ON STRONGLY DEGENERATE CONVECTION-DIFFUSION PROBLEMS MODELING SEDIMENTATION-CONSOLIDATION PROCESSES

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Abstract. We investigate initial-boundary value problems for a quasilinear strongly degenerate convection-diffusion equation with a discontinuous diffusion coefficient. These problems come from the mathematical modeling of certain sedimentation-consolidation processes. Existence of entropy solutions belonging to BV is shown by the vanishing viscosity method. The existence proof for one of the models includes a new regularity result for the integrated diffusion coefficient. New uniqueness proofs for entropy solutions are also presented. These proofs rely on a recent extension to second order equations of Kružkov's method of "doubling of the variables". The application to a sedimentation-consolidation model is illustrated by two numerical examples.

1. Introduction

In this paper, we consider quasilinear strongly degenerate parabolic equations of the type

$$\partial_t u + \partial_x(q(t)u + f(u)) = \partial_x^2 A(u), \quad (x,t) \in Q_T, \quad A(u) := \int_0^u a(s)\,ds, \quad a(u) \geq 0, \qquad (1.1)$$

where $Q_T := \Omega \times \mathcal{T}$, $\Omega := (0,1)$ and $\mathcal{T} := (0,T)$. In general, we allow that the diffusion coefficient $a(u)$ vanishes on intervals of solution values u, where (1.1) is then of hyperbolic type; therefore this equation is also called hyperbolic-parabolic. Although equations of this type occur in a variety of applications, we here focus on the application to sedimentation-consolidation processes [3, 8, 9], which leads to an initial-boundary value problem (IBVP) with mixed Dirichlet-flux boundary conditions ("Problem A") or alternatively to an IBVP with two flux conditions ("Problem B"). It is well known that solutions of (1.1) develop discontinuities due to the nonlinearity of the flux density function $f(u)$ and the degeneracy of the diffusion coefficient. Therefore one has to consider entropy solutions in order to have a well-posed problem. Moreover, in regions where (1.1) is hyperbolic, solution values propagate along straight-line characteristics which might intersect the lateral boundaries of Q_T from the interior and require the treatment of Dirichlet boundary conditions as entropy boundary conditions [2, 7]. A review of properties and known existence and uniqueness results related to the concept of entropy solutions for equation (1.1), as well as an overview of numerical methods for strongly degenerate parabolic equations, is provided in [11].

Our particular application justifies various assumptions on the coefficients of (1.1) and on the initial and boundary data. Most notably, many constitutive equations proposed for these processes imply that $a(u) = 0$ for $u \leq u_c$ and that $a(u)$ jumps at u_c to a positive value, where u_c is a given constant, the so-called critical concentration. We therefore insist on using a discontinuous diffusion coefficient $a(u)$. This case had not been covered by the previous existence and uniqueness analysis of Problem A by Bürger and Wendland [6], which relies on relatively strong assumptions on the regularity of the coefficients of equation (1.1) and on the initial and boundary data; in particular, $a(u)$ is assumed to be continuously differentiable. We point out that the previous analysis [6] was limited to Problem A, and that Problem B has not been treated so far.

Key words and phrases. Degenerate convection-diffusion equation, entropy solutions, discontinuous diffusion coefficient, sedimentation-consolidation processes, BV solutions.

AMS subject classifications. 35D05, 35D10, 35K65, 76T05.

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The first objective of this paper is to show existence of entropy solutions belonging to $BV(Q_T)$ for these problems when the diffusion coefficient is discontinuous. We show that smoothing out the jumps of $a(u)$ and of the initial and boundary data by a standard mollifier technique will not cause new singularities when the smoothing parameter tends to zero in the vanishing viscosity method. As a part of the existence proof of Problem B, we show that the integrated diffusion coefficient $A(u)$ belongs to the Hölder space $C^{1,1/2}(\overline{Q_T})$. This is a significantly better regularity property compared to the result $\partial_x A(u) \in L^2(Q_T)$ valid for Problem A.

The second objective of this paper is to present new uniqueness proofs for both problems based on the technique known as "doubling of the variables". This technique was introduced in Kružkov's pioneering work [13] as a tool for proving the L^1 contraction principle for entropy solutions of scalar conservation laws and very recently extended elegantly by Carrillo [10] to a class of degenerate parabolic equations. It is the extension in [10] that we adopt here to our initial-boundary value problems. We emphasize that these uniqueness proofs merely require that the functions $f(u)$ and $A(u)$ are locally Lipschitz continuous ($a(u)$ may be discontinuous) and that they are not based on deriving jump conditions as in [24]. In fact, continuity of $a(u)$ has been assumed in previous papers [6, 24, 25] in order to derive such jump conditions. Furthermore, the jump conditions — and thus the corresponding uniqueness proof — derived by Wu and Yin [25] (see also [7]) have at present no multidimensional analogue, whereas the uniqueness approach presented here also works in multidimensions [4]. Having said this, some new results dedicated to the solution of this problem are available, see Vol'pert [20].

We mention that to produce an entropy solution belonging to $BV(Q_T)$, it is necessary to require that the initial function u_0 belongs to the class $\mathcal{B}$ of functions for which $\mathrm{TV}(\partial_x A(u))$ is uniformly bounded with respect to regularization. This condition is rather restrictive but satisfied by most initial data occurring in the context of the sedimentation-consolidation problems. Our problems are also solvable for $u_0 \notin \mathcal{B}$ (say $u_0 \in BV(\Omega)$), but then it is only possible to show existence of an entropy solution in the larger class $C^{1/2}(\mathcal{T}; L^1(\Omega)) \cap L^\infty(\mathcal{T}; BV(\Omega))$, also referred to as $BV_{1,1/2}(Q_T)$ [22]. In this larger class, one can not assume a priori that the traces of the entropy solution at the boundaries of Q_T exist. To resolve this problem one needs a reformulation of the concept of solution that avoids these traces. Such a solution concept has been employed by Wu [22], but will not be considered here since it is not obvious how to prove uniqueness of such solutions.

This paper is organized as follows. In § 2, we recall some properties of mollifiers and related functions, state the initial-boundary value problems with the respective pertaining assumptions on the data and formulate definitions of entropy solutions. In § 3, existence of entropy solutions is shown by the vanishing viscosity method and the improved regularity result valid for entropy solutions of Problem B is derived. Uniqueness of entropy solutions is shown in § 4. In Section 5 we present two numerical solutions of the IBVP modeling sedimentation with compression, in which the assumptions for the existence of BV solutions are satisfied.

2. Mathematical preliminaries and definition of entropy solutions

2.1. **Mollifiers and related functions.** Let $\omega \in C_0^\infty(\mathbb{R})$ be a function satisfying $\omega \geq 0$, $\operatorname{supp}\omega \subset (-1,1)$ and $\|\omega\|_{L^1(\mathbb{R})} = 1$, and define a standard mollifier [16] with support in $(-h,h)$ by $\omega_h(x) = \omega(x/h)/h$. A C^∞ regularization of a bounded function $b(u)$ is then given by the convolution

$$(b * \omega_h)(u) := \int_{-h}^{h} b(u-v)\omega_h(v)\,dv.$$

Moreover, we define for sufficiently small $h > 0$ the functions

$$\varrho_h(x) := \int_{-\infty}^{x} \omega_h(\xi)\,d\xi, \quad \mu_h(x) := 1 - \varrho_h(x-2h), \quad \nu_h(x) := \varrho_h\big(x-(1-2h)\big), \qquad (2.1)$$

which have the property stated in the following lemma given in [24].

Lemma 1. *Let $v \in L^1(\mathcal{T}; L^\infty(\Omega))$. If the traces $\gamma_0 v := (\gamma v)(0,t)$ and $\gamma_1 v := (\gamma v)(1,t)$ exist a.e. in $\mathcal{T}$, then we have for $\varphi \in C^\infty(Q_T)$*

$$\lim_{h \downarrow 0} \iint_{Q_T} \partial_x \Big(\varphi(x,t)\big(\mu_h(x) + \nu_h(x)\big)\Big) v(x,t)\, dxdt = \int_0^T \big(\varphi(1,t)\gamma_1 v - \varphi(0,t)\gamma_0 v\big)\, dt.$$

2.2. **Statement of Problem A.** We consider the IBVP

$$\partial_t u + \partial_x(q(t)u + f(u)) = \partial_x^2 A(u), \quad (x,t) \in Q_T, \tag{A1}$$

$$u(x,0) = u_0(x), \quad x \in \overline{\Omega}, \tag{A2}$$

$$u(1,t) = \varphi_1(t), \quad t \in (0,T], \tag{A3}$$

$$f(u(0,t)) - \partial_x A(u(0,t)) = 0, \quad t \in (0,T], \tag{A4}$$

where we assume

$$f \text{ is continuous and piecewise differentiable}, \quad f \leq 0, \quad \operatorname{supp} f \subset [0, u_{\max}], \quad \|f'\|_\infty \leq \infty, \tag{2.2}$$

$$a(u) \geq 0, \quad \operatorname{supp} a \subseteq \operatorname{supp} f, \quad a(u) = 0 \text{ for } u \leq u_{\mathrm{c}},\ 0 < u_{\mathrm{c}} \leq u_{\max}, \tag{2.3}$$

$$q(t) \leq 0 \quad \forall t \in \overline{\mathcal{T}}, \quad \mathrm{TV}_{\mathcal{T}}(q) < \infty, \quad \mathrm{TV}_{\mathcal{T}}(q') < \infty, \tag{2.4}$$

Since A is monotonically non-increasing, $\operatorname{sgn}(k_1 - k_2)(A(k_1) - A(k_2)) = |A(k_1) - A(k_2)|$. Defining

$$a_\varepsilon(u) := ((a + \varepsilon) * \omega_\varepsilon)(u), \quad A_\varepsilon(u) := \int_0^u a_\varepsilon(s) ds, \quad \varepsilon > 0$$

and $\mathcal{U}_\varepsilon := [-\varepsilon, u_{\max} + \varepsilon]$ for $\varepsilon \geq 0$, we can state the regularity assumption on u_0 as

$$u_0 \in \mathcal{B} := \Big\{ u \in BV(\Omega) :\ u(x) \in \mathcal{U}_0\ \forall x \in \overline{\Omega};\ \mathrm{TV}_\Omega(\partial_x A_\varepsilon(u)) < M_0 \text{ uniformly in } \varepsilon \Big\}, \tag{2.5}$$

We comment on the assumption (2.5). First note that, if $u_0 \in \mathcal{B}$, then also $\widetilde{u_0} \in \mathcal{B}$. This requirement is needed to show that the entropy solution of the initial-boundary value problem is L^1 Lipschitz continuous in time. It might be difficult to verify whether a given initial function belongs to $\mathcal{B}$, but, for example, all piecewise constant functions defined on Ω do so.

The boundary datum φ_1 is assumed to satisfy

$$0 \leq \varphi_1(t) \leq u_{\max}, \quad t \in \overline{\mathcal{T}}, \tag{2.6}$$

$$\varphi_1 \text{ changes its monotonicity behaviour at most a finite number of times.} \tag{2.7}$$

In particular, we admit that the functions u_0 and φ_1 may possess jumps, and note that (2.6) and (2.7) imply that $\mathrm{TV}_{\mathcal{T}}(\varphi_1) < \infty$. The assumptions (2.2)–(2.7) are essential in the proof of existence of an entropy solution belonging to $BV(Q_T)$.

Remark 1. *Of course, if $a(\cdot)$ is sufficiently smooth, then the requirement that $u_0 \in \mathcal{B}$ can be replaced by the requirement $\mathrm{TV}_\Omega(\partial_x u_0)) < M_0$. In particular, the existence analysis conducted in* [6] *is contained in the analysis presented below.*

2.3. **Definition of entropy solutions of Problem A.** Since weak, possibly discontinuous solutions are in general not unique, we need a selection principle or entropy criterion. This is included in the following solution concept. Wherever notationally convenient, we set $g(u,t) := q(t)u + f(u)$.

Definition 1. *A function $u \in L^\infty(Q_T) \cap BV(Q_T)$ is an* entropy solution *of Problem A if the following conditions are satisfied:*

$$\partial_x A(u) \in L^2(Q_T); \tag{2.8}$$

$$\forall \varphi \in C^\infty\left((0,1] \times \overline{\mathcal{T}}\right),\ \varphi \geq 0, \operatorname{supp} \varphi \subset (0,1] \times \mathcal{T}, \forall k \in \mathbb{R}:$$

$$\iint_{Q_T} \Big\{ |u - k| \partial_t \varphi + \operatorname{sgn}(u-k)\big[g(u,t) - g(k,t) - \partial_x A(u)\big] \partial_x \varphi \Big\}\, dxdt$$

$$+ \int_0^T \Big\{ -\operatorname{sgn}(\varphi_1(t) - k)\big[g(\gamma_1 u, t) - g(k,t) - \gamma_1 \partial_x A(u)\big] \varphi(1,t)$$

$$+\big[\operatorname{sgn}(\gamma_1 u - k) - \operatorname{sgn}(\varphi_1(t) - k)\big]\big[A(\gamma_1 u) - A(k)\big]\partial_x\varphi(1,t)\Big\}\,dt \geq 0; \tag{2.9}$$

$$\text{for almost all } t \in \mathcal{T},\ \gamma_0\left(f(u) - \partial_x A(u)\right) = 0; \tag{2.10}$$

$$\text{for almost all } x \in \overline{\Omega},\ \ \lim_{t\downarrow 0} u(x,t) = u_0(x). \tag{2.11}$$

Entropy inequalities like (2.9) go back to the pioneering papers of Vol'pert [19] and Kružkov [13] for first order equations and Vol'pert and Hudjaev [21] for second order equations.

2.4. **Statement of Problem B.** We also consider Problem B which is obtained from Problem A if the boundary condition at $x = 1$, (A3), is replaced by the total flux condition

$$\big(g(u,t) - \partial_x A(u)\big)(1,t) = \Psi(t), \ \ t \in (0,T]. \tag{B3}$$

We have to assume that

$$\text{either } \Psi \equiv 0 \quad \text{or} \quad \exists \xi > 0, M_g > 0 : \xi a(u) - (q(t) + f'(u)) \geq M_g. \tag{2.12}$$

All other assumptions (2.2), (2.3) and (2.5) remain valid. Here, we assume that

$$\forall t \in \overline{\mathcal{T}}: \ \min_{u\in\mathcal{U}_0}\big(g(u,t)\big) \leq \Psi(t) \leq 0, \ \ \Psi(t) \geq g(u_{\max},t); \ \ \mathrm{TV}_{\mathcal{T}}(\Psi) < \infty. \tag{2.13}$$

Problem B is also of interest in the application of the sedimentation-consolidation model, since frequently the feed flux Ψ rather than a boundary concentration φ_1 is prescribed.

