Large-time behavior for conservation laws with source in a bounded domain

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to appear in Journal of Differential Equations

1 Introduction

The purpose of the present paper is to analyze qualitative properties of the solutions of the Dirichlet problem

$$\partial_t u + \partial_x f(u) = g(x,u) \qquad x \in [0,L],\, t > 0, \tag{1}$$

$$u(x,0) = u_0(x) \in BV(0,L) \qquad x \in [0,L], \tag{2}$$

$$u(0,t) = u_-, \quad u(L,t) = u_+ \qquad t > 0. \tag{3}$$

Here $u = u(x,t), u_\pm \in \mathbb{R}$. Precise assumptions will be given later. We are, in particular, interested in the asymptotic behavior of solutions to problem (1)–(3).

Equation of the form (1) is called *reaction-convection equation* or *balance law*. The correct setting to deal with for such equations is the one given by entropy solutions, in the sense of Kružkov in [15]. In such class it is well known that the Cauchy problem has a unique solution, continuously depending on initial data in L^1-norm.

In this article we consider boundary conditions and we analyze the influence of such conditions on the whole solution as time goes on. Such conditions for equations of the form (1) were considered in [1]. It turns out that, in order to have existence, uniqueness and continuous dependence, conditions (3) have to be interpreted in a non classical fashion. Following [1], we look for entropy solutions of (1), such that their boundary values belong to appropriate sets (details will be given later on).

The main novelties of the paper are exactly boundary conditions and their influence on the large-time behavior of solutions to problem (1)–(3). Such kind of subject is (more or less) new with respect to previous literature on reaction-convection equations (some interesting considerations were made in [7]). Indeed, concerning equation (1), many papers have been devoted to the Cauchy problem, [9],[19], [22] and references, [24]. In these works results on asymptotic behavior were proved for different choices of initial data (periodic, with bounded support, of perturbed Riemann type). It has been shown that the solutions for large time are represented by opportune combination of travelling wave solutions of the same equation (1). In the case of bounded domains we cannot expect that the solution is given by travelling waves with nonzero speed, since such solutions would interact with the boundary after finite time. In fact we are able to prove that the asymptotic picture is completely described by stationary solutions.

Equation (1) can be regarded as a simplified model for Euler equations when dealing with source and/or reaction phenomena. The presence of such terms is represented by the function $g = g(x,u)$. The same kind of equation appears naturally when considering the nozzle case, i.e. the case of a fluid flowing in a duct with variable size (see [7, 18]).

Coming back to the mathematical interest of problem (1)–(3), we stress the fact that equation (1) can be seen as the singular limit of the second order parabolic equation

$$\partial_t u + F(x,u)\partial_x u = \epsilon\, \partial_x^2 u + G(x,u), \qquad (\epsilon > 0), \tag{4}$$

(in fact the entropy solution of (1) can be obtained via the so-called vanishing viscosity method, that is as limit of solutions to (4)). Equation (4) is usually called *reaction-diffusion-convection equation*; and the Dirichlet problem for such equations was investigated in [2, 3, 4],[12]–[14], [17] under different assumptions on F, G, ϵ. Some different problems for the same equation have been considered in [8] and in [11]. We do not want to go into details of all such papers; let us just stress some interesting connections of our hyperbolic paper with some of the cited parabolic works.

In [2] it has been considered the Burgers-Sivashinsky equation as a simplified model for flame propagation (see also [10])

$$\partial_t u + \partial_x \left(\frac{1}{2}u^2\right) = \epsilon\, \partial_x^2 u + u,$$

with homogeneous boundary conditions, $\epsilon \in \mathbb{R}$ small. Results announced in the paper appear very natural if compared with properties of entropy solutions of (1) (see e.g. [22]); actually they seem to be inheredited by the underlying hyperbolic structure of the singular limit of the equation. This suggests that a complete analysis of problem (1)–(3) could be very useful in order to understand the complete picture for the viscous equation (4), at least for small ϵ.

In [13] sufficient conditions guaranteeing asymptotic stability of the steady solutions of (4) are given. In this direction we note that assumption (A2) of [13] excludes the existence of internal shocks for the corresponding unviscous problem (i.e. equation (4) with $\epsilon = 0$). On the contrary, in this paper, we deal with case showing the presence of discontinuous stationary solutions to (1). Such discontinuities correspond to internal layer in the viscous case ($\epsilon > 0$).

Properties of the global attractor for (4) (with $F(x,u) = F(u)$ and $G(x,u) = G(u)$) as $\epsilon \to 0$ are considered in [8, 11]. In [8] the Cauchy problem with periodic data has been studied. In [11] the case of Neumann boundary conditions on a bounded domain has been investigated (for connections with the hyperbolic case, see the Discussion at the end of [11]).

Finally let us note that in [16] and in [23] it has been shown for equation (4) with $G \equiv 0$ that (for opportune initial data) the solution rapidly generates a viscous shock, corresponding to the one of the unviscous equations. The unviscous shock is stationary for (1), on the contrary, due to diffusion effects, the viscous wave moves to an equilibrium point. Nevertheless such motion is, for small ϵ, very slow. Therefore we can conjecture that the qualitative behavior of solutions to equation (4) is approximately given by the corresponding behavior of the solutions to the unviscous equation (1) for intermediate time, and that the transport effect caused by the presence of the diffusion becomes important for larger time. Hence our analysis should give information on “intermediate properties” of solutions to (4).

In order to prove large-time behavior results, we assume the flux function f to be convex. This assumption has been made in the majority of the articles concerning the Cauchy problem for (1). Under such assumption, in [5], it was build up the theory of generalized characteristics. This tool is very useful to analyze solutions of reaction-convection equations and it can also be used in the case of bounded domains. The nonconvex case was considered in few papers, either extending the method of generalized characteristics to the case of one-change of convexity [19], or constructing explicit solutions to some specific problems [25], or by a broad use of comparison principle [20, 21]. In any case, nowadays, the detailed analysis, guaranteed by generalized characteristic for convex fluxes, is not available for the nonconvex case. Nevertheless it seems natural to expect that many of the result still hold in the general case.

Concerning the reaction function g, apart from hypothesis guaranteeing that no blow-up occur, we assume

$$f'(s) = 0 \qquad \Rightarrow \qquad g(x,s) \neq 0 \qquad \forall x \in [0,L]. \tag{5}$$

Such assumption permits us to give a complete picture on existence and multiplicity of the steady-states for problem (1)–(3) and on the asymptotic behavior of solutions. Assumption (5) yields that there is no characteristic curve for equation (1) completely contained in the half-strip $[0,L]\times[0,+\infty)$ of the $(x,t)-$plane. Such assumption is suggested by the fact that the presence of global characteristic in $[0,L]\times[0,+\infty)$ would possibly imply the existence of oscillating steady solutions (see also [24]). This kind of solutions have been well-studied for the Cauchy problem; moreover in [22] it has been proved that, generically, such solutions do not appear in the asymptotic behavior. Finally let us stress the fact that assumptions (H3) of [11] corresponds to (5). Thus (5) essentially controls the dimension of the global attractor for problem (1)–(3).

Now we sketch the content of our main theorem (precise statement is given in the following of the paper, see Theorem 4.4). We prove that, after a finite time (not depending on the initial datum) the evolution of the solution to problem (1)–(3) becomes one-dimensional. More precisely we show that there exists T^* such that for $t \geq T^*$ the solution is of the form

$$u(x,t) = \begin{cases} \phi_l(x) & \text{for } x < y, \\ \phi_r(x) & \text{for } x \geq y, \end{cases}$$

where ϕ_l and ϕ_r are opportune stationary solutions of (1) and y solves the o.d.e.

$$\frac{dy}{dt} = \hat{F}(y) := \begin{cases} 0 & \text{if } y \in \{0, L\}, \\ \dfrac{f(\phi_l(y)) - f(\phi_r(y))}{\phi_l(y) - \phi_r(y)} & \text{elsewhere.} \end{cases}$$

The function y describes the evolution of the (unique) layer. Moreover there is a one-to-one correspondence between the critical points of such o.d.e. and the stationary solutions of problem (1)–(3).

Once it has been proved such property, an opportune mapping from initial data of (1)–(3) to the interval $[0,L]$ is defined. By use of this mapping, results concerning the structure of the attractor are proved.

The paper is structured as follows. In Section 2 we recall the mathematical tools used for proving the results. Section 3 concernes with the analysis of entropic steady states of the problem. In Section 4 we deal with the time-dependent problem. There we state and prove the main theorem on large-time behavior. Finally, in Section 5, we generalize some results and analyze, as an example, the case of a space-dependent source.

2 Mathematical Tools

In this Section we introduce the mathematical background and recall main properties of the techniques used in this paper.

Let f, g be continuous functions of their arguments.

