Granular avalanche flow down a smoothly varying slope: the existence of entropy solutions

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Abstract

We consider a model developed by Savage and Hutter [SH91] which describes the flow of granular avalanches down a smoothly varying slope. The system of two nonstrictly hyperbolic conservation equations for height and momentum of the avalanche resembles the shallow water equations.

The existence of entropy solutions to this model is proved using the vanishing viscosity method, where we make extensive use of a generalised version of the invariant region theorem in order to prove *a priori* estimates. Existence can generally only be guaranteed for a finite time depending on the initial data and the geometry of the slope, though in some cases we can say more.

A feature of the model is that it has a discontinuous source term. In order to prove the existence of the viscosity limit, we must also use a monotonicity argument to deal with this term.

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1 Introduction

We shall investigate the existence of entropy solutions to a one-dimensional continuum model for the flow of a granular avalanche down a smoothly varying slope. In [SH89], Savage and Hutter derived a model for a one-dimensional avalanche flowing down a rough incline. The same authors then improved this model in [SH91] to allow a smoothly varying slope. Other works ([GKH94],[KGH94], [GT97],[TG97]) describe the derivation of some other related models, as well as some experimental and numerical results.

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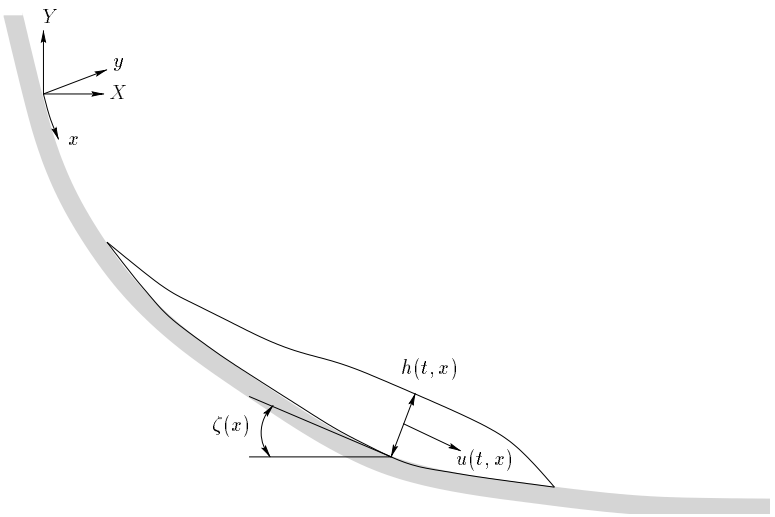

Figure 1: A finite mass granular avalanche with depth-averaged velocity $u(t,x)$ and perpendicular height $y = h(t,x)$ above a curved base with variable slope angle $\zeta(x)$.

The system we shall consider forms a 2×2 nonstrictly hyperbolic system of equations which describe the conservation of mass and the conservation of momentum. The system is related to other hyperbolic systems such as the shallow water equations, traffic flow equations, and the system of isentropic gas dynamics. The difficulty here lies in the position dependence of the pressure coefficient as well as in the form of the source term. We will prove the existence of weak entropy solutions to the system and derive a lower bound on the existence time which depends upon the initial data and the geometry of the slope.

We now present the Savage-Hutter model for granular avalanche flow. More details regarding the derivation of the model can be found in [SH91]. Denoting by (X, Y) the usual Cartesian coordinates, we define a new system of orthogonal curvilinear coordinates (x, y) such that y is constant on lines parallel to the slope ($y = 0$) and x is constant on lines perpendicular to the slope. At a point on the slope $(x, 0)$, the angle to the horizontal is denoted by $\zeta(x)$. Given the initial height $y = h_0(x)$ and (height-averaged) velocity $u_0(x)$ of the avalanche we wish to determine the height $h(t,x)$, and velocity, $u(t,x)$ at later times (see Figure 1).

Using a Mohr-Coulomb yield criterion to model the pressure as well as scalings similar to the shallow water scalings (see [SH91]), Savage and Hut-

ter obtain a nonstrictly hyperbolic system of two equations for h and hu corresponding to the conservation of mass and momentum respectively.

$$\begin{aligned} h_t + (hu)_x &= 0 \\ (hu)_t + (hu^2 + \tfrac{1}{2}\beta h^2)_x &= h\, g(x,u), \end{aligned} \tag{1.1}$$

with initial conditions:

$$\begin{aligned} h(0,x) &= h_0(x) \geq 0 \\ u(0,x) &= u_0(x), \end{aligned} \tag{1.2}$$

where the function β is strictly positive and takes the form

$$\beta(x,u_x) = \epsilon k_{\text{act,pass}}(u_x)\cos\zeta(x),$$

and $k_{\text{act,pass}}(u_x)$ is the earth pressure coefficent.

The dependence on u_x only exhibits itself as a switching between two positive states depending on whether u_x is positive or negative. In this paper we shall concentrate on the simpler system obtained by supposing that these two states are equal and thus assuming that $\beta = \beta(x)$. Physically, this corresponds to the situation where the base is composed of the same material as the avalanche flowing over it. This assumption holds for flows which are initiated in a pile of granular material, and is also true for snow avalanches. We shall make the further assumption that that the slope is smoothly varying and bounded, by demanding that β is differentiable and that $\beta'/\beta \in L^\infty(\mathbb{R})$.

In the Savage-Hutter model, the source term g takes the form

$$g(x,u) = \sin\zeta(x) - \frac{u}{|u|}\tan\delta_F(x)\cos\zeta(x), \quad u \neq 0. \tag{1.3}$$

where $\delta_F(x)$ is the dynamic friction angle at the position x. The model is only valid in regions where the velocity is almost everywhere different from zero. In particular, two important features of avalanche flows, namely, the initiation mechanism and the configuration of the final rest state, cannot be determined using this model, though numerical results indicate that it is an excellent model for the dynamics. In this work we shall extend the function g by defining

$$g(x,0) := \sin\zeta(x), \tag{1.4}$$

though what one might actually expect in a physical flow is that $g(x,0)$ be a multivalued function: $g(x,0) \in [g(x,0^+), g(x,0^-)]$, with the additional

restriction that the height should be a Lipschitz function in the rest state. Some interesting articles by Evans *et al* [AEW96, EFG97] discuss the growth of sandpiles from point sources and the collapse of sandpiles in an initially unstable configuration. While in this article we can describe the dynamics without being able to properly model the rest state, these articles describe the rest state but ignore the dynamics.

It is helpful to transform the system so that the x-dependence in the flux term is transferred to the source term. To this aim we introduce the new unknowns, $\rho = \beta h$, $m = \beta h u$ and define the function $\gamma = \beta'/\beta$. We then derive the following system:

$$v_t + \nabla F(v) v_x = G(v) \quad \text{in } Q_T, \tag{1.5}$$

$$v(0,x) = v_0(x) \tag{1.6}$$

where $v = (\rho, m)$, $v_0 = (\beta h_0, \beta h_0 u_0) =: (\rho_0, m_0)$, $F(v) = (m, m^2/\rho + \frac{1}{2}\rho^2)$ and $G(v,x) = (\gamma m, \gamma\,(m^2/\rho + \frac{1}{2}\rho^2) + \rho\, g(x, m/\rho))$.

This is a nonstrictly hyperbolic system with a parabolic degeneracy at $\rho = 0$. In general, we expect shocks to develop in finite time regardless of how smooth we choose the initial data. To allow nonsmooth solutions we shall seek weak solutions of (1.5)-(1.6) which also satisy an entropy condition.

DEFINITION 1.1
Suppose that $\eta = \eta(v)$, $q = q(v)$ are scalar C^1 functions satisfying,

$$\nabla\eta\,\nabla F = \nabla q.$$

Such functions η and q are called Entropy-Flux Pairs. *If $\eta(\rho{=}0){=}0$ then (η, q) is a* Weak Entropy-Flux Pair. *If η is convex then (η, q) is a* Convex Entropy-Flux Pair.