2.5. **Definition of entropy solutions of Problem B.** Here, the definition of entropy solution is analogous to Definition 1:

Definition 2. *A function $u \in L^\infty(Q_T) \cap BV(Q_T)$ is an* entropy solution *of Problem B if* (2.8) *and* (2.11) *in Definition 1 are satisfied, if for all $\varphi \in C_0^\infty(Q_T)$, $\varphi \geq 0$ and $k \in \mathbb{R}$ the inequality*

$$\iint_{Q_T} \Big\{|u-k|\partial_t\varphi + \operatorname{sgn}(u-k)\big[g(u,t) - g(k,t) - \partial_x A(u)\big]\partial_x\varphi\Big\}\,dt\,dx \geq 0 \tag{2.14}$$

holds, if the boundary condition (2.10) *is valid, and if*

$$\gamma_1\left(g(u,t) - \partial_x A(u)\right) = \Psi(t) \text{ for almost all } t \in \mathcal{T}. \tag{2.15}$$

3. Existence of entropy solutions

3.1. **Existence of entropy solutions of Problem A.** Existence of entropy solutions is proved here by the vanishing viscosity method. Therefore we replace $a(u)$ by $a_\varepsilon(u^\varepsilon)$ and use the regularizations $f_\varepsilon(u) := (f * \omega_\varepsilon)(u)$, $q_\varepsilon(t) := (q * \omega_\varepsilon)(t)$ and $g_\varepsilon(u,t) := f_\varepsilon(u) + q_\varepsilon(t)u$. Note that (2.2) implies that $\operatorname{supp} f_\varepsilon \subset \mathcal{U}_\varepsilon$. The functions u_0 and φ_1 are replaced by smooth approximations u_0^ε and φ_1^ε with $u_0^\varepsilon \to u_0$ in $L^1(\Omega)$ and $\varphi_1^\varepsilon \to \varphi_1$ in $L^1(\mathcal{T})$ for $\varepsilon \downarrow 0$. The solution to the degenerate IBVP is then obtained as the limit for $\varepsilon \downarrow 0$ of the family $\{u^\varepsilon\}_{\varepsilon>0}$ of smooth solutions of the regularized parabolic IBVP (referred to as Problem A^ε):

$$\partial_t u^\varepsilon + \partial_x\left(q_\varepsilon(t)u^\varepsilon + f_\varepsilon\left(u^\varepsilon\right)\right) = \partial_x^2 A_\varepsilon\left(u^\varepsilon\right), \ \ (x,t) \in Q_T, \tag{A$^\varepsilon$.1}$$

$$u^\varepsilon(x,0) = u_0^\varepsilon(x), \ \ x \in \Omega, \tag{A$^\varepsilon$.2}$$

$$u^\varepsilon(1,t) = \varphi_1^\varepsilon(t), \ \ t \in (0,T], \tag{A$^\varepsilon$.3}$$

$$\big(f_\varepsilon(u^\varepsilon) - \partial_x A_\varepsilon(u^\varepsilon)\big)(0,t) = 0, \ \ t \in (0,T]. \tag{A$^\varepsilon$.4}$$

To ensure the existence of a smooth solution of Problem A^ε for any fixed value $\varepsilon > 0$, the functions u_0^ε and φ_1^ε have to satisfy first order compatibility conditions:

$$u_0^\varepsilon(1) = \varphi_1^\varepsilon(0), \tag{3.1a}$$

$$-\Big[q_\varepsilon(0) + f_\varepsilon'\big(u_0^\varepsilon(1)\big)\Big]\big(u_0^\varepsilon\big)'(1) - \alpha'\big(u_0^\varepsilon(1)\big)\big[(u_0^\varepsilon)'(1)\big]^2\alpha\big(u_0^\varepsilon(1)\big)(u_0^\varepsilon)''(1) = \left(\varphi_1^\varepsilon\right)'(0), \tag{3.1b}$$

$$\alpha\big(u_0^\varepsilon(0)\big)(u_0^\varepsilon)'(0) - f_\varepsilon\big(u_0^\varepsilon(0)\big) = 0, \tag{3.1c}$$

where $\alpha(u) = a_\varepsilon(u)$. In [6] it is required that the functions $a(u)$, u_0 and φ_1 do already satisfy the smoothness conditions necessary for the existence of a smooth solution of the Problem A^ε.

The compatibility conditions are established there by setting $\varphi_1^\varepsilon \equiv \varphi_1$ and $u_0^\varepsilon(x) = u_0(x) + h^\varepsilon(x)$, where h^ε satisfies $\|h^\varepsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\varepsilon)$ with $\operatorname{supp} h^\varepsilon \subset [0,\varepsilon] \cup [1-\varepsilon, 1]$. Moreover, in that paper, the functions u_0 and φ_1 are assumed to satisfy a priori the compatibility conditions (3.1) with respect to $\alpha(u) = a(u)$, and the choice of h^ε ensures that (3.1) remains valid for $\alpha(u) = a(u) + \varepsilon$. Here, the regularity assumptions made in [6] are relaxed to (2.5) and (2.6). We set

$$x^\varepsilon := (x - 2\varepsilon)/(1 - 4\varepsilon), \quad t^\varepsilon := \big((t - 2\varepsilon)T\big)/(T - 2\varepsilon), \tag{3.2}$$

$$\widetilde{u_0}(x) := \begin{cases} u_0(1) & \text{for } x \geq 1 - 2\varepsilon, \\ u_0(x^\varepsilon) & \text{for } 2\varepsilon < x < 1 - 2\varepsilon, \\ -\varepsilon & \text{for } x \leq 2\varepsilon, \end{cases} \quad \widetilde{\varphi_1}(t) := \begin{cases} u_0(1) & \text{for } t \leq 2\varepsilon, \\ \varphi_1(t^\varepsilon) & \text{for } 2\varepsilon < t < T, \end{cases}$$

and define the regularized initial and boundary data by $u_0^\varepsilon(x) := (\widetilde{u_0} * \omega_\varepsilon)(x)$ for $x \in \overline{\Omega}$ and $\varphi_1^\varepsilon(t) := (\widetilde{\varphi_1} * \omega_\varepsilon)(t)$ for $t \in \overline{\mathcal{T}}$.

Lemma 2. *The functions u_0^ε and φ_1^ε satisfy the regularity assumptions necessary for the existence of a smooth solution of Problem $\mathcal{A}^\varepsilon$ and the first order compatibility conditions* (3.1)*. They also satisfy $u_0^\varepsilon(x) \in \mathcal{U}_\varepsilon$ for $x \in \overline{\Omega}$, $\varphi_1^\varepsilon(t) \in \mathcal{U}_0$ for $t \in \mathcal{T}$,*

$$\mathrm{TV}_{\mathcal{T}}(\varphi_1^\varepsilon) \leq \mathrm{TV}_{\mathcal{T}}(\varphi_1) + |u_0(1) - \varphi_0(0)|, \quad \mathrm{TV}_\Omega(u_0^\varepsilon) \leq \mathrm{TV}_\Omega + u_0(0) + \varepsilon.$$

Proof. The compatibility conditions (3.1) follow from

$$u_0^\varepsilon(1) = \varphi_1^\varepsilon(0) = u_0(1), \quad (u_0^\varepsilon)'(1) = (\varphi_1^\varepsilon)'(1) = 0, \quad (u_0^\varepsilon)'(0) = 0, \quad f_\varepsilon\big(u_0^\varepsilon(0)\big) = f_\varepsilon(-\varepsilon) = 0.$$

From $\omega_\varepsilon \geq 0$, we obtain $\varphi_1^\varepsilon \geq 0$ and $u_0^\varepsilon \geq -\varepsilon$. Since $u_{\max} - \widetilde{u_0}(x) \geq 0$ for all $x \in \mathbb{R}$, we have

$$0 \leq \big((u_{\max} - \widetilde{u_0}) * \omega_\varepsilon\big)(x) = u_{\max} - (\widetilde{u_0} * \omega_\varepsilon)(x) \quad \text{for } x \in \mathbb{R},$$

that is, $u_0^\varepsilon \leq u_{\max}$, and by the same argument, $\varphi_1^\varepsilon \leq u_{\max}$. Furthermore, we have

$$\begin{aligned} \mathrm{TV}_{\mathbb{R}}(u_0^\varepsilon) &= \mathrm{TV}_\Omega(u_0^\varepsilon), \quad \mathrm{TV}_{\mathbb{R}}(\varphi_1^\varepsilon) = \mathrm{TV}_{\mathcal{T}}(\varphi_1^\varepsilon), \\ \mathrm{TV}_{\mathbb{R}}(\widetilde{\varphi_1}) &= \mathrm{TV}_{\mathcal{T}}(\widetilde{\varphi_1}) = \mathrm{TV}_{\mathcal{T}}(\varphi_1) + |u_0(1) - \varphi_1(0)|, \\ \mathrm{TV}_{\mathbb{R}}(\widetilde{u_0}) &= \mathrm{TV}_\Omega(\widetilde{u_0}) = \mathrm{TV}_\Omega(u_0) + u_0(0) + \varepsilon. \end{aligned}$$

Following [16], we show that mollifying the functions $\widetilde{u_0}$ and $\widetilde{\varphi_1}$ does not increase their total variation, respectively. Recall that the total variation of a function $g \in L^1(\mathbb{R})$ can be expressed as

$$\mathrm{TV}(g) = \sup_{\varphi \in \mathcal{D}} \int_{\mathbb{R}} g(x)\varphi'(x)\,dx, \quad \mathcal{D} := \Big\{\varphi \in C_0^1(\mathbb{R}) : \ \|\varphi\|_\infty \leq 1\Big\}.$$

Then we have

$$\mathrm{TV}_{\mathbb{R}}(u_0^\varepsilon) = \sup_{\varphi \in \mathcal{D}} \int_{\mathbb{R}} \int_{-\varepsilon}^{\varepsilon} \widetilde{u_0}(x-y)\omega_\varepsilon(y)\varphi'(x)\,dydx = \sup_{\varphi \in \mathcal{D}} \int_{-\varepsilon}^{\varepsilon} \int_{\mathbb{R}} \widetilde{u_0}(x-y)\partial_x\big(\omega_\varepsilon(y)\varphi(x)\big)\,dxdy.$$

Using the substitution $\tilde{x} := x - y$, $\tilde{y} := -y$, the symmetry $\omega_\varepsilon(-y) = \omega_\varepsilon(y)$ and finally replacing $\tilde{x}$ by x and $\tilde{y}$ by y and setting $\varphi_\varepsilon(x) := (\varphi * \omega_\varepsilon)(x)$, we obtain from this

$$\mathrm{TV}_{\mathbb{R}}(u_0^\varepsilon) = \sup_{\varphi \in \mathcal{D}} \int_{-\varepsilon}^{\varepsilon} \int_{\mathbb{R}} \widetilde{u_0}(x)\partial_x\big(\omega_\varepsilon(y)\varphi(x-y)\big)\,dxdy = \sup_{\varphi \in \mathcal{D}} \int_{\mathbb{R}} \widetilde{u_0}(x)\varphi_\varepsilon'(x)\,dx,$$

and hence, noting that $\Big\{\varphi * \omega_\varepsilon : \varphi \in C_0^1(\mathbb{R}), \ \|\varphi\|_\infty \leq 1\Big\} \subset \mathcal{D}$,

$$\mathrm{TV}_{\mathbb{R}}(u_0^\varepsilon) \leq \sup_{\varphi \in \mathcal{D}} \int_{\mathbb{R}} \widetilde{u_0}(x)\varphi'(x)\,dx = \mathrm{TV}_{\mathbb{R}}(\widetilde{u_0}). \tag{3.3}$$

The inequality $\mathrm{TV}_{\mathbb{R}}(\varphi_1^\varepsilon) \leq \mathrm{TV}_{\mathbb{R}}(\widetilde{\varphi_1})$ follows in the same way. □

Lemma 3. *Let u^ε be a smooth solution of Problem $\mathcal{A}^\varepsilon$. Then there exist positive constants M_1, M_2 and M_3 such that the following estimates hold uniformly with respect to ε:*

$$\|u^\varepsilon(x,t)\|_{L^\infty(Q_T)} \leq M_1, \tag{3.4}$$

$$\|\partial_x u^\varepsilon(\cdot,t)\|_{L^1(\Omega)} \leq M_2 \text{ for all } t \in \mathcal{T}, \tag{3.5}$$

$$\|\partial_t u^\varepsilon\|_{L^1(Q_T)} \leq M_3. \tag{3.6}$$

Proof. For every $\varepsilon > 0$, Problem A^ε has a unique solution $u^\varepsilon \in C^{2+\beta,1+\beta/2}(Q_T) \subset C^{2,1}(Q_T)$, $\beta > 0$. This is shown in [6] by applying the well-known results from [17]. The uniform boundedness of u^ε can be shown in a standard way by rewriting Problem A^ε in terms of $\exp(-Kt)u^\varepsilon(x,t)$ and $\exp(-Kt)\left(u_{\max} + \varepsilon - u^\varepsilon(x,t)\right)$, where $K > 0$ is an arbitrary constant, and showing that these functions are nonnegative on $\overline{Q_T}$, see [6]. Hence we have $u^\varepsilon(x,t) \in \mathcal{U}_\varepsilon$ for $(x,t) \in \overline{Q_T}$; in particular, (3.4) is valid. Estimate (3.5) can be established here by following the derivation in [6], where more regularity was assumed on the initial and boundary data, and by arguing additionally that mollifying the data does not increase their respective one-dimensional total variations with respect to x and t. Similarly, estimate (3.6) can be proved by following the derivation in [6], where assumption (2.7) is required. Here, the derivation of these estimates will be performed in detail for Problem B (see § 3.2). $\square$

Estimates (3.4), (3.5) and (3.6) imply that the family $\{u^\varepsilon\}_{\varepsilon>0}$ of solutions of Problem A^ε is bounded in $W^{1,1}(Q_T) \subset BV(Q_T)$. Since $BV(Q_T)$ is compactly imbedded in $L^1(Q_T)$, there exists a sequence $\varepsilon = \varepsilon_n \downarrow 0$ such that $\{u^{\varepsilon_n}\}$ converges in $L^1(Q_T)$ to a function $u \in L^\infty(Q_T) \cap BV(Q_T)$. To show that u is an entropy solution of Problem A, we must show that the integrated diffusion function $A(u)$ possesses the required regularity.

Lemma 4. *The limit u of solutions u^ε of Problem A^ε satisfies condition* (2.8).

