (4.16) imply directly that (c)-(d) hold. For the proof of (e), one has to use the time-L^1 Lipschitz continuity of $\overline{R}_i$, we refer to [25] for a similar analysis and omit the details. $\square$

The solution to the linear transport equations is unique. We do not need an entropy inequality in order to impose uniqueness. Still, we may use a discrete version of the entropy inequalities in order to study the convergence rate of $\overline{R}_i$ to R_i as $\Delta x \downarrow 0$.

Since $\overline{R}_i(y,\tau)$ is a weak solution (also an entropy solution due to its uniqueness) of the system (4.11) in (t_n, t_{n+1}), the Kružkov-type entropy formulation is valid. This means that

$$\begin{aligned}\int_{t_n}^{t_{n+1}}\int\sum_{i=1}^{2}|\overline{R}_i(y,\tau)-q_i|[\phi_\tau+\lambda_i\phi_y]dyd\tau+\int\sum_{i=1}^{2}|\overline{R}_i(y,t_n^+)-q_i|\phi(y,t_n)dy\\ -\int\sum_{i=1}^{2}|\overline{R}_i(y,t_{n+1}^-)-q_i|\phi(y,t_{n+1})dy \ge 0\end{aligned} \tag{4.17}$$

for all values $(q_1,q_2) \in K_{\rho_0}$ and all nonnegative C^∞-function ϕ with compact support in $\mathbb{R} \times [0,T]$.

On the other hand, from the step (iii) above we observe that for any $(q_1,q_2) \in K_{\rho_0}$, $i = 1,2$

$$\begin{aligned}|\overline{R}_i(y,t_n^+)-q_i| &= \big((\overline{R}_i(y,t_n)-q_i)+\kappa[M_i(\bar{u}(y,t_n^+))-\overline{R}_i(y,t_n^+)]\big)\,\mathrm{sgn}\big(\overline{R}_i(y,t_n^+)-q_i\big)\\ &\le |\overline{R}_i(y,t_n)-q_i|+\kappa\big[M_i(\bar{u}(y,t_n^+))-\overline{R}_i(y,t_n^+)\big]\mathrm{sgn}\big(\overline{R}_i(y,t_n^+)-q_i\big).\end{aligned} \tag{4.18}$$

Inserting (4.18) into (4.17), summing over n from $n = 0$ to $N-1$ and using $\overline{R}_i(y,t_0^+) =$

$\overline{R}_i(y,t_0^-) = \overline{R}_i(y,t_0)$, we obtain the following entropy inequality for the discrete solution

$$\begin{aligned}
&\int_0^T \int \sum_{i=1}^2 |\overline{R}_i(y,\tau) - q_i|[\phi_\tau + \lambda_i \phi_y] dy d\tau \\
&+ \sum_{n=0}^{N-1} \int \sum_{i=1}^2 \Big[|\overline{R}_i(y,t_n) - q_i| - |\overline{R}_i(y,t_n^-) - q_i| \Big] \phi(y,t_n) dy \\
&+ \int \sum_{i=1}^2 |\overline{R}_i(y,0) - q_i| \phi(y,0) dy - \int \sum_{i=1}^2 |\overline{R}_i(y,t_N^-) - q_i| \phi(y,T) dy \\
\geq & -\kappa \sum_{n=1}^{N} \int \sum_{i=1}^2 \mathrm{sgn}(\overline{R}_i(y,t_n^+) - q_i) \big[M_i(\bar{u}(y,t_n^+)) - \overline{R}_i(y,t_n^+) \big] \phi(y,t_n) dy. \qquad (4.19)
\end{aligned}$$