Definition 2.1 *Let $u_0 \in BV(0,L)$, $a_0, b_0 \in BV(0,T)$. A function $u \in BV((0,L)\times(0,T)) \cap L^\infty((0,L)\times(0,T))$ is an* entropy solution *to the problem*

$$\begin{cases} \partial_t u + \partial_x f(u) = g(x,u) & \text{in } (0,L)\times(0,T) \\ u(x,0) = u_0(x) & \text{in } (0,L) \\ u(0,t) = a_0(t) \quad u(L,t) = b_0(t) & \text{in } (0,T), \end{cases} \tag{6}$$

if the following holds

i) for any $\phi \in C^1((0,L)\times[0,T))$, $\phi \geq 0$ and for any $k \in \mathbb{R}$

$$\int_0^T \int_0^L |u-k|\partial_t \phi + \operatorname{sgn}(u-k)(f(u)-f(k))\partial_x \phi + \operatorname{sgn}(u-k) g(x,u)\phi \, dx\, dt + \\ + \int_0^L |u_0 - k|\phi \, dx \geq 0;$$

ii) for almost any $t \in (0,T)$

$$\max\{\operatorname{sgn}(u(0,t)-a_0(t))(f(u(0,t))-f(k)) \, : \, k \in I(a_0(t), u(0,t))\} = 0,$$

$$\min\{\operatorname{sgn}(u(L,t)-b_0(t))(f(u(L,t))-f(k)) \, : \, k \in I(b_0(t), u(L,t))\} = 0.$$

where $I(a,b) := (\min\{a,b\}, \max\{a,b\})$.

(Here $u(0,t)$ and $u(L,t)$ represent the traces of the function $u = u(x,t)$ on the boundary. Such traces are well-defined since u is of bounded variation).

In order to guarantee uniqueness and global existence, let us assume:

Hypothesis G:

(i) for any $M > 0$ there exists $K_M > 0$ such that for any $x \in [0,L]$ and for any $u, v \in \mathbb{R}$ with $|u|, |v| \leq M$ there holds

$$|g(x,u) - g(x,v)| \leq K_M |u - v|;$$

(ii) there exist $A, B > 0$ such that for any $x \in [0,L]$ for any $u \in \mathbb{R}$ there holds

$$|g(x,u)| \leq A + B|u|;$$

The following result is due to Bardos, Le Roux and Nedelec [1].

Theorem 2.2 *Assume hypothesis (G) and let* $f \in C^2(\mathbb{R})$. *Given* u_0, a_0 *and* b_0 *as in Definition 2.1, then for any* $T > 0$ *there exists a unique entropy solution* $u \in BV((0,L) \times (0,T)) \cap L^\infty([0,T), BV(0,L)) \cap C([0,T), L^1(0,L))$ *to problem (6).*

Definition 2.3 *Let* u_0, a_0, b_0 *as in Definition 2.1.*

A function $\underline{u} \in L^\infty((0,L) \times (0,T)) \cap BV((0,L) \times (0,T))$ *is an* entropy subsolution *(respectively* $\overline{u} \in L^\infty((0,L) \times (0,T)) \cap BV((0,L) \times (0,T))$ *is an* entropy supersolution *of problem (6)), if* $\underline{u}$ *(resp.* $\overline{u}$*) enjoies properties i), ii), iii), of Definition 2.1 with* $[\,\cdot\,]_+$ *(resp.* $[\,\cdot\,]_-$*) replacing* $|\cdot|$ *and* $H^+(\,\cdot\,)$ *(resp.* $H^-(\,\cdot\,)$*) replacing* $\mathrm{sgn}(\cdot)$.

(Here $[s]_+ = \max(s,0)$, $[s]_- = -\min(s,0)$, $H^\pm(s) = (\mathrm{sgn}(s) \pm 1)/2$*).*

For entropy sub- and supersolution comparison property holds.

Theorem 2.4 *Let* $\underline{u}, \overline{u} \in BV((0,L) \times (0,T)) \cap L^\infty([0,T), BV(0,L)) \cap C([0,T), L^1(0,L))$ *be, respectively, a subsolution and a supersolution of problem (6) with data* $\underline{u}_0$, $\underline{a}_0$, $\underline{b}_0$, *and* $\overline{u}_0$, $\overline{a}_0$, $\overline{b}_0$.

Then, for any $t \in (0,T)$, *it holds*

$$\int_0^L [\underline{u}(x,t) - \overline{u}(x,t)]_+ dx \le e^{ct} \left[\int_0^L [\underline{u}_0(x) - \overline{u}_0(x)]_+ \, dx + M \left(\int_0^t [\underline{a}_0(\tau) - \overline{a}_0(\tau)]_+ \, d\tau + \int_0^t [\underline{b}_0(\tau) - \overline{b}_0(\tau)]_+ \, d\tau \right) \right], \tag{7}$$

where $M := \sup\{|f'(u)| \ : \ |u| \le \max(|\underline{u}|_\infty, |\overline{u}|_\infty)\}$ *and* c *is the lipschitz constant of the function* g.

Proof. For any $\phi \in C_0^1((0,L) \times (0,T))$, $\phi \ge 0$ we get

$$\int_0^L \int_0^T [\underline{u} - \overline{u}]_+ \partial_t \phi + H^+(\underline{u} - \overline{u})(f(\underline{u}) - f(\overline{u})) \partial_x \phi_x \, dx dt \ge \int_0^L \int_0^T H^+(\underline{u} - \overline{u})(g(\underline{u}, x) - g(\overline{u}, x)) \phi \, dx dt.$$

Taking a sequence of smooth function approximating the characteristic function of the set $(0,L) \times (0,t)$, we obtain that, for any $t \in (0,T)$,

$$\begin{aligned} \int_0^L [\underline{u}(x,t) - \overline{u}(x,t)]_+ dx \le & \int_0^L [\underline{u}_0(x) - \overline{u}_0(x)]_+ \, dx + \\ & \int_0^t H^+(\underline{u}(0,\tau) - \overline{u}(0,\tau))(f(\underline{u}(0,\tau)) - f(\overline{u}(0,\tau))) \, d\tau + \\ & \int_0^t H^+(\underline{u}(L,\tau) - \overline{u}(L,\tau))(f(\overline{u}(L,\tau)) - f(\underline{u}(L,\tau))) \, d\tau - \\ & \int_0^t \int_0^L H^+(\underline{u} - \overline{u})(g(\underline{u}, x) - g(\overline{u}, x)) \, dx dt. \end{aligned}$$

Then (see [26])

$$\int_0^L [\underline{u} - \overline{u}]_+ \, dx \le \int_0^L [\underline{u}_0(x) - \overline{u}_0(x)]_+ \, dx + M \int_0^t [\underline{a}_0(\tau) - \overline{a}_0(\tau)]_+ \, d\tau + \\ M \int_0^t [\underline{b}_0(\tau) - \overline{b}_0(\tau)]_+ \, d\tau + c \int_0^t \int_0^L [\underline{u} - \overline{u}]_+ \, dx \, d\tau.$$

and the conclusion follows from Gronwall inequality.

Corollary 2.5 *Let u and v be entropy solutions of the problem (6) with data u_0, a_0, b_0 and v_0, a_0', b_0', respectively. Assume*

$$u_0 \le v_0 \quad \text{a.e. in } (0, L), \qquad a_0 \le a_0' \quad \text{and} \quad b_0 \le b_0' \quad \text{a.e. in } (0, T).$$

Then

$$u \le v \quad \text{a.e. in } (0, L) \times (0, T).$$

Since the solution u to problem (6) is in the class $L^\infty([0, T), BV(0, L))$ it is possible to consider left and right limits of function $u(\cdot, t)$. Given $(\overline{x}, \overline{t})$, set

$$u(\overline{x}\pm, \overline{t}) := \lim_{\epsilon \to 0+} u(\overline{x} \pm \epsilon, \overline{t}).$$

In proving results on large-time behavior, we assume also

Hypothesis F: the function $f \in C^2(\mathbb{R})$, is strictly convex and $f(\pm\infty) = +\infty$.

In this case the admissibility condition at a discontinuity point is the following

$$u(x-, t) \ge u(x+, t).$$

Moreover, in the convex case, we make wide use of the technique of the generalized characteristic. Such theory was introduced in [5], and it has been applied in order to obtain results on asymptotic behavior for scalar balance law (see [22] and references therein).

Here we recall the main properties of such theory.

Definition 2.6 *A* generalized characteristic *associated to equation (1) is a Lipschitz curve $\xi : [a, b] \subset (0, \infty) \to \mathbb{R}$ such that*

$$\xi'(s) \in [f'(u(\xi(s)+, s)), f'(u(\xi(s)-, s))].$$

A characteristic ξ is genuine *in the interval (a, b), if $u(\xi(s)-, s) = u(\xi(s)+, s)$ for any $s \in (a, b)$.*

Fixed $(\overline{x}, \overline{t})$, a characteristic curve ξ, is a backward (resp. forward) characteristic for $(\overline{x}, \overline{t})$ if $\xi(\overline{t}) = \overline{x}$ and ξ is defined in $[\overline{t} - \epsilon, \overline{t}]$ (resp. $[\overline{t}, \overline{t} + \epsilon]$), for some $\epsilon > 0$.