Proof. Multiply equation (A^ε.1) by $\left(u^\varepsilon - \varphi_1^\varepsilon(t)\right)$, integrate over Q_T and use the boundary conditions to obtain (see also [6])

$$\iint_{Q_T} a_\varepsilon(u^\varepsilon)(\partial_x u^\varepsilon)^2\,dtdx = -\int_0^1 \frac{1}{2}(u^\varepsilon)^2 - \varphi_1^\varepsilon(t)u^\varepsilon\Big|_0^T\,dx - \iint_{Q_T} u^\varepsilon\left(\varphi_1^\varepsilon\right)'(t)\,dtdx$$
$$+\iint_{Q_T}\partial_x u^\varepsilon g_\varepsilon(u^\varepsilon,t)\,dtdx + \int_0^T\left(u^\varepsilon(0,t) - \varphi_1^\varepsilon(t)\right)q_\varepsilon(t)u^\varepsilon(0,t)\,dt. \quad (3.7)$$

It is easy to see that (3.7) implies that

$$\iint_{Q_T} a_\varepsilon(u^\varepsilon)(\partial_x u^\varepsilon)^2\,dtdx \le 5M_1^2 + M_1M_3 + \|g_\varepsilon\|_\infty TM_2 + \|q_\varepsilon\|_\infty TM_1^2,$$

so that $\|a_\varepsilon^{1/2}(u^\varepsilon)\partial_x u^\varepsilon\|_{L^2(Q_T)}$ is uniformly bounded with respect to ε. However, since $a(u^\varepsilon)$ is bounded, we can conclude that $\|\partial_x A_\varepsilon(u^\varepsilon)\|_{L^2(Q_T)}$ is also bounded. Therefore, passing if necessary to a subsequence, $A_\varepsilon(u^\varepsilon) \to \overline{A}$ in $L^2(Q_T)$ and $\partial_x A_\varepsilon(u^\varepsilon) \rightharpoonup \partial_x\overline{A}$ weakly in $L^2(Q_T)$ as $\varepsilon \downarrow 0$. Since $A_\varepsilon(u^\varepsilon) \to A(u)$ a.e. as $\varepsilon \downarrow 0$, we conclude that $\overline{A} = A(u)$ a.e. and thus condition (2.8) holds. $\square$

Lemma 5. *The vanishing viscosity limit u of solutions u^ε of Problem A^ε satisfies the entropy inequality* (2.9) *and the boundary condition* (2.10).

For the proof, we need the following lemma given in [24].

Lemma 6. *If $v \in L^1(Q_T)$ and $\partial_x v$ is an absolutely continuous measure, then for $\varphi \in C^\infty\left(\overline{Q_T}\right)$ with* $\operatorname{supp}\varphi \subset \overline{\Omega}\times\mathcal{T}$ *there holds*

$$\iint_{Q_T}\partial_x\varphi v\,dtdx = \int_0^T\left(\gamma_1 v\varphi(1,t) - \gamma_0 v\varphi(0,t)\right)dt - \iint_{Q_T}\varphi\partial_x v\,dtdx.$$

Proof of Lemma 5. We multiply the viscous equation (A^ε.1) by $\operatorname{sgn}_\eta(u^\varepsilon - k)\varphi$, $\varphi \in C^\infty(Q_T)$, $\varphi \ge 0$, $\operatorname{supp}\varphi \subset (0,1]\times\mathcal{T}$, $k \in \mathbb{R}$, integrate over Q_T and use integration by parts to obtain

$$\begin{aligned}
&-\iint_{Q_T}|u^\varepsilon - k|_\eta\partial_t\varphi\,dtdx + \int_0^T \operatorname{sgn}_\eta\left(u^\varepsilon(1,t)-k\right)\left[g_\varepsilon(u^\varepsilon(1,t),t) - g_\varepsilon(k,t)\right]\varphi(1,t)\,dt\\
&-\iint_{Q_T}\operatorname{sgn}_\eta(u^\varepsilon - k)\left(g_\varepsilon(u^\varepsilon,t) - g_\varepsilon(k,t)\right)\partial_x\varphi\,dtdx \qquad (3.8)\\
&-\iint_{Q_T}\operatorname{sgn}_\eta'(u^\varepsilon - k)\partial_x u^\varepsilon\left(g_\varepsilon(u^\varepsilon,t) - g_\varepsilon(k,t)\right)\varphi\,dtdx
\end{aligned}$$

$$= \int_0^T \operatorname{sgn}_\eta\big(u^\varepsilon(1,t)-k\big)\partial_x\big(A_\varepsilon(u^\varepsilon)\big)(1,t)\varphi(1,t)\,dt \tag{3.9}$$
$$- \iint_{Q_T} \operatorname{sgn}_\eta(u^\varepsilon-k)\partial_x\big(A_\varepsilon(u^\varepsilon)-A_\varepsilon(k)\big)\partial_x\varphi\,dtdx - \iint_{Q_T} \operatorname{sgn}'_\eta(u^\varepsilon-k)(\partial_x u^\varepsilon)^2 a_\varepsilon(u^\varepsilon)\varphi\,dtdx.$$

The last integral on the left-hand side of (3.9) vanishes when $\eta\downarrow 0$, while the last one is nonnegative. Noting that $\nu_h(1)=1$ and $\nu'_h(1)=0$, we obtain from this when $\eta\downarrow 0$ the inequality

$$\iint_{Q_T}\Big\{|u^\varepsilon-k|\,\partial_t\varphi + \operatorname{sgn}(u^\varepsilon-k)\big[g_\varepsilon(u^\varepsilon,t)-g_\varepsilon(k,t)\big]\partial_x\varphi\Big\}\,dtdx$$
$$-\int_0^T \operatorname{sgn}\big(\varphi_1^\varepsilon(t)-k\big)\big(g_\varepsilon(u^\varepsilon(1,t),t)-g_\varepsilon(k,t)-\partial_x A_\varepsilon(u^\varepsilon)(1,t)\big)\varphi(1,t)\nu_h(1)\,dt \tag{3.10}$$
$$-\int_0^T \operatorname{sgn}\big(\varphi_1^\varepsilon(t)-k\big)\big[A_\varepsilon(u^\varepsilon)-A_\varepsilon(k)\big]\partial_x\big(\varphi(x,t)\nu_h(x)\big)(1,t)\,dt$$
$$+\iint_{Q_T}\operatorname{sgn}\,(u^\varepsilon-k)\,\big(A_\varepsilon(u^\varepsilon)-A_\varepsilon(k)\big)\partial_x^2\varphi\,dtdx \ge 0.$$

We have

$$\iint_{Q_T}\operatorname{sgn}\,(u^\varepsilon-k)\,\big(A_\varepsilon(u^\varepsilon)-A_\varepsilon(k)\big)\partial_x^2\varphi\,dtdx = \int_0^T \operatorname{sgn}\,(u^\varepsilon(1,t)-k)\,\times$$
$$\times\,\big(A_\varepsilon(u^\varepsilon(1,t))-A_\varepsilon(k)\big)(\partial_x\varphi)(1,t)\,dt - \iint_{Q_T}\partial_x\Big(\operatorname{sgn}(u^\varepsilon-k)\big(A_\varepsilon(u^\varepsilon)-A_\varepsilon(k)\big)\Big)\partial_x\varphi\,dtdx,$$

which, by using Lemma 1 and the fact that

$$\partial_x\Big(\operatorname{sgn}(u-k)\big(A(u)-A(k)\big)\Big) = \operatorname{sgn}(u-k)\partial_x\big(A(u)-A(k)\big)\ \text{ in the sense of measures,}$$

yields

$$\iint_{Q_T}\operatorname{sgn}(u^\varepsilon-k)\big(A_\varepsilon(u^\varepsilon)-A_\varepsilon(k)\big)\partial_x^2\varphi\,dtdx$$
$$\xrightarrow{\varepsilon\downarrow 0}\int_0^T \operatorname{sgn}(\gamma_1 u-k)\big(A(\gamma_1 u)-A(k)\big)(\partial_x\varphi)(1,t)\,dt - \iint_{Q_T}\operatorname{sgn}(u-k)\partial_x\big(A(u)-A(k)\big)\partial_x\varphi\,dtdx.$$

Moreover,

$$-\int_0^T \operatorname{sgn}\big(\varphi_1^\varepsilon(t)-k\big)\big(A_\varepsilon(u^\varepsilon)-A_\varepsilon(k)\big)\partial_x\big(\varphi(x,t)\nu_h(x)\big)(1,t)\,dt$$
$$= -\iint_{Q_T}\operatorname{sgn}\big(\varphi_1^\varepsilon(t)-k\big)\big(A_\varepsilon(u^\varepsilon)-A_\varepsilon(k)\big)\partial_x^2(\varphi\nu_h)\,dtdx$$
$$-\iint_{Q_T}\partial_x\Big(\operatorname{sgn}\big(\varphi_1^\varepsilon(t)-k\big)\big(A_\varepsilon(u^\varepsilon)-A_\varepsilon(k)\big)\Big)\partial_x(\varphi\nu_h)\,dtdx$$
$$\xrightarrow{\varepsilon\downarrow 0} -\iint_{Q_T}\operatorname{sgn}\big(\varphi_1(t)-k\big)\big(A(u)-A(k)\big)\partial_x^2(\varphi\nu_h)\,dtdx$$
$$-\iint_{Q_T}\operatorname{sgn}\big(\varphi_1(t)-k\big)\partial_x\big(A(u)-A(k)\big)\partial_x(\varphi\nu_h)\,dtdx.$$

Taking the limits $\varepsilon\downarrow 0$ and $h\downarrow 0$ and using Lemma 6, we obtain inequality (2.9) from (3.10). To verify that the limit satisfies the boundary condition (2.10), we multiply equation (A$^\varepsilon$.4) by a test function $\Phi\in C_0^\infty(\mathcal{T})$ and integrate over $\mathcal{T}$ to obtain

$$0 = \int_0^T \big(f_\varepsilon(u^\varepsilon)-\partial_x A_\varepsilon(u^\varepsilon)\big)(0,t)\Phi(t)\,dt$$
$$= -\iint_{Q_T}\partial_x\big(f^\varepsilon(u^\varepsilon)-\partial_x A_\varepsilon(u^\varepsilon)\big)\Phi(t)\mu_h(x)\,dtdx - \iint_{Q_T}\big(f_\varepsilon(u^\varepsilon)-\partial_x A(u^\varepsilon)\big)\Phi(t)\mu'_h(x)\,dtdx$$

$$= \iint_{Q_T} \left(\partial_t u^\varepsilon + q_\varepsilon(t)\partial_x u^\varepsilon\right)\Phi(t)\mu_h(x)\,dtdx - \iint_{Q_T} \left(f_\varepsilon(u^\varepsilon) - \partial_x A_\varepsilon(u^\varepsilon)\right)\Phi(t)\mu_h(x)\,dtdx. \quad (3.11)$$

The first integral on the right-hand side of (3.11) vanishes for $h \downarrow 0$. The boundary condition at $x = 0$ follows then from

$$-\iint_{Q_T} \left(f_\varepsilon(u^\varepsilon) - \partial_x A_\varepsilon(u^\varepsilon)\right)\Phi(t)\mu_h(x)\,dtdx$$

$$\xrightarrow{\varepsilon\downarrow 0} -\iint_{Q_T} \left(f(u) - \partial_x A(u)\right)\Phi(t)\mu_h'(x)\,dtdx \xrightarrow{h\downarrow 0} \int_0^T \gamma_0\left(f(u) - \partial_x A(u)\right)\Phi(t)\,dt. \qquad \square$$

Lemma 7. *The limit function u of solutions u^ε of Problem A^ε satisfies the initial condition* (2.11).

To prove Lemma 7, we need the following variant of Kružkov's lemma [13] proved in [12].

Lemma 8. *Assume that there exist finite constants c_1 and c_2 such that the function $u : \Omega\times\overline{\mathcal{T}} \to \mathbb{R}$ satisfies $\|u(\cdot,t)\|_\infty \le c_1$ and $\mathrm{TV}_\Omega(u(\cdot,t)) \le c_2$ for all $t \in \overline{\mathcal{T}}$, and that $u(x,t)$ is weakly Lipschitz continuous in the time variable in the sense that*

$$\left|\int_0^1 \left(u(x,t_2) - u(x,t_1)\right)\varphi(x)\,dx\right| \le \mathcal{O}(t_2 - t_1)\sum_{i=0}^n \left\|\varphi^{(i)}\right\|_{L^\infty(\Omega)} \quad \forall\varphi \in C_0^n(\Omega), \quad 0 \le t_1 \le t_2 \le T.$$

Then there exists a constant c, depending in particular on c_1 and c_2, such that the following interpolation result is valid:

$$\|u(\cdot,t_2) - u(\cdot,t_2)\|_{L^1(\Omega)} \le c(t_2 - t_1)^{1/(n+1)}, \quad 0 \le t_1 \le t_2 \le T. \quad (3.12)$$

Proof of Lemma 7. Multiplying equation (A^ε.1) with a test function $\varphi \in C_0^2(Q_T)$ and using integration by parts it is easy to see that the statement of Lemma 8 holds with $n = 2$, i.e. there exists a constant c such that $\|u^\varepsilon(\cdot,\tau) - u_0^\varepsilon\|_{L^1(\Omega)} \le c\tau^{1/3}$ holds uniformly in ε for sufficiently small $\tau > 0$. This implies for $\tau \downarrow 0$ and $\varepsilon \downarrow 0$ that the initial condition (2.11) satisfied. $\square$

As a consequence of Lemmas 2 to 5 and 7, we obtain

Theorem 1. *Under the assumptions* (2.2)–(2.7)*, Problem A admits an entropy solution u.*

3.2. **Existence of entropy solutions of Problem B.** To show existence of entropy solutions of Problem B, we consider the regularized parabolic IBVP B^ε, which is obtained from Problem A^ε if the boundary condition (A^ε.3) is replaced by

$$\left(q_\varepsilon(t)u^\varepsilon + f_\varepsilon(u^\varepsilon) - \partial_x A_\varepsilon(u^\varepsilon)\right)(1,t) = \Psi_\varepsilon(t), \quad t \in (0,T]. \quad (B^\varepsilon.3)$$

Here f_ε and A_ε denote the same regularizations as before. Obviously, the definition of u_0^ε has to be modified slightly; here we set $u_0^\varepsilon(x) := (\widetilde{u_0} * \omega_\varepsilon)(x)$ and $q_\varepsilon(t) := (\tilde{q} * \omega_\varepsilon)(t)$, where

$$\widetilde{u_0}(x) := \begin{cases} -\varepsilon & \text{for } x \ge 1 - 2\varepsilon, \\ u_0(x^\varepsilon) & \text{for } 2\varepsilon < x < 1 - 2\varepsilon, \\ -\varepsilon & \text{for } x \le 2\varepsilon, \end{cases} \quad \tilde{q}(t) := \begin{cases} q(0) & \text{for } t \le 2\varepsilon, \\ q(t^\varepsilon) & \text{for } 2\varepsilon < t < T, \end{cases}$$

and $x^\varepsilon(x)$ and $t^\varepsilon(t)$ are defined in (3.2). The first-order compatibility conditions appropriate for Problem B^ε are then given by (3.1c) and the condition

$$q_\varepsilon(0)u_0^\varepsilon(1) + f_\varepsilon\left(u_0^\varepsilon(1)\right) - a_\varepsilon\left(u_0^\varepsilon(1)\right)\left(u_0^\varepsilon\right)'(1) = \Psi_\varepsilon(0) \quad (3.13)$$

valid at $x = 1$, $t = 0$. This condition is satisfied if we set $\Psi_\varepsilon(t) := (\widetilde{\Psi} * \omega_\varepsilon)(t)$, where

$$\widetilde{\Psi}(t) := \begin{cases} 0 & \text{for } t \le 2\varepsilon, \\ \Psi(t^\varepsilon) & \text{for } 2\varepsilon < t < T. \end{cases}$$

As in the previous case, mollifying the functions $\widetilde{u_0}$, $\tilde{q}$ and $\widetilde{\Psi}$ does not increase their respective total variations. By the classical theory of quasilinear parabolic equations, also Problem B^ε has a smooth solution $u^\varepsilon \in C^{2+\beta,1+\beta/2}(Q_T)$ for a fixed value of $\varepsilon > 0$.