Inspired by the papers [30] and [25] of Tveito and others, we use the following Kružkov-type inequality for our problem since the exact solution $R_i(x,t)$ is the unique weak solution of (1.8),

$$\begin{aligned}
&\int_0^T \int \sum_{i=1}^2 |R_i(x,t) - q_i|[\psi_t + \lambda_i \psi_x] dx dt \\
&+ \int \sum_{i=1}^2 |R_i(x,0) - q_i| \psi(x,0) dx - \int \sum_{i=1}^2 |R_i(x,T) - q_i| \psi(x,T) dx \\
\geq & -\frac{1}{\epsilon} \int_0^T \int \sum_{i=1}^2 \mathrm{sgn}(R_i(x,t) - q_i) \big[M_i(u(x,t)) - R_i(x,t) \big] \psi(x,t) dx dt \qquad (4.20)
\end{aligned}$$

for any constants $(q_1, q_2) \in \Omega_{\rho_0}$ and all $\psi \in C_0^\infty$ with $\psi \geq 0$. As in Kružkov [11] any weak solution to (1.8) satisfies (4.20).

Equipped with the above variational inequalities, we return to estimate the error bound of $||R_i(\cdot,T) - \overline{R}_i(\cdot,t_N^+)||_1$.

Proof of Theorem 2.3. Our proof in this section is inspired by the one given by Schroll et al. [25]. They show an L^1-error bound for a model arising in chromatography. Their argument is in the spirit of the work by Kružkov[11], Kuznetsov[15] and Lucier[18].

To obtain the desired error bound we need to combine the inequality (4.19) with (4.20). To this end we define a mollifier function

$$\eta_\delta(x) = \frac{1}{\delta} \eta\left(\frac{x}{\delta}\right)$$

where η is any nonnegative smooth function with support in $[-1,1]$, even, i.e. $\eta(x) = \eta(-x)$, and unit mass. This mollifier therefore satisfies

$$\int \eta_\delta(x) dx = 1, \quad \int |x| \eta_\delta(x) dx \leq \delta, \quad \int |\eta'_\delta(x)| dx \leq \frac{1}{\delta}$$

and $\mathrm{supp}(\eta_\delta) \subseteq [-\delta, \delta]$. Let $T > 0$ be given. We proceed by selecting the constants (q_1, q_2) and the test functions ϕ, ψ. First taking $q_i = R_i(x,t)$, $\phi(y,\tau) = \eta_\delta(x-y)\eta_\delta(t-\tau)$ in (4.19) and integrating in x and t, we obtain using $\overline{R}_i := \overline{R}_i(y,\tau), \quad R_i := R_i(x,t)$

$$
\begin{aligned}
&-\int_0^T \int \int_0^T \int \sum_{i=1}^{2} |\overline{R}_i - R_i| \Big[\eta_\delta(x-y)\eta_\delta'(t-\tau) + \lambda_i \eta_\delta'(x-y)\eta_\delta(t-\tau) \Big] dy d\tau dx dt \\
&+ \int_0^T \int \sum_{n=0}^{N-1} \int \sum_{i=1}^{2} \left[|\overline{R}_i(y,t_n) - R_i| - |\overline{R}_i(y,t_n^-) - R_i| \right] \eta_\delta(x-y)\eta_\delta(t-t_n) dy dx dt \\
&+ \int_0^T \int \int \left[\sum_{i=1}^{2} |\overline{R}_i(y,0) - R_i| \eta_\delta(t) - \sum_{i=1}^{2} |\overline{R}_i(y,t_N^-) - R_i| \eta_\delta(t-T) \right] \eta_\delta(x-y) dy dx dt \\
&\geq -\frac{\kappa}{\Delta t} \int_0^T \int \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int \sum_{i=1}^{2} \mathrm{sgn}\left(\overline{R}_i(y,t_n^+) - R_i \right) \left[M_i(\bar{u}(y,t_n^+)) - \overline{R}_i(y,t_n^+) \right] \\
&\qquad \cdot \eta_\delta(x-y)\eta_\delta(t-t_n) dy d\tau dx dt.
\end{aligned} \tag{4.21}
$$