DEFINITION 1.2
We call $v \in L^\infty_+(Q_T) \times L^\infty(Q_T)$ a Weak Entropy Solution *to the system* (1.5) *with initial conditions* (1.6) *if it is a weak solution of* (1.5),(1.6), *i.e.:*

$$\iint_{t>0} (\, v \cdot \psi_t + F(v) \cdot \psi_x \, + G \cdot \psi \,)\; dx\, dt = \int_{t=0} v_0 \cdot \psi \, dx$$

for all test functions $\psi \in (C_0^1([0,T) \times \mathbb{R}))^2$, and the entropy inequality

$$\iint_{t>0} (\, \eta\phi_t + q\phi_x \, + \nabla\eta\, G\, \phi \,)\; dx\, dt \geq \int_{t=0} \eta\phi \, dx$$

holds for all test functions $\phi \in C_0^1([0,T) \times \mathbb{R})$ and all convex weak entropy flux pairs (η, q) such that either $\eta_m \geq 0$, or $\eta_m \rho m/|m|$ is a locally Lipschitz function of ρ and m.

The condition on η_m above is nonstandard and arises because of the form of the source term g. As we shall see, we can only guarantee that the entropy inequality is satisfied for this family of entropies. This is nevertheless a large class.

In this paper we shall prove the following result:

THEOREM 1.3
Let $\beta : \mathbb{R} \to \mathbb{R}_+$ be a strictly positive differentiable function with $\gamma := \beta'/\beta \in L^\infty(\mathbb{R})$. Let ρ_0, $m_0 \in L^\infty(\mathbb{R})$ with $\rho_0 \geq 0$ and ρ_0, $m_0 \to 0$ as $|x| \to \infty$.Let $g : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto g(x, y)$ be the function defined by (1.3), (1.4). Let $P = \frac{5}{8}||\gamma||_\infty$, $Q = ||g||_\infty$ and $M_0 = \max(2\|\sqrt{\rho_0}\|_\infty, \|m_0/\rho_0\|_\infty)$.

1. *If $\gamma = 0$* (i.e. constant slope) *then the hyperbolic system* (1.5) *has a weak entropy solution in the sense of Definition 1.2 for all times $t < \infty$. The transformed height ρ grows at most quadratically as $t \to \infty$ and the velocity $u = m/\rho$ grows at most linearly as $t \to \infty$. If in addition the slope angle is less than the dynamic friction angle,* i.e. *$|\zeta| \leq \delta_F$, then ρ and u are uniformly bounded, independent of time. This bound depends only on the quantity M_0.*

2. *If $\gamma \neq 0$* (i.e. curved slope) *then there exists a time T_{max} such that* (1.5) *has a weak entropy solution for all $t < T_{max}$. If $|\zeta| \leq \delta_F$, then it is sufficient to take $T_{max} = 1/M_0 P$; otherwise it is sufficient to take*

$$T_{max} = \frac{1}{PQ}\left(\frac{\pi}{2} - \tan^{-1} M_0\sqrt{P/Q}\right).$$

3. *If $\gamma \leq 0$, $m_0/\rho_0 \geq 2\sqrt{\rho_0}$ and $\zeta \geq \delta_F$* (i.e. constant or steepening slope with sufficiently large downslope initial velocity and slope angle greater or equal to the dynamic friction angle) *then* (1.5) *has a weak entropy solution for all times $t < \infty$. The transformed height ρ grows at most quadratically as $t \to \infty$ and the velocity $u = m/\rho$ grows at most linearly as $t \to \infty$. Furthermore, the velocity remains in the down slope direction for all time and satisfies $m/\rho \geq 2\sqrt{\rho}$.*

We remark that there is a one-to-one correspondence between weak entropy solutions of (1.5)-(1.6) and weak entropy solutions of (1.1)-(1.2) obtained via the transformations $\rho = \beta h$, $m = \beta h u$ and $\gamma = \beta'/\beta$ (the Rankine-Hugoniot conditions are the same for both systems). Also, if v is a C^1 weak solution of (1.5)-(1.6), then v is a classical solution of (1.5)-(1.6).

Our strategy for finding weak entropy solutions is the vanishing viscosity method and compensated compactness. First of all we add small second order

terms to the system and approximate the data in order to make it strictly parabolic. Standard local existence results for parabolic systems furnish us with a classical local solution of the approximate problem. Next we obtain various *a priori* estimates which allow us to extend this local solution to the existence times guaranteed by the theorem. For general slopes, the solution may grow too rapidly to allow us to prove global existence results.

Further estimates on the entropies of the system are then required in order that we may pass to the vanishing viscosity limit. Then we are able to apply the div-curl lemma and results of Lions, Perthame and Souganidis [LPS96] to show the strong convergence of the approximate solutions.

Finally we show that we obtain a weak solution in the limit, and that the entropy inequality is satisfied for all entropies in Definition 1.2. This does not follow directly since the source term $g(x,u)$ is discontinuous in u, though the result can be obtained using a monotonicity argument.

2 Approximation by a parabolic system

Let $\rho_0^\varepsilon, m_0^\varepsilon$ be smooth functions such that $\rho_0^\varepsilon \to \rho_0$ and $m_0^\varepsilon \to m_0$ pointwise a.e. as $\varepsilon \to 0$, and that $\rho_0^\varepsilon \geq \delta(\varepsilon) > 0$. We consider the parabolic system:

$$\begin{aligned} v_t^\varepsilon + \nabla F(v^\varepsilon) v_x^\varepsilon = G^\varepsilon(v^\varepsilon) + \varepsilon v_{xx}^\varepsilon \quad \text{in } Q_T, \\ v^\varepsilon(0,x) = (\rho_0^\varepsilon(x), m_0^\varepsilon(x)),\ x \in \mathbb{R} \\ \rho^\varepsilon \to \delta,\ m^\varepsilon \to 0 \text{ as } |x| \to \infty \end{aligned} \tag{2.1}$$

where $v^\varepsilon = (\rho^\varepsilon, m^\varepsilon)$ and $G^\varepsilon(v,x) = \chi_{(-1/\varepsilon, 1/\varepsilon)}(x) G(v,x)$.

In order to prove existence results for the above problem, we first consider a finite domain problem for each fixed $\varepsilon > 0$. We approximate ρ^ε, m^ε by smooth functions $\rho_0^{\varepsilon,L} \geq \delta(\varepsilon)$ and $m_0^{\varepsilon,L}$, such that $\operatorname{supp}(\rho_0^{\varepsilon,L} - \delta), \operatorname{supp}(m_0^{\varepsilon,L}) \subset \Omega_L := (-L, L)$. We suppose $\rho_0^{\varepsilon,L} \to \rho_0^\varepsilon$ and $m_0^{\varepsilon,L} \to m_0^\varepsilon$ in $\mathbb{R}$, pointwise almost everywhere as $L \to \infty$. We study the following parabolic system:

$$\begin{aligned} v_t^{\varepsilon,L} + \nabla F(v^{\varepsilon,L}) v_x^{\varepsilon,L} = G^\varepsilon(v^{\varepsilon,L}) + \varepsilon v_{xx}^{\varepsilon,L} \quad \text{in } (0,T) \times \Omega_L, \\ v^{\varepsilon,L}(0,x) = (\rho_0^{\varepsilon,L}(x), m_0^{\varepsilon,L}(x)),\ x \in \Omega_L \\ \rho^{\varepsilon,L}(t, \pm L) = \delta(\varepsilon),\ m^{\varepsilon,L}(t, \pm L) = 0,\ t \in (0,t). \end{aligned} \tag{2.2}$$

The function g^ε is a Lipschitz approximation of g such that $g^\varepsilon \to g$ pointwise a.e. as $\varepsilon \to 0$. Furthermore, we assume that for each $\varepsilon > 0$, g^ε has bounded support (as a function of x). More specifically, we define

$$g^\varepsilon(x, m^\varepsilon/\rho^\varepsilon) = \chi_\varepsilon(x) \big(\sin\zeta - \operatorname{sgn}_\varepsilon(m^\varepsilon/\rho^\varepsilon) \tan\delta_F \cos\zeta \big), \tag{2.3}$$

where χ_ε is the characteristic function of the set $(-1/\varepsilon, 1/\varepsilon)$ and $\mathrm{sgn}_\varepsilon(u)$ is the Lipschitz approximation of $u/|u|$ defined by:

$$\mathrm{sgn}_\varepsilon(u) = \begin{cases} u/\varepsilon, & \text{if } u \in (-\varepsilon, \varepsilon), \\ u/|u|, & \text{otherwise,} \end{cases} \tag{2.4}$$

In the following section we shall show that a classical solution to this problem will exist for some small time. The existence time can be extended to the maximum interval in which $||\rho^{\varepsilon,L}||_\infty$, $||m^{\varepsilon,L}||_\infty$ and $||1/\rho^{\varepsilon,L}||_\infty$ remain bounded.