Lemma 9. *Let u^ε be a solution of Problem B^ε. Then there exist positive constants C_1 and C_2 independent of ε satisfying*

$$-C_1\varepsilon \le u^\varepsilon(x,t) \le u_{\max} + C_2\varepsilon \quad \text{for } (x,t) \in \overline{Q_T}. \tag{3.14}$$

In particular, there exists a constant M_4 such that $\|u^\varepsilon\|_{L^\infty(Q_T)} \le M_4$ holds uniformly in ε.

Proof. The maximum principle can be applied in a similar way as for Problem A and as in [6], but the treatment at the boundary $x = 1$ is different. Suppose that u^ε assumes a maximum at $(x = 1, t = t_0)$, $0 < t_0 \le T$. Then $\partial_x u^\varepsilon(1,t_0) \ge 0$ must be valid; without loss of generality we may assume that $\partial_x u^\varepsilon(1,t_0) > 0$. Inserting this assumption into (B^ε.3), which can be expressed as

$$\partial_x u^\varepsilon(1,t) = \big[g_\varepsilon\big(u^\varepsilon(1,t),t\big) - \Psi_\varepsilon(t)\big]/a_\varepsilon\big(u^\varepsilon(1,t)\big), \tag{3.15}$$

reveals that then $g_\varepsilon(t)\big(u^\varepsilon(1,t_0),t_0\big) > \Psi_\varepsilon(t_0)$ holds. Due to the regularity assumptions on $f(u)$ and $q(t)$, we may conclude from this that

$$g\big(u^\varepsilon(1,t_0),t_0\big) > \Psi(t_0) + \mathcal{O}(\varepsilon). \tag{3.16}$$

Since $g(u,t) \le \Psi(t)$ for $u \ge u_{\max}$, inequality (3.16) implies that $u^\varepsilon(1,t_0) \le u_{\max} + \mathcal{O}(\varepsilon)$.

Now assume that u^ε assumes a local minimum at $(1,t_0)$, this implies $\partial_x u^\varepsilon(1,t_0) \le 0$; again we have to consider only the case $\partial_x u^\varepsilon(1,t_0) < 0$. This assumption yields $g_\varepsilon\big(u^\varepsilon(1,t_0),t_0\big) < \Psi_\varepsilon(t_0)$. In view of $\Psi(t) \le q(t)u = g(u,t)|_{u<0}$, this can not hold for $-u > \mathcal{O}(\varepsilon)$, and we conclude that $u(1,t_0) \ge \mathcal{O}(\varepsilon)$ is valid. These arguments, combined with the discussion of extrema of u^ε on the remaining parts of $\overline{Q_T}$ following the analysis of Problem A, imply that estimate (3.14) is valid. □

To derive estimates on the derivatives of u^ε, we first need to prove the following lemma.

Lemma 10. *Let u be the limit function of solutions u^ε of Problem B^ε. Then $\partial_x A(u) \in L^2(Q_T)$.*

Proof. Multiplying equation (A^ε.1) by u^ε and integrating over Q_T, we obtain

$$\begin{aligned}\iint_{Q_T} a_\varepsilon(u^\varepsilon)(\partial_x u^\varepsilon)^2\,dtdx &= \int_0^T u^\varepsilon\big(a_\varepsilon(u^\varepsilon)\partial_x u^\varepsilon - g_\varepsilon(u^\varepsilon,t)\big)\Big|_0^1\,dt - \frac{1}{2}\iint_{Q_T}\partial_t(u^\varepsilon)^2\,dtdx \\ &\quad - \iint_{Q_T} g_\varepsilon(u^\varepsilon,t)\partial_x u^\varepsilon\,dtdx \\ &= \int_0^T \big(u^\varepsilon(1,t)\Psi_\varepsilon(t) + q_\varepsilon(t)u^\varepsilon(0,t)\big)\,dt - \frac{1}{2}\int_0^1 (u^\varepsilon)^2\Big|_0^T dx - \int_0^T G_\varepsilon(u^\varepsilon,t)\Big|_0^1 dt,\end{aligned}$$

where $G_\varepsilon(u,t) := \int_0^u g_\varepsilon(s,t)\,ds$. Obviously, we have the uniform estimate

$$\iint_{Q_T} a_\varepsilon(u^\varepsilon)(\partial_x u^\varepsilon)^2\,dtdx \le TM_4\big(\|\Psi_\varepsilon\|_\infty + \|q_\varepsilon\|_\infty + M_4 + 2\|G_\varepsilon\|_\infty\big) =: M_5, \tag{3.17}$$

hence $\partial_x A_\varepsilon(u^\varepsilon) \in L^2(Q_T)$ independently of ε and the conclusion of Lemma 10 follows as in the proof of Lemma 4. □

We note that the regularity result expressed in Lemma 10 will be significantly improved in § 3.3.

Lemma 11. *Let u^ε be a solution of Problem B^ε.*

a) *In the case where $\Psi \equiv 0$, there exists a constant M_6 such that the following estimate holds uniformly in ε:*

$$\|\partial_x u^\varepsilon(\cdot,t)\|_{L^1(\Omega)} \le M_6 \text{ for all } t \in \mathcal{T}. \tag{3.18}$$

b) *In the case where $\xi a(u) - \big(q(t) + f'(u)\big) \ge M_g$ for some positive constants ξ, M_g, there exists a constant M_6 such that the following estimate holds uniformly in ε:*

$$\|\partial_x u^\varepsilon\|_{L^1(Q_T)} \le M_6. \tag{3.19}$$

In both cases, there exists a constant M_7 such that the following uniform estimate is valid:

$$\|\partial_t u^\varepsilon(\cdot,t)\|_{L^1(\Omega)} \le M_7 \text{ for all } t \in \mathcal{T}. \tag{3.20}$$

Proof. Let approximations sgn_η and $|\cdot|_\eta$ of the sign and modulus functions be given by

$$\mathrm{sgn}_\eta(\tau) := \begin{cases} \mathrm{sgn}(\tau) & \text{if } |\tau| > \eta, \\ \tau/\eta & \text{if } |\tau| \le \eta, \end{cases} \quad |x|_\eta = \int_0^x \mathrm{sgn}_\eta(\xi)\, d\xi, \quad \eta > 0. \tag{3.21}$$

We first consider the estimate on $\partial_t u^\varepsilon$. We define $v^\varepsilon := \partial_t u^\varepsilon$ and $w^\varepsilon := \partial_x u^\varepsilon$ and differentiate equation (A^ε.1) with respect to t to obtain

$$\partial_t v^\varepsilon = \partial_t \Big(\partial_x \big(-g_\varepsilon(u^\varepsilon, t) + a_\varepsilon(u^\varepsilon) w^\varepsilon \big) \Big) = \partial_x \Big(\partial_t \big(-g_\varepsilon(u^\varepsilon, t) + a_\varepsilon(u^\varepsilon) w^\varepsilon \big) \Big). \tag{3.22}$$

Multiplying (3.22) by $\mathrm{sgn}_\eta(\varphi^\varepsilon)$, integrating over $Q_{T_0} := \Omega \times (0, T_0)$, $0 < T_0 \le T$, integrating by parts and using the boundary conditions yields

$$\begin{aligned} \iint_{Q_{T_0}} \partial_t |v^\varepsilon|_\eta \, dt dx &\le \int_0^{T_0} \mathrm{sgn}_\eta(v^\varepsilon) \big(-\Psi_\varepsilon'(t) + q_\varepsilon'(t) u^\varepsilon(1,t) + q_\varepsilon(t) v^\varepsilon(1,t) \big)\, dt \\ &- \iint_{Q_{T_0}} \mathrm{sgn}_\eta'(v^\varepsilon) \partial_x v^\varepsilon \big(a_\varepsilon'(u^\varepsilon) w^\varepsilon - f_\varepsilon'(u^\varepsilon) - q_\varepsilon(t) \big) v^\varepsilon \, dt dx - \iint_{Q_{T_0}} \mathrm{sgn}_\eta'(v^\varepsilon) a_\varepsilon(u^\varepsilon) (\partial_x v^\varepsilon)^2 \, dt dx \\ &+ \int_0^{T_0} \mathrm{sgn}_\eta(v^\varepsilon) q_\varepsilon(t) u^\varepsilon \Big|_0^1 \, dt - \iint_{Q_{T_0}} \mathrm{sgn}_\eta(v^\varepsilon) q_\varepsilon'(t) w^\varepsilon \, dt dx =: I_\eta^1 + I_\eta^2 + I_\eta^3 + I_\eta^4 + I_\eta^5. \end{aligned} \tag{3.23}$$

Observe that

$$I_\eta^1 \xrightarrow{\eta \downarrow 0} \int_0^{T_0} \mathrm{sgn}\big(v^\varepsilon(1,t)\big) \big(\Psi_\varepsilon'(t) + q_\varepsilon'(t) \big)\, dt + \int_0^{T_0} q_\varepsilon(t) |v^\varepsilon(1,t)| \, dt \le \mathrm{TV}_{\mathcal{T}}(\Psi_\varepsilon) + \mathrm{TV}_{\mathcal{T}}(q_\varepsilon).$$

By Saks' lemma, $I_\eta^2 \xrightarrow{\eta \downarrow 0} 0$; and $I_\eta^3 \le 0$. Finally, we have $|I_\eta^4| \le 2T \|q_\varepsilon\|_\infty (u_{\max} + \varepsilon)$ and

$$I_\eta^5 \xrightarrow{\eta \downarrow 0} I_0^5 := \int_0^{T_0} q_\varepsilon(t) \int_0^1 \mathrm{sgn}(v^\varepsilon) w^\varepsilon \, dx dt. \tag{3.24}$$

To evaluate the integral I_0^5, we have to derive the estimate on $\partial_x u^\varepsilon$.

a) Let $\Psi \equiv 0$. To obtain an estimate on $\partial_x u^\varepsilon$, differentiate (A^ε.1) with respect to x. Setting $w^\varepsilon := \partial_x u^\varepsilon$, we get

$$\partial_t w^\varepsilon + \partial_x^2 \big(q_\varepsilon(t) u^\varepsilon + f_\varepsilon(u^\varepsilon) \big) = \partial_x^2 \left(a_\varepsilon(u^\varepsilon) w^\varepsilon \right). \tag{3.25}$$

Multiplying equation (3.25) by $\mathrm{sgn}_\eta(w^\varepsilon)$, integrating over Q_T and using integration by parts yields

$$\begin{aligned} \iint_{Q_T} \mathrm{sgn}_\eta(w^\varepsilon) \partial_t w^\varepsilon \, dt dx &= \iint_{Q_T} \mathrm{sgn}_\eta(w^\varepsilon) \partial_x^2 \big(a_\varepsilon(u^\varepsilon) w^\varepsilon - g_\varepsilon(u^\varepsilon, t) \big) \, dt dx \\ = \int_0^T \mathrm{sgn}_\eta(w^\varepsilon) &\partial_x \big(a_\varepsilon(u^\varepsilon) w^\varepsilon - g_\varepsilon(u^\varepsilon, t) \big) \Big|_0^1 \, dt + \iint_{Q_T} \mathrm{sgn}_\eta'(w^\varepsilon) \partial_x w^\varepsilon (\partial_u g_\varepsilon)(u^\varepsilon, t) w^\varepsilon \, dt dx \\ &- \iint_{Q_T} \mathrm{sgn}_\eta'(w^\varepsilon) \partial_x w^\varepsilon a_\varepsilon'(u^\varepsilon) \left(w^\varepsilon \right)^2 \, dt dx - \iint_{Q_T} \mathrm{sgn}_\eta'(w^\varepsilon) a_\varepsilon(u^\varepsilon) \left(\partial_x w^\varepsilon \right)^2 \, dt dx. \end{aligned}$$

From the nonnegativity of the last integral, from the initial condition and from equation (A^ε.1), we obtain

$$\begin{aligned} \int_0^1 \left| w^\varepsilon(x, T) \right|_\eta \, dx &\le \int_0^1 \left| (u_0^\varepsilon)'(x) \right|_\eta \, dx + \int_0^T \mathrm{sgn}_\eta \left(w^\varepsilon(1,t) \right) \partial_t u^\varepsilon(1,t) \, dt \\ &- \int_0^T \left(w^\varepsilon(0,t) \right) \partial_t u^\varepsilon(0,t) \, dt + \iint_{Q_T} \mathrm{sgn}_\eta'(w^\varepsilon) \partial_x w^\varepsilon (\partial_u g_\varepsilon)(u(u^\varepsilon, t)) w^\varepsilon \, dt dx \\ &- \iint_{Q_T} \mathrm{sgn}_\eta'(w^\varepsilon) \partial_x w^\varepsilon \partial_x a_\varepsilon(u^\varepsilon) w^\varepsilon \, dt dx =: I_\eta^6 + I_\eta^7 + I_\eta^8 + I_\eta^9 + I_\eta^{10}. \end{aligned} \tag{3.26}$$

From Saks' lemma (see [2, 18]) we infer that $I_\eta^9 \to 0$ and $I_\eta^{10} \to 0$ for $\eta \downarrow 0$. By the boundary condition (B^ε.3), we have for $\Psi \equiv 0$ that $w^\varepsilon(1,t) = g_\varepsilon\big(u^\varepsilon(1,t), t\big) / a_\varepsilon\big(u^\varepsilon(1,t)\big)$. We have therefore either $w^\varepsilon(1,t) < 0$ or $w^\varepsilon(1,t) = 0$. However, the latter is true if and only if $u^\varepsilon(1,t)$ assumes the constant value $-\varepsilon$ or $u_{\max}+\varepsilon$. Letting $E=\{t\in[0,T]: u^\varepsilon(1,t)=-\varepsilon \text{ or } u^\varepsilon(1,t)=u_{\max}+\varepsilon\}$, we note that $\partial_t u^\varepsilon(1,t)=0$ a.e. in E. We therefore conclude that

$$I_\eta^7 \xrightarrow{\eta\downarrow 0} \int_0^T \operatorname{sgn}(w^\varepsilon(1,t))\partial_t u^\varepsilon(1,t)\,dt = -\int_0^T \partial_t u^\varepsilon(1,t)\,dt = u^\varepsilon(1,0)-u^\varepsilon(1,T). \tag{3.27}$$