In the inequality (4.20) we set $q_i = \overline{R}_i(y,t_n^+)$ and $\psi(x,t) := \eta_\delta(x-y)\eta_\delta(t-\tau)$, integrate over $(y,\tau) \in \mathrm{IR} \times]t_{n-1}, t_n[$, and sum n from 1 to N

$$
\begin{aligned}
&\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int \int_0^T \int \sum_{i=1}^{2} |R_i - \overline{R}_i(y,t_n^+)| \Big[\eta_\delta(x-y)\eta_\delta'(t-\tau) \\
&\qquad\qquad + \lambda_i \eta_\delta'(x-y)\eta_\delta(t-\tau) \Big] dx dt dy d\tau \\
&+ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int \int \sum_{i=1}^{2} \left| R_i(x,0) - \overline{R}_i(y,t_n^+) \right| \eta_\delta(x-y)\eta_\delta(-\tau) dx dy dt \\
&- \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int \int \sum_{i=1}^{2} \left| R_i(x,T) - \overline{R}_i(y,t_n^+) \right| \eta_\delta(x-y)\eta_\delta(T-\tau) dx dy d\tau \\
&\geq -\frac{1}{\epsilon} \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int \int_0^T \int \sum_{i=1}^{2} \mathrm{sgn}\big(R_(x,t) - \overline{R}_i(y,t_n^+) \big) \left[M_i(u(x,t)) - R_i(x,t) \right] \\
&\qquad \cdot \eta_\delta(x-y)\eta_\delta(t-\tau) dx dt dy d\tau.
\end{aligned}
$$

Adding this inequality to (4.21) and suitably grouping the terms we obtain an inequality that we write in the short hand form

$$
L(\delta) \leq \sum_{i=1}^{4} E_i(\delta). \tag{4.22}
$$

The individual expressions are

$$
\begin{aligned}
L(\delta) :=& \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int\int \sum_{i=1}^{2} |R_i(x,T) - \overline{R}_i(y,t_n^+)| \eta_\delta(x-y)\eta_\delta(T-\tau)dxdyd\tau \\
&+ \int_0^T \int\int \sum_{i=1}^{2} |\overline{R}_i(y,t_N^-) - R_i(x,t)|\eta_\delta(x-y)\eta_\delta(t-T)dydxdt, \\
E_1(\delta) :=& \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int\int \sum_{i=1}^{2} |R_i(x,0) - \overline{R}_i(y,t_n^+)| \eta_\delta(x-y)\eta_\delta(\tau)dxdyd\tau \\
&+ \int_0^T \int\int \sum_{i=1}^{2} |\overline{R}_i(y,0) - R_i(x,t)|\eta_\delta(x-y)\eta_\delta(t)dxdydt, \\
E_2(\delta) :=& \int_0^T \int \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int \left(\sum_{i=1}^{2} \Big[|R_i - \overline{R}_i(y,t_n^+)| - |\overline{R}_i - R_i| \Big] \eta_\delta(x-y)\eta_\delta'(t-\tau) \right. \\
&\left. + \sum_{i=1}^{2} \lambda_i \big[|R_i - \overline{R}_i(y,t_n^+)| - |\overline{R}_i - R_i| \big] \eta_\delta'(x-y)\eta_\delta(t-\tau) \right) dyd\tau dxdt, \\
E_3(\delta) :=& \frac{1}{\epsilon} \int_0^T \int \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int \sum_{i=1}^{2} \operatorname{sgn}(R_i - \overline{R}_i(y,t_n^+)) \Big[(M_i(u) - R_i)\eta_\delta(t-\tau) \\
&\qquad - (M_i(\bar{u}(y,t_n^+)) - \overline{R}_i(y,t_n^+))\eta_\delta(t-t_n) \Big] \eta_\delta(x-y)dyd\tau dxdt, \\
E_4(\delta) :=& \int_0^T \int \sum_{n=0}^{N-1} \int \sum_{i=1}^{2} \Big[|\overline{R}_i(y,t_n) - R_i| - |\overline{R}_i(y,t_n^-) - R_i| \Big] \eta_\delta(x-y)\eta_\delta(t-t_n)dydxdt.
\end{aligned}
$$

Here $\overline{R}_i = \overline{R}_i(y,\tau)$, $\quad R_i = R_i(x,t)$. For these expressions we have the following bounds and postpone their proof for the moment.

