First Order Singular Perturbations of Quasilinear Nonconvex Type

Corrado MASCIA

Dipartimento di Matematica "G. Castelnuovo"
Università degli Studi di Roma "La Sapienza"
P.le Aldo Moro 5, I–00185 Roma, Italy
e-mail: mascia@mat.uniroma1.it

1 Introduction

In this paper we deal with the following singular perturbation problem: let $u^\varepsilon = u^\varepsilon(x,t)$ be the entropy solution of

$$\begin{cases} \varepsilon\,(\partial_t u + \partial_x f(u)) = g(u), & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{1}$$

where u_0 is a given initial datum. Then, what happens to the family of solutions u^ε as $\varepsilon \to 0^+$? And what relation has the limit u^0 of such a family (if it exists) with the (formal) limiting equation $g(u) = 0$?

To start with let us assume that g has isolated zeros (precise assumptions will be given later). It is natural to expect that where the limit u^0 is regular the limiting equation is satisfied in a classical sense, and therefore that u^0 is constant in such regions. So the natural idea is that the function u^0 is piecewise constant (at least if the initial datum u_0 does not oscillate too much). Any curve between two different values of u^0 corresponds to a transition layer of the singular limit. Some questions arise naturally: how does the transition layer propagate? Can we predict which constant states are chosen by the limiting process?

Such problems are widely studied in different areas of differential equations. In some context, they are called *singular perturbation problems* since the complete equation ($\varepsilon > 0$) can be seen as a perturbation of higher order of the reduced equation ($\varepsilon = 0$). In the case of o.d.e. many interesting phenomenon already come into view: transitional layers, non-uniformity of convergence. Here we deal with a p.d.e., hence the appeareance of a second (space-)variable complicates the problem, since the layers may move while the time-variable increases.

Another field of research dealing with similar questions considers *relaxation problems* (actually, a relaxation problem is a singular perturbation problem). There is a very large amount of papers concerning such kind of problems. Here we just stress that the equation considered in this paper can be seen as a scalar case of the hyperbolic system considered in [1, 5] (for a recent review on hyperbolic relaxation problems see [10]).

Fixed $\varepsilon > 0$, (1) is a Cauchy problem for an hyperbolic conservation law with reaction term. Hence, through all the paper, we deal with entropy solution of (1), whose existence and uniquenesse was proved in [4].

The starting point of this work is the paper [3]. There the authors consider the same problem presented at the beginning under the following assumptions: the source term g is of the form $g(u) = -W'(u)$ where W is the double well potential (e.g. $W = k(u^2 - 1)^2$ with $k > 0$); the flux $f(u)$ is a *convex* function; $u_0 \in C^1$ has finitely many zeroes a_j ($j = 1, \cdots, n$) with $u_0'(a_j) \neq 0$ for any j. Now we recall the statement of their theorem under a stronger assumption on the initial datum in order to have a unique layer in the limit.

Theorem [3]. *Let f and g satisfy the above assumptions; assume $u_0 \in C^1$ such that $u_0(x) = 0$ if and only if $x = 0$, $u'(0) \neq 0$.*

Then the limit $u^0(x,t) := \lim_{\varepsilon \to 0} u^\varepsilon(x,t)$ exists for almost all $(x,t) \in \mathbb{R} \times (0,\infty)$ and it holds $u^0(x,t) = sgn\ u_0(x - \gamma t)$ with $\gamma = f'(0)$ if $u_0'(0) > 0$, $\gamma = (f(1) - f(-1))/2$ if $u_0'(0) < 0$.

To get the complete statement of the result in [3], observe that any simple zero of the initial datum u_0 generates a layer propagating with a speed depending on the sign of the first derivative of u_0 at that point, as described above in the case of a single layer. The final picture is obtained by noting that such layers may interact and that each interaction causes a cancellation of these discontinuity curves.

In this paper we generalize the result in [3] by weakening the assumption on the flux function f. Our hypothesis are the following:

Hypotheses F: $f \in C^2([-1,1])$ with a finite number of inflection points;

Hypotheses G: $g \in C^1([-1,1])$ such that $g(0) = g(\pm 1) = 0$ and $s \cdot g(s) > 0$ for any $s \in (-1,0) \cup (0,1)$;

In order to have stronger results on convergence, we will make the following additional assumption;
Hypothesis H: if $f(s) \geq f(0) + f'(0)s$ for any $s \in [-1,1]$ then

$$\frac{f'(\cdot) - f'(0)}{g(\cdot)} \in L^1(-\delta, \delta) \quad \text{for some } \delta > 0.$$

(Hypothesis (H) is always satisfied if $g'(0) \neq 0$.)

Now we are ready to state the main theorem of this paper.

Theorem 1.1 *Assume (F), (G). Let $u_0 \in C(\mathbb{R})$ such that $x \cdot u_0(x) > 0$ for any $x \in \mathbb{R} \setminus \{0\}$, $u_0(\mathbb{R}) \subset [-1,1]$ and $\liminf_{x \to \pm\infty} |u_0(x)| > 0$.*

Then there exists a unique $\gamma \in \mathbb{R}$ such that for almost all $(x,t) \in \mathbb{R} \times (0,\infty)$

$$\lim_{\varepsilon \to 0} u^\varepsilon(x,t) = u^0(x,t) := sgn\ \ (x - \gamma t),$$

the convergence being uniform in $\Sigma_{\tau,T} := \{(x,t) \in \mathbb{R} \times [\tau, T] \ : \ |x - ct| \geq \tau\}$ for any $0 < \tau < T$. The speed γ is characterized by the following property

$$\min_{y \in [-1,0]} (f(y) - \gamma y) = \min_{y \in [0,1]} (f(y) - \gamma y) . \tag{2}$$

Moreover, if it does hold also assumption (H), then the convergence is uniform in $\Sigma_\tau := \{(x,t) \in \mathbb{R} \times [\tau, +\infty) \ : \ |x - ct| \geq \tau\}$ for any $\tau > 0$.

Remarks.