Applying the same argument to the boundary condition (A^ε.4), we have

$$I_\eta^8 \xrightarrow{\eta\downarrow 0} -\int_0^T \operatorname{sgn}(w^\varepsilon(0,t))\partial_t u^\varepsilon(0,t)\,dt = \int_0^T \partial_t u^\varepsilon(0,t)\,dt = u^\varepsilon(0,T)-u^\varepsilon(0,0). \tag{3.28}$$

From (3.26) we obtain then for $\eta\downarrow 0$:

$$\|\partial_x u^\varepsilon(\cdot,T)\|_{L^1(\Omega)} \le \|(u_0^\varepsilon)'\|_{L^1(\Omega)} + u^\varepsilon(0,T)-u^\varepsilon(1,T) \le \|(u_0^\varepsilon)'\|_{L^1(\Omega)} + u_{\max}+\varepsilon,$$

which proves estimate (3.18). Inserting this into (3.24) shows that $I_0^5 \le T_0\|q_\varepsilon\|_\infty M_6$. Consequently, the right-hand part of the limit for $\eta\downarrow 0$ of (3.23) is uniformly bounded in ε. Estimate (3.20) follows since $u_0\in\mathcal{B}$ and hence $\|v^\varepsilon(\cdot,0)\|_{L^1(\Omega)}$ is uniformly bounded.

b) In this part of the proof, we follow Wu [23]. We now assume that the second alternative of (2.12) holds, from which we may infer that

$$\xi a_\varepsilon(u^\varepsilon) - \big(q_\varepsilon + f_\varepsilon'(u^\varepsilon)\big) \ge \tilde M_g,\quad \tilde M_g = M_g + \mathcal{O}(\varepsilon) > 0. \tag{3.29}$$

Multiplying equation (A^ε.1) by $-\operatorname{sgn}_\eta(w^\varepsilon)$ and integrating over Ω yields

$$\begin{aligned}\int_0^1 \operatorname{sgn}_\eta(w^\varepsilon)\big(-q_\varepsilon(t)-f_\varepsilon(u^\varepsilon)\big)w^\varepsilon\,dx &= -\operatorname{sgn}\big(w^\varepsilon(1,t)\big)\big(g_\varepsilon(u^\varepsilon(1,t))-\Psi_\varepsilon(t)\big)\\ &\quad + \operatorname{sgn}_\eta\big(w^\varepsilon(0,t)\big)f_\varepsilon\big(u^\varepsilon(0,t)\big) + \int_0^1 \operatorname{sgn}_\eta'(w^\varepsilon)\partial_x w^\varepsilon a_\varepsilon(u^\varepsilon)w^\varepsilon\,dx + \int_0^1 \operatorname{sgn}_\eta(w^\varepsilon)v^\varepsilon\,dx.\end{aligned} \tag{3.30}$$

Note that first integral on the right-hand part of (3.30) vanishes due to Saks' lemma. For $\eta\downarrow 0$, we then obtain from (3.30)

$$\int_0^1 \big(-q_\varepsilon(t)-f_\varepsilon'(u^\varepsilon)\big)|w^\varepsilon|\,dx \le 2\|f_\varepsilon\|_\infty + \|q_\varepsilon\|_\infty + \|\Psi_\varepsilon\|_\infty + \int_0^1 |v^\varepsilon|\,dx. \tag{3.31}$$

Integrating (3.31) over $[0,T_0]$, we obtain

$$\iint_{Q_{T_0}} \big(-q_\varepsilon(t)-f_\varepsilon'(u^\varepsilon)\big)|w^\varepsilon|\,dtdx \le T_0\big(2\|f_\varepsilon\|_\infty + \|q_\varepsilon\|_\infty + \|\Psi_\varepsilon\|_\infty\big) + \iint_{Q_{T_0}} |v^\varepsilon|\,dtdx. \tag{3.32}$$

From (3.17) we obtain

$$\begin{aligned}\iint_{Q_{T_0}} a_\varepsilon(u^\varepsilon)|w^\varepsilon|\,dtdx &\le \left(\iint_{Q_T} a_\varepsilon(u^\varepsilon)\,dtdx\right)^{1/2}\left(\iint_{Q_T} a_\varepsilon(u^\varepsilon)(\partial_x u^\varepsilon)^2\,dtdx\right)^{1/2}\\ &\le \big(T\|a_\varepsilon\|_\infty\big)^{1/2} M_5^{1/2} =: M_8.\end{aligned} \tag{3.33}$$

Consequently, adding $\iint_{Q_{T_0}} \xi a_\varepsilon(u^\varepsilon)|w^\varepsilon|\,dtdx$ to both sides of (3.32) yields

$$\begin{aligned}&\iint_{Q_{T_0}} \big(\xi a_\varepsilon(u^\varepsilon) - q_\varepsilon(t) - f_\varepsilon'(u^\varepsilon)\big)|w^\varepsilon|\,dtdx\\ &\qquad\le \xi M_8 + T_0\big(2\|f_\varepsilon\|_\infty + \|q_\varepsilon\|_\infty + \|\Psi_\varepsilon\|_\infty\big) + \iint_{Q_{T_0}} |v^\varepsilon|\,dtdx.\end{aligned}$$

In view of (3.29), we finally obtain

$$\iint_{Q_{T_0}} |w^\varepsilon|\,dtdx \le M_9 + \frac{1}{\tilde M_g}\iint_{Q_{T_0}} |v^\varepsilon|\,dtdx, \tag{3.34}$$

where $M_9 := \big[\xi M_8 + T_0\big(2\|f_\varepsilon\|_\infty + \|q_\varepsilon\|_\infty + \|\Psi_\varepsilon\|_\infty\big)\big]/\tilde{M}_g$. Using (3.34) we obtain from (3.24) that

$$I_0^5 \leq \|q_\varepsilon\|_\infty \left(M_9 + \frac{1}{\tilde{M}_g} \iint_{Q_{T_0}} |v^\varepsilon|\, dt dx \right).$$

Using this estimate in (3.23) and sending $\eta \downarrow 0$, we see that v^ε satisfies the inequality

$$\int_0^1 |v^\varepsilon(x,T_0)|\, dx \leq \int_0^1 |v^\varepsilon(x,0)|\, dx + M_{10} + M_{11} \int_0^{T_0} \int_0^1 |v^\varepsilon(x,t)|\, dt dx, \tag{3.35}$$

for some suitable constants M_{10} and M_{11}. Note that the first integral on the right-hand side is bounded since we assume $u_0 \in \mathcal{B}$. Using Gronwall's lemma, we obtain from (3.35) the desired estimate (3.20). Finally, using (3.20) in (3.34) for $T_0 = T$ shows that (3.19) is also valid. □

Remark 2. *Note that we have not been able to establish that* $\|\partial_x u^\varepsilon(\cdot,t)\|_{L^1(\Omega)}$ *is uniformly bounded when the second alternative of* (2.12) *holds.*

As in § 3.1, we may conclude from the estimates established by Lemmas 9 to 11 that there exists a sequence $\varepsilon = \varepsilon_n \downarrow 0$ such that the sequence of solutions $\{u^{\varepsilon_n}\}$ of solutions of Problem B^ε converges in $L^1(Q_T)$ to a function $u \in L^\infty(Q_T) \cap BV(Q_T)$. We now prove:

Lemma 12. *The viscosity limit function u of solutions u^ε of Problem B^ε satisfies inequality* (2.14) *for all $\varphi \in C_0^\infty(Q_T)$, $\varphi \geq 0$ and $k \in \mathbb{R}$ and the boundary and initial conditions* (2.10) *and* (2.15).

Proof. To show that u satisfies the integral inequality (2.14), we follow the first part of the proof of Lemma 5 by multiplying equation (A^ε.1) by $\mathrm{sgn}_\eta(u^\varepsilon - k)\varphi$, $\varphi \in C_0^\infty(Q_T)$, $\varphi \geq 0$ and $k \in \mathbb{R}$, and letting $\eta \downarrow 0$ and $\varepsilon \downarrow 0$. Note that in this case, no boundary terms appear. The verification of boundary condition (2.10) is, of course, exactly as in the second part of the proof of Lemma 5. Using the function ν_h instead of μ_h and starting from

$$0 = \int_0^T \big(g_\varepsilon(u^\varepsilon(1,t)) - \partial_x A_\varepsilon(u^\varepsilon(1,t)) - \Psi_\varepsilon(t)\big)\Phi(t) = 0,$$

the boundary condition (2.15) can be verified in the same way. As for Problem A, the initial condition (2.10) can be inferred from estimate (3.20). □

Summarizing, we have:

Theorem 2. *If* (2.2)–(2.5) *and* (2.13) *hold, then Problem B admits an entropy solution u.*

3.3. **An improved regularity result for entropy solutions of Problem B.** In Lemma 10, we proved that the vanishing viscosity solution u of Problem B satisfies $\partial_x A(u) \in L^2(Q_T)$, as required by the definition of entropy solution. The purpose of this section is to show that $A(u)$ is actually more regular than this; namely, we have that $A(u)$ is Hölder continuous on $\overline{Q_T}$.

Lemma 13. *Let u^ε be a solution of Problem B^ε. Then there exists a constant $M_{12} > 0$ such that the following estimate holds uniformly with respect to ε:*

$$\|\partial_x A_\varepsilon(u^\varepsilon)\|_{L^\infty(\overline{Q_T})} \leq M_{12}. \tag{3.36}$$

Proof. Define $V^\varepsilon := -q_\varepsilon(t)u^\varepsilon - f_\varepsilon(u^\varepsilon) + a_\varepsilon(u^\varepsilon)\partial_x u^\varepsilon$. Equation ($\mathrm{A}^\varepsilon$.1) can then be written as $\partial_t u^\varepsilon = \partial_x V^\varepsilon$. Inserting this into (3.22), we obtain

$$\partial_x(\partial_t V^\varepsilon) + \partial_x\Big(\big[q_\varepsilon(t) + f_\varepsilon'(u^\varepsilon)\big]\partial_x V^\varepsilon + q_\varepsilon'(t)u^\varepsilon\Big) = \partial_x^2\big(a_\varepsilon(u^\varepsilon)\partial_x V^\varepsilon\big), \tag{3.37}$$

which implies that V^ε satisfies an equation of the type

$$\partial_t V^\varepsilon + \big[q_\varepsilon(t) + f_\varepsilon'(u^\varepsilon)\big]\partial_x V^\varepsilon + q_\varepsilon' u^\varepsilon = \partial_x\big(a_\varepsilon(u^\varepsilon)\partial_x V^\varepsilon\big) + C(t). \tag{3.38}$$

Evaluating (3.38) at $x = 0$ and using the boundary condition (A^ε.4) yields $C(t) \equiv 0$. In view of Problem B^ε, V^ε can be considered as the solution of the linear IBVP with Dirichlet boundary conditions

$$\partial_t V^\varepsilon + \big[q_\varepsilon(t) + f_\varepsilon'(u^\varepsilon)\big]\partial_x V^\varepsilon + q_\varepsilon u^\varepsilon = \partial_x\big(a_\varepsilon(u^\varepsilon)\partial_x V^\varepsilon\big), \quad x \in \Omega,\ t \in (0,T], \tag{3.39a}$$

$$V^\varepsilon(x,0) = -q_\varepsilon(0)u_0^\varepsilon(x) - f_\varepsilon\big(u_0^\varepsilon(x)\big) + a_\varepsilon\big(u_0^\varepsilon(x)\big)(u_0^\varepsilon)'(x), \quad x \in \overline{\Omega}, \tag{3.39b}$$

$$V^\varepsilon(1,t) = -\Psi_\varepsilon(t), \quad t \in (0,T], \tag{3.39c}$$
$$V^\varepsilon(0,t) = -q_\varepsilon(t)u^\varepsilon(0,t), \quad t \in (0,T]. \tag{3.39d}$$

Since $u_0 \in \mathcal{B}$, the right-hand part of equation (3.39b) is uniformly bounded in ε, and so are those of (3.39c) and (3.39d). Thus, the maximum principle implies that there exists a constant $\tilde{M}_{12}$ such that the uniform estimate $\|V^\varepsilon\|_{L^\infty(\overline{Q_T})} \le \tilde{M}_{12}$ holds; as a consequence, we have shown (3.36). ☐

Theorem 3. *Assume that $u^\varepsilon \to u$ a.e. in Q_T as $\varepsilon \downarrow 0$. Then there exists a subsequence $\varepsilon_n \downarrow 0$ such that $A(u^{\varepsilon_n}) \to A(u)$ uniformly on $\overline{Q_T}$ and*

$$A(u) \in C^{1,1/2}(\overline{Q_T}). \tag{3.40}$$

Proof. We shall estimate the L^1 continuity in time of V^ε applying Lemma 8. Integrating equation (3.39a) against a function $\varphi \in C_0^1(\Omega)$, and exploting the relation $\partial_x V^\varepsilon = \partial_t u^\varepsilon$ and Lemma 11, we obtain for $0 \le t_1 < t_2 \le T$:

$$\begin{aligned}
&\left| \int_0^1 \left(V^\varepsilon(x,t_2) - V^\varepsilon(x,t_1)\right)\varphi(x)\,dx \right| \\
&= \left| \int_{t_1}^{t_2} \int_0^1 \Big(-\left[q_\varepsilon(t) + f_\varepsilon'(u^\varepsilon)\right]\partial_x V^\varepsilon - q_\varepsilon' u^\varepsilon + \partial_x\left(a_\varepsilon(u^\varepsilon)\partial_x V^\varepsilon\right)\Big)\varphi(x)\,dx dt \right| \\
&= \left| \int_{t_1}^{t_2} \int_0^1 \Big\{ -q_\varepsilon' u^\varepsilon \varphi(x) + \Big(\left[q_\varepsilon(t) + f_\varepsilon'(u^\varepsilon)\right]V^\varepsilon - a_\varepsilon(u^\varepsilon)\partial_t u^\varepsilon\Big)\varphi'(x)\Big\}\,dx dt \right| \\
&\le (t_2 - t_1)\Big\{ \|q_\varepsilon'\|_\infty M_1 \|\varphi\|_\infty + \Big((\|q_\varepsilon\|_\infty + \|f_\varepsilon'\|_\infty)\tilde{M}_{12} + \|a^\varepsilon\|_\infty M_7 \Big) \|\varphi'\|_\infty \Big\}.
\end{aligned}$$