3 Local existence for the parabolic system (2.2)

We now prove that the parabolic system (2.2) has a unique classical solution for some small time interval. This is essentially standard.

We write (2.2) in the form

$$\frac{dw(t)}{dt} = Aw(t) + B(w(t))$$

where $w(t) = (\rho^{\varepsilon,L}(t) - \delta, m^{\varepsilon,L}(t))$, $A = -\varepsilon\Delta$, $\mathcal{D}(A) = H^2(\Omega_L) \cap H_0^1(\Omega_L)$ and $B(w) = G^\varepsilon(v^{\varepsilon,L}) - \nabla F(v^{\varepsilon,L}) v_x^{\varepsilon,L}$. The operator A with the domain $\mathcal{D}(A)$ generates an analytic semigroup and we can define its fractional powers. For $\alpha \in [0,1]$, we define X_α to be the Banach space $\mathcal{D}(A^\alpha)$ equipped with the graph norm. We define the open set

$$U = \{w = (\rho^{\varepsilon,L} - \delta, m^{\varepsilon,L}) \in \mathcal{D}(A^\alpha) \cap L^\infty(\Omega_L); \rho^{\varepsilon,L} > 0\}.$$

Let $w_1, w_2 \in U_\alpha$, $w_i = (\rho_i^{\varepsilon,L} - \delta, m_i^{\varepsilon,L})$, $i = 1, 2$. If we suppose that $\alpha \in (\frac{1}{2}, 1]$ then we have the continuous embeddings $X_\alpha \hookrightarrow H^1(\Omega_L)$ and $X_\alpha \hookrightarrow L^\infty(\Omega_L)$. Using these embeddings, one can show that B satisfies a local Lipschitz condition of the form

$$||B(w_1) - B(w_2)||_{L^2(\Omega\varepsilon)} \leq C||A^\alpha(w_1 - w_2)||_{L^2(\Omega_L)}.$$

where

$$C = C(||\rho_i^{\varepsilon,L}||_{L^\infty(\Omega_L)}, ||m_i^{\varepsilon,L}||_{L^\infty(\Omega_L)}, ||1/\rho_i^{\varepsilon,L}||_{L^\infty(\Omega_L)}).$$

Hence for data in U and some small enough $t_0 > 0$, there exists a unique strong local solution of (2.2) (e.g. [Paz83]),

$$(\rho^{\varepsilon,L}, m^{\varepsilon,L}) \in C^0([0, t_0), L^2(\Omega_L)) \cap C^1((0, t_0), L^2(\Omega_L)).$$

Regularity arguments allow us to conlude *a posteriori* that $(\rho^{\varepsilon,L}, m^{\varepsilon,L})$ is in fact a classical solution of (2.2).

To extend this local solution to a global solution, we need to ensure that solutions cannot leave the set U (*n.b.* the size of the interval of existence does not depend on the initial data). This means that we require estimates for the quantities $\rho^{\varepsilon,L}$, $m^{\varepsilon,L}$ and $1/\rho^{\varepsilon,L}$ in $L^\infty(\Omega_L)$. In order to be able to pass to the limit as $L \to \infty$ these estimates should be independent of L. Furthermore, to pass to the limit as $\varepsilon \to 0$, the estimates for $\rho^{\varepsilon,L}$ and $m^{\varepsilon,L}$ should be independent of both L and ε.

4 An *a priori* estimate for $\rho^{\varepsilon,L}$

Before we can prove that $\rho^{\varepsilon,L}$ is strictly positive for all $\varepsilon > 0$, we shall first need to prove a slightly weaker estimate. Namely, in this section we show that $\rho^{\varepsilon,L} \geq 0$. This result will be required in the next section, where we prove L^∞ bounds for $\rho^{\varepsilon,L}$ and $m^{\varepsilon,L}$.

Suppose that ρ is a classical solution of the equation

$$\begin{aligned} \rho_t + (\rho u)_x &= \gamma \rho u + \varepsilon \rho_{xx} \\ \rho(0,x) &= \rho_0(x) \geq 0, \end{aligned} \tag{4.1}$$

where u is a given function in $C^1((0,T)\times\Omega_L) \cap L^\infty((0,T)\times\Omega_L)$. Notice that if $(\rho^{\varepsilon,L}, m^{\varepsilon,L})$ is a classical solution of (2.2) and $m^{\varepsilon,L}/\rho^{\varepsilon,L}$ is bounded, then $\rho = \rho^{\varepsilon,L}$ satisfies the above conditions with with $u = m^{\varepsilon,L}/\rho^{\varepsilon,L}$.

Multiply the equation by $\phi = \min(0,\rho)$ and integrate over the region $(0,T)\times\Omega_L$ to get

$$\begin{aligned} &\tfrac{1}{2}\int_{\Omega_L} \phi(T)^2\,dx + \varepsilon \int_0^T\!\!\int_{\Omega_L} \phi_x^2\,dxdt \\ &= \int_0^T\!\!\int_{\Omega_L} u\phi\phi_x\,dxdt + \int_0^T\!\!\int_{\Omega_L} \gamma u\phi^2\,dxdt \\ &\leq ||u||_\infty \int_0^T\!\!\int_{\Omega_L} |\phi|\,|\phi_x|\,dxdt + ||\gamma u||_\infty \int_0^T\!\!\int_{\Omega_L} \phi^2\,dxdt \\ &\leq \varepsilon \int_0^T\!\!\int_{\Omega_L} \phi_x^2\,dxdt + \left(\frac{||u||_\infty^2}{4\varepsilon} + ||\gamma u||_\infty\right)\int_0^T\!\!\int_{\Omega_L} \phi^2\,dxdt \end{aligned}$$

Hence

$$\tfrac{1}{2}\int_{\Omega_L} \phi(T)^2\,dx \leq \left(\frac{||u||_\infty^2}{4\varepsilon} + ||\gamma u||_\infty\right)\int_0^T\!\!\int_{\Omega_L} \phi^2\,dxdt \tag{4.2}$$

and so $\phi = 0$ by Gronwall's lemma. In other words, $\rho \geq 0$.

5 An invariant region theorem

In this section we modify the well-known invariant region theorem of Chueh, Conley and Smoller [CCS77], and use it to prove L^∞-estimates for $\rho^{\varepsilon,L}$ and $m^{\varepsilon,L}$, at least for some finite time which can be explicitly calculated in terms of the initial data and the slope geometry.

DEFINITION 5.1
Let $R_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, k$, and $M_i : [0,T] \to \mathbb{R}$ be smooth functions. We define the region

$$\Sigma(t) = \bigcap_{i=1}^{k} \{v \in \mathbb{R}^n : R_i(v) \leq M_i(t)\}. \tag{5.1}$$

If $v : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^n$, we call Σ an invariant region for v *if $v(0,x) \in \Sigma(0)$ implies that $v(t,x) \in \Sigma(t)$ for $t > 0$.*

We now present the modified invariant region theorem (*c.f.* [CCS77] and [FI97] which are the cases $M_i(t) = M_0$ and $M_i(t) = M_0 e^{\omega t}$ respectively):

THEOREM 5.2
Let $v : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^n$ be a classical solution of the $n \times n$ system:

$$v_t + A(v,x)v_x = B(v,x,t) + \varepsilon C(v,x)v_{xx} \tag{5.2}$$

and Σ be defined by (5.1). Suppose that whenever (t_0, v_0) satisfies $R_i(v_0) = M_i(t_0)$ for some i, the following conditions hold:

1. *∇R_i at v_0 is a left eigenvector of $C(v_0,x)$ and $A(v_0,x)$ for all $x \in \mathbb{R}$.*
2. *If $\nabla R_i \, C(v_0,x) = \mu \nabla R_i$ and $\mu \neq 0$, then R_i is quasiconvex at v_0 (i.e. $\nabla^2 R_i(r,r) \geq 0$ whenever $\nabla R_i \, r = 0$).*
3. *$\nabla R_i \, B < M_i'(t_0)$ at $(t,v) = (t_0, v_0)$, for all $x \in \mathbb{R}$.*

Then Σ is an invariant region for v.