1. In the case of $u_0 \in C(\mathbb{R})$ such that $x \cdot u_0(x) < 0$ for any $x \in \mathbb{R} \setminus \{0\}$, a similar result hold with a different speed γ' and with the singular limit being $\lim_{\varepsilon \to 0} u^\varepsilon(x,t) = u^0(x,t) := -\text{sgn}\ \ (x - \gamma' t)$. The speed γ' is characterized by

$$\max_{y \in [-1,0]} (f(y) - \gamma' y) = \max_{y \in [0,1]} (f(y) - \gamma' y) . \tag{3}$$

2. Concerning the regularity of the initial datum u_0 what is really needed for the proof is that conclusion of Lemma 5.5 does hold. So the same result of Theorem 1.1 holds for other classes of initial data. As an example, increasing monotonicity can replace the continuity assumption.

3. It is interesting to note that the convergence as $\varepsilon \to 0$ to a traveling profile with speed γ holds even if assumption (H) is not satisfied. In this case it can be proved that the solution of the Cauchy problem (1) does not converge to a traveling wave with speed γ as $t \to +\infty$. This motivates the lack of uniformity of the convergence as $\varepsilon \to 0$ in Σ_τ. Moreover this is an interesting example of qualitative difference between limit $\varepsilon \to 0$ and limit $t \to +\infty$. For more details, see Appendix.

By the charaterization of the speed γ, we deduce that if $f(s) \geq f(0) + f'(0)s$ for any $s \in [-1, 1]$ (this condition holds if f is convex), it can be checked that that $\gamma = f'(0)$. If $f(s) \leq f(-1) + (f(1) - f(-1))(s+1)/2$ for any $s \in [-1, 1]$ (this condition holds if f is concave), then $\gamma = (f(1) - f(-1))/2$, i.e. γ is given by the Rankine-Hugoniot condition. Similar results holds for γ'. Note that such limiting speed are exactly the same found in [3] in the case of convex flux f.

In the statement of the Theorem we consider an initial datum u_0 with a unique zero (at $x = 0$). This is assumed in order to give a simpler proof of the Theorem. Nevertheless similar results can be obtained by the same technique in the case of more general u_0, introducing more complicated sub- and supersolutions.

The general picture is very similar to the one sketched in [3]. Any point x_0 such that $(x - x_0) \cdot u(x) > 0$ in a neighborhood of x_0 will generate, in the limit, a layer with values -1 at the left and $+1$ at the right of it propagating with speed γ, and any point x_0 such that $(x - x_0) \cdot u(x) < 0$ in a neighborhood of x_0 generate a layer with values $+1$ at the left and -1 at the right of it propagating with speed γ'. If the speeds γ and γ' are different, then interaction of layers may occur. In this case, any one of them cause a cancellation of such discontinuities as can be seen by the following consequence of Theorem 1.1.

Corollary 1.2 *Assume (F) and (G) and let $u_0 \in X$ be such that $|u_0(x)| \leq 1$ for any $x \in \mathbb{R}$ and $\limsup_{x\to\pm\infty} u_0(x) < 0$. Finally assume that $\gamma > \gamma'$, where γ and γ' are defined as in formulas (2) and (3). Then there exists $T > 0$ such that*

$$\lim_{\varepsilon\to 0} u^\varepsilon(x,t) = -1 \qquad \forall x \in \mathbb{R}, \forall t > T.$$

Throughout the paper we assume g has a very simple structure. What happens for more general functions g? As will be seen later on, the key argument in the proof is the fact that the problem (1) exhibits traveling wave solutions connecting the stable states of g, -1 and $+1$. In the case of many more zeros between -1 and $+1$, then we cannot expect existence of traveling wave connecting such states. In fact, the structures we have to work with are the so-called *non-interacting wave patterns*, that are family of traveling waves with ordered speeds. In the case of convex flux f such patterns always exists and can be explicitly constructed (see [9]), giving the picture presented in [3]. In the case of non-convex f, the existence of non-interacting wave pattern has not yet been proved in the general case (see [6] for some partial result in this direction). In any case, it is natural to conjecture that such patterns always exist and that they give the complete structure of the singular limit.

Results similar to the one proved in [3] and to Theorem 1.1 had been the starting point for a very rich field of research in recent years: front propagation for reaction-diffusion system (see [2], also [12] and references). In that context, there has been widely studied the singular limit of the following equations

$$\left.\begin{aligned}&\partial_t u - \varepsilon\Delta u + \varepsilon^{-1}W'(u) = 0,\\ &\partial_t u - \Delta u + \varepsilon^{-2}W'(u) = 0,\end{aligned}\right.\qquad (W \text{ double-well potential})$$

as well as their generalizations with a general second-order elliptic operator in place of Δ and more complicate reaction term W.

It is interesting to note that the strong analogy of the hyperbolic results proved in this paper and the ones for reaction-diffusion equations are due to the fact that same properties hold for both equations: existence of traveling waves and comparison results. These properties are the main tool in our proofs, unlike the results in [3], obtained by using generalized characteristic theory, a typical hyperbolic object.

The paper has five further Sections. In Section 2, some heuristic arguments are given to indicate the important rôle played by traveling waves. Section 3 is devoted to the construction of traveling waves for the equation. In order to prove the results on convergence we need an opportune topology that is introduced in Section 4. In Section 5, once we have all the tools we need, we prove the main result of the paper. Finally, in Section 6, we consider slightly different assumption on the reaction function g such that the equilibrium set has a one-dimensional component. This causes the possible appearance in the limit u^0 of non-constant smooth pieces.

2 Heuristics

This Section aims to explain heuristically the main result of this paper. In particular we want to motivate the fact that the limiting speed of propagation of layers connecting the stable states -1 and $+1$ is given by the speed of traveling waves for the singularly perturbed equation.