It can be proved that, fixed an entropy solution of (1) globally bounded, given $(\overline{x}, \overline{t})$, there exist at least one forward characteristic and one backward characteristic. Moreover any characteristic is confined in between a minimal and a maximal characteristic.

Dafermos results' can be applied even in bounded domain$(0, L) \times (0, T)$. The following statement summarizes the content of Theorem 3.1–4 and Corollaries of [5].

Theorem 2.7 *Assume hypothesis F and G. Let u be an entropy solution of problem (1)–(3), and let $(\overline{x},\overline{t}) \in (0,L)\times(0,+\infty)$.*

i) Let $\xi:(a,b)\to(0,L)$ $(0<a<b<\infty)$ be a generalized characteristic. Then, at any point $s\in(a,b)$ of differentiability ξ, it holds

$$\xi'(s)=\begin{cases} f'(u(\xi(s)-,s)) & \text{if } u(\xi(s)-,s)=u(\xi(s)+,s),\\[2ex] \dfrac{f(u(\xi(s)-,s))-f(u(\xi(s)+,s))}{u(\xi(s)-,s)-u(\xi(s)+,s)} & \text{if } u(\xi(s)-,s)>u(\xi(s)+,s).\end{cases}$$

ii) From any point $(\overline{x},\overline{t})$ a backward minimal characteristic $\xi_-=\xi_-(t;\overline{x},\overline{t})$ and a backward maximal one $\xi_+=\xi_+(t;\overline{x},\overline{t})$ start and both are genuine.

In their domain of definition, such characteristics are solution of

$$\begin{cases} \xi_\pm'(s)=f'(v_\pm(s)) & \xi(\overline{t})=\overline{x},\\ v_\pm'(s)=g(v_\pm(s),\xi_\pm(s)) & v(\overline{t})=\overline{v}_\pm.\end{cases}$$

where $\overline{v}_\pm=u(\overline{x}\pm,\overline{t})$. Moreover, setting

$$t_\pm(\overline{x},\overline{t}):=\inf\{t\in[0,\overline{t})\;:\;\xi_\pm(s;\,\overline{x},\overline{t})\in(0,L),\,\forall s\in[t,\overline{t}]\}. \tag{8}$$

the following holds

$$u(\xi_\pm(s),s)=v_\pm(s)\qquad \forall s\in(t_\pm(\overline{x},\overline{t}),\overline{t}).$$

iii) Let ξ_1,ξ_2 be two generalized characteristic defined in $[a,b]$, such that $\xi_1(b)<\xi_2(b)$. Then $\xi_1(s)<\xi_2(s)$ for any $s\in(a,b]$.

iv) There exists a unique forward characteristic $\eta=\eta(t;\overline{x},\overline{t})$ through $(\overline{x},\overline{t})$.

Before ending this Section we give a final useful Lemma. This result essentially guarantees backward uniqueness for ordered entropy solution.

Lemma 2.8 *Let $\xi,\eta:[t_1,t_2]\subset[0,T]\to(0,L)$ be Lipschitz functions such that $\xi(\tau)\le\eta(\tau)$ for any $\tau\in[t_1,t_2]$. Let $u,v,$ be entropy solutions of (1) in $(0,L)\times(0,T)$ satisfying*

$$\begin{array}{l} u\le v \quad \text{in } D^{\xi,\eta}_{t_1,t_2}:=\{(x,t)\in(0,L)\times[0,T]\;:\;t_1\le t\le t_2,\;\xi(t)\le x\le\eta(t)\}\\ u(\cdot,t_2)\equiv v(\cdot,t_2) \qquad \text{in } [\xi(t_2),\eta(t_2)]\\ u(\xi(t)+,t)=v(\xi(t)+,t) \quad \text{and} \quad u(\eta(t)-,t)=v(\eta(t)-,t) \qquad \forall t\in[t_1,t_2].\end{array}$$

Then

$$u\equiv v \qquad a.e.\ \text{in} \quad D^{\xi,\eta}_{t_1,t_2}.$$

Proof. Applying Lemma 3.2 of [5], we get

$$\int_{\xi(t)}^{\eta(t)}u(x,t)\,dx-\int_{\xi(s)}^{\eta(s)}u(x,s)\,dx=\int_t^s\int_{\xi(\tau)}^{\eta(\tau)}g(x,u(x,\tau))\,dxd\tau+$$
$$+\int_t^s\{f(u(\xi(\tau)-,\tau))-\xi'(\tau)u(\xi-,\tau)\}\,d\tau-\int_t^s\{f(u(\eta(\tau)+,\tau))-\eta'(\tau)u(\eta+,\tau)\}\,d\tau.$$

The same holds substituting u with v.

For any $t \in [t_1, t_2]$ define

$$F(t) := \int_{\xi(t)}^{\eta(t)} [v(x,t) - u(x,t)]\, dx.$$

From the previous relation we obtain that, for any $t \in [t_1, t_2]$,

$$0 \leq F(t) = \int_{D_{t,t_2}^{\xi,\eta}} [g(x,v) - g(x,u)]\, dx\, d\tau \leq c \int_t^{t_2} F(\tau)\, d\tau,$$

where c is a Lipschitz constant for g.

Then

$$\left(e^{ct} \int_t^{t_2} F(\tau)\, d\tau \right)' \geq 0.$$

Therefore, by standard calculations, we get $F(\tau) = 0$ for any $\tau \in (t, t_2)$ and the conclusion follows.

3 Entropic Steady States

Throughout this Section we make the following assumptions on functions f and g:

Hypothesis F': $f \in C^1(\mathbb{R})$, there exists $s_0 \in \mathbb{R}$ such that f is strictly decreasing (strictly increasing) in $(-\infty, s_0)$ (in $(s_0, +\infty)$) and $f(\pm\infty) = +\infty$;

Hypothesis G:

(i) for any $M > 0$ there exists $K_M > 0$ such that for any $x \in [0, L]$ and for any $u, v \in \mathbb{R}$ with $|u|, |v| \leq M$ there holds $|g(x,u) - g(x,v)| \leq K_M |u - v|$;

(ii) there exists $A, B > 0$ such that for any $x \in [0, L]$ for any $u \in \mathbb{R}$ there holds $|g(x,u)| \leq A + B|u|$;

Hypothesis H: for any $x \in [0, L]$ there holds $g(x, s_0) \neq 0$.

Notations. Let us define three useful functions of a real variable: the first one is real valued, the others are set-valued. Let f be a function satisfying hypothesis F.

We set $v_f : \mathbb{R} \to \mathbb{R}$, where $v_f = v_f(u)$ is given by

$$v_f(u) := \begin{cases} w & \text{if } \exists w \neq u \text{ s.t. } f(w) = f(u), \\ s_0 & \text{if } u = s_0, \end{cases} \tag{9}$$

Note that v_f is decreasing and $v_f(v_f(u)) = u$.

Given $u \in \mathbb{R}$, set $I\pm : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ (here $\mathcal{P}(\mathbb{R})$ denotes the sets of all subsets of $\mathbb{R}$) as follows

$$I_-(u) := \begin{cases} (-\infty, v_f(u)] \cup \{u\} & u \geq s_0, \\ (-\infty, s_0] & u < s_0; \end{cases} \tag{10}$$

$$I_+(u) := \begin{cases} \{u\} \cup [v_f(u), +\infty) & u \le s_0, \\ [s_0, +\infty) & u > s_0. \end{cases} \tag{11}$$

Finally given $u_\pm \in \mathbb{R}$, for shortness, we denote $I_-(u_-)$ (resp. $I_+(u_+)$) with I_- (resp. I_+).

Definition 3.1 *Assume f satisfies F' and let g be a continuous function. Given $u_\pm \in \mathbb{R}$, a function $\phi : [0, L] \to \mathbb{R}$ is an* entropy solution *of the problem*

$$\begin{cases} \big(f(v)\big)' = g(x, v) & x \in [0, L], \\ v(0) = u_-, \quad v(L) = u_+, \end{cases} \tag{12}$$

if the following hold

(i) there exist $\xi_1, \cdots, \xi_N \in (0, L)$ such that $\phi \in C^1((0, L) \setminus \{\xi_1, \cdots, \xi_N\})$;

(ii) $f'(\phi(x))\phi'(x) = g(x, \phi)$ for any $x \in (0, L) \setminus \{\xi_1, \cdots, \xi_N\}$;

(iii) for any $i = 1, \cdots, N$ there exist $\phi(\xi_i\pm)$, and there hold $f(\phi(\xi_i-)) = f(\phi(\xi_i+))$, $\phi(\xi_i-) \ge \phi(\xi_i+)$;

(iv) there exist $\phi(0+)$, $\phi(L-)$ and $\phi(0+) \in I_-$, $\phi(L-) \in I_+$.

Entropy solutions of problem (12) given in Definition 3.1 correspond to entropy stationary solution of problem (1)–(3) that are piecewise smooth and satisfy boundary conditions in the sense of [1]. This motivates the definitions of the functions $I_\pm$.