PROOF

The proof is *mutatis mutandis* the same as the proofs in [CCS77] and [FI97], though we include it for completeness.

Let us suppose that for some i, $R_i(v(t,x)) < M_i(t)$ for $0 \leq t < t_0$ and $R_i(v(t_0,x_0)) = M_i(t_0)$ for some $x_0 \in \mathbb{R}$. It suffices to show that

$$\frac{\partial}{\partial t} R_i(v(t_0,x_0)) < M_i'(t_0).$$

Now under the assumptions of the theorem, at $v_0 = v(t_0, x_0)$ we have

$$\nabla R_i \, A(v_0, x) v_x = \lambda(x) \nabla R_i \, v_x = \lambda(x)(R_i)_x = 0.$$

(To see this define $h(x) = R_i(v(t_0, x))$, so $h(x_0) = M_i(t_0)$. If $h'(x_0) > 0$ then by smoothness in x, $h(x) > M_i(t_0)$ for $|x - x_0| < \delta_0$, and δ_0 sufficiently small. Hence $R_i(v(t_0, x)) > M_i(t_0)$ for $|x - x_0| < \delta_0$, so by smoothness in t, $R_i(v(t, x)) > M_i(t)$ for $|x - x_0| < \delta_0$ and $|t - t_0| < \delta_1$, δ_1 sufficiently small. But this is a contradiction. Similarly if we assume $h'(x_0) < 0$. Hence $h'(x_0) = 0$.)

We also have:

$$\nabla R_i \, v_{xx} = (R_i)_{xx} - \nabla^2 R_i(v_x, v_x) \leq 0.$$

(To see this, with h as above, suppose that $h''(x_0) > 0$, then $h(x) > h(x_0) = M_i(t_0)$, for $|x - x_0| < \delta$, $x \neq x_0$ and δ sufficiently small, so we have a contradiction as before. Hence $h''(x_0) \leq 0$. This accounts for the first term above, and the second is nonpositive due to the quasiconvexity assumption.)

Hence

$$\begin{aligned}
\frac{\partial}{\partial t} R_i(v_0) &= \nabla R_i(v_0) \, v_t \\
&= \nabla R_i(v_0) \, (\varepsilon C v_{xx} - A v_x + B) \\
&= \varepsilon \mu \nabla R_i(v_0) \, v_{xx} - \lambda R_i(v_0) \, v_x + R_i(v_0) \, B \\
&\leq \nabla R_i(v_0) \, B(v_0, x_0, t_0) \\
&< M_i'(t_0).
\end{aligned}$$

□

We now apply this result to the system (2.2). For ease of notation we shall drop the ε, L superscripts except on the function g^ε. We choose the R_i to be the Riemann invariants of the system $R_\pm$ given by:

$$R_\pm = \pm \frac{m}{\rho} + 2\sqrt{\rho},$$

so that the first condition of the theorem is satisfied. It is also easy to verify that $R_\pm$ are quasiconvex. For the final condition of the theorem, we notice that in our case $B(\rho, m, x) = (\gamma m, \gamma m^2/\rho + \frac{1}{2}\gamma\rho^2 + \rho g^\varepsilon)$ for $x \in (-1/\varepsilon, 1/\varepsilon)$ and $B(\rho, m, x) = 0$ otherwise. Hence it suffices to only consider $x \in (-1/\varepsilon, 1/\varepsilon)$. In this case we have:

$$\nabla R_\pm \cdot B = \frac{\gamma m}{\sqrt{\rho}} \pm \tfrac{1}{2}\gamma\rho \pm g$$

Suppose that for $t \leq t_0$, $R_\pm \leq M_\pm(t)$, for some functions $M_\pm(t)$ which are to be chosen. Assume also that $\rho \geq 0$ for $t \leq t_0$. From these conditions we derive:

$$0 \leq 2\sqrt{\rho} \leq \tfrac{1}{2}(M_+(t) + M_-(t)) \tag{5.3}$$

$$-M_-(t) \leq \frac{m}{\rho} \leq M_+(t). \tag{5.4}$$

We shall now consider the three cases from Theorem 1.3 separately: (i) $|\zeta| \leq \delta_F$, (ii) $|\zeta| \not> \delta_F$ and (iii) $\gamma \leq 0$, $\zeta \geq \delta_F$ and $m_0/\rho_0 \geq 2\sqrt{\rho_0}$.

Case (i): $|\zeta| \leq \delta_F$

We define the quantity $\alpha = |\tan\zeta|/\tan\delta_F$. and choose $M_- = M_+ - \alpha\varepsilon$. Note that $0 \leq \alpha \leq 1$ by assumption.

Suppose that at time t_0, we have $R_+ = M_+(t_0)$ at some point x_0. This implies that $m/\rho = M_+ - 2\sqrt{\rho} \geq \alpha\varepsilon$ at (t_0, x_0). In particular, $\operatorname{sgn}_\varepsilon(m/\rho) \geq \alpha$ at this point and so,

$$\begin{aligned} g^\varepsilon &= \sin\zeta - \operatorname{sgn}_\varepsilon(m/\rho)\cos\zeta\tan\delta_F \\ &\leq \sin\zeta - \alpha\cos\zeta\tan\delta_F \\ &\leq 0. \end{aligned}$$

This yields

$$\nabla R_+ \cdot B \leq \frac{\gamma m}{\sqrt{\rho}} + \tfrac{1}{2}\gamma\rho.$$

Similarly, if $R_- = M_-(t_0)$ at some point x_0, we have $m/\rho = 2\sqrt{\rho} - M_- \leq \alpha\varepsilon$ at (t_0, x_0). In this case, $\operatorname{sgn}_\varepsilon(m/\rho) \leq \alpha$ and it follows that

$$\begin{aligned} -g^\varepsilon &= -\sin\zeta + \operatorname{sgn}_\varepsilon(m/\rho)\cos\zeta\tan\delta_F \\ &\leq -\sin\zeta + \alpha\cos\zeta\tan\delta_F) \\ &\leq 0. \end{aligned}$$

Hence at (t_0, x_0) we have

$$\nabla R_- \cdot B \leq \frac{\gamma m}{\sqrt{\rho}} - \tfrac{1}{2}\gamma\rho.$$

Using the conditions (5.3) and (5.4), we see that the third condition of the theorem can be satisfied if M_+ is a function which satisfies

$$\frac{dM_+}{dt} > \tfrac{5}{8}\|\gamma\|_\infty M_+^2.$$

In particular, for a flat slope $\gamma \equiv 0$, so here we can choose M_+ to be constant. In other words, both ρ and m/ρ are uniformly bounded, independent of time.

For a variable slope, however, we see that M_+ must blow up in finite time. In fact we can integrate to find $M_+(t) > M_0/(1 - M_0 P t)$ where $P = \frac{5}{8}\|\gamma\|_\infty$ and $M_0 = M_+(0)$. We thus see that any region we choose will blow up by $t = T_{\max} := 1/M_0 P$. Since we can choose $M_+(t) = M_0/(1 - M_0 \tilde{P} t)$ for any $\tilde{P} > P$, we see that $\|\rho^\varepsilon\|_\infty$ and $\|m\|_\infty$ will be bounded for any $t < T_{\max}$. Of course, the blow-up of the region does not necessarily imply the blow-up of ρ and m/ρ, but it means that we no longer have any control of these quantities at times greater than $T_{\max}$.

Case (ii): $|\zeta| \not\leq \delta_F$

If there are regions where $|\zeta| \geq \delta_F$ then we no longer have control over the sign of g^ε and we can only estimate it using $Q := \|g\|_\infty$. Physically this means that if the slope is too steep, then friction will no longer be enough to prevent the flow from continuing to accelerate. This time we choose $M_+(t) = M_-(t) = M(t)$, where $M(t)$ must be chosen so that it satisfies

$$\frac{dM}{dt} > PM^2 + Q.$$

In the flat slope case $\gamma = 0$, we can choose $M(t) = M_0 + \tilde{Q}t$, for any $\tilde{Q} > Q$. This time we have a bounded region for any bounded time, hence $\|\rho\|_\infty$ and $\|m\|_\infty$ will be bounded for any bounded time.