Let $u^\varepsilon = u^\varepsilon(x,t)$ be the solution of

$$\begin{cases} \varepsilon\left(\partial_t u + \partial_x f(u)\right) = g(u), & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = u_0(x) & x \in \mathbb{R}, \end{cases}$$

where u_0 is a given initial datum. Then it is easy to see that

$$u^\varepsilon(x,t) = v^\varepsilon\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right),$$

where $v^\varepsilon = v^\varepsilon(x,t)$ is solution of the Cauchy problem

$$\begin{cases} \partial_t v + \partial_x f(v) = g(v), & (x,t) \in \mathbb{R} \times (0,\infty), \\ v(x,0) = u_0(\varepsilon x) & x \in \mathbb{R}. \end{cases} \tag{4}$$

Assume that there exist $\phi \in C(\mathbb{R})$ and $c \in \mathbb{R}$ such that the function $\phi(x-ct)$ is an entropy solution of problem

$$\partial_t u + \partial_x f(u) = g(u),$$

such that

$$\phi(-\infty) = -1, \qquad \phi(0) = 0, \qquad \phi(+\infty) = 1.$$

Moreover assume that the solution v of the Cauchy problem (4) converges to $\phi(x-ct)$ as $t \to +\infty$, uniformly in ε, in the following sense:

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} |v^\varepsilon(x,t) - \phi(x-ct)| = 0. \tag{5}$$

Then it follows that

$$\lim_{\varepsilon\to0+} u^\varepsilon(x,t) = \begin{cases} -1 & x < ct, \\ +1 & x > ct \end{cases}$$

uniformly in the set $\Sigma_\tau := \{(x,t) \in \mathbb{R} \times [\tau,+\infty) \; : \; |x-ct| \geq \tau\}$ for any $\tau > 0$. Indeed, let $\tau > 0$ be given, then by assumption (5) we have that for any $\eta > 0$ there exists T_η such that

$$\sup_{t\geq\tau} \sup_{x\in\mathbb{R}} \left| u^\varepsilon(x,t) - \phi\left(\frac{x-ct}{\varepsilon}\right) \right| < \eta \qquad \forall \varepsilon < \tau/T_\eta.$$

Since

$$\lim_{\varepsilon\to0+} \phi\left(\frac{x-ct}{\varepsilon}\right) = \begin{cases} -1 & x < ct, \\ +1 & x > ct \end{cases}$$

uniformly in the set Σ_τ, the conclusion follows.

This result shows some interesting features of the problem.

1. The speeds of propagation of the layers for the limit problem ($\varepsilon = 0$) are determined by the speed of propagation of the traveling waves connecting the stable zeros for the problem with $\varepsilon = 1$. Hence the first step is to build traveling waves in the general case (see Section 3).

2. The convergence is uniform in Σ_τ. The fact that we have to take $t \geq \tau$ is consequence of an *initial layer phenomenon*, well-known even for ordinary differential equation. Moreover, since the limit solution connects different stable states, some *internal layer* is expected too.

3. The assumption (5) is too strong since we are working with entropy solution, i.e. solution exhibiting discontinuities. Therefore we cannot expect uniform convergence to some asymptotic (possible discontinuous) profile. It is necessary to work with some different topology, taking in account the possibility of jumps (see Section 4).

4. Some results have been proved showing convergence to a traveling wave as $t \to +\infty$ as in assumptions (5) fixing the initial datum of the Cauchy problem. In order to catch the behavior for $\varepsilon = 0$ we need *uniformity with respect to* ε, hence such results are unuseful and a finer analysis is needed (see Section 5).

3 Existence of Traveling Waves

In this Section we consider the existence of entropy traveling wave for the problem

$$\varepsilon\left(\partial_t u + \partial_x f(u)\right) = g(u), \tag{6}$$

connecting the stable zeros of the function g, -1 and $+1$. Such construction is needed to find the speed of propagations for the limiting equation as noted in the previous Section. Moreover the traveling wave we build here will be the main tool to prove the singular limit result, by means of comparison principle in Section 5.

A traveling wave is a particular solution of the form

$$u(x,t) = \phi(x - ct),$$

hence it can be seen as a couple (ϕ, c) where $\phi : \mathbb{R} \to \mathbb{R}$ is a function called *profile*, and $c \in \mathbb{R}$ is the *speed* of the wave. A traveling wave is *monotone* if its profile ϕ is monotone. Moreover, by saying that *the traveling wave connects* -1 *with +1* we mean

$$\lim_{y\to-\infty}\phi(y)=-1, \qquad \lim_{y\to+\infty}\phi(y)=+1.$$

Hereafter we look for traveling waves with at most a finite number of discontinuities, more precisely we assume that ϕ is a function belonging to $C^1(\mathbb{R}\setminus\{\xi_1,\dots,\xi_N\})$ for some $\xi_1,\dots,\xi_N\in\mathbb{R}$.

First of all let us note that a traveling wave of equation (6) is always of the form

$$u^\varepsilon(x,t)=\phi\left(\frac{x-ct}{\varepsilon}\right),$$

where (ϕ,c) is a traveling wave of the equation

$$\partial_t u+\partial_x f(u)=g(u). \tag{7}$$

Therefore it is enough to study existence of traveling wave for equation (7).

Existence of traveling waves for general source terms g and convex flux f was considered in [8]. In the case of nonconvex fluxes it was proved by the author the following result.

Theorem [7]. *Assume $f\in C([0,1])$ be such that there exist $0\equiv a_0<a_1<\dots<a_{n+1}\equiv 1$ such that $f\in C^1([a_i,a_{i+1}])$ and f' is either strictly monotone or constant in any interval (a_i,a_{i+1}); assume $g\in Lip([0,1])$ such that $g(0)=g(1)=0$ and $g(s)>0$ for any $s\in(0,1)$. Finally let c_* be defined by*

$$c_*:=\inf_{s\in(0,1]}\frac{f(s)-f(0)}{s-0}.$$

Then there exist monotone traveling waves connecting 0 and 1 if and only if $c\le c_*$. Moreover, fixing the speed c, the waves are unique (up to translations).

If $g<0$ in $(0,1)$, then there is existence of traveling waves if and only if the speed c is such that

$$c\ge c^*:=\sup_{s\in(0,1]}\frac{f(s)-f(0)}{s-0}.$$

General results in the case of nonconvex fluxes are not known.

Here we are interested in monotone traveling waves, so we aim to prove a result similar to the one above stated for the case of g resembles $-W'$, where W is the double well potential.

On the functions f and g we assume the following.