Proposition 3.2 *Assume F',G,H and let ϕ be an entropy solution of problem (12).*
Then the following hold.

(i) ϕ has at most one point of discontinuity, say $\xi_0 \in (0, L)$ and

$$\phi(\xi_0+) < s_0 < \phi(\xi_0-).$$

(ii) If $\phi(0+) < s_0$ then $\phi(x) < s_0$ for any $x \in (0, L)$ and $\phi \in C^1(0, L)$.

(iii) If $\phi(L-) > s_0$ then $\phi(x) > s_0$ for any $x \in (0, L)$ and $\phi \in C^1(0, L)$.

Proof. (i) Assume by contradiction that there exists a solution ϕ to problem (12) with two internal discontinuities at points ξ_1 and ξ_2 with $\xi_1 < \xi_2$. Then there is $\xi^* \in (\xi_1, \xi_2)$ with $\phi(\xi^*) = s_0$. Hence the function ϕ is such that

$$f'(\phi(\xi))\phi(\xi) = g(\xi, \phi(\xi)) \qquad \forall \xi \in I_\delta(\xi^*) \setminus \{\xi^*\},$$

and $\phi(\xi^*) = s_0$. This contradicts assumption H.

Assertions (ii) and (iii) are immediate consequences of part (i).

Theorem 3.3 *Assume hypothesis F',G,H.*
Then problem (12) has at least one entropy solution.

Moreover there exist $x_l, x_r \in [0,1]$ with $x_r \leq x_l$ and two functions $\phi_l : [0, x_l] \to \mathbb{R}$ and $\phi_r : [x_r, 1] \to \mathbb{R}$ such that

$$\begin{cases} f'(\phi_i(x))\phi_i'(x) = g(x, \phi(x)) & \forall x \in J_i \quad i \in \{l, r\}, \\ \phi_l(0) = \max I_-, & \phi_l(x) > s_0 \quad \forall x \in J_l, \\ \phi_r(L) = \min I_+, & \phi_r(x) < s_0 \quad \forall x \in J_r, \end{cases}$$

where $J_l := (0, x_l)$ and $J_r := (x_r, 1)$, and any entropy solution ϕ of problem (12) is of the form

$$\phi(x) := \begin{cases} \phi_l(x) & x < x_0, \\ \phi_r(x) & x > x_0, \end{cases} \tag{13}$$

for some $x_0 \in [0, L]$. Such x_0 is either 0, or L, or zero of the function $h(x) := f(\phi_l(x)) - f(\phi_r(x))$.

Finally there is a one-to-one correspondence between the discontinuous solutions to problem (12) and zeros of function h in (x_r, x_l).

At the boundary of $[0, L]$ we expect that the behavior of solutions to (1)–(3) depends on the directions of characteristic curves; therefore it is useful to distinguish different cases depending on the sign of $f'(u\pm)$ (i.e. depending on whether the classical characteristics enter the domain or not). We use the following notation:

Compressive case (C): $u_+ < s_0 < u_-$;
Expansive case (E): $u_- \leq s_0 \leq u_+$;
Left-wind case (L): $u_- \leq s_0$ and $u_+ < s_0$;
Right-wind case (R): $u_- > s_0$ and $u_+ \geq s_0$.

Corollary 3.4 *If one of the following assumption holds*
(i) either $u_- \leq s_0 \leq u_+$,
(ii) or $u_\pm \geq s_0$ and $g(x, s_0) > 0$ for any $x \in (0, L)$,
(iii) or $u_\pm \leq s_0$ and $g(x, s_0) < 0$ for any $x \in (0, L)$,
then there exists a unique entropy solution ϕ to problem (12) and $\phi \in C^1(0, L)$.

Proof of Theorem 3.3. We give the proof for the compressive case, thus we assume $u_+ < s_0 < u_-$ (the other cases can be managed in a similar way).

In this situation we have

$$I_- = (-\infty, v_f(u_-)] \cup \{u_-\} \qquad I_+ = \{u_+\} \cup [v_f(u_+), +\infty).$$

Let ϕ_l (respectively ϕ_r) be classical solution of

$$f'(\phi)\phi' = g(x, \phi) \tag{14}$$

satisfying $\phi(0) = u_-$ (resp. $\phi_r(L) = u_+$) and let ϕ_l (resp. ϕ_r) be defined in $[0, x_l)$ (resp. $(x_r, L]$). Then $\phi_l > s_0$ and $\phi_r < s_0$.

If $x_l \in (0,L)$, then $\phi_l(x_l-) = s_0$. Since ϕ_l is a classical solution of equation (14), then we deduce $g(x,s_0) < 0$ for any $x \in [0,L]$. Analogously, if $x_r \in (0,L)$, then $g(x,s_0) > 0$ for any $x \in [0,L]$.

Therefore, either $J_l = (0,L)$ or $J_r = (0,L)$. Without restriction, let us assume $x_l = L$ and $x_r \in [0,L)$. Then $\phi_r(x_r) \le s_0$.

If $\phi_l(L) \ge v_f(u_+)$, then ϕ_l is an entropy solution of problem (12). Similarly, if $x_r = 0$ and $\phi_r(0) \le v_f(u_-)$ then ϕ_r is an entropy solution of problem (12). Therefore let us assume that this is not the case. Then $\phi_l(L) < v_f(L)$ and either $x_r = 0$ and $\phi_r(0) > v_f(u_-)$, or $x_r > 0$ and $\phi_r(x_r) = s_0$. Properties of v_f guarantee

$$h(x_r) = f(\phi_l(x_r)) - f(\phi_r(x_r)) = f(\phi_l(x_r)) - f(v_f(\phi_r(x_r))) > 0,$$

indeed if $x_r = 0$, since $v_f(\phi_r(0)) < v_f(v_f(u_-)) = u_-$, then $h(x_r) > f(\phi_l(0)) - f(u_-) = 0$; if $x_r > 0$, then $h(x_r) = f(\phi_l(x_r)) - s_0 > 0$.

Moreover, since $\phi_l(L) < v_f(u_+)$, $v_f(\phi_l(L)) > v_f(v_f(u_+))$, therefore

$$h(L) = f(v_f(\phi_l(L))) - f(u_+) < 0.$$

Hence there exists $x_0 \in (x_r, L)$ such that $h(x_0) = 0$. Then the function

$$\phi(x) := \begin{cases} \phi_l(x) & x < x_0, \\ \phi_r(x) & x > x_0, \end{cases}$$

is an entropy solution of (12). Any other zero of h defines another stationary solution.

In order to complete the proof it is enough to show that such construction gives any stationary solution. Let ϕ be a stationary solution. By definition $\phi(0) \in (-\infty, v_f(u_-)] \cup \{u_-\}$. If $\phi(0) \in (-\infty, v_f(u_-)]$ we can apply Proposition 3.2(ii) to conclude that $\phi(L) = u_+$ and ϕ is given by (13) with $x_0 = 0$. If $\phi(0) = u_-$, then the solution is either regular (and it coincides with ϕ_l), or discontinuous (and it is of the form (13)).

Proof of Corollary 3.4. (i) If $u_- \le s_0 \le u_+$, then $I_- = (-\infty, s_0]$, $I_+ = [s_0, +\infty)$, then $x_l > 0$ if and only if $g(x,s_0) > 0$, and $x_r < L$ if and only if $g(x,s_0) < 0$. Therefore there is a unique solution given by formula (13) with $x_0 = L$ if $g(x,s_0) > 0$ or $x_0 = 0$ if $g(x,s_0) < 0$.

(ii) If $u_\pm \ge s_0$, then $I_+ = [s_0, +\infty)$. Since $g(x,s_0) > 0$, then $x_r = L$ and the conclusion follows.

(iii) Similar to (ii).

4 Large–Time Dynamic

In this Section we assume the flux function f to be convex. Summarizing, we make the following assumptions:

Hypothesis F: the function $f \in C^2(\mathbb{R})$, strictly convex, $f'(s_0) = 0$ and $f(\pm\infty) = +\infty$.

Hypothesis G:

(i) for any $M > 0$ there exists $K_M > 0$ such that for any $x \in [0, L]$ and for any $u, v \in \mathbb{R}$ with $|u|, |v| \leq M$ there holds $|g(x, u) - g(x, v)| \leq K_M |u - v|$;

(ii) there exists $A, B > 0$ such that for any $x \in [0, L]$ for any $u \in \mathbb{R}$ there holds $|g(x, u)| \leq A + B|u|$;

Hypothesis H: for any $x \in [0, L]$ there holds $g(x, s_0) \neq 0$.

Under assumption F, the boundary conditions (3) are satisfied in the sense of Definition 2.1(ii) if and only if

$$u(0-, t) \in I_- \qquad \text{and} \qquad u(0+, t) \in I_+,$$

where $I_\pm$ are defined in (9), (10) and (11).