For a variable slope the region will again blow up in finite time. This time we integrate to get

$$M(t) > \sqrt{\frac{Q}{P}} \tan\left(C + t\sqrt{PQ}\right), \quad C = \tan^{-1}(M_0\sqrt{P/Q})$$

This time we have that $\|\rho\|_\infty$ and $\|m\|_\infty$ are bounded for any $t < T_{\max}$, where

$$T_{\max} := \frac{1}{PQ}\left(\frac{\pi}{2} - \tan^{-1} M_0\sqrt{P/Q}\right).$$

Case (iii): $\gamma \leq 0$, $\zeta \geq \delta_F$ **and** $m_0/\rho_0 \geq 2\sqrt{\rho_0}$

This case corresponds to the situation where the avalanche is initially moving in the downslope direction and the slope angle is steep enough that the flow will not be stopped by friction. If the initial velocity is high enough then the avalanche will continue to move in the downslope direction for all time.

Here we take $M_-(t) = 0$. Suppose that at time t_0, we have $R_+ = M_+(t_0)$ at some point x_0. This implies that $m/\rho = M_+ - 2\sqrt{\rho} \geq M_+/2$ at (t_0, x_0). When $M_+ > 2\varepsilon$ we have $\mathrm{sgn}_\varepsilon(m/\rho) = 1$ at this point. We thus have

$$\begin{aligned}
\nabla R_+ \cdot B &= \frac{\gamma m}{\sqrt{\rho}} + \tfrac{1}{2}\gamma\rho + \sin\zeta - \cos\zeta\tan\delta_F \\
&\leq \sin\zeta - \cos\zeta\tan\delta_F \\
&\leq C,
\end{aligned}$$

for some positive constant C.

If instead $R_- = 0$ at (t_0, x_0), then we have $m/\rho = 2\rho \geq 0$. Hence

$$\begin{aligned} \nabla R_- \cdot B &= \frac{\gamma m}{\sqrt{\rho}} + \tfrac{1}{2}\gamma\rho - \sin\zeta + \cos\zeta \tan\delta_F \\ &\leq \frac{\gamma m}{\sqrt{\rho}} + \tfrac{1}{2}\gamma\rho = \tfrac{3}{2}\gamma\rho \leq 0. \end{aligned}$$

Hence we can apply the invariant region theorem by choosing $M_+(t) = \tilde{C}t + M_0$ for any $\tilde{C} > C$. In particular, the condition $R_- \leq M_- = 0$ implies that the velocity is bounded from below for all later times according to $m/\rho \geq 2\sqrt{rho}$.

Summary: Notice that in all cases the invariant region is bounded independent of both L and ε. In the flat slope case we either have a global bound independent of time if the slope is less than the friction angle, and a bound which grows with time, but is finite for all finite times if there are regions in which the slope angle exceeds the friction angle. In the case of variable slope, we can only obtain a maximum time $T_{\max}$ for which solutions of the parabolic approximation are guaranteed to be bounded, or equivalently, $T_{\max}$ is a lower bound on the blow-up time. We can control $T_{\max}$ by the initial data (M_0) and the quantity P, which roughly corresponds to the curvature. $T_{\max}$ will be large if either M_0 or P is small. P is small when $||\gamma||_\infty$ is small. Since $\gamma(x) = \beta'(x)/\beta(x)$, and taking $\beta(x) = k\cos\zeta(x)$, this gives $\gamma(x) = -\zeta'(x)\tan\zeta(x)$. In other words, demanding that $||\gamma||_\infty$ be small is equivalent to demanding that the slope must vary slowly unless the slope angle is small. An example of a curve satisfying $||\gamma||_\infty = 0.001$ is shown in Figure 2.

In the paper [GT97] additional terms were added to the source term g to model the effects of drag. Clearly, in a physical situation the velocity cannot become unbounded even if the slope angle exceeds the friction angle. Such a term acts in the direction which opposes the direction, like the friction term, but increases in magnitude with the velocity. Since the drag has the right sign, by appropriately choosing the drag coefficient we can thus find a uniform bound, independent of time, in the flat slope case even when the slope angle is greater than the friction angle. In the variable slope case, such a term is still not sufficient to prevent the invariant region from becoming unbounded in finite time.

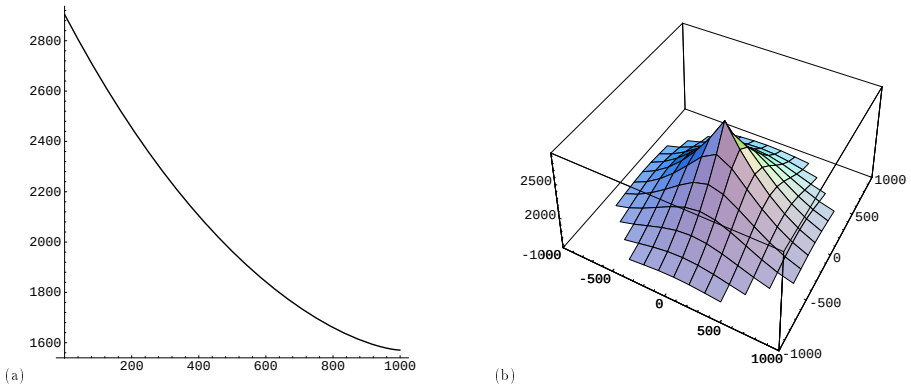

Figure 2: (a) The curve $f(x) = \delta^{-1} e^{1-\delta x}\sqrt{1-e^{2(\delta x-1)}} + \delta^{-1}\sin^{-1}\left(e^{-1+\delta x}\right)$, where $\delta = 0.001$. Setting $\zeta(x) = \tan^{-1} f'(x)$, we find that $\gamma(x) = -\zeta'(x)\tan\zeta(x) = \delta$.
(b) A mountain generated by rotating the curve in (a).

6 An L-independent H^1-estimate

We shall write $w^{\varepsilon,L} = (\rho^{\varepsilon,L} - \delta, m^{\varepsilon,L})$ and $v^{\varepsilon,L} = (\rho^{\varepsilon,L}, m^{\varepsilon,L})$. If $v^{\varepsilon,L}$ is a classical solution of (2.2), then $w^{\varepsilon,L}$ satisfies:

$$w_t^{\varepsilon,L} + (F(v^{\varepsilon,L}))_x = G^\varepsilon(x, v^{\varepsilon,L}) + \varepsilon w_{xx}^{\varepsilon,L}.$$

Multiplying on the left by $w^{\varepsilon,L}$ and integrating over Ω_L, we have,

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_L} |w^{\varepsilon,L}(t)|^2\,dx + \varepsilon\int_{\Omega_L} |w_x^{\varepsilon,L}(t)|^2\,dx \\ = \int_{\Omega_L} w^{\varepsilon,L}(t)\cdot G^\varepsilon(v^{\varepsilon,L}(t))\,dx + \int_{\Omega_L} w_x^{\varepsilon,L}(t)\cdot F(v^{\varepsilon,L}(t))\,dx.\end{aligned}$$

The first term on the right hand side is bounded independent of L, since by Hölder's inequality we have

$$\begin{aligned}\int_{\Omega_L} w^{\varepsilon,L}(t)\cdot G^\varepsilon(v^{\varepsilon,L}(t))\,dx &= \int_{\Omega_L\cap(-1/\varepsilon,1/\varepsilon)} w^{\varepsilon,L}(t)\cdot G^\varepsilon(v^{\varepsilon,L}(t))\,dx \\ &\le \frac{2}{\varepsilon}\,\|w^{\varepsilon,L}\|_\infty \|G(v^{\varepsilon,L})\|_\infty,\end{aligned}$$

and

$$\begin{aligned}\|w^{\varepsilon,L}\|_\infty &\le \delta + \|\rho^{\varepsilon,L}\|_\infty + \|m^{\varepsilon,L}\|_\infty \\ \|G(v^{\varepsilon,L})\|_\infty &\le ||\gamma||_\infty(||m^{\varepsilon,L}||_\infty + ||m^{\varepsilon,L}/\rho^{\varepsilon,L}||_\infty^2||\rho^{\varepsilon,L}||_\infty + \tfrac{1}{2}||\rho^{\varepsilon,L}||_\infty^2) \\ &\quad + ||\rho^{\varepsilon,L}||_\infty||g||_\infty. \end{aligned} \tag{6.1}$$

These quantities are all bounded independent of L due to estimates (5.3) and (5.4).