Hypotheses F: $f \in C^1([-1,1])$ with a finite number of inflection points;

Hypotheses G: $g \in C^1([-1,1])$ such that $g(0) = g(\pm 1) = 0$ and $s \cdot g(s) > 0$ for any $s \in (-1,0) \cup (0,1)$;

Hypothesis H: if $f(s) \geq f(0) + f'(0)s$ for any $s \in [-1,1]$ then

$$\frac{f'(\cdot) - f'(0)}{g(\cdot)} \in L^1(-\delta, \delta),$$

for some $\delta > 0$.

Note that, if $g'(0) \neq 0$, then (H) is satisfied.
Here we prove the following Theorem on existence of traveling waves.

Theorem 3.1 *Assume hypothesis (F), (G) and (H). Then there is a unique speed $\gamma \in \mathbb{R}$ such that there exist monotone traveling waves (ϕ, γ) connecting -1 with $+1$.*

Moreover

$$\gamma \in \left[\inf_{s \in (0,1]} \frac{f(s) - f(0)}{s - 0}, \sup_{s \in [-1,0)} \frac{f(s) - f(0)}{s - 0} \right],$$

and γ is characterized by the following condition

$$\min_{y \in [-1,0]} \left(f(y) - \gamma y \right) = \min_{y \in [0,1]} \left(f(y) - \gamma y \right).$$

In general we do not have uniqueness (up to translations) of the profile ϕ, as can be shown by the following example.

Example. Let $f(s) = \frac{1}{2}s^2$ and $g(s) = s(1 - s^2)$. Then the family of functions parametrized by $l \in \mathbb{R}$

$$\phi_l(x) := \begin{cases} \tanh(x + l), & x < -l, \\ 0, & -l \leq x \leq l, \\ \tanh(x - l), & x > l, \end{cases}$$

is such that the couple $(\phi_l, 0)$ is a traveling wave connecting -1 to $+1$ for any $l \in \mathbb{R}$.

Now we turn our attention to the proof of Theorem 3.1. First of all let us introduce an auxiliary function and prove some of its properties.

Proposition 3.2 *For any $c \in \mathbb{R}$ define*

$$\underline{f}(x;c) := \begin{cases} f_l(x;c) & x \in [-1,0], \\ \\ f_r(x;c) & x \in (0,1], \end{cases}$$

where

$$f_l(x;c) := \min_{y\in[-1,x]} (f(y) - cy) + cx, \quad f_r(x;c) := \min_{y\in[x,1]} (f(y) - cy) + cx.$$

Moreover set

$$c_0 := \inf_{s\in(0,1]} \frac{f(s) - f(0)}{s - 0} \le f'(0) \le c^0 := \sup_{s\in[-1,0)} \frac{f(s) - f(0)}{s - 0}.$$

Then

(i) $\underline{f}(x;c) \le f(x)$ *for any* $x \in [-1,1]$ *and for any* $c \in \mathbb{R}$*;*

(ii) $f_l(0;c) = f(0)$ *if and only if* $c \ge c^0$*, and* $f_r(0;c) = f(0)$ *if and only if* $c \le c_0$*;*

(iii) if $d < c$ *then* $f_l(x;d) \le f_l(x;c)$ *for any* $x \in [-1,0]$*, equality holding if and only if* $f_l(x;d) = f_l(x;c) = f(x)$*;*

(iv) if $d < c$ *then* $f_r(x;c) \le f_r(x;d)$ *for any* $x \in [0,1]$*, equality holding if and only if* $f_r(x;d) = f_r(x;c) = f(x)$*;*

(v) if $f_l(0;c_1) < f_r(0;c_1)$ *then* $f_l(0;c_2) < f_r(0;c_2)$ *for any* $c_2 < c_1$*, and if* $f_l(0;c_1) > f_r(0;c_1)$ *then* $f_l(0;c_2) > f_r(0;c_2)$ *for any* $c_2 > c_1$*.*

Proof. (i) It is immediate consequence of definition.

(ii) Since $f_l(0;c) = f(0)$ if and only if $f(y) - cy \ge f(0)$ for any $y \in [-1,0]$, then

$$f_l(0;c) = f(0) \qquad \Longleftrightarrow \qquad c \ge \frac{f(y) - f(0)}{y - 0} \quad \forall y \in [-1,0).$$

Similarly it can be proved that $f_r(0;c) = f(0)$ if and only if $c \le c_0$.

(iii) Let $d < c$, then $f(y) - dy \le f(y) - cy$ for any $y \in [-1,0]$ and the first part of the conclusion follows.

Next let $d < c$ and assume by contradiction that $f_l(x;d) = f_l(x;c) < f(x)$ for some $x \in [-1,0]$. Then there exists $y_0 \in [-1,x)$ such that $f_l(x;c) = f(y_0) - cy_0 + cx$. Hence

$$f_l(x;d) \le f(y_0) - dy_0 + dx = f_l(x;c) + (c-d)(y_0 - x) < f_l(x;c),$$

contradicting the assumption.

(iv) Similar to (iii).

(v) It is consequence of (iii) and (iv).

The proof is complete.

Note that the functions f_l and f_r are continuous with respect to their arguments. To show this claim, it is enough to prove that the functions $\min_{y\in[-1,x]}(f(y)-cy)$ and $\min_{y\in[x,1]}(f(y)-cy)$ are continuous. In order to do this for the first one, one can introduce the following auxiliary function

$$g(y,c,s) := \begin{cases} f(y)-cy & -1\le y\le s, \\ f(s)-cs & s\le y\le 1, \end{cases}$$

and note that $\min_{y\in[-1,x]}(f(y)-cy) = \min_{y\in[-1,1]} g(y,c,s)$. Similar procedure can be performed for the other function. Hence the function $\underline{f}$ is continuous in $[-1,1]\setminus\{0\}\times\mathbb{R}$.

Next we show that the problem of existence of entropy traveling waves is equivalent to the problem of finding $c\in\mathbb{R}$ such that the function $\underline{f}(\,\cdot\,;c)$ is continuous at 0. Such function $\underline{f}(\,\cdot\,;c)$ plays the rôle of an *effective flux* for the traveling wave, in the sense that the wave is solution of a reaction-convection equation with flux $\underline{f}(\,\cdot\,;c)$, instead of $f(\,\cdot\,)$.