Applying Lemma 8, we obtain

$$\exists M_{13} > 0: \quad \|V^\varepsilon(\cdot,t_2) - V^\varepsilon(\cdot,t_1)\|_{L^1(\Omega)} \le M_{13}\sqrt{t_2 - t_1}. \tag{3.41}$$

We use this to obtain a continuity in time estimate of $A_\varepsilon(u^\varepsilon)$. From the definition of V^ε we obtain

$$\begin{aligned}
A_\varepsilon\left(u^\varepsilon(x,t_2)\right) - A_\varepsilon\left(u^\varepsilon(x,t_1)\right) &= \int_0^x \Big\{ a_\varepsilon\left(u^\varepsilon(\xi,t_2)\right)\partial_x u^\varepsilon(\xi,t_2) - a_\varepsilon\left(u^\varepsilon(\xi,t_1)\right)\partial_x u^\varepsilon(\xi,t_1)\Big\} d\xi \\
&= \int_0^x \Big\{ \left[q_\varepsilon(t) + f_\varepsilon'(\tilde{u})\right]\left(u(\xi,t_2) - u(\xi,t_1)\right) + V^\varepsilon(\xi,t_2) - V^\varepsilon(\xi,t_1)\Big\}\, d\xi,
\end{aligned}$$

and using the L^1 continuity in time estimates (3.20) and (3.41),

$$\left|A_\varepsilon\left(u^\varepsilon(x,t_2)\right) - A_\varepsilon\left(u^\varepsilon(x,t_1)\right)\right| \le \left(\|q_\varepsilon\|_\infty + 2\|q_\varepsilon'\|_\infty + \|f_\varepsilon'\|_\infty\right)(t_2 - t_1) + M_{13}\sqrt{t_2 - t_1}. \tag{3.42}$$

In view of Lemma 13 and (3.42), there exists a constant $M_{14} > 0$ independent of ε such that

$$\left|A_\varepsilon\left(u^\varepsilon(x_2,t_2)\right) - A_\varepsilon\left(u^\varepsilon(x_1,t_1)\right)\right| \le M_{14}\Big(|x_2 - x_1| + \sqrt{|t_2 - t_1|}\Big), \qquad \forall (x_1,t_1),(x_2,t_2) \in \overline{Q_T}.$$

The Ascoli-Arzelà compactness theorem then yields the existence of a subsequence of $\{A(u^{\varepsilon_n})\}$ converging uniformly on $\overline{Q_T}$ to a limit $\bar{A} \in C^{1,1/2}(\overline{Q_T})$ and we conclude easily that $\bar{A} = A(u)$. ☐

Remark 3. *If one could prove for the solution u^ε of Problem A^ε that $\varepsilon\partial_x u^\varepsilon(1,t)$ is bounded uniformly in ε, then, under some additional technical assumptions, it is easy to see that Theorem 3 would also be valid for Problem A.*

4. Uniqueness of entropy solutions

4.1. **General results.** We consider Problem A or B and assume only that f and A are locally Lipschitz continuous functions. Observe that if u is an entropy solution of Problem A or B, then it is easy to see that the equality

$$\iint_{Q_T} \Big\{ u\partial_t\varphi + \left[g(u,t) - g(k,t) - \partial_x A(u)\right]\partial_x\varphi \Big\}\, dx dt = 0 \tag{4.1}$$

holds for all $\varphi \in C_0^\infty(Q_T)$. An approximation argument will reveal that (4.1) holds also for all $\varphi \in L^2(\mathcal{T}; H_0^1(\Omega)) \cap W^{1,1}(\mathcal{T}; L^\infty(\Omega))$. This immediately implies $\partial_t u$ may be viewed as an element in $L^2(\mathcal{T};H^{-1}(\Omega))$, since $\partial_x A(u)\in L^2(Q_T)$ and obviously $u,g(u,t),A(u)\in L^p(Q_T)$ for all p. In what follows, we let $\langle\cdot,\cdot\rangle$ denote the usual pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

For later use, introduce the function

$$\mathcal{A}_\psi(k)=\int_0^k \psi\big(A(r)\big)\,dr,$$

where $\psi:\mathbb{R}\to\mathbb{R}$ is a nondecreasing and Lipschitz continuous function. Recall that $A(u)=0$ for $u\le u_{\mathrm{c}}$, $A(\cdot)$ is increasing in $(u_{\mathrm{c}},u_{\max})$, and $A(u)=A(u_{\max})=:A_{\max}$ for $u\ge u_{\max}$. Thus the range of $A(\cdot)$ is the interval $[0,A_{\max}]$ and therefore $|\mathcal{A}_\psi(k)|$ is bounded by $|k|\psi(A_{\max})$.

We shall need the following "weak chain rule" (see [1, 10]), which is here properly adapted to our problem.

Lemma 14. *Let $u:Q_T\to\mathbb{R}$ be a measurable function satisfying the following four conditions: (a) $u\in L^\infty(Q_T)\cap C(\mathcal{T};L^1(\Omega))$, (b) $u(0)=u_0\in L^\infty(\Omega)$, (c) $\partial_t u\in L^2\big(\mathcal{T};H^{-1}(\Omega)\big)$, and (d) $A(u)\in L^2\big(\mathcal{T};H^1(\Omega)\big)$. Then, for a.e. $s\in\mathcal{T}$ and every nonnegative $\varphi\in C_0^\infty(\overline{Q_T})$ for which $\partial_x^p\varphi|_{x=0,1}=0$ for $p=0,1,2,\dots$, we have*

$$-\int_0^s \big\langle \partial_t u,\psi\big(A(u)\big)\varphi\big\rangle\,dt=\int_\Omega\int_0^s \mathcal{A}_\psi(u)\partial_t\varphi\,dtdx+\int_\Omega \mathcal{A}_\psi(u_0)\varphi(x,0)\,dx-\int_\Omega \mathcal{A}_\psi(u(x,s))\varphi(x,s)\,dx.$$

Proof. In the sequel let φ be as in Lemma 14. We can assume without loss of generality that $\psi(0)=0$. If $\psi(0)\neq 0$, we simply replace ψ by $\tilde\psi=\psi-\psi(0)$ and note that $\mathcal{A}_{\tilde\psi}(u)=\mathcal{A}_\psi(u)-\psi(0)u$.

Note that $\mathcal{A}_\psi$ is a nonnegative and convex function. Convexity implies that for a.e. $(x,t)\in Q_T$, we have

$$\mathcal{A}_\psi\big(u(x,t)\big)-\mathcal{A}_\psi\big(u(x,t-\tau)\big)\le\big(u(x,t)-u(x,t-\tau)\big)\psi\big(A(u(x,t))\big),$$

where we define $u(t)=u_0$ for $t\in(-\tau,0)$. Multiplying this inequality by $\varphi(x,t)$ yields

$$\begin{aligned}&\mathcal{A}_\psi\big(u(x,t)\big)\varphi(x,t)-\mathcal{A}_\psi\big(u(x,t-\tau)\big)\varphi(x,t-\tau)+\mathcal{A}_\psi\big(u(x,t-\tau)\big)\big(\varphi(x,t-\tau)-\varphi(x,t)\big)\\&\quad=\mathcal{A}_\psi\big(u(x,t)\big)\varphi(x,t)-\mathcal{A}_\psi\big(u(x,t-\tau)\big)\varphi(x,t)\le\big(u(x,t)-u(x,t-\tau)\big)\psi\big(A(u(x,t))\big)\varphi(x,t).\end{aligned}\tag{4.2}$$

Note that $u_0,\mathcal{A}_\psi(u_0)\in L^1(\Omega)$ and $u,\mathcal{A}_\psi(u)\in L^\infty\big(\mathcal{T};L^1(\Omega)\big)$. Dividing (4.2) by τ and integrating over $\Omega\times(0,s)$, we get

$$\begin{aligned}&\frac{1}{\tau}\int_\Omega\int_{s-\tau}^s \mathcal{A}_\psi\big(u(x,t)\big)\varphi(x,t)\,dtdx-\frac{1}{\tau}\int_\Omega\int_0^\tau \mathcal{A}_\psi\big(u_0(x)\big)\varphi(x,t-\tau)\,dtdx\\&\qquad+\frac{1}{\tau}\int_\Omega\int_0^s \mathcal{A}_\psi\big(u(x,t-\tau)\big)\big(\varphi(x,t-\tau)-\varphi(x,t)\big)\,dxdt\\&\quad\le\frac{1}{\tau}\int_\Omega\int_0^s\big(u(x,t)-u(x,t-\tau)\big)\psi\big(A(u(x,t))\big)\varphi(x,t)\,dtdx.\end{aligned}\tag{4.3}$$

Since $\varphi\in C_0^\infty(Q_T)$ and $\partial_x A(u)\in L^2(Q_T)$, we have $\psi(A(u))\varphi\in L^2\big(\mathcal{T};H_0^1(\Omega)\big)$. Therefore, exploiting that $u\in C\big(\mathcal{T};L^1(\Omega)\big)$ and $\partial_t u\in L^2\big(\mathcal{T};H^{-1}(\Omega)\big)$, we can let $\tau\downarrow 0$ in (4.3) and obtain

$$\begin{aligned}&\int_\Omega \mathcal{A}_\psi\big(u(x,s)\big)\varphi(x,s)\,dx-\int_\Omega\mathcal{A}_\psi(u_0)\varphi(x,0)\,dx\\&\qquad-\int_\Omega\int_0^s\mathcal{A}_\psi(u)\partial_t\varphi\,dtdx\le\int_0^s\big\langle\partial_t u,\psi\big(A(u)\big)\varphi\big\rangle\,dt,\end{aligned}$$

for a.e. $s\in\mathcal{T}$. Convexity implies also that for a.e. $(x,t)\in Q_T$ and $t>\tau$, we have

$$\mathcal{A}_\psi\big(u(x,t)\big)-\mathcal{A}_\psi\big(u(x,t-\tau)\big)\ge\big(u(x,t)-u(x,t-\tau)\big)\psi\big(A(u(x,t-\tau))\big).$$

Multiplying this inequality by $\varphi(x,t-\tau)$ yields

$$\begin{aligned}&\mathcal{A}_\psi\big(u(x,t)\big)\varphi(x,t)-\mathcal{A}_\psi\big(u(x,t-\tau)\big)\varphi(x,t-\tau)+\mathcal{A}_\psi\big(u(x,t)\big)\big(\varphi(x,t-\tau)-\varphi(x,t)\big)\\&\quad=\mathcal{A}_\psi\big(u(x,t)\big)\varphi(x,t-\tau)-\mathcal{A}_\psi\big(u(x,t-\tau)\big)\varphi(x,t-\tau)\\&\qquad\ge\big(u(x,t)-u(x,t-\tau)\big)\psi\big(A(u(x,t-\tau))\big)\varphi(x,t-\tau).\end{aligned}\tag{4.4}$$

After dividing (4.4) by τ and integrating over $\Omega\times(\tau,s)$, we obtain

$$\begin{aligned}\frac{1}{\tau}\int_\Omega\int_{s-\tau}^{s}\mathcal{A}_\psi\big(u(x,t)\big)\varphi(x,t)\,dtdx-\frac{1}{\tau}\int_\Omega\int_0^{\tau}\mathcal{A}_\psi\big(u(x,t)\big)\varphi(x,t)\,dtdx\\ +\frac{1}{\tau}\int_\Omega\int_\tau^{s}\mathcal{A}_\psi\big(u(x,t)\big)\big(\varphi(x,t-\tau)-\varphi(x,t)\big)\,dxdt\\ \geq\frac{1}{\tau}\int_\Omega\int_\tau^{s}\big(u(x,t)-u(x,t-\tau)\big)\psi\big(A(u(x,t-\tau))\big)\varphi(x,t-\tau)\,dtdx.\end{aligned}\tag{4.5}$$

Finally, similar to the case (4.3), letting $\tau\downarrow 0$ in (4.5), we get, for a.e. $s\in\mathcal{T}$,

$$\begin{aligned}\int_\Omega\mathcal{A}_\psi\big(u(x,s)\big)\varphi(x,s)\,dx-\int_\Omega\mathcal{A}_\psi(u_0)\varphi(x,0)\,dx\\ -\int_\Omega\int^{s}\mathcal{A}_\psi(u)\partial_t\varphi\,dtdx\geq\int_0^{s}\langle\partial_t u,\psi\big(A(u)\big)\varphi\rangle\,dt.\end{aligned}$$

This concludes the proof of the lemma. □

The following lemma is an adaption to our problem of Carrillo's [10] main observation:

Lemma 15. *Let u be an entropy solution of Problem A or B. Then, for any nonnegative $\varphi\in C_0^\infty(Q_T)$ and $k\in(u_{\mathrm{c}},u_{\max})$, we have*

$$\begin{aligned}\iint_{Q_T}\Big\{|u-k|\partial_t\varphi+\operatorname{sgn}(u-k)\big[g(u,t)-g(k,t)-\partial_x A(u)\big]\partial_x\varphi\Big\}\,dt\,dx\\ =\lim_{\eta\downarrow 0}\iint_{Q_T}\big(\partial_x A(u)\big)^2\operatorname{sgn}_\eta'\big(A(u)-A(k)\big)\varphi\,dtdx.\end{aligned}\tag{4.6}$$

Proof. In what follows, we always let φ,k be as in the lemma and use the approximation sgn_η (see (3.21)) for the sign function. Introduce the function $\psi_\eta(z)=\operatorname{sgn}_\eta\big(z-A(k)\big)$ and note that it satisfies the hypothesis of Lemma 14, so that

$$-\int_0^T\big\langle\partial_t u,\operatorname{sgn}_\eta\big(A(u)-A(k)\big)\varphi\big\rangle\,dt=\int_\Omega\int_0^T\mathcal{A}_{\psi_\eta}(u)\partial_t\varphi\,dtdx.$$

Since u satisfies (4.1) and $\operatorname{sgn}_\eta(A(u)-A(k))\varphi\in L^2(\mathcal{T};H_0^1(\Omega))$, we have

$$\begin{aligned}-\int_0^T\big\langle\partial_t u,\operatorname{sgn}_\eta\big(A(u)-A(k)\big)\varphi\big\rangle\,dt\\ +\iint_{Q_T}\big[g(u,t)-g(k,t)-\partial_x A(u)\big]\partial_x\big(\operatorname{sgn}_\eta\big(A(u)-A(k)\big)\varphi\big)\,dtdx=0,\end{aligned}$$

which implies

$$\iint_{Q_T}\mathcal{A}_{\psi_\eta}(u)\partial_t\varphi\,dtdx+\iint_{Q_T}\big(g(u,t)-g(k,t)-\partial_x A(u)\big)\partial_x\big(\operatorname{sgn}_\eta\big(A(u)-A(k)\big)\varphi\big)\,dtdx=0.\tag{4.7}$$