For the second term, we have

$$\begin{aligned} w_x^{\varepsilon,L}\cdot F(v^{\varepsilon,L}) &= \rho_x^{\varepsilon,L}m^{\varepsilon,L} + m_x^{\varepsilon,L}\frac{(m^{\varepsilon,L})^2}{\rho^{\varepsilon,L}} + \tfrac{1}{2}m_x^{\varepsilon,L}(\rho^{\varepsilon,L})^2 \\ &= \rho_x^{\varepsilon,L}m^{\varepsilon,L} + m_x^{\varepsilon,L}\frac{(m^{\varepsilon,L})^2}{\rho^{\varepsilon,L}} + \tfrac{1}{2}m_x^{\varepsilon,L}(\rho^{\varepsilon,L}-\delta)^2 \\ &\quad + \delta m_x^{\varepsilon,L}(\rho^{\varepsilon,L}-\delta) - \tfrac{1}{2}\delta^2 m_x^{\varepsilon,L}. \end{aligned}$$

and so (noting that the integral of the final term vanishes):

$$\begin{aligned}\int_{\Omega_L} w_x^{\varepsilon,L}\cdot F(v^{\varepsilon,L})\,dx &\le \int_{\Omega_L} |w_x^{\varepsilon,L}|\,|w^{\varepsilon,L}|(1+\|\frac{m^{\varepsilon,L}}{\rho^{\varepsilon,L}}\|_\infty + \tfrac{1}{2}\|\rho^{\varepsilon,L}\|_\infty + \tfrac{3}{2}\delta)\,dx \\ &\le \tfrac{1}{2}\varepsilon\int_{\Omega_L}|w_x^{\varepsilon,L}(t)|^2\,dx + C(\varepsilon)\int_{\Omega_L}|w^{\varepsilon,L}(t)|^2\,dx,\end{aligned}$$

where the final inequality is due to Young's inequality and estimates (5.3) and (5.4). The constant $C(\varepsilon)$ is independent of L.

Using Gronwall's inequality we can now derive estimates

$$||w^{\varepsilon,L}(t)||_{L^2(\Omega_L)} \le C(\varepsilon) \tag{6.2}$$

$$||w_x^{\varepsilon,L}||_{L^2((0,T)\times\Omega_L)} \le C(\varepsilon) \tag{6.3}$$

for some $C(\varepsilon)$, independent of L. From these estimates, we see that we can extend w by 0 for x outside Ω_L. In other words, we can extend $(\rho^{\varepsilon,L}, m^{\varepsilon,L})$ to the whole of $(0,T)\times\mathbb{R}$ by setting $(\rho^{\varepsilon,L}(t,x), m^{\varepsilon,L}(t,x)) = (\delta, 0)$ for $(t,x)\in (0,T)\times(\mathbb{R}\backslash\Omega_L)$.

A standard compactness and regularity argument now shows that we can pass to the limit $L\to\infty$ and obtain a classical solution $(\rho^\varepsilon, m^\varepsilon)$ to problem (2.1). Notice that all of the estimates we have proved so far are also valid for $(\rho^\varepsilon, m^\varepsilon)$. However, the only estimates we have available which are also independent of ε are (5.3) and (5.4).

We now have another important consequence of (6.2). We define the set:

$$J(t) = \left\{x\in\Omega_L \;\middle|\; \rho^{\varepsilon,L} < \delta/2\right\}. \tag{6.4}$$

We thus have,

$$(\delta/2)^2|J(t)| = \int_{\rho^{\varepsilon,L}<\delta/2} (\delta/2)^2\,dx \le \int_{\rho^{\varepsilon,L}<\delta/2} (\rho^{\varepsilon,L}-\delta)^2\,dx \le \int_{\Omega_L} |w^{\varepsilon,L}|^2\,dx \le C(\varepsilon). \tag{6.5}$$

In other words, the set J has finite measure for each $\varepsilon > 0$. This result will be very useful in the next section since it allows us to apply the methods of Fang and Ito [FI97] in a situation where their assumptions (C2) and (C4) do not hold.

7 Proof that $\rho^\varepsilon \ge C(\varepsilon) > 0$

We now have the existence of a solution $(\rho^\varepsilon, m^\varepsilon)$ to system (2.1) for some small time. In order to extend our solution up to the time $T_{\max}$ we still need to ensure that $\rho^\varepsilon \ge C(\varepsilon) > 0$ for $T < T_{\max}$. We sketch an argument of Fang and Ito [FI97], slightly modified to this situation. Let $r = 1/e^{\mu t}\rho^{\varepsilon,L}$ and $u = m^\varepsilon/\rho^\varepsilon$. Then

$$r_t + ur_x - u_x r + (\mu + \gamma u)r = \varepsilon(r_{xx} - 2r_x^2/r) \tag{7.1}$$

In [FI95] Lemma 2.3, Fang and Ito prove the following result:

LEMMA 7.1
Suppose $\phi \ge 0$ is a nonincreasing function on $[c_0, \infty)$, $\phi(c_0) < \infty$ and there are positive constants K, a and b so that

$$\phi(c_2) \le K c_1^a (c_2 - c_1)^{-a} \phi(c_1)^{1+b}, \quad \text{for all } c_2 > c_1 \ge c_0.$$

Then there exists a $c^* < \infty$ such that

$$\phi(c^*) = 0.$$

We apply this lemma to

$$\phi(c) := \sup_{t\in[0,T]} \left| \left\{ x \in \mathbb{R} \;\middle|\; r(t,x) > c \right\} \right|.$$

Because of estimate (6.5), there exists $c_0 < \infty$ so that $\phi(c_0) < \infty$ ($c_0 = 1/\delta$ will do). Now define $\xi = \max(0, r - c_1)$ for some $c_1 > c_0$ (large enough so that $\xi(0,x) = 0$). Multiply (7.1) by ξ^3 and integrate over Q_T to get:

$$\begin{aligned} \tfrac{1}{4}\int_{\mathbb{R}} \xi^4(T)\,dx + \ (\mu+\gamma u)\int_0^T\!\!\int_{\mathbb{R}} (\xi^4 + c_1\xi^3)\,dx\,dt & \\ + \frac{3\varepsilon}{4}\int_0^T\!\!\int_{\mathbb{R}} |(\xi^2)_x|^2\,dx\,dt & \\ = -2\varepsilon\int_0^T\!\!\int_{\mathbb{R}} \frac{\xi_x^2}{\xi + c_1}\,dx\,dt - \int_0^T\!\!\int_{\mathbb{R}} (5\xi^3 + 3c_1\xi^2)u\xi_x\,dx\,dt & \end{aligned} \tag{7.2}$$

Young's inequality yields

$$-\int_0^T\int_{\mathbb{R}} 5\xi^3 u\xi_x\,dx\,dt \le \frac{\varepsilon}{4}\int_0^T\int_{\mathbb{R}} |(\xi^2)_x|^2\,dx\,dt + \frac{25}{4\varepsilon}\|u\|_\infty^2\int_0^T\int_{\mathbb{R}}\xi^4\,dx\,dt$$

and

$$-\int_0^T\int_{\mathbb{R}} 3c_1\xi^2 u\xi_x\,dx\,dt \le \frac{\varepsilon}{4}\int_0^T\int_{\mathbb{R}} |(\xi^2)_x|^2\,dx\,dt + \frac{9c_1^2}{4\varepsilon}\|u\|_\infty^2\phi(c_1)^{2/3}\int_0^T\|\xi(t)\|_6^2\,dt$$

Now choose

$$\mu = \frac{\varepsilon}{4} + \frac{25}{4\varepsilon}\|u\|_\infty^2 + K$$

where $K \ge \|\gamma u\|_\infty$. We thus derive

$$\int_{\Omega_L}\xi^4(T)\,dx + \varepsilon\int_0^T\int_{\mathbb{R}}(\xi^4 + |(\xi^2)_x|^2)\,dx\,dt \le \frac{9c_1^2}{\varepsilon}\|u\|_\infty^2\phi(c_1)^{2/3}\int_0^T\|\xi(t)\|_6^2\,dt$$

Sobolev's embedding theorem gives $\|\xi(t)\|_{L^6(\mathbb{R})}^2 \le C\|\xi(t)^2\|_{H^1(\mathbb{R})}$. Combining this with another application of Young's inequality we thus derive

$$\int_{\mathbb{R}}\xi^4(T)\,dx \le \frac{\alpha c_1^4\phi(c_1)^{4/3}T}{\varepsilon}$$

for some constant α. Furthermore, by definition of ξ, for $c_2 > c_1$ we have

$$\int_{\mathbb{R}}\xi^4(T)\,dx \ge (c_2 - c_1)^4\phi(c_2)$$

and so

$$\phi(c_2) \le Kc_1^4(c_2 - c_1)^{-4}\phi(c_1)^{4/3}$$

Hence there exists $c^* < \infty$ such that $\phi(c^*) = 0$, and so $r > c^*$ for all $t < T_{\max}$, and hence

$$\rho^\varepsilon(t,x) \ge 1/c^* e^{\mu t} > 0, \tag{7.3}$$

where the constants c^* and μ depend on ε.