Here we use the following notation: given a nondecreasing function h, we assume it to be multivalued at any discontinuity point by the position

$$h(x_0) = [h(x_0-), h(x_0+)];$$

moreover we always define h^{-1} as the inverse function of h, that is multivalued at any y_0 such that $h\equiv y_0$ in some interval.

Proposition 3.3 *(I) Let (ϕ,c) be a monotone traveling wave of (6) connecting -1 with $+1$, then the function*

$$f_{\phi,c}(s) := f(-1) + c(s+1) + \int_{-\infty}^{\xi_0} g(\phi(\xi))\,d\xi \quad \textit{with } \phi(\xi_0)=s$$

is well-defined in $[-1,1]$, continuous, and it holds

$$f_{\phi,c}(s) = \underline{f}(s;c) \qquad \forall s\in[-1,1].$$

(II) Conversely, assume that $\underline{f}(\cdot;c)$ is continuous at 0, then, defining

$$\phi^{-1}(s) := \int_0^s \frac{\underline{f}'(\sigma;c)-c}{g(\sigma)}\,d\sigma, \tag{8}$$

the couple (ϕ,c) is a monotone traveling wave of (6) connecting -1 with $+1$.

Proof. (I) Since f has a finite number of inflection points, ϕ has at most a finite number of discontinuities. Let us denote such points with $\xi_1 < \xi_2 < \cdots < \xi_n$. Moreover set

$$\begin{aligned} &\alpha_i := \phi(\xi_i-), \quad \beta_i := \phi(\xi_i+) \qquad i = 1, \cdots, n, \\ &\xi_0 := -\infty, \quad \beta_0 := -1, \quad \alpha_{n+1} := 1. \end{aligned}$$

If $\alpha_1 = 0$, then there exists ξ_1 such that $\phi(\xi) = -1$ for any $\xi \leq \xi_1$; hence the integral

$$\int_{-\infty}^{\xi_0} g(\phi(\xi))\,\mathrm{d}\xi = \int_{\xi_1}^{\xi_0} g(\phi(\xi))\,\mathrm{d}\xi$$

is convergent. If $\alpha_1 > 0$, then there exists ξ_2 such that ϕ is regular in $(-\infty, \xi_2)$, $\phi(\xi_2-) = \alpha_1$ and $\phi(\xi_2+) = \beta_1$. In this case

$$\int_{-\infty}^{\xi_2} g(\phi(\xi))\,\mathrm{d}\xi = \int_{-1}^{\alpha_1} (f'(s) - c)\,\mathrm{ds} = f(\alpha_1) - f(-1) - c(\alpha_1 + 1),$$

and the integral in the definition of $f_{\phi,c}$ converges.

The function ϕ^{-1} is multivalued at s if there exists an interval $I \subset \mathbb{R}$ such that $\phi(\xi) = s$ for any $\xi \in I$. Since ϕ is regular over i, it holds

$$g(\phi(\xi)) = (f'(\phi(\xi)) - c)\phi'(\xi) = 0 \qquad \forall \xi \in I,$$

hence the integral $\int_{-\infty}^{\xi_0} g(\phi)\,\mathrm{d}\xi$ does not depend on the choice of $\xi_0 \in \phi^{-1}(s)$.

The following representation for $f_{\phi,c}$ holds:

$$f_{\phi,c}(s) = \begin{cases} f(s) & s \in (\beta_k, \alpha_{k+1}) \qquad (k = 0, \cdots, n), \\ f(\alpha_k) + c(s - \alpha_k) & s \in [\alpha_k, \beta_k] \qquad (k = 1, \cdots, n). \end{cases}$$

Hence $f_{\phi,c}$ is continuous and differentiable at any $s \notin \{\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n\}$. and $f'_{\phi,c}(s)g(s) \geq c$ for any $s \neq 0$, where $f'_{\phi,c}$ exists.

Moreover it holds for any $s \leq 0$

$$\min_{-1 \leq \sigma \leq s} (f(\sigma) - c\sigma) \leq f_{\phi,c}(s) - cs.$$

If there exist $-1 < \sigma_0 < s_0 \leq 0$ such that $f(\sigma_0) - c\sigma_0 < f_{\phi,c}(s_0) - cs_0$ then

$$f_{\phi,c}(s_0) = f_{\phi,c}(\sigma_0) + \int_{\sigma_0}^{s_0} f'_{\phi,c}(s)\,\mathrm{ds} \leq f_{\phi,c}(\sigma_0) + c(s_0 - \sigma_0),$$

since $f'_{\phi,c}(s) \leq c$ a.e.. Therefore

$$f_{\phi,c}(s_0) - cs_0 \leq f(\sigma_0) - c\sigma_0,$$

contradicting the assumption on s_0 and σ_0. This shows that $f_{\phi,c} \equiv \underline{f}$ in $[-1, 0]$.

The proof for $s \in [0, 1]$ can be obtained by similar arguments.

(II) Assume $\underline{f}(\cdot; c)$ continuous at 0 and define

$$\Xi(s) := \int_0^s \frac{\underline{f}'(s; c) - c}{g(s)} \, \mathrm{ds}.$$

Such function is well defined. Indeed either $\underline{f}'(s; c) = c$ for any $s \in (-\varepsilon, e)$ for some $\varepsilon > 0$, or $\underline{f}(0; c) = f(0)$. Then, by Proposition 3.2(ii), $c = f'(0)$ and the integral in the definition of Ξ converges by hypothesis (H).

The profile of the traveling wave is obtained by inverting the function Ξ:

$$\phi(\xi) := \Xi^{-1}(\xi),$$

hence ϕ is implicitly defined by

$$\int_0^{\phi(\xi)} \frac{\underline{f}'(s; c) - c}{g(s)} \, \mathrm{ds} = \xi.$$

This completes the proof.

By applying the previous Propositions we can prove the existence theorem.

Proof of Theorem 3.1. It is enough to show that the following holds. *Claim.* There exists a unique $\gamma \in \mathbb{R}$ such that $f_l(0; \gamma) = f_r(0; \gamma)$; moreover $\gamma \in [c_0, c^0]$.