Lemma 4.3. *For any $T > 0$, there exists a positive constant C, independent of step sizes, relaxation time ϵ, and δ such that*

(*i*) $|L(\delta) - \sum_{i=1}^{2} \|R_i(\cdot,T) - \overline{R}_i(\cdot,t_N^+)\|_1| \le C(\delta + \Delta t)$,
(*ii*) $|E_1(\delta) - \sum_{i=1}^{2} \|R_i(\cdot,0) - \overline{R}_i(\cdot,0)\|_1| \le C(\delta + \Delta t)$,
(*iii*) $|E_2(\delta)| \le C\frac{\Delta t}{\delta}$,
(*iv*) $E_3(\delta) \le C\frac{\Delta t}{\delta}$,
(*v*) $|E_4(\delta)| \le C\frac{\Delta t}{\delta}$.

Equipped with these above estimates we continue the proof of the Theorem 2.3. Using

$L(\delta) \geq 0$, (4.22) and Lemma 4.3 we find for a suitably large constant $C > 0$

$$\begin{aligned}
\sum_{i=1}^{2} \|R_i(\cdot, T) - \overline{R}_i(\cdot, t_N^+)\|_1 &\leq |L(\delta) - \sum_{i=1}^{2} \|R_i(\cdot, T) - \overline{R}_i(\cdot, t_N^+)\|_1| + L(\delta) \\
&\leq C(\delta + \Delta t) + \sum_{i=1}^{4} E_i(\delta) \\
&\leq \sum_{i=1}^{2} \|R_i(\cdot, 0) - \overline{R}_i(\cdot, 0)\|_1 + C(\delta + \Delta t + \frac{\Delta t}{\delta}). \qquad (4.23)
\end{aligned}$$

Using an obvious bound for the initial error like (2.11), we have

$$\sum_{i=1}^{2} \|R_i(\cdot, 0) - \overline{R}_i(\cdot, 0)\|_1 \leq \sum_{i=1}^{2} TV(R_i(\cdot, 0))\Delta x \leq C\Delta x \qquad (4.24)$$

Inserting this into (4.23) and picking $\delta = \sqrt{\Delta t}$ to minimize the right hand side of the relation (4.23), we have

$$\sum_{i=1}^{2} \|R_i(\cdot, T) - \overline{R}_i(\cdot, t_N^+)\| \leq C(\Delta x + \Delta t + \sqrt{\Delta t}).$$

Using the CFL condition $\Delta t \leq \Delta x / \sqrt{a}$ we have

$$\sum_{i=1}^{2} \|R_i(\cdot, T) - \overline{R}_i(\cdot, t_N^+)\| \leq C\sqrt{\Delta x}. \qquad (4.25)$$

Returning to the original variables (u, v) and noting the fact that we have

$$u_\Delta^N = \sum_{i=1}^{2} R_{i,\Delta}^N = \sum_{i=1}^{2} \overline{R}_i(y, t_N^+),$$

we obtain

$$\|u(\cdot, T) - u_\Delta^N\| + \|v(\cdot, T) - v_\Delta^N\| \leq (1 + \sqrt{a}) \sum_{i=1}^{2} \|R_i(\cdot, T) - \overline{R}_i(\cdot, t_N^+)\| \leq C\sqrt{\Delta x}.$$

This completes the proof of Theorem 2.3. □

Finally, we return the proof of Lemma 4.3 in order to conclude this paper.