8 Entropy estimates

Estimates (5.3) and (5.4) are independent of ε, hence there exist functions $(\bar{h}, \bar{m})$ satisfying the same estimates, and for a subsequence,

$$(\rho^\varepsilon, m^\varepsilon) \overset{*}{\rightharpoonup} (\bar{\rho}, \bar{m}) \quad \text{in } L^\infty((0,T)\times\mathbb{R}) \text{ as } \varepsilon \to 0.$$

Young's theorem also tells us that this is equivalent to convergence in the sense of measures. To prove that we actually have strong convergence to an entropy solution we shall apply the method of compensated compactness. For this we additionally require some estimates on the entropy. We shall also need to improve the H^1 estimate (6.3). Of course, if it were possible to find an ε-independent H^1 estimate, then by compactness we would immediately have strong convergence. This is too much to expect, though we shall see that we can prove that $\sqrt{\varepsilon} v_x^\varepsilon \in L^2_{\mathrm{loc}}(Q_T)$.

DiPerna [DiP83] showed that every weak entropy-flux pair (η, q) can be expressed in the form of a convolution:

$$\eta(\rho, m) = \int_{\mathbb{R}} f(\xi)\chi(\rho, \xi - m/\rho)\, d\xi \tag{8.1}$$

$$q(\rho, m) = \tfrac{1}{2}\int_{\mathbb{R}} f(\xi)(\xi + m/\rho)\chi(\rho, \xi - m/\rho)\, d\xi \tag{8.2}$$

where

$$\chi(\rho, z) = \sqrt{(4\rho - z^2)_+}$$

Furthermore Lions, Perthame and Tadmor [LPT94] showed that η is a convex weak entropy exactly when f is convex. In particular, setting $f(\xi) = \xi^2/2$ yields a strictly convex entropy which represents the mechanical energy:

$$\eta_E = m^2/2\rho + \rho^2/2.$$

The corresponding flux is given by

$$q_E = \frac{\rho}{2}\left(\frac{m}{\rho}\right)^3 + m\rho.$$

Differentiating the convolution integral for η, one can easily derive the following estimates for any C^2 weak entropy:

$$|\nabla \eta| \le C(||f||_{W^{1,\infty}}, ||\rho||_\infty, ||m||_\infty) \tag{8.3}$$

$$|\nabla^2 \eta| \le C(||f||_{W^{2,\infty}}, ||\rho||_\infty, ||m||_\infty) \tag{8.4}$$

If (η, q) is an entropy-flux pair, then multiplying (2.1) by $\nabla\eta$ gives:

$$\eta_t + q_x = \varepsilon\big(\, \eta_{xx} - \nabla^2\eta(v_x^\varepsilon, v_x^\varepsilon)\,\big) + \nabla\eta\, G^\varepsilon \tag{8.5}$$

Now multiply by a non-negative test function $\phi \in C_0^\infty(\mathbb{R})$, and integrate over Q_T. We have,

$$\begin{aligned}\int_{\mathbb{R}} \phi\eta\,\big|_{t=T} dx + \varepsilon\int_0^T\!\!\int_{\mathbb{R}} \phi\nabla^2\eta(v_x^\varepsilon, v_x^\varepsilon)\, dx\, dt \\ = \int_{\mathbb{R}} \phi\eta\,\big|_{t=0} dx + \int_0^T\!\!\int_{\mathbb{R}} (\varepsilon\phi_{xx}\,\eta + \phi_x\, q)\, dx\, dt + \int_0^T\!\!\int_{\mathbb{R}} \phi\nabla\eta\, G^\varepsilon dx\, dt.\end{aligned}$$

Now suppose that (η, q) is a non-negative, strictly convex entropy-flux pair (e.g. (η_E, q_E)). Then

$$\int_{\mathbb{R}} \phi\eta \,\big|_{t=T} dx + \varepsilon \int_0^T\!\!\int_{\mathbb{R}} \phi \nabla^2\eta(v_x^\varepsilon, v_x^\varepsilon)\, dx\, dt \geq K\varepsilon \int_0^T\!\!\int_{\mathbb{R}} \phi |v_x^\varepsilon|^2\, dx\, dt.$$

and the L^∞ estimates for $\nabla\eta$, ρ^ε and m^ε yield:

$$\varepsilon \int_0^T\!\!\int_{\mathbb{R}} \phi |v_x^\varepsilon|^2\, dx\, dt \leq C\left(1 + \int_0^T\!\!\int_{\mathbb{R}} |\phi\, G| dx\, dt\right).$$

We also have $\|G\|_\infty \leq K(T)$ by (6.1). Hence we have

$$\varepsilon \int_0^T\!\!\int_{\mathbb{R}} \phi |v_x^\varepsilon|^2\, dx\, dt \leq C < \infty$$

where the constant depends on ϕ and $T < T_{\max}$, but not on $\varepsilon > 0$. In other words, we have shown that $(\sqrt{\varepsilon} v_x^\varepsilon)_{\varepsilon>0}$ is bounded independent of ε in $L^2_{\mathrm{loc}}(Q_T)$.

9 Compensated compactness

Since $(v^\varepsilon)_{\varepsilon>0}$ takes values in a bounded set B, there is a subsequence (v^{ε_j}) and a measurable family $(t,x) \mapsto \nu_{t,x}$ of probability measures (Young measures) with $\operatorname{supp} \nu_{t,x} \subset \bar{B}$ such that

$$f \circ v^{\varepsilon_j} \overset{*}{\rightharpoonup} \langle f \rangle \quad \text{in } L^\infty(Q_T)$$

$\langle f \rangle$ is the average with respect to the Young measure:

$$\langle f \rangle(t,x) := \int_{Q_T} f\, d\nu_{t,x}$$

Recall the following result of Murat:

LEMMA 9.1 (MURAT, [MUR81])
Let U be an open bounded subset of $\mathbb{R}^n$ and $(v^\varepsilon)_{\varepsilon>0}$, a bounded sequence in $W^{-1,\infty}(U)$ which is contained in a set of the form $B+C$ where B is a bounded subset of $\mathcal{M}(U)$ (the space of bounded measures) and C is a compact subset of $H^{-1}(U)$. Then $(v^\varepsilon)_{\varepsilon>0}$ is relatively compact in $H^{-1}(U)$.

Let U be an arbitrary open, bounded subset of Q_T, $0 < T < T_{\max}$. Recall equation (8.5):

$$\eta_t + q_x = \varepsilon\eta_{xx} - \varepsilon\nabla^2\eta(v_x^\varepsilon, v_x^\varepsilon) + \nabla\eta\, G^\varepsilon$$

Looking at each term in turn, we find:

- $(\eta(v^\varepsilon) + q(v^\varepsilon))_{\varepsilon>0}$ is uniformly bounded in $L^\infty(Q_T)$, and hence also in $W^{-1,\infty}(Q_T)$.
- $\varepsilon\eta_x = \sqrt{\varepsilon}\cdot\nabla\eta\cdot\sqrt{\varepsilon}\,v_x^\varepsilon \longrightarrow 0$ in $L^2(U)$ as $\varepsilon\to 0$. Hence $(\varepsilon\eta_{xx})_{\varepsilon>0}$ is relatively compact in $H^{-1}(U)$.
- $\varepsilon\nabla^2\eta(v_x^\varepsilon, v_x^\varepsilon) \le |\nabla^2\eta|\,\varepsilon|v_x^\varepsilon|^2$, so $(\varepsilon\nabla^2\eta(v_x^\varepsilon, v_x^\varepsilon))_{\varepsilon>0}$ is uniformly bounded in $L^1_{\mathrm{loc}}(Q_T)$, and hence in $\mathcal{M}(U)$.
- $(\nabla\eta\, G^\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^\infty(Q_T)$ and hence in $\mathcal{M}(U)$.