To prove uniqueness, let us assume by contradiction that there exists $\gamma < \gamma'$ such that $f_l(0; \gamma) = f_r(0; \gamma)$ and $f_l(0; \gamma') = f_r(0; \gamma')$. Then by property (iii) and (iv) of Proposition 3.2 it follows

$$f_l(0; \gamma) \leq f_l(0; \gamma') = f_r(0; \gamma') \leq f_r(0; \gamma) = f_l(0; \gamma),$$

hence $f_l(0; \gamma) = f_l(0; \gamma')$ and $f_r(0; \gamma) = f_r(0; \gamma')$. Applying Proposition 3.2(iv), we get

$$f_l(0; \gamma) = f_l(0; \gamma') = f_r(0; \gamma) = f_r(0; \gamma') = f(0).$$

Finally, by (ii), we deduce that $\gamma = \gamma' = c_0 = c^0 = f'(0)$, contradicting $\gamma < \gamma'$.

To complete the proof of the claim we observe that, since there hold

$$f_l(0; c_0) = f(0) \geq f_r(0; c_0) \quad \text{and} \quad f_l(0; c^0) \leq f(0) = f_r(0; c^0),$$

existence of γ follows from continuity of $f_l(0; \cdot)$ and $f_r(0; \cdot)$.

4 Convergence of Graphs

The aim of this Section is to introduce a useful topology in the space of functions. Let $\mathcal{X}$ be the set of measurable functions from $\mathbb{R}$ to $\mathbb{R}$ such that at any $x_0 \in \mathbb{R}$ there exists

$$\lim_{\varepsilon \to 0^+} f(x \pm \varepsilon) = f(x\pm).$$

Let us introduce the following set in the plane:

$$\Gamma_f := \{(x,y) \in \mathbb{R}^2 \,:\, y \in [f(x-) \wedge f(x+), f(x-) \vee f(x+)]\},$$

where $a \wedge b := \min\{a,b\}$ and $a \vee b := \max\{a,b\}$ $(a, b \in \mathbb{R})$. We call Γ_f the *extended graph* of f. Note that denoting by f^* (respectively f_*) the upper (resp. lower) semicontinuous envelope of f, defined by

$$f^*(x) := \limsup_{y \to x} f(y) \qquad f_*(x) := \liminf_{y \to x} f(y),$$

it holds

$$(x,y) \in \Gamma_f \quad \Longleftrightarrow \quad f_*(x) \le y \le f^*(x).$$

Given $\delta > 0$, set

$$\Gamma_f^\delta := \bigcup_{P \in \Gamma_f} J_\delta(P) \subset \mathbb{R}^2,$$

where $J_\delta(P)$ denotes the usual neighborhood of $P \in \mathbb{R}^2$ with radius δ. Finally we define neighborhood I_δ of functions f in $\mathcal{X}$ as follows

$$I_\delta(f) := \{g \in \mathcal{X} \,:\, \Gamma_g \subset \Gamma_f^\delta\} \qquad (\delta > 0).$$

Definition 4.1 *Given a family of function $f_\varepsilon \in \mathcal{X}$ for $\varepsilon > 0$ and $f \in \mathcal{X}$ we say that f_ε* converges to f in the sense of the extended graph as $\varepsilon \to 0$ *if f_ε converges to f in the above topology, i.e. for any $\eta > 0$ there exists $\varepsilon_\eta > 0$ such that $f_\varepsilon \in I_\eta(f)$ for any $\varepsilon \in (0, \varepsilon_\eta)$.*

In the following $f_\varepsilon \xrightarrow{gr} f$ means that f_ε converges to f in the sense of the extended graph.

Given $f, g \in \mathcal{X}$, we say that *g is a translation of f* if and only if $g(x) = f(x-a)+b$ for some $(a,b) \in \mathbb{R}^2$. Let us note that *translation is a continuous operation with respect to the topology of the extended graph.* Indeed, setting $f_{(a,b)}(x) := f(x-a)+b$, the assertion easily follows from

$$\text{dist}\,((x, f_{(a,b)}(x)), (x-a, f(x))) = |(a,b)|.$$

Let us state and prove a result concerning the connection between the convergence in the sense of the graph and some other kinds of convergence.

Proposition 4.2 *Let $f_\varepsilon, f \in \mathcal{X}$ $(\varepsilon > 0)$.*

(i) If $f_\varepsilon \to f$ in $L^\infty(\mathbb{R})$ then $f_\varepsilon \xrightarrow{gr} f$;

(ii) if $f_\varepsilon \xrightarrow{gr} f$ then $f_\varepsilon \to f$ uniformly in any set I such that f is uniformly continuous in $I_r := \{x \in \mathbb{R} : \text{dist}\,(x, I) \leq r\}$ for some $r > 0$;

(iii) if $f_\varepsilon \xrightarrow{gr} f$ then $f_\varepsilon \to f$ in $L^1_{loc}(\mathbb{R})$.

Property (ii) implies that the converse of (i) holds if f is uniformly continuous over $\mathbb{R}$. If f is continuous and not uniformly continuous then it can be easily constructed an example with $f_\varepsilon \xrightarrow{gr} f$ and f_ε not converging to f in L^∞.

Proof. Properties (i) and (iii) are direct consequences of definition. Let us prove (ii). Assume f be uniformly continuous in I_r for some $I \subset \mathbb{R}$ and $r > 0$. Then for any $\eta > 0$ there exists $\delta_\eta > 0$ such that $|f(x) - f(y)| < \eta/2$ if $|x - y| < \eta$, with $x, y \in I_r$.

Since $f_\varepsilon \xrightarrow{gr} f$, there exists ε_0 such that for any $\varepsilon \in (0, \varepsilon_0)$, for any $x \in I$ there exists $y \in I_r$ such that

$$\text{dist}\,((x, f_\varepsilon(x)), (y, f(y))) < \min\{\eta/2, \delta_\eta, r\}.$$

Therefore, given $\eta > 0$, there exists ε_0 such that for any $x \in I$,

$$|f_\varepsilon(x) - f(x)| \leq |f_\varepsilon(x) - f(y)| + |f(x) - f(y)| < \frac{\eta}{2} + \frac{\eta}{2} = \eta.$$

This completes the proof.

5 Singular Limit

In this Section we prove of Theorem 1.1. We assume without restriction that the speed γ of the traveling wave connecting -1 with $+1$ is equal to 0 and denote the profile of the wave with ϕ.