Since $A(r)>A(k)$ if and only if $r>k$, $\operatorname{sgn}_\eta(A(r)-A(k))\to 1$ as $\eta\downarrow 0$ for any $r>k$. Similarly, $\operatorname{sgn}_\eta(A(r)-A(k))\to -1$ as $\eta\downarrow 0$ for any $r<k$. Consequently, $\mathcal{A}_{\psi_\eta}(u)\to|u-k|$ a.e. in Q_T as $\eta\downarrow 0$. Moreover, we have $|\mathcal{A}_{\psi_\eta}(u)|\leq|u|$, so by Lebegue's dominated convergence theorem

$$\lim_{\eta\downarrow 0}\iint_{Q_T}\mathcal{A}_{\psi_\eta}(u)\partial_t\varphi\,dtdx=\iint_{Q_T}|u-k|\partial_t\varphi\,dtdx.$$

We have

$$\begin{aligned}\lim_{\eta\downarrow 0}\iint_{Q_T}\big[g(u,t)-g(k,t)-\partial_x A(u)\big]\partial_x\big(\operatorname{sgn}_\eta\big(A(u)-A(k)\big)\varphi\big)\,dtdx\\ =\lim_{\eta\downarrow 0}\iint_{Q_T}\big[g(u,t)-g(k,t)-\partial_x A(u)\big]\partial_x\operatorname{sgn}_\eta\big(A(u)-A(k)\big)\varphi\,dtdx\end{aligned}$$

$$
\begin{aligned}
&+\lim_{\eta\downarrow 0}\iint_{Q_T}\big[g(u,t)-g(k,t)-\partial_x A(u)\big]\operatorname{sgn}_\eta\big(A(u)-A(k)\big)\partial_x\varphi\,dtdx\\
&=\lim_{\eta\downarrow 0}\iint_{Q_T}\big[g(u,t)-g(k,t)\big]\operatorname{sgn}_\eta'\big(A(u)-A(k)\big)\partial_x A(u)\varphi\,dtdx\\
&\quad-\lim_{\eta\downarrow 0}\iint_{Q_T}\big(\partial_x A(u)\big)^2\operatorname{sgn}_\eta'\big(A(u)-A(k)\big)\varphi\,dtdx\\
&\quad+\lim_{\eta\downarrow 0}\iint_{Q_T}\big[g(u,t)-g(k,t)-\partial_x A(u)\big]\operatorname{sgn}_\eta\big(A(u)-A(k)\big)\partial_x\varphi\,dtdx\\
&=: I_1-\lim_{\eta\downarrow 0}\iint_{Q_T}\big(\partial_x A(u)\big)^2\operatorname{sgn}_\eta'\big(A(u)-A(k)\big)\varphi\,dtdx+I_2.
\end{aligned}
$$

One can easily check that

$$
I_1=\lim_{\eta\downarrow 0}\iint_{Q_T}\big[g(u,t)-g(k,t)\big]\operatorname{sgn}_\eta'\big(A(u)-A(k)\big)\partial_x A(u)\varphi\,dtdx=0.
$$

Using that $\operatorname{sgn}(u-k)=\operatorname{sgn}\big(A(u)-A(k)\big)$ a.e. in Q_T,

$$
\begin{aligned}
I_2&=\lim_{\eta\downarrow 0}\iint_{Q_T}\big[g(u,t)-g(k,t)-\partial_x A(u)\big]\operatorname{sgn}_\eta\big(A(u)-A(k)\big)\partial_x\varphi\,dtdx\\
&=\iint_{Q_T}\operatorname{sgn}\big(u-k\big)\big[g(u,t)-g(k,t)-\partial_x A(u)\big]\partial_x\varphi\,dtdx.
\end{aligned}
$$

Consequently, letting $\eta\downarrow 0$ in (4.7), we obtain the desired equality (4.6). □

Theorem 4. *If u and v are two entropy solutions of Problem A or B, then we have for any $\varphi\in C_0^\infty(Q_T)$, $\varphi\ge 0$:*

$$
\iint_{Q_T}\Big\{|u-v|\partial_t\varphi+\operatorname{sgn}(u-v)\big[g(u,t)-g(v,t)-\big(\partial_x A(u)-\partial_x A(v)\big)\big]\partial_x\varphi\Big\}\,dtdx\ge 0. \tag{4.8}
$$

Proof. Let $\varphi\in C_0^\infty(Q_T\times Q_T)$, $\operatorname{supp}\varphi\subset Q_T\times Q_T$, $\varphi=\varphi(x,t,y,s)\ge 0$, $u=u(x,t)$, and $v=v(y,s)$. Observe that

$$
\begin{gathered}
\partial_x A(u)=0\ \text{ a.e. in } \mathcal{O}_u:=\Big\{(x,t)\in Q_T: u(x,t)\le u_{\mathrm{c}}\ \text{ or }\ u(x,t)\ge 1\Big\},\\
\partial_y A(v)=0\ \text{ a.e. in } \mathcal{O}_v:=\Big\{(y,s)\in Q_T: v(y,s)\le u_{\mathrm{c}}\ \text{ or }\ v(y,s)\ge 1\Big\},\\
\operatorname{sgn}(u-v)=\operatorname{sgn}\big(A(u)-A(v)\big)\ \text{ a.e. in } \big[Q_T\times(Q_T\setminus\mathcal{O}_u)\big]\cup\big[(Q_T\setminus\mathcal{O}_v)\times Q_T\big].
\end{gathered}
$$

From the definitions of entropy solutions and Lemma 15, we easily derive

$$
\begin{aligned}
&\iiiint\limits_{Q_T\times Q_T}\Big\{|u-v|\partial_t\varphi+\operatorname{sgn}(u-v)\big[g(u,t)-g(v,t)-\partial_x A(u)\big]\partial_x\varphi\Big\}\,dtdxdsdy\\
&\quad\ge\lim_{\eta\downarrow 0}\iiiint\limits_{(Q_T\setminus\mathcal{O}_v)\times Q_T}\big(\partial_x A(u)\big)^2\operatorname{sgn}_\eta'\big(A(u)-A(v)\big)\varphi\,dtdxdsdy \qquad (4.9)\\
&\quad=\lim_{\eta\downarrow 0}\iiiint\limits_{(Q_T\setminus\mathcal{O}_v)\times(Q_T\setminus\mathcal{O}_u)}\big(\partial_x A(u)\big)^2\operatorname{sgn}_\eta'\big(A(u)-A(v)\big)\varphi\,dtdxdsdy, \qquad (4.10)\\
&\iiiint\limits_{Q_T\times Q_T}\Big\{|v-u|\partial_s\varphi+\operatorname{sgn}(v-u)\big[g(v,s)-g(u,s)-\partial_y A(v)\big]\partial_y\varphi\Big\}\,dtdxdsdy \qquad (4.11)\\
&\quad=\lim_{\eta\downarrow 0}\iiiint\limits_{(Q_T\setminus\mathcal{O}_u)\times(Q_T\setminus\mathcal{O}_v)}\big(\partial_y A(v)\big)^2\operatorname{sgn}_\eta'\big(A(v)-A(u)\big)\varphi\,dtdxdsdy. \qquad (4.12)
\end{aligned}
$$

Observe that for a.e. $(x,t) \in Q_T$,

$$\iint_{Q_T} \partial_x A(u)\partial_y \left(\mathrm{sgn}_\eta \left(A(u) - A(v)\right)\varphi\right) dsdy = 0,$$

or if one prefers

$$-\iint_{Q_T} \mathrm{sgn}_\eta \left(A(u) - A(v)\right)\partial_x A(u)\partial_y \varphi \, dsdy = \iint_{Q_T} \partial_y \mathrm{sgn}_\eta \left(A(u) - A(v)\right)\partial_x A(u)\varphi \, dsdy. \tag{4.13}$$

Similarly, for a.e. $(y,s) \in Q_T$,

$$-\iint_{Q_T} \mathrm{sgn}_\eta \left(A(v) - A(u)\right)\partial_y A(u)\partial_x \varphi \, dtdx = \iint_{Q_T} \partial_x \mathrm{sgn}_\eta \left(A(v) - A(u)\right)\partial_y A(u)\varphi \, dtdx. \tag{4.14}$$

Now using (4.13), we find that

$$\begin{aligned}
&-\iiiint\limits_{Q_T \times Q_T} \mathrm{sgn}(u-v)\partial_x A(u)\partial_y \varphi \, dtdxdsdy \\
&\quad = -\iiiint\limits_{(Q_T \setminus \mathcal{O}_v)\times Q_T} \mathrm{sgn}\left(A(u) - A(v)\right)\partial_x A(u)\partial_y \varphi \, dtdxdsdy \\
&\quad = -\lim_{\eta\downarrow 0} \iiiint\limits_{(Q_T \setminus \mathcal{O}_v)\times Q_T} \mathrm{sgn}_\eta \left(A(u) - A(v)\right)\partial_x A(u)\partial_y \varphi \, dtdxdsdy \\
&\quad = -\lim_{\eta\downarrow 0} \iiiint\limits_{(Q_T \setminus \mathcal{O}_v)\times Q_T} \partial_y A(v)\partial_x A(u)\mathrm{sgn}_\eta' \left(A(u) - A(v)\right)\varphi \, dtdxdsdy. \\
&\quad = -\lim_{\eta\downarrow 0} \iiiint\limits_{(Q_T \setminus \mathcal{O}_v)\times (Q_T \setminus \mathcal{O}_u)} \partial_y A(v)\partial_x A(u)\mathrm{sgn}_\eta' \left(A(u) - A(v)\right)\varphi \, dtdxdsdy.
\end{aligned} \tag{4.15}$$

Similarly, using (4.14), we find that

$$\begin{aligned}
&-\iiiint\limits_{Q_T \times Q_T} \mathrm{sgn}(v-u)\partial_y A(v)\partial_x \varphi \, dtdxdsdy \\
&\quad = -\lim_{\eta\downarrow 0} \iiiint\limits_{(Q_T \setminus \mathcal{O}_v)\times (Q_T \setminus \mathcal{O}_u)} \partial_x A(u)\partial_y A(v)\mathrm{sgn}_\eta' \left(A(v) - A(u)\right)\varphi \, dtdxdsdy.
\end{aligned} \tag{4.16}$$

Adding (4.9) and (4.15) yields

$$\begin{aligned}
&\iiiint\limits_{Q_T \times Q_T} \Big\{ |u-v|\partial_t \varphi + \mathrm{sgn}(u-v)\Big[\left(g(u,t) - g(v,t)\right)\partial_x \varphi - \partial_x A(u)(\partial_x \varphi + \partial_y \varphi)\Big]\Big\} \, dtdxdsdy \\
&\quad = \lim_{\eta\downarrow 0} \iiiint\limits_{(Q_T \setminus \mathcal{O}_v)\times (Q_T \setminus \mathcal{O}_u)} \left[\left(\partial_x A(u)\right)^2 - \partial_y A(u)\partial_x A(v)\right]\mathrm{sgn}_\eta' \left(A(u) - A(v)\right)\varphi \, dtdxdsdy.
\end{aligned} \tag{4.17}$$

Adding (4.11) and (4.16) yields

$$\begin{aligned}
&\iiiint\limits_{Q_T \times Q_T} \Big\{ |v-u|\partial_s \varphi + \mathrm{sgn}(v-u)\Big[\left(g(v,s) - g(u,s)\right)\partial_y \varphi - \partial_y A(v)(\partial_y \varphi + \partial_x \varphi)\Big]\Big\} \, dtdxdsdy \\
&\quad = \lim_{\eta\downarrow 0} \iiiint\limits_{(Q_T \setminus \mathcal{O}_v)\times (Q_T \setminus \mathcal{O}_u)} \left[\left(\partial_y A(v)\right)^2 - \partial_x A(v)\partial_y A(v)\right]\mathrm{sgn}_\eta' \left(A(v) - A(u)\right)\varphi \, dtdxdsdy.
\end{aligned} \tag{4.18}$$

Using that $\mathrm{sgn}(-r) = -\mathrm{sgn}(r)$ and $\mathrm{sgn}_\eta'(-r) = \mathrm{sgn}_\eta'(r)$ a.e. in $\mathbb{R}$, adding (4.17) and (4.18) gives

$$\iiiint\limits_{Q_T \times Q_T} \Big\{ |u-v|(\partial_t \varphi + \partial_s \varphi) + \mathrm{sgn}(u-v)\Big[g(u,t) - g(v,s) - \left(\partial_x A(u) - \partial_y A(v)\right)\Big](\partial_x \varphi + \partial_y \varphi)$$

$$+\operatorname{sgn}(u-v)\Big[\big(g(u,s)-g(u,t)\big)\partial_y\varphi+\big(g(v,s)-g(v,t)\big)\partial_x\varphi\Big]\Big\}\,dtdxdsdy$$

$$=\lim_{\eta\downarrow 0}\iiiint\limits_{(Q_T\setminus \mathcal{O}_v)\times(Q_T\setminus\mathcal{O}_u)}\big(\partial_x A(v)-\partial_y A(v)\big)^2\operatorname{sgn}'_\eta\big(A(u)-A(v)\big)\varphi\,dtdxdsdy\geq 0. \tag{4.19}$$

Let $\varphi\in C_0^\infty(Q_T)$ be nonnegative and let $\{\delta_h\}_{h>0}$ be a standard regularizing sequence in $\mathbb{R}$. We then introduce the test function

$$\varphi_h(x,t,y,s)=\varphi\Big(\frac{x+y}{2},\frac{t+s}{2}\Big)\delta_h\Big(\frac{x-y}{2}\Big)\delta_h\Big(\frac{t-s}{2}\Big).$$

Observe that

$$\partial_t\varphi_h+\partial_s\varphi_h=\partial_t\varphi\Big(\frac{x+y}{2},\frac{t+s}{2}\Big)\delta_h\Big(\frac{x-y}{2}\Big)\delta_h\Big(\frac{t-s}{2}\Big),$$
$$\partial_x\varphi_h+\partial_y\varphi_h=\partial_x\varphi\Big(\frac{x+y}{2},\frac{t+s}{2}\Big)\delta_h\Big(\frac{x-y}{2}\Big)\delta_h\Big(\frac{t-s}{2}\Big).$$

Using φ_h as test function in (4.19), we get

$$\begin{aligned}\iiiint\limits_{Q_T\times Q_T}&\Big\{|u-v|\partial_t\varphi\Big(\frac{x+y}{2},\frac{t+s}{2}\Big)+\operatorname{sgn}(u-v)\Big[g(u,t)-g(v,s)\\&-\big(\partial_xA(u)-\partial_yA(v)\big)\Big]\partial_x\varphi\Big(\frac{x+y}{2},\frac{t+s}{2}\Big)\delta_h\Big(\frac{x-y}{2}\Big)\delta_h\Big(\frac{t-s}{2}\Big)\\&+\operatorname{sgn}(u-v)\Big[\big(g(u,s)-g(u,t)\big)\partial_y\varphi_h+\big(g(v,s)-g(v,t)\big)\partial_x\varphi_h\Big]\Big\}\,dtdxdsdy\geq 0.\end{aligned}\tag{4.20}$$

It is now classical (see Kružkov [13]) to take the limit $h\downarrow 0$ in (4.20) to obtain (4.8). □

Remark 4. *One should note that* (4.8) *is valid under significantly less regularity than* $u\in BV(Q_T)$.