Given $\tau \geq 0$, let us denote with $\xi(\cdot; y, \tau, \sigma) : [\tau, \infty) \to \mathbb{R}$, the solution of

$$\begin{cases} \xi'(t) = f'(v(t)) & v'(t) = g(v(t)) \\ \xi(\tau) = y \in [0, L] & v(\tau) = \sigma \in \mathbb{R} \end{cases}$$

Moreover let us set

$$\Sigma_t := [0, L] \times [0, t) \qquad (t \in [0, +\infty]).$$

Definition 4.1 *Given $A \subset \Sigma_\infty$, set*

$$K(A) := \{(x, t) \in \Sigma_\infty \,:\, \exists\, (y, \tau) \in A, \sigma \in \mathbb{R}, \tau \in [0, t] \text{ s.t. } \xi(t; y, \tau, \sigma) = x \\ \text{and } \xi(s; y, \tau, \sigma) \in [0, L] \quad \forall s \in [\tau, t]\},$$

and

$$\overline{T}(A) := \sup\{t > 0 \,:\, \exists x \in [0, L] \text{ s.t. } (x, t) \in K(A)\}.$$

From Definition 4.1, it immediately follows that

$$A \subset B \quad \Rightarrow \quad K(A) \subset K(B) \quad \text{and} \quad \overline{T}(A) \leq \overline{T}(B).$$

Proposition 4.2 *Assume hypotheses F,G,H.*

If $A \subset \Sigma_t$ for some $t \geq 0$, then there exists $T^* = T^*(f, g, L)$ such that

$$\overline{T}(A) \leq T^* + t < +\infty.$$

Moreover

$$\overline{T}([0, L] \times \{0\}) \equiv T^* < +\infty.$$

First of all, let us state and prove a lemma.

Lemma 4.3 *Let $H = (H_1, \cdots, H_N) : \mathbb{R} \to \mathbb{R}^N$, and $G : \mathbb{R}^{N+1} \to \mathbb{R}$ be locally Lipschitz functions, let $C \subset \mathbb{R}^N$ be a compact set ($N \geq 1$). Assume the following*

(i) $\liminf\limits_{Y \to \pm\infty} |H_i(Y)| > 0$ *for any i,*

(ii) if $H_i(Y_*) = 0$ for some i and for some $Y_* \in \mathbb{R}$, then $K(X, Y_*) \neq 0$ for any $X \in C$. Given $(X_0, Y_0) \in C \times \mathbb{R}$, let $(X(t), Y(t))$ be the solution of

$$\begin{cases} X' = H(Y), & Y' = K(X,Y), \\ X(0) = X_0, & Y(0) = Y_0. \end{cases}$$

Then

$$T_0 := \sup\{T \geq 0 \,:\, \exists (X_0, Y_0) \in C \times \mathbb{R} \text{ s.t. } X(t) \in C \quad \forall t \in [0,T]\} < +\infty.$$

Proof of Lemma 4.3. Assume that there exist $i \in \{1, \cdots, N\}$ and $Y_* \in \mathbb{R}$ such that $H_i(Y_*) = 0$ (if this is not the case the proof is easy). Define

$$\alpha := \inf\{Y \in \mathbb{R} \,:\, \exists i \in \{1, \cdots, N\} \text{ s.t. } H_i(Y) = 0\},$$
$$\beta := \sup\{Y \in \mathbb{R} \,:\, \exists i \in \{1, \cdots, N\} \text{ s.t. } H_i(Y) = 0\}.$$

From assumptions (i), it follows

$$-\infty < \alpha \leq \beta < +\infty.$$

Step 1. There exists ϵ_0, δ_0 such that for any $Y \notin J := [\alpha - \delta_0, \beta + \delta_0]$ it holds

$$|H_i(Y)| \geq \epsilon_0 \qquad \forall i.$$

Moreover $K(X, \alpha - \delta_0), K(X, \beta + \delta_0) \neq 0$ for any $X \in C$.

Step 2. Assume $X_0 \in C$ and $Y_0 \notin J$. If $Y(t) \notin J$ for any $t \geq 0$ such that $X(t) \in C$, then $|H_i(Y(t))| \geq \epsilon_0$. Assume $H_i > 0$ for some i, then

$$X_i(t) \geq \epsilon_0 t + X_{i,0}$$

for any t such that $X(t) \in C$. Then it follows that $X(t)$ exits C at a time t less than T_1, where

$$T_1 := \frac{\text{diam } C}{\epsilon_0}.$$

Step 3. Let $X_0 \in C$ and $Y_0 \in J$, then there exists $T = T(X_0, Y_0)$ such that $(X(t), Y(t))$ exits $C \times J$ at time T.

In fact, assume by contradiction that $(X(t), Y(t)) \in C \times J$ for any $t \geq 0$. If $\inf Y(t) = \sup Y(t)$, then $Y(t) = Y_0$ for any t. Hence $K(X(t), Y_0) = 0$, and therefore $H_i(Y_0) \neq 0$ for any i, by assumption (i).

Assume $\inf Y(t) < \sup Y(t)$, and let $\theta \in (\inf Y(t), \sup Y(t))$. Then there exist $t_1, t_2 > 0$ such that $Y(t_1) = Y(t_2) = 0$ and $K(X(t_1), \theta) \leq 0 \leq K(X(t_2), \theta)$. By assumption (ii), it follows $H_i(\theta) \neq 0$ for any i. Therefore we get

$$H_i(\theta) \neq 0 \qquad \forall i \quad \forall \theta \in [\inf Y(t), \sup Y(t)].$$

Hence the trajectory $X(t)$ exits C in finite time.

We set
$$T_2 := \max\{T(X_0,Y_0)\,:\,(X_0,Y_0)\in C\times J\}.$$

Final step. By Step 1, any trajectory $Y(t)$ cannot cross twice the values $\alpha-\delta_0$ and $\beta+\delta_0$. Since any trajectory stays a finite time, depending only on H,K and C, in any of the regions $\{Y<\alpha-\delta_0\}$, J, $\{Y>\beta+\delta_0\}$, then any trajectory exits at a time T such that
$$T\le 2T_1+T_2.$$
Taking the supremum, we get $T_0\le 2T_1+T_2<+\infty$.

Proof of Proposition 4.2. Since $A\subset\Sigma_t$, it follows
$$\overline{T}(A)\le\overline{T}(\Sigma_t).$$
By Lemma 4.3,
$$\overline{T}(\Sigma_t)\le T^*+t,$$
and the conclusion follows.

Now we can state the following Theorem, showing that the dynamic of the problem (1)–(3) becomes one-dimensional after a finite time.

Theorem 4.4 *Assume hypotheses F, G, H. Let u be the entropy solution of problem (1)–(3). Then there exists $T^*=T^*(f,g,L)$ such that for any $t\ge T^*$ there exists $\overline{x}=\overline{x}(t)\in[0,L]$, $\overline{x}(T^*)\in[x_r,x_l]$ such that*
$$u(x,t)=\begin{cases}\phi_l(x) & \text{in}\quad 0\le x\le\overline{x}(t)\\ \phi_r(x) & \text{in}\quad \overline{x}(t)<x\le 1,\end{cases}\tag{15}$$
where ϕ_l,ϕ_r,x_l,x_r are defined in Theorem 3.3.

Moreover
$$\frac{d\overline{x}}{dt}(t)=\hat{F}(\overline{x}(t))\qquad\forall t\ge T^*,\tag{16}$$
where
$$\hat{F}(y):=\begin{cases}0 & \text{if } y\in\{0,L\},\\[2ex] \dfrac{f(\phi_l(y))-f(\phi_r(y))}{\phi_l(y)-\phi_r(y)} & \text{elsewhere.}\end{cases}$$

Remark. If $x_r=L$ then $\overline{x}(t)\equiv L$; hence the solution becomes stationary in finite time. Analogously, if $x_l=0$, then $\overline{x}(t)\equiv 0$ and the same conclusion holds. Therefore, from now on, we assume $x_r<L$ and $x_l>0$, the other case being trivial.

Proof of Theorem 4.4. Let $x_0\in(0,L)$ and $t_0\ge T^*$ be fixed with T^* as in Proposition 4.2. We denote by $\xi_-=\xi_-(t;x_0,t_0)$ and $\xi_+=\xi_+(t;x_0,t_0)$, the minimal and the maximal backward generalized characteristic through (x_0,t_0). Such curves are genuine characteristics.

Let $t_\pm = t_\pm(x_0, t_0)$ be defined as in (8); then the characteristics $\xi_\pm$ exit the domain $[0, L] \times (0, t_0]$.

By definition of T^* it follows that $t_\pm(x_0, t_0) > 0$. Hence $\xi_\pm(t_\pm) \in \{0, L\}$.