Now we briefly recall some definitions for the reaction convection equation

$$\varepsilon\,(\partial_t u + \partial_x f(u)) = g(u), \tag{9}$$

and the corresponding Cauchy problem (1).

Let Q be an open subset of $\mathbb{R} \times (0, \infty)$ and set

$$[s]_\pm := \frac{|s| \pm s}{2}, \qquad H_\pm(s) := \frac{1 \pm \text{sgn}\,(s)}{2}.$$

Definition 5.1 *A function $u \in BV_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})(Q)$ is an* entropy subsolution *of equation (9) in Q if for any $\theta \in C_0^\infty(Q)$, $\theta \geq 0$ and $k \in \mathbb{R}$ it holds*

$$\int\int_Q \Big\{\varepsilon\,[u-k]_+ \partial_t\theta + H_+(u-k)\Big[\varepsilon\,(f(u)-f(k))\partial_x\theta + g(u)\theta\Big]\Big\}\,dx\,dt \geq 0 \quad (10)$$

An entropy supersolution *of equation (9) in Q is defined replacing $[\,\cdot\,]_+$ and H_+ by $[\,\cdot\,]_-$ and H_- in (10).*

Definition 5.2 *Let $u_0 \in BV_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})(\mathbb{R})$. An* entropy subsolution of the problem (1) *in $\mathbb{R} \times (0,\infty)$ is an entropy subsolution of (9) in $\mathbb{R} \times (0,\infty)$, such that for any $r > 0$*

$$\operatorname*{ess\,lim}_{t\to 0^+} \int_{|x|\leq r} [u(x,t) - u_0(x)]_+ \,dx = 0. \quad (11)$$

An entropy supersolution of the problem (1) *in $\mathbb{R} \times (0,\infty)$ is an entropy supersolution of (9) in $\mathbb{R} \times (0,\infty)$, such that for any $r > 0$ (11) holds with $[\,\cdot\,]_+$ replaced by $[\,\cdot\,]_-$.*

Comparison result holds for entropy sub- and supersolution (see [11]).

Theorem 5.3 *Let u^- and u^+ be an entropy subsolution and supersolution of problem (1) in $\mathbb{R} \times (0,\infty)$, with initial data u_0^- and u_0^+ respectively. Assume $u_0^\pm \in BV_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$. If $u_0^- \leq u_0^+$ a.e. in $\mathbb{R} \times (0,\infty)$, then $u^- \leq u^+$ a.e. in $\mathbb{R} \times (0,\infty)$.*

Proof of Theorem 1.1 is based on the construction of suitable sub- and supersolution of problem (1). The following proposition characterizes piecewise smooth stationary sub- and supersolutions.

Proposition 5.4 *Let $W \in C^1(\mathbb{R} \setminus \{x_1,\ldots,x_n\})$ for some $x_1,\ldots,x_n \in \mathbb{R}$ and assume that there exists finite $W_i^\pm := \lim_{\varepsilon\to 0^+} W(x_i \pm \varepsilon)$. Then the following holds.*

1. *W is a subsolution in $\mathbb{R} \times (0,\infty)$ if and only if*

$$\varepsilon\,\partial_x f(W) \leq g(W) \qquad \text{in } \mathbb{R} \setminus \{x_1,\ldots,x_n\}, \quad (12)$$

$$f(k) \leq f(W_i^-) \text{ if } W_i^+ \leq W_i^-, \quad f(k) \geq f(W_i^+) \text{ if } W_i^- < W_i^+. \quad (13)$$

for any k in between W_i^- and W_i^+ for any $i = 1,\ldots,n$.

2. W is a supersolution in $\mathbb{R}\times(0,\infty)$ if and only if

$$\varepsilon\,\partial_x f(W) \geq g(W) \qquad \text{in } \mathbb{R}\setminus\{x_1,\dots,x_n\}, \tag{14}$$

$$f(k) \leq f(W_i^+) \text{ if } W_i^+ \leq W_i^-, \quad f(k) \geq f(W_i^-) \text{ if } W_i^- < W_i^+, \tag{15}$$

for any k in between W_i^- and W_i^+ for any $i=1,\dots,n$.

The proof of the above proposition is quite classical. We just sketch it for completeness.

Proof of Proposition 5.4. We just consider case 1. Let $P\equiv(x_0,t_0)\in\mathbb{R}\times(0,\infty)$ be fixed and let $\theta_n=\theta_n(x,t)\geq 0$ be a sequence of functions in C_0^∞ with support in $B(P,1/n)$ such that $\theta_n\to\delta_P$ as $n\to+\infty$, where δ denotes classical Dirac distribution.

Let $x_0\neq x_i$ for any $i=1,\dots,n$. Choosing $k<-\|u\|_\infty$ in (10) and integrating by parts we obtain

$$\int\int_Q(\varepsilon\,\partial_x f(u)-g(u))\theta_n\,dx\,dt\leq 0.$$

Passing to the limit as $n\to+\infty$ we get property (i).

Now let $x_0=x_i$ for some $i=1,\dots,n$. Set $B(P,1/n)=B_-\cup B_+$, where $B_\pm=\{(x,t)\in B(P,1/n)\,:\,\pm(x-x_1)\geq 0\}$.

Then the left hand side of (10) can be written as the sum of two integrals over B_- and B_+. Integrating by parts both the terms and passing to the limits as $n\to+\infty$ we get

$$H_+(W_i^+-k)(f(W_i^+)-f(k))-H_+(W_i^--k)(f(W_i^-)-f(k))\leq 0 \qquad \forall k\in\mathbb{R}.$$

Let $W_i^+<W_i^-$. If $k\geq W_i^-$, then the previous relation is trivially satisfied. For $W_i^+\leq k\leq W_i^-$ then we get property (ii) of Proposition 5.4. If $k\leq W_i^+$ we get (ii) with $k=W_i^+$.

The case $W_i^-<W_i^+$ and the prove of sufficiency of properties (i) and (ii) can be dealth with similarly.

Before proving Theorem 1.1, let us prove the following Lemma.