4.2. **Uniqueness of entropy solutions of Problem A.**

Corollary 1. *Let u,v be two entropy solutions of Problem A with initial data u_0,v_0, respectively. Then*

$$\|u(\cdot,t)-v(\cdot,t)\|_{L^1(\Omega)}\leq\|u_0-v_0\|_{L^1(\Omega)}. \tag{4.21}$$

In particular, Problem A has at most one entropy solution.

For the proof of Corollary 1, we need the following lemma.

Lemma 16. *Let u be an entropy solution of Problem A. Then condition* (2.9) *is satisfied if and only if the integral inequality* (2.14) *holds for all nonnegative $\varphi\in C_0^\infty(Q_T)$ and $k\in\mathbb{R}$; if $a(s)=0$ is valid for all $s\in\mathcal{I}(\varphi_1(t),\gamma_1u):=[\min\{\varphi_1(t),\gamma_1u\},\max\{\varphi_1(t),\gamma_1u\}]$; and if the following entropy boundary inequality is satisfied:*

$$\Big[\operatorname{sgn}\big((\gamma_1u)(1,t)-k\big)-\operatorname{sgn}\big(\varphi_1(t)-k\big)\Big]\Big[g(\gamma_1u,t)-g(k,t)-\gamma_1\partial_xA(u)\Big]\geq 0. \tag{4.22}$$

Proof of Lemma 16. Set $\varphi(x,t)=\tilde\varphi(x)\nu_h(x)\Phi(t)$ in inequality (2.9), where $\tilde\varphi\geq 0$, $\varphi\in C_0^\infty\big(\overline{\Omega}\big)$, $\Phi\geq 0$, $\Phi\in C_0^\infty\big(\overline{\mathcal{T}}\big)$ and ν_h is defined in (2.1) and let $h\downarrow 0$. See [7] for details. □

Proof of Corollary 1. In inequality (4.8) we choose $\varphi(x,t)=((1-\mu_h(x)-\nu_h(x))\Phi(t)$ with $\Phi\in C_0^\infty(\mathcal{T})$, $\Phi\geq 0$ and μ_h and ν_h from (2.1). Taking the limit $h\downarrow 0$, we obtain from Lemma 1, using the boundary condition at $x=0$ and the nonpositivity of q:

$$\begin{aligned}\iint_{Q_T}|u-v|\Phi'(t)\,dt&=\int_0^T\Big\{\operatorname{sgn}(\gamma_1u-\gamma_1v)\big[g(\gamma_1u,t)-g(\gamma_1v,t)-\big(\gamma_1\partial_xA(u)-\gamma_1\partial_xA(v)\big)\big]\\&\qquad\qquad-\operatorname{sgn}(\gamma_0u-\gamma_0v)q(t)\,(\gamma_0u-\gamma_0v)\Big\}\,dt\\&\geq\int_0^T\Big\{\operatorname{sgn}(\gamma_1u-\gamma_1v)\big[g(\gamma_1u,t)-g(\gamma_1v,t)-\big(\gamma_1\partial_xA(u)-\gamma_1\partial_xA(v)\big)\big]\Big\}\,dt.\end{aligned}\tag{4.23}$$

Note that

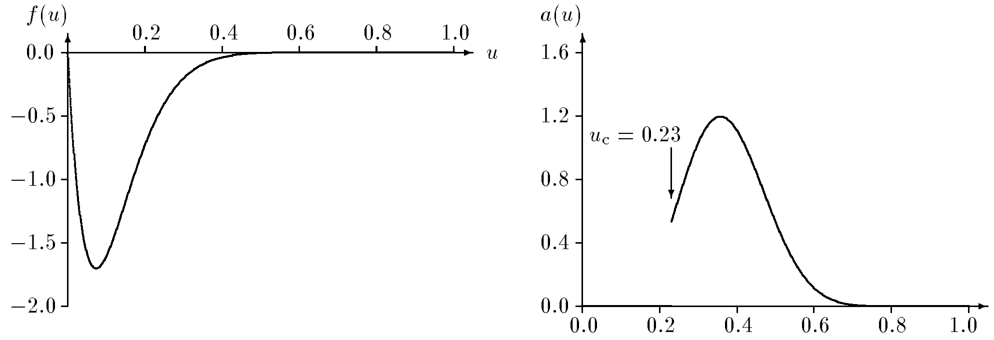

FIGURE 1. The model functions $f(u)$ and $a(u)$ for the sedimentation-consolidation problem. The units are 10^{-5} [m/s] for $f(u)$ and 10^{-5} [m^2/s] for $a(u)$.

$$\operatorname{sgn}(\gamma_1 u-\gamma_1 v)\Big[g(\gamma_1 u,t)-g(\gamma_1 v,t)-\big(\gamma_1\partial_x A(u)-\gamma_1\partial_x A(v)\big)\Big]=\operatorname{sgn}(\gamma_1 u-\gamma_1 v)\times$$
$$\times\big[g(\gamma_1 u,t)-g(k,t)-\gamma_1\partial_x A(u)\big]+\operatorname{sgn}(\gamma_1 v-\gamma_1 u)\big[g(\gamma_1 v,t)-g(k,t)-\gamma_1\partial_x A(v)\big]. \quad (4.24)$$

Choosing in a standard fashion

$$k(t)=\begin{cases}\gamma_1 u & \text{if } \gamma_1 u\in\mathcal{I}\big(\varphi_1(t),\gamma_1 v\big),\\ \varphi_1(t) & \text{if } \varphi_1(t)\in\mathcal{I}\big(\gamma_1 u,\gamma_1 v\big),\\ \gamma_1 v & \text{if } \gamma_1 v\in\mathcal{I}\big(\varphi_1(t),\gamma_1 u\big)\end{cases}$$

in the entropy boundary inequality (4.22) and its analogue for v, we see that both summands on the right-hand part of (4.24) are nonnegative. Consequently,

$$\iint_{Q_T}|u-v|\Phi'(\tau)\,d\tau dx\geq 0. \quad (4.25)$$

Now let $\Phi(\tau)=\varrho_h(\tau)-\varrho_h(\tau-t)$, where ϱ_h is given in (2.1). Corollary 1 follows by taking $h\downarrow 0$. □

4.3. **Uniqueness of entropy solutions of Problem B.** We note that by the boundary condition (B3), the right-hand part of (4.23) is zero, so that inequality (4.25) follows immediately. Summarizing, we may conclude:

Corollary 2. *Let u, v be two entropy solutions of Problem A with initial data u_0, v_0, respectively. Then (4.21) holds. In particular, Problem B has at most one entropy solution.*

Remark 5. *We point out that for both initial-boundary value problems A and B, the stability proof essentially depends on the nonpositivity of q. In other words, stability relies on reducing the total flux $g(u,t)-\partial_x A(u)$ to its convective part $q(t)u$ at the 'outflow' boundary of $\overline{Q_T}$ only.*

5. APPLICATION TO GRAVITATIONAL SEDIMENTATION-CONSOLIDATION PROCESSES

5.1. **Statement of the problem.** The study of degenerate convection-diffusion equations of type (1.1) is in part motivated by a model of sedimentation-consolidation processes of flocculated suspensions in an idealized sedimentation vessel, here considered to be of height 1 [m]. In that application, $u=u(x,t)$ denotes the local volumetric solid concentration, $q(t)\leq 0$ is the average flow velocity of the mixture which can be controlled externally, $f(u)$ is a given nonlinear function relating the local solid-fluid relative velocity to the local solids concentration, and

$$a(u)=-f(u)\sigma_{\mathrm{e}}'(u)/(\Delta\varrho g u), \quad (5.1)$$

where $\Delta\varrho>0$ denotes the solid-fluid mass density difference, g is the acceleration of gravity, and $\sigma_{\mathrm{e}}'(u)\geq 0$ is the derivative of the solid effective stress function. The material behaviour of the suspension is described by the functions $f(u)$ and $\sigma_{\mathrm{e}}(u)$. Condition (A2) corresponds to a given initial concentration distribution, condition (A3) to prescribing a concentration value at $x=1$ due to dilution of feed suspension which enters the container continuously, and condition (A4) is then

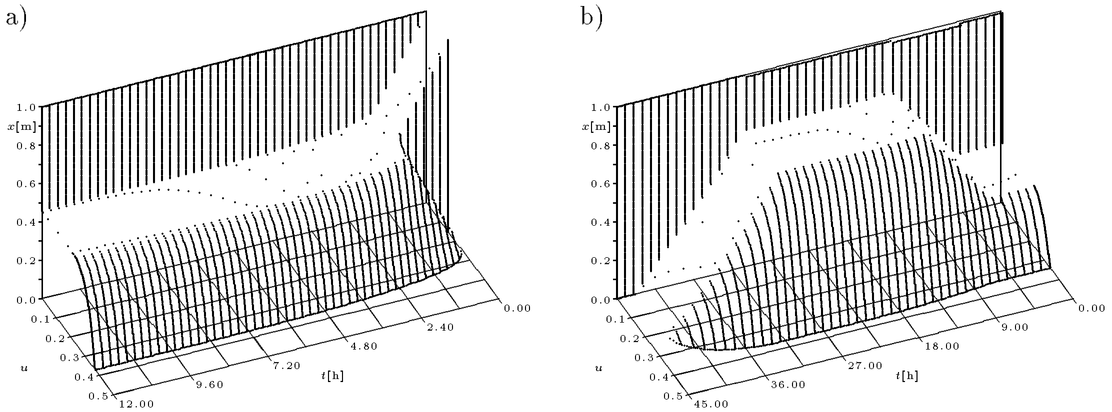

FIGURE 2. Numerical solutions of Problem A applied to the sedimentation-consolidation model: a) batch settling, b) continuous sedimentation-consolidation

equivalent to reducing the solid volume flux density at the bottom of the vessel to its convective part $q(t)u(0,t)$. This sedimentation-consolidation model is described in detail in [3, 8, 9].

The property which is of interest here is that most researchers (see, e.g., [15]) assume that σ_{e} is constant for u not exceeding a critical value u_{c}, at which the solid flocs are assumed to touch each other, and that σ_{e} is strictly increasing for $u > u_{\mathrm{c}}$. Consequently, $a(u) = 0$ for $u \leq u_{\mathrm{c}}$ and $a(u) > 0$ for $u > u_{\mathrm{c}}$ wherever $f(u) < 0$. Most notably, many constitutive equations for σ_{e} imply a jump of σ_{e}' at $u = u_{\mathrm{c}}$, which makes $a(u)$ discontinuous.

5.2. **Numerical examples.** We calculate entropy solutions of Problem A in this application by using the finite-difference operator splitting scheme described in [5].

We employ a flux density function of the well-known Richardson and Zaki type with parameters which were determined for a suspension of copper ore tailings (see [9]):

$$f(u) = -6.05 \times 10^{-4} u(1-u)^{12.59}\,[\mathrm{m/s}].$$

The function $a(u)$ is given by (5.1) with $\Delta\varrho = 1500\,[\mathrm{kg/m^3}]$, $\sigma_{\mathrm{e}}'(u) = 0$ for $u \leq u_{\mathrm{c}} = 0.23$ and

$$\sigma_{\mathrm{e}}'(u) = \tfrac{d}{du}\big(100(u/u_{\mathrm{c}})^8 - 1)\big)\,[\mathrm{Pa}] \text{ for } u > u_{\mathrm{c}},$$

see e.g. [15]. Figure 1 shows the resulting model functions $f(u)$ and $a(u)$.

In the first example, see Figure 2a), we consider the settling of an initially homogeneous suspension of concentration $u_0 = 0.15$ in a closed column, i.e. $\varphi_1 \equiv 0, q \equiv 0$. Observe that the discontinuity between $u = 0$ and $u = u_0$ is a shock. In the second example, we set $q = -1.5 \times 10^{-5}\,[\mathrm{m/s}]$ and start with a steady state: the function $u_0(x)$ is obtained by setting $u_0(0) = 0.34$, by integrating the time-independent version of equation (1.1) using this boundary condition until $u = u_{\mathrm{c}}$ is reached at a certain level x_{c} and setting $u_0(x) = \Phi_1(u_0(0))$ above, where Φ_1 is obtained from solving $q\Phi_1 + f(\Phi_1) = qu_0(0)$, yielding $\Phi_1(0.34) = 0.00922$ and $\Phi_1(0.37) = 0.01014$. Setting

$$\varphi_1(t) = \begin{cases} \Phi_1(0.34) & \text{for } 0 < t \leq 5\,[\mathrm{h}], \qquad \Phi_1(0.37) \quad \text{for } 12\,[\mathrm{h}] < t \leq 30[\mathrm{h}], \\ 0.02 & \text{for } 5\,[\mathrm{h}] < t \leq 12[\mathrm{h}], \quad 0 \qquad\qquad \text{for } t > 30\,[\mathrm{h}], \end{cases}$$

we obtain the numerical solution depicted in Figure 2b). This is a successive simulation of the operation at steady state, rise of the sediment level, convergence to the next steady state and emptying of the sedimentation vessel.

Note that $u_0 \in \mathcal{B}$ in both examples. This is obvious for $u_0 = const.$, while in the second case

$$\partial_x A\big(u_0(x)\big) = q\big(u_0(x) - u_0(0)\big) + f\big(u_0(x)\big) \text{ for } 0 \leq x \leq x_{\mathrm{c}}, \tag{5.2}$$

where $u_0(0)$ was chosen such that the right-hand part of (5.2) is nonpositive, therefore $u_0'(x) \leq 0$ for $0 \leq x \leq x_{\mathrm{c}}$. We have $u_0 \in \mathcal{B}$, since we can conclude from the jump condition [7] that

$$\mathrm{TV}_\Omega(\partial_x A(u_0)) = \int_0^{x_c} \left|\partial_x^2 A\bigl(u_0(x)\bigr)\right| dx + \left|\lim_{x\uparrow x_c} \partial_x A\bigl(u_0(x)\bigr)\right|$$
$$\leq \bigl(\|f'\|_\infty - q\bigr)\bigl(u_0(0) - u_c\bigr) + \left|f(u_c) - f\bigl(\Phi_1(u_0(0))\bigr)\right| < \infty.$$

Acknowledgements

We acknowledge support by the Sonderforschungsbereich 404 at the University of Stuttgart and by the Applied Mathematics in Industrial Flow Problems (AMIF) programme of the European Science Foundation (ESF).

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