If $\xi_\pm(t_\pm) = 0$, we deduce from boundary condition that

$$u(0, t_\pm) \in I_-.$$

Since a classical characteristic starting from $(0, t_\pm)$ enters the strip $[0, L] \times (0, t_0]$, then

$$u(0, t_\pm) = \max I_-.$$

Analogously, if $\xi_\pm(t_\pm) = L$, then

$$u(L, t_\pm) = \min I_+.$$

Set

$$x_- := \sup\{x_0 \in (0, L) \, : \, \xi_+(t_+; x_0, t_0) = 0\};$$
$$x_+ := \inf\{x_0 \in (0, L) \, : \, \xi_-(t_-; x_0, t_0) = L\},$$

putting $x_- = 0$ if $\{x \in (0, L) \, : \, \xi_+(x, t_+) = 0\} = \emptyset$ and $x_+ = L$ if $\{x \in (0, L) \, : \, \xi_-(x, t_-) = L\} = \emptyset$.

Since classical characteristics cannot intersect, $x_- \leq x_+$. Now we show that $x_- = x_+$. Assume by contradiction $x_- < x_+$, then take z and y such that $x_- < z < y < x_+$. Therefore, by definition of x_+,

$$\xi_-(t_-(y, t_0)) = 0,$$

hence, since classical characteristic cannot cross each other,

$$\xi_+(t_+(z, t_0)) = 0,$$

contradicting the minimality of x_-. Hence $x_- = x_+ =: \overline{x}$.

From definition of $\overline{x}$, it follows that, at time t_0, the solution is given by formula (15). This completes the proof.

By Theorem 4.4, after finite time t_0, the dynamic of the problem (1)–(3) becomes one-dimensional, and it is given by (15)–(16). In general the time t_0 depends on the initial data u_0. Nevertheless $t_0 \leq \overline{T}([0, L] \times \{0\}) = T^*$ for any $u_0 \in BV(0, L)$.

Therefore, fixed the boundary value $u_\pm$, we can define a mapping from the set of the initial data $BV(0, L)$ to the interval $[0, L]$ as follows

$$u_0 \to \mathcal{F}(u_0) := \overline{x}(T^*),$$

with $\overline{x}$ as in Theorem 4.4. Then the dynamic of problem (1)–(3), for $t \geq T^*$, is equivalent to the dynamic of the following problem

$$\frac{dy}{dt}(t) = \hat{F}(y), \qquad y(T^*) = \overline{x}(T^*).$$

The following result concernes with properties of the mapping $\mathcal{F}$.

Theorem 4.5 *Assume hypotheses F, G, H.*

(i) (Continuity of $\mathcal{F}$) For any $u_{0,n}, u_0 \in BV(0,L)$ such that

$$\lim_{n\to+\infty} \|u_{0,n} - u_0\|_1 = 0,$$

it holds

$$\lim_{n\to+\infty} \mathcal{F}(u_{0,n}) = \mathcal{F}(u_0).$$

(ii) (Monotonicity of $\mathcal{F}$) For any $u_0, v_0 \in BV(0,L)$ are such that $u_0 \le v_0$ (i.e. $u_0(x) \le v_0(x)$ a.e. in $[0,L]$), it holds

$$\mathcal{F}(u_0) \le \mathcal{F}(v_0).$$

Proof. (i) Let u_n and u be the entropy solution associated to the initial data $u_{0,n}$ and u_0. Then, applying inequality (7), we get

$$\int_0^L |u_n(x,T^*) - u(x,T^*)|\,dx \le e^{cT^*} \int_0^L |u_{0,n}(x) - u_0(x)|\,dx.$$

Since, for any $t = T^*$, the solutions u_n, u have the structure given in (15), with $\overline{x}(T^*)$ given, respectively, by $\mathcal{F}(u_{0,n})$ and $\mathcal{F}(u_0)$, we have

$$\lim_{n\to\infty} \left| \int_{\mathcal{F}(u_0)}^{\mathcal{F}(u_{0,n})} \big(\phi_l(x) - \phi_r(x)\big)\,dx \right| = 0.$$

Suppose, by contradiction, that there exists a subsequence of $\{u_{0,n}\}$ (for simplicity, we again denote it by $\{u_{0,n}\}$) such that $|\mathcal{F}(u_{0,n}) - \mathcal{F}(u_0)| \ge \epsilon_0$, for some $\epsilon_0 > 0$. Without restriction, we can assume $\mathcal{F}(u_{0,n}) \ge \mathcal{F}(u_0) + \epsilon_0$. Since $\phi_l(x) - \phi_r(x) > 0$ in (x_r, x_l), then

$$\int_{\mathcal{F}(u_0)}^{\mathcal{F}(u_{0,n})} \big(\phi_l(x) - \phi_r(x)\big)\,dx \ge \int_{\mathcal{F}(u_0)}^{\mathcal{F}(u_0)+\epsilon_0} \big(\phi_l(x) - \phi_r(x)\big)\,dx > 0.$$

Passing to the limit, as $n \to +\infty$, we get a contradiction.

(ii) Let u and v be the entropy solution associated to the initial data u_0 and v_0. From Corollary 2.5, it follows that $u(x,t) \le v(x,t)$ a.e. in $(0,L) \times (0,\infty)$. Then

$$u(x,T^*) \le v(x,T^*).$$

Since, by Theorem 4.4,

$$u(x,T^*) = \begin{cases} \phi_l(x) & \text{in } 0 \le x \le \mathcal{F}(u_0) \\ \phi_r(x) & \text{in } \mathcal{F}(u_0) < x \le 1, \end{cases} \qquad v(x,T^*) = \begin{cases} \phi_l(x) & \text{in } 0 \le x \le \mathcal{F}(v_0) \\ \phi_r(x) & \text{in } \mathcal{F}(v_0) < x \le 1, \end{cases}$$

the conclusion follows from the property $\phi_r \le s_0 \le \phi_l$.

Remark. If $\alpha := \min[\phi_l(x) - \phi_r(x)] > 0$, it can be proved that the map $\mathcal{F}$ is Lipschitz continuous with constant $\alpha \cdot [x_l - x_r]$.

Moreover, from definitions of ϕ_l and ϕ_r, if $\alpha = 0$, either $g(x, s_0) > 0$ and $u_- \leq s_0$ or $g(x, s_0) < 0$ and $u_+ \geq s_0$. In particular, in the compressive case $\mathcal{F}$ is always a Lipschitz function.

From Theorem 4.5 some interesting properties concerning L^1-stability of steady states and asymptotic behavior for problem (1)–(3) follow.

Corollary 4.6 *Assume hypotheses F, G, H.*
(i) If ϕ_l (respectively ϕ_r) is a stationary solution of problem (12) in the sense of Definition 3.1, then ϕ_l (resp. ϕ_r) is unstable if and only if $x_r = 0$ (resp. $x_l = L$) and $f(\phi_l(x)) - f(\phi_r(x)) > 0$ (resp. < 0) for $x \in (0, \epsilon)$ (resp. $x \in (L - \epsilon, L)$) for some $\epsilon > 0$.
(ii) Let ϕ be a discontinuous steady state with jump point $x_0 \in (x_r, x_l)$ such that

$$g(x_0, \phi_l(x_0)) - g(x_0, \phi_r(x_0)) < 0 \quad (> 0).$$

Then ϕ is asymptotically stable (unstable).
Moreover the domain of attraction of ϕ is open (closed) in L^1.
(iii) If there exists a unique stationary solution ϕ then it is globally attractive.
Moreover if $\phi \in C(0, L)$ then there exists $T_1 < +\infty$ such that for any u_0 it holds

$$u(x, T_1) = \phi(x).$$

Remark. Note that the stability condition given in Corollary 4.6(ii) is the same found in [18] and in [7]. Infact, if g is smooth,

$$g(x_0, \phi_l(x_0)) - g(x_0, \phi_r(x_0)) = g_u(x_0, \xi)(\phi_l(x_0) - \phi_r(x_0)).$$

In the nozzles case, the function g is of the form $g(x, u) = a(x)k(u)$, therefore the condition reads

$$a(x)k'(u) < 0 \quad (> 0) \qquad \Rightarrow \qquad \phi \text{ is asymptotically stable} \quad \text{(is unstable)}.$$

Proof of Corollary 4.6. (i) The proof is similar to the one of (ii).
(ii) Set

$$h(x) := f(\phi_l(x)) - f(\phi_r(x)) \qquad \forall x \in [x_r, x_l].$$

By Theorem 4.4 the dynamic for $t \geq T^*$ is given by the o.d.e. (16). It is easy to see that the stability character of the stationary solution of (16) is given by the sign of h near the jump point x_0. More precisely

$$h'(x_0) < 0 \quad (> 0) \quad \Rightarrow \quad \phi \text{ is asymptotically stable} \quad \text{(is unstable)}$$

Since

$$h'(x_0) = f'(\phi_l(x_0))\phi_l'(x_0) - f'(\phi_r(x_0))\phi_r'(x_0) = g(x_0, \phi_l(x_0)) - g(x_0, \phi_r(x_0)),$$

claim on stability properties follows.

The second part of the statement is an immediate consequence of continuity of the mapping $\mathcal{F}$.