Hence by Murat's lemma, the sequence $(\eta(v^\varepsilon)_t + q(v^\varepsilon)_x)_\varepsilon$ is relatively compact in $H^{-1}_{\mathrm{loc}}(Q_T)$, and so we are able to apply the following lemma due to Tartar.

LEMMA 9.2 (TARTAR, [TAR79])
If for all weak entropy-flux pairs, the sequence

$$(\eta(v^\varepsilon)_t + q(v^\varepsilon)_x)_{\varepsilon>0}$$

is relatively compact in $H^{-1}_{\mathrm{loc}}(Q_T)$, then for all weak entropy-flux pairs (η_j, q_j), $j = 1, 2$, the following equation holds:

$$\langle\eta_1 q_2 - \eta_2 q_1\rangle = \langle\eta_1\rangle\langle q_2\rangle - \langle\eta_2\rangle\langle q_1\rangle. \tag{9.1}$$

We can then apply results of Lions, Perthame and Souganidis [LPS96] to show that $v^{\varepsilon_j} \to v$ (strong) in $L^p(U)$, for all open, bounded $U \subset Q_T$, $p \in [1,\infty)$.

10 Convergence to an entropy solution

We now show that $\bar v := (\bar\rho, \bar m)$ is a weak solution of (1.1). Although we have strong convergence of $(\rho^\varepsilon, m^\varepsilon) \to (\bar\rho, \bar m)$ in L^p, there is still something to prove here because the source term $g(x,u) = \sin\zeta(x) - u/|u|\ \cos\zeta(x)\tan\delta_F(x)$ is discontinuous in u. We are saved by the antimonotonicity of g and the approximation g^ε. Namely,

$$(g^\varepsilon(x,u) - g^\varepsilon(x,v))(u-v) \le 0, \quad u, v \in \mathbb{R}. \tag{10.1}$$

Let $w \in C_0^1(0,T)\times\mathbb{R}$ be fixed, and define the function χ_w by

$$\chi_w(x,t) = \begin{cases} 1, & \text{if } (x,t) \in \operatorname{supp} w, \\ 0, & \text{otherwise,} \end{cases} \tag{10.2}$$

Now define $r = \chi_w \bar{m}/\bar{\rho} + \lambda w$, for some $\lambda \in \mathbb{R}\backslash\{0\}$.

We then have

$$\begin{aligned} 0 \geq & \big(\rho^\varepsilon\, g^\varepsilon(\cdot, \chi_w m^\varepsilon/\rho^\varepsilon) - \rho^\varepsilon\, g^\varepsilon(\cdot, r), \chi_w m^\varepsilon/\rho^\varepsilon - r\big) \\ = & \big(\rho^\varepsilon\, g^\varepsilon(\cdot, \chi_w m^\varepsilon/\rho^\varepsilon), \chi_w m^\varepsilon/\rho^\varepsilon - \chi_w \bar{m}/\bar{\rho}\big) \\ & - \lambda\big(\rho^\varepsilon\, g^\varepsilon(\cdot, \chi_w m^\varepsilon/\rho^\varepsilon), w\big) - \big(\rho^\varepsilon\, g(\cdot, r), \chi_w m^\varepsilon/\rho^\varepsilon - r\big) \end{aligned}$$

The first of these terms converges to 0 as $\varepsilon \to 0$, since

$$\begin{aligned} & \big|\big(\rho^\varepsilon\, g^\varepsilon(\cdot, \chi_w m^\varepsilon/\rho^\varepsilon), \chi_w m^\varepsilon/\rho^\varepsilon - \chi_w \bar{m}/\bar{\rho}\big)\big| \\ \leq & \iint_{\operatorname{supp} w} |g^\varepsilon(x, \chi_w m^\varepsilon/\rho^\varepsilon)|\, |m^\varepsilon - \bar{m}\rho^\varepsilon/\bar{\rho}| dx\, dt \\ \leq & \|\rho^\varepsilon\, g^\varepsilon\|_{L^\infty(\operatorname{supp} w)} \left(\|m^\varepsilon - \bar{m}\|_{L^1(\operatorname{supp} w)} + \|\bar{m}/\bar{\rho}\|_{L^\infty(\operatorname{supp} w)} \|\rho^\varepsilon - \bar{\rho}\|_{L^1(\operatorname{supp} w)}\right) \\ & \longrightarrow 0 \quad \text{as } \varepsilon \to 0. \end{aligned}$$

For the second term we have

$$\begin{aligned} & - \lambda\big(\rho^\varepsilon\, g^\varepsilon(\cdot, \chi_w m^\varepsilon/h^\varepsilon), w\big) \\ = & \lambda \iint_{\operatorname{supp} w} \left(m^\varepsilon w_t + \left((m^\varepsilon)^2/\rho^\varepsilon + \beta(\rho^\varepsilon)^2/2\right) w_x + \varepsilon m^\varepsilon_x w_x\right) dx\, dt \\ \longrightarrow & \lambda \iint_{\operatorname{supp} w} \left(\bar{m} w_t + \left(\bar{m}^2/\bar{\rho} + \beta\bar{\rho}^2/2\right) w_x\right) dx\, dt. \end{aligned}$$

Similarly, it is easy to show that the third term satisfies

$$\begin{aligned} & - \big(\rho^\varepsilon\, g^\varepsilon(\cdot, r), \chi_w m^\varepsilon/\rho^\varepsilon - r\big) \\ = & \iint_{\operatorname{supp} w} \rho^\varepsilon\, g^\varepsilon(x, r)\, (r - m^\varepsilon/\rho^\varepsilon)\, dx\, dt \\ \to & \iint_{\operatorname{supp} w} \bar{\rho}\, g(x, r)\, (r - \bar{m}/\bar{\rho})\, dx\, dt \\ = & \lambda \iint_{\operatorname{supp} w} \bar{\rho}\, g(x, r)\, w\, dx\, dt. \end{aligned}$$

We have thus shown

$$\lambda \iint_{\operatorname{supp} w} \left(\bar{m} w_t + \left(\bar{m}^2/\bar{\rho} + \beta\bar{\rho}^2/2\right) w_x + \bar{\rho}\, g(x, \bar{m} + \lambda w)\, w\right) dx\, dt \leq 0. \tag{10.3}$$

Suppose for the moment that $\lambda > 0$. We then have

$$w \operatorname{sgn}(\bar{m} + \lambda w) \geq w \operatorname{sgn}(\bar{m})$$

and hence

$$\iint_{\operatorname{supp} w} \bar{\rho}\, g(x,\bar{m})dx\,dt \geq \iint_{\operatorname{supp} w} \bar{\rho}\, g(x,\bar{m}+\lambda w)dx\,dt.$$

It thus follows that

$$\iint_{\operatorname{supp} w} \left(\bar{m} w_t + \left(\bar{m}^2/\bar{\rho} + \beta\bar{\rho}^2/2\right) w_x + \bar{\rho}\, g(x,\bar{m})\, w\right) dx\,dt \leq 0. \qquad (10.4)$$

If we set $\lambda < 0$ we get the reverse inequality. We have thus shown that $(\bar{\rho},\bar{m})$ is a weak solution of equation (1.1).

It remains to show that the entropy inequality is satisfied in the limit. Clearly this will be true if $\rho\eta_m m/|m|$ is a locally Lipschitz function of ρ and m. This is certainly true for a very large class of entropies. For example, this holds for all entropies generated by the convex functions $f(\xi) = \xi^{2i}$, $i \in \mathbb{N}$ in equation (8.1). We can also pass to the limit if η_m is a nonnegative function. In this case the monotonicity argument presented above can be applied. An entropy of this form is generated by the function $f(\xi) = e^{\xi}$.

Hence $(\bar{\rho},\bar{m})$ is an entropy solution of equation (1.1) in the sense of Definition 1.2.

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