Proof of Lemma 4.3. The proof of (i)-(iii) and (v) can be done by an analogous analysis as in the proof for the chromatography model given by Schroll et al.[25]. We next estimate the term $E_3(\delta)$. Let us analogously as above use the notations

$$Z_i(x, t) := M_i(u(x, t)) - R_i(x, t), \qquad \overline{Z}_i(y, \tau) := M_i(\bar{u}(y, \tau)) - \overline{R}_i(y, \tau),$$

then $E_3(\delta)$ can be rewritten as

$$
\begin{aligned}
E_3(\delta) =& \frac{1}{\epsilon}\int_0^T \int \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \int \sum_{i=1}^2 \mathrm{sgn}(R_i(x,t) - \overline{R}_i(y,t_n^+))\Big[Z_i(x,t) - \overline{Z}_i(y,t_n^+)\Big] \\
& \qquad\qquad \cdot \eta_\delta(t-t_n)\eta_\delta(x-y)dyd\tau dxdt \\
&+ \frac{1}{\epsilon}\int_0^T \int \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \int \sum_{i=1}^2 \mathrm{sgn}(R_i(x,t) - \overline{R}_i(y,t_n^+)) Z_i(x,t) \\
& \qquad\qquad \cdot [\eta_\delta(t-\tau) - \eta_\delta(t-t_n)]\,\eta_\delta(x-y)dyd\tau dxdt.
\end{aligned} \tag{4.26}
$$

The first term in $E_3(\delta)$ is nonpositive due to the relations (3.3) and (3.6)

$$
\begin{aligned}
\sum_{i=1}^2 \mathrm{sgn}(R_i(x,t) - \overline{R}_i)[Z_i - \bar{Z}_i] \le& \sum_{i=1}^2 |M_i(u) - M_i(\bar{u})| - \sum_{i=1}^2 |R_i - \overline{R}_i| \\
=& |u-\bar{u}| - \sum_{i=1}^2 |R_i - \overline{R}_i| \le 0.
\end{aligned}
$$

Therefore

$$
E_3(\delta) \le \frac{1}{\epsilon}\int_0^T \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \sum_{i=1}^2 \|Z_i(\cdot,t)\|_1 |\eta_\delta(t-\tau) - \eta_\delta(t-t_n)| d\tau dt.
$$

Due to Lemma 3.2 the sum $\sum_{i=1}^2 \|Z_i(\cdot,t)\|_1$ is bounded by $C[\omega(\epsilon)e^{-\frac{t}{\epsilon}} + \epsilon(1-e^{-\frac{t}{\epsilon}}]$. The integral of this sum over $[0,T]$ is clearly bounded by $C\epsilon$. Noting that

$$
\eta_\delta(t-\tau) - \eta_\delta(t-t_n) = \int_{t-t_n}^{t-\tau} \eta_\delta'(s)ds,
$$

we find the estimates

$$
E_3(\delta) \le C\frac{\Delta t}{\epsilon}\int_0^T \int_0^T \sum_{i=1}^2 \|Z_i(\cdot,t)\|_1 |\eta_\delta'(s-t)| dsdt \le C\frac{\Delta t}{\delta},
$$

which completes the proof. □

Remark 2. The above proof uses the exponential rate $e^{-\frac{t}{\epsilon}}$ without resorting to the restriction $\omega(\epsilon) = \epsilon$ on the initial error. To see this, note that from

$$
\sum_{i=1}^2 \|Z_i(\cdot,t)\|_1 \le C[\omega(\epsilon)e^{-\frac{t}{\epsilon}} + \epsilon(1-e^{-\frac{t}{\epsilon}}],
$$

we can easily deduce the bound

$$
\frac{1}{\epsilon}\int_0^T \sum_{i=1}^2 \|Z_i(\cdot,t)\|_1 d\tau \le C.
$$

Here, we do not use any specific choice of $\omega(\epsilon)$. We mention that the authors in [25] obtained the same error bound in the discretization step by assuming that $\omega(\epsilon) = \epsilon$. □

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