(iii) Assume ϕ is discontinuous at x_0. Then there are two cases: either $x_l = L$ or $x_l < L$. In the first case, $s_0 \le \phi_l(L) < v_f(u_+), u_+ < s_0$ and

$$h(L) = f(\phi_l(L)) - f(\phi_r(L)) = f(\phi_l(L)) - f(u_+) < 0.$$

Moreover $h(x_r) > 0$. In fact, if $x_r > 0$ then $\phi_r(x_r) = s_0$ and the assertion follows; if $x_r = 0$, then $h(0) = f(\max I_-) - f(\phi_r(0)) > 0$, since we are assuming that ϕ_r is not a steady state (hence it does not satisfy boundary condition).

Since ϕ is the unique solution $h(x) = 0$ if and only if $x = x_0$. Hence

$$h(x) > 0 \quad \iff \quad x < x_0.$$

On the other hand, if $x_l < L$, then $\phi_l(x_l) = s_0, x_r = 0$ and $v_f(u_-) < \phi_r(0) \le s_0$. Therefore

$$h(x_l) = f(s_0) - f(\phi_r(x_l)) < 0, \quad h(0) = f(u_-) - f(\phi_r(0)) > 0,$$

and the conclusion follows similarly.

The case of $\phi \in C(0, L)$ can be treated in the same way.

Concerning unstable solution with internal layer we can also prove a result showing the so-called *hair trigger effect.*

Proposition 4.7 *(Hair-Trigger Effect) Let ϕ be a discontinuous steady-state of (1)–(3), with jump point $x_0 \in [x_r, x_l]$ such that, for some $\epsilon > 0$,*

$$\begin{cases} h(x_0) < 0 & \forall\, x \in (x_0 - \epsilon, x_0), \\ h(x_0) > 0 & \forall\, x \in (x_0, x_0 + \epsilon). \end{cases}$$

Given $u_0 \in BV(0, L)$, let u be the entropy solution of problem (1)–(3).

Then, if $u_0 \le \phi$ ($u_0 \ge \phi$), either $u_0 \equiv \phi$ either u converges to a steady state $\psi \le \phi$ ($\psi \ge \phi$), different from ϕ.

Proof. Suppose that u converges to ϕ, as $t \to \infty$. From assumptions on ϕ and from Theorem 4.4, it follows that u coincides with ϕ for any $T \ge T^*$. Let t_0 be defined by

$$t_0 := \inf\{t > 0 \,:\, u(x, t) = \phi(x) \text{ in } [0, L]\} \le T^*.$$

Assume, by contradiction, $t_0 > 0$. Then, by continuity, $u(x, t_0) = \phi(x)$. Let $\zeta_- = \zeta_-(t)$ be the minimal backward characteristic of solution u starting from (x_0, t_0), and let $\zeta_+ = \zeta_+(t)$ be the maximal one. Since u and ϕ coincide at $t = t_0$ such characteristics are the minimal and the maximal from (x_0, t_0) even for ψ. Therefore, for small $\epsilon > 0$, u coincides with ψ in Ω, where

$$\Omega := \{(x, t) \in \Sigma_\infty \,:\, t_0 - \epsilon < t < t_0,\, 0 < x < \zeta_-(t) \text{ or } \zeta_+(t) < x < L\}.$$

Applying Lemma 2.8 to u and ϕ in the region

$$D := \{(x,t) \in \Sigma_\infty \ : \ t_0 - \epsilon < t < t_0, \ \zeta_-(t) < x < \zeta_+(t), \},$$

we deduce that u coincides with ϕ in D. Hence u coincides with ϕ for $t \in (t_0 - \epsilon, t_0)$, contradicting the minimality assumption on t_0.

Before ending this section, we state and prove one more result. This gives a sharp characterization of ordered solutions of problem (1)–(3) coinciding one each other after finite time. Apart from the case of solutions becoming ϕ_l or ϕ_r, rarefactions of (1) turn out to be the main point in such analysis.

Proposition 4.8 *Let $u_0, v_0 \in BV(0,L)$ such that $u_0 \le v_0$ and meas$\{x \in (0,L) \ : \ u_0(x) = v_0(x)\} = 0$. Let u, respectively v solution of problem (1)–(3) with initial data u_0, respectively v_0. Suppose that, for some $\bar{t}$, $u(\cdot,\bar{t}) \equiv v(\cdot,\bar{t})$, then one of the following holds.*
i) $u(\cdot,\bar{t}) \equiv v(\cdot,\bar{t}) \equiv \phi_l$;
ii) $u(\cdot,\bar{t}) \equiv v(\cdot,\bar{t}) \equiv \phi_r$;
iii) there exist $x_0 \in (0,L)$, $p,q \in \mathbb{R}$, $p < q$, l_0, $l_1 \in [0,L]$ such that

$$u(\cdot,\tilde{t}) \equiv v(\cdot,\tilde{t}) \equiv u(x,\tilde{t}) = v(x,\tilde{t}) = \begin{cases} \phi_l(x) & \text{if } x \in [0,l_0) \\ R(x_0,p,q;x,\tilde{t}) & \text{if } x \in [l_0,l_1) \\ \phi_r(x) & \text{if } x \in [l_1,L], \end{cases}$$

where

$$\tilde{t} := \inf\{t \in [0,\bar{t}) \ : \ u(\cdot,t) = v(\cdot,t)\}$$

and $R(x_0,p,q;\cdot,\cdot)$ is the solution of the Riemann problem

$$\partial_t u + \partial_x f(u) = g(x,u) \qquad x \in \mathbb{R}, \, t > 0,$$

$$u(x,0) = \begin{cases} p & \text{if } x \le x_0 \\ q & \text{if } x > x_0. \end{cases}$$

Proof. Let

$$y_- := \sup\{x \in (0,L) \ : \ \zeta_+(t_-;x,\tilde{t}) = 0\}$$

$$y_+ := \inf\{x \in (0,L) \ : \ \zeta_-(t_+;x,\tilde{t}) = L\}$$

putting $y_- = 0$ if $\{x \in (0,L) \ : \ \zeta_+(t_-;x,\tilde{t}) = 0\} = \emptyset$ and $y_+ = L$ if $\{x \in (0,L) \ : \ \zeta_-(t_+;x,\tilde{t}) = L\} = \emptyset$.

If $y_- = L$ or $y_+ = 0$ we are respectively in the case i) or ii).

Suppose that $y_+(y_- - L) \neq 0$. By definition, $y_- \le y_+$. Moreover, by Lemma 2.8, $y_- \neq y_+$ and $y_-(y_+ - L) = 0$.

Let $y_- = 0$ and $y_+ \in (0,L]$, then for every $x \in (0,y_+)$ we obtain $t_\pm(x,\tilde{t}) = 0$. Suppose that there exist $x,y \in (0,y_+)$, $x \le y$ such that $\zeta_-(0;x,\tilde{t}) < \zeta_+(0;y,\tilde{t})$. Then, by Lemma 2.8, we obtain

$$u_0 \equiv v_0 \qquad \text{in } (\zeta_-(0;x,\tilde{t}), \zeta_+(0;y,\tilde{t})$$

which contradicts the hypothesis. Therefore there exists $x_0 \in (0, L)$ such that

$$\zeta_\pm(0; x, \tilde{t}) = x_0 \qquad \text{for every } x \in (0, y_+).$$

Then the conclusion follows. Indeed

$$u(x, \tilde{t}) = v(x, \tilde{t}) = \begin{cases} R(x_0, v_1(0), v_2(0); x, \tilde{t}) & \text{if } x \in (0, y_+) \\ \phi_r(x) & \text{if } x \in [y_+, L), \end{cases}$$

where the functions v_i satisfy

$$\begin{cases} v_i'(t) = g(v_i(t), \zeta_i(t)), & \zeta_i'(t) = f'(v_i(t)) \\ v_1(\tilde{t}) = u(0+, \tilde{t}), & v_2(\tilde{t}) = u(y_+-, \tilde{t}) \\ \zeta_1(\tilde{t}) = 0, & \zeta_2(\tilde{t}) = y_+. \end{cases}$$

The case $y_- \in (0, L)$, $y_+ = L$ is similar to the previous one.

5 Examples and Generalizations

Nonconstant boundary data

All of our results concern with constant boundary data. Nevertheless all the theorems can be extended to a class of nonconstant data. In fact boundary conditions are satisfied in the sense of [1], i.e. $u(0, t)$ and $u(L, t)$ belong to an appropriate set depending on the boundary data. Such set of admissibility are defined in (10), (11). From these definitions it follows that the results still hold for problem (1)–(2) with boundary conditions given by

$$u(0, t) = a_0(t), \qquad u(L, t) = b_0(t) \qquad \forall t > 0,$$

if $a_0(t) \leq s_0$ and $b_0(t) \geq s_0$ for any t; indeed in such case

$$I_-(a_0(t)) = (-\infty, s_0], \qquad I_+(b_0(t)) = [s_0, +\infty) \qquad \forall t > 0.$$

A result on asymptotic behavior for nonconvex flux

In Section 3, we have proved results on existence of steady states without assuming convexity on the flux function f. On the contrary, in Section 4, we have used the hypothesis that f is strictly convex. This assumption allows us to use the technique of generalized characteristics, therefore it is just a technical hypothesis. Here we want to show, by proving another result, that we can expect that the same behavior holds even in the absence of convexity. Anyway, note that this result is not as general as Theorem 4.4. For simplicity we consider space-independent source term, hence we consider the equation

$$\partial_t u + \partial_x f(u) = g(u). \tag{17}$$

Let us assume the following

hypothesis G': $g \in C^1(\mathbb{R})$, $g(s) \geq \gamma_0 > 0$ for any $s \in [-\epsilon, \epsilon]$ for some $\epsilon > 0$, and there exists $N > 0$ such that $g(s)s < 0$ for any s with $|s| \geq N$;

hypothesis F'': $f \in C^2(\mathbb{R})$, $f(0) = 0$, $f'(s)s > 0$ for any $s \in \mathbb{R}$ and $f''(0) > 0$.

Moreover we consider the expansive case, therefore we assume

$$u_- \leq 0 \leq u_+. \tag{18}$$

From Corollary 3.4, we deduce that there exists a unique stationary solution ϕ. Moreover, since g is positive in a neighborhood of 0, we also deduce that ϕ is such that

$$\phi \in C^1(0, L),\ \phi(0) = 0 \ \text{ and } \ \phi \text{ increasing}.$$

Then the following result holds, the proof being based on the construction of appropriate sub- and supersolutions.

Theorem 5.1 *Assume hypothesis F'' and G'. Let u be the entropy solution of problem (17), (2),(3), with $u_\pm$ satisfying (18). Let ϕ be the unique steady state of the same problem.*
Then

$$\lim_{t \to +\infty} \|u(\cdot, t) - \phi(\cdot)\|_\infty = 0.$$

Proof. Let us set $M := \max\{\|u_0\|_\infty, N\}$.
Step 1. Supersolution. Let us introduce the following function

$$\hat{f}(s, \sigma) := \begin{cases} \dfrac{f(s) - f(\sigma)}{s - \sigma} & s \neq \sigma, \\ \\ f'(s) & s = \sigma. \end{cases}$$

Then $\hat{f}$ is continuous.
For any $\eta > 0$, small enough, let us set

$$R_\eta := [\phi(\eta), M] \qquad \text{and} \qquad c_\eta := \min\{\hat{f}(s, \sigma) \ : \ s, \sigma \in R_\eta\}.$$

From assumption F$''$, it follows that $c_\eta > 0$.
Then consider the following one-parameter family of functions

$$V_\eta := \begin{cases} M & x \geq c_\eta t \text{ and } x \leq L, \\ \phi(x + \eta) & x < c_\eta t \text{ and } x \in [0, L]. \end{cases}$$

We claim that, for any $\eta > 0$, V_η is a supersolution. From Definition 2.3, it follows that V_η is a supersolution if and only if

$$c_\eta \leq \frac{f(M) - f(\phi(ct + \eta))}{M - \phi(ct + \eta)},$$

for any t such that $\phi(c_\eta t + \eta) \le \phi(L+\eta)$. Such properties follows from the definition of c_η.

Therefore, since $u_0 \le V_\eta(x,0)$, we deduce by comparison principle

$$u(x,t) \le \phi(x+\eta) \qquad \forall t \ge M/c_\eta.$$

Hence

$$\limsup_{t\to+\infty} u(x,t) \le \phi(x) \qquad \forall x \in (0,L).$$

Step 2. First Subsolution. Let $\delta > 0$ be such that $f''(-\delta) > 0$ and $g(s) > 0$ for any $s \in (-\delta, 0)$. Let

$$c_1 := \sup\{\hat{f}(s,\sigma) \, : \, s,\sigma \in [-M,-\delta]\}.$$

Since $c_1 < 0$, it follows that the function

$$W_1(x,t) := \begin{cases} -M & 0 < x < c_1 t + L, \\ -\delta & c_1 t + L < x < L, \end{cases}$$

is a subsolution and that $W_1(x,t) = -\delta$ for any $t > t_1 := L/|c_1|$. Therefore

$$u(x,t) \ge -\delta \qquad \forall t > t_1 := L/|c_1|.$$

Step 3. Second Subsolution. By Step 2, we can assume $u_0 \ge -\delta$, with δ as above. Let $U(\sigma;t)$ be the solution of

$$\partial_t U = g(U) \qquad U(\sigma;0) = \sigma.$$

Let t_2 be the unique value such that $U(-\delta;t_2) = 0$. Then, defining $\xi(t) := \phi^{-1}(U(-\delta;t))$ for $t \ge t_2$, the function

$$W_2(x,t) := \begin{cases} \phi(x) & x \le \xi(t),\ t \ge t_2, \\ U(-\delta;t) & \text{elsewhere}, \end{cases}$$

is a subsolution of the problem. Hence

$$u(x,t) \ge \phi(x) \qquad \forall t \ge t_3,$$

where t_3 is such that $U(-\delta;t_3) = \phi(L)$.

Joining together this estimate with the one at the end of Step 1, we get the conclusion.

An example with source depending only on the space variable

Consider the problem

$$\begin{cases} \partial_t u + \partial_x f(u) = g(x) & x \in [0,L], \\ u(0,t) = u_-, \quad u(L,t) = u_+ & t > 0, \end{cases} \tag{19}$$

under the assumption $g(x) > 0$ for any $x \in [0, L]$, hypothesis F' and $f(0) = 0$. Let f_+^{-1} (respectively f_-^{-1}) be the inverse function of f over $[0, +\infty)$ (resp. over $(-\infty, 0]$), and let $w_- = \max I_-$, $w_+ = \min I_+$. Set

$$\phi_l(x) := f_+^{-1}\left(f(w_-) + \int_0^x g(\xi)\, d\xi\right),$$

$$\phi_r(x) := f_-^{-1}\left(f(w_+) - \int_x^L g(\xi)\, d\xi\right),$$

$$F(u_+, u_-) := f(w_-) - f(w_+) + \int_0^L g(\xi)\, d\xi.$$

The function ϕ_l is always defined over all $[0, L]$, while the function ϕ_r is defined in $[x_r, L]$, where

$$x_r := \inf\left\{x \in [0, L] \,:\, f(w_+) \geq \int_x^L g(\xi)\, d\xi\right\}.$$

After easy calculations, we get

$$F(u_+, u_-) = 0 \qquad \Longleftrightarrow \qquad u_+ = f_-^{-1}\left(f(w_-) + \int_0^L g(\xi)\, d\xi\right).$$

The following result holds

Proposition 5.2 *Problem (19) has a unique stationary solution if and only if $F(u_+, u_-) \neq 0$. Moreover if $F(u_+, u_-) > 0$ the unique solution is given by ϕ_l, if $F(u_+, u_-) < 0$ the solution is ϕ_r.*

Finally if $F(u_+, u_-) = 0$ the function

$$\phi(x) := \begin{cases} \phi_l(x) & x < x_0, \\ \phi_r(x) & x > x_0, \end{cases}$$

is a stationary solution of problem (19) for any $x_0 \in [0, L]$.

Proof. First of all let us prove that ϕ_l is a solution if and only if $F(u_+, u_-) \geq 0$.

If $u_+ \geq 0$, then $w_+ = 0$. Therefore $\phi_l(L) \in I_+$ and ϕ_l is a solution. Moreover

$$F(u_+, u_-) = f(w_-) + \int_0^L g(\xi)\, d\xi > 0.$$

Assume $u_+ < 0$. Then $w_+ = u_+$ and

$$F(u_+, u_-) = f(w_-) - f(u_+) + \int_0^L g(\xi)\, d\xi.$$

The function ϕ_l satisfies the boundary condition at $x = L$ if and only if $f(\phi_l(L)) - f(u_+) \geq 0$. By definition of ϕ_l

$$f(\phi_l(L)) - f(u_+) = F(u_+, u_-),$$

and the conclusion follows.

Analogously ϕ_r is a solution if and only if $F(u_+,u_-)\leq 0$. Indeed assume that ϕ_r is a solution Since ϕ_r is defined in $[0,L]$ if and only if $f(w_+)\geq \int_0^L g(\xi)\,d\xi>0$, it holds $w_+=u_+<0$. Then

$$0\geq f(u_-)-f(\phi_r(0))=f(u_-)-f(u_+)+\int_0^L g(\xi)\,d\xi=F(u_+,u_-)+f(u_-)-f(w_-),$$

implying $F(u_+,u_-)\leq 0$. Similarly it can be proved that if $F(u_+,u_-)\leq 0$, then ϕ_r is a solution.

In order to complete the proof we have only to show that discontinuous solutions exist if and only if $F(u_+,u_-)=0$, and that every point x_0 of the interval $[0,L]$ can be a jump point. From Theorem 3.3 it follows that jump point x_0 must satisfy $f(\phi_l(x_0))-f(\phi_r(x_0))=0$. Then we get

$$0=f(\phi_l(x_0))-f(\phi_r(x_0))=f(w_-)-f(w_+)+\int_0^L g(\xi)\,d\xi=F(u_+,u_-),$$

hence the conclusion.

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