Lemma 5.5 *Let $u_0\in C(0,+\infty)$ be such that $u_0(x)>0$ for any $x>0$ and $\liminf_{x\to+\infty}u_0(x)>0$. For any $\sigma>0$, set $x_\sigma:=\inf I_\sigma\in[0,+\infty]$ where $I_\sigma:=\{y>0\,:\,u_0(x)>\sigma\quad\forall x>y\}$. Then x_σ is an increasing function of σ and*

$$\lim_{\sigma\to 0^+}x_\sigma=0. \tag{16}$$

Proof. The monotonicity of x_σ is immediate since $\sigma_1 < \sigma_2$ implies $I_{\sigma_2} \subset I_{\sigma_1}$.

If $\sigma < \liminf_{x\to+\infty} u_0(x)$, then $I_\sigma \neq \emptyset$ and $0 \le x_\sigma < +\infty$. Therefore there exists $x_0 \in [0,+\infty)$ such that $x_0 = \lim_{\sigma\to 0+} x_\sigma$.

From assumptions on u_0 it follows that, for any $y > 0$, $\inf_{[y,+\infty)} u_0(x) > 0$. Hence for any $y > 0$ there exists $\sigma_0 > 0$ such that $y \in I_\sigma$ for any $0 < \sigma \le \sigma_0$, therefore $x_\sigma \le y$ for any $0 < \sigma \le \sigma_0$. Passing to the limit as $\sigma \to 0$, we get that $x_0 \le y$ for any $y > 0$, and the conclusion follows.

Let us denote by $U^\varepsilon(\,\cdot\,;\sigma)$ the (unique) solution of the Cauchy problem

$$\varepsilon\,\partial_t U = g(U), \qquad U(0;\sigma) = \sigma \in \mathbb{R},$$

and let $Y(\,\cdot\,;\sigma) := U^1(\,\cdot\,;\sigma)$. Then $U^\varepsilon(t;\sigma) = Y(t/\varepsilon;\sigma)$ for any t.

Lemma 5.6 *Given $\varepsilon > 0$, $\sigma \in [-1,1]$ and $y, c \in \mathbb{R}$ set $u^R(x) := sgn\ x$, and*

$$\underline{u}^\varepsilon(x,t;\sigma,y,c) := \begin{cases} -1 & x < y + ct, \\ U^\varepsilon(t;\sigma) & x > y + ct. \end{cases}$$

Then for any $\delta > 0$, $\sigma \in (0,1]$, $c \in \mathbb{R}$ and y such that $|y| < \delta$ then there exists ε_0 such that

$$\underline{u}^\varepsilon(\cdot, \varepsilon T_{\delta,\sigma};\sigma,y) \in I_\delta(u^R) \qquad \forall \varepsilon \in (0,\varepsilon_0),$$

where $T_{\delta,\sigma}$ is such that $Y(T_{\delta,\sigma};\sigma) = 1 - \delta/2$.

Proof. From the definition of $\underline{u}^\varepsilon$ the conclusion holds if and only if

$$U^\varepsilon(\varepsilon T_{\delta,\sigma};\sigma) > 1 - \delta \quad \text{and} \quad |y + c\varepsilon T_{\delta,\sigma}| < \delta.$$

Since $U^\varepsilon(t;\sigma) = Y(t/\varepsilon;\sigma)$, then

$$U^\varepsilon(\varepsilon T_{\delta,\sigma};\sigma) = Y(T_{\delta,\sigma};\sigma) = 1 - \delta/2 > 1 - \delta.$$

The second inequality is satisfied by choosing $\varepsilon_0 := (\delta - y \cdot \mathrm{sgn}\ c)/(|c| T_{\delta,\sigma})$. Indeed, assume $c > 0$ (the other case can be dealt with similarly). Then $0 < \varepsilon < \varepsilon_0$, with ε_0 chosen as above, implies $-\delta < y + c\,\varepsilon T_{\delta,\sigma} < \delta$. This concludes the proof of the Lemma.

Remark. If $c \ge \sup f'$ then the function $\underline{u}^\varepsilon$ is an entropy subsolution of the reaction-convection equation. Moreover, given an initial datum u_0

satisfying the assumption of Theorem 1.1, by Lemma 5.5, for any $\sigma > 0$ it is possible to choose $y := x_\sigma$ such that u^ε is a subsolution for problem (1). Similar construction can be made for supersolution. Hence we get that for any $\delta > 0$ there exist T, independent of ε such that at time $t_\varepsilon := \varepsilon T$ the entropy solution of the problem (1) is close to the function sgn x (ε sufficiently small). Therefore essentially in order to have the proof of the main Theorem we have to study the behavior of the solution of the Riemann problem with data -1 and $+1$.

Lemma 5.7 *Assume hypothesis (F), (G) and (H). Then there exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ there exists a function $w = w(x)$ that is a subsolution of equation (9) with the properties:*

(i) w is a nondecreasing function and $w(0) = 0$,

(ii) there exists $x_0 > 0$ such that $w(x) = -1$ for any $x < -x_0$ and $w(x) = 1 - \delta$ for any $x > x_0$.

Proof. Let $\phi = \phi(x)$ be an entropy traveling wave connecting -1 with $+1$ (see Section 3). Using the same notations of Section 3, we denote with $\xi_1 < \ldots < \xi_n$ the (eventual) discontinuities of ϕ and we set

$$\begin{array}{ll} \alpha_i := \phi(\xi_i-), \quad \beta_i := \phi(\xi_i+) & i = 1, \cdots, n, \\ \xi_0 := -\infty, \quad \beta_0 := -1, \quad \alpha_{n+1} := 1. & \end{array}$$

Choose δ_0 as follows if ϕ has no discontinuities $\delta_0 := 1/2$ otherwise:

$$\delta_0 := \begin{cases} \min\{\alpha_1 + 1, 1 - \beta_n\} & \text{if } -1 < \alpha_1 < \beta_n < 1, \\ \min\{\alpha_1 + 1, 1 - \alpha_n\} & \text{if } -1 < \alpha_1 < \beta_n = 1, \\ \min\{\beta_1 + 1, 1 - \beta_n\} & \text{if } -1 = \alpha_1 < \beta_n < 1, \\ \min\{\beta_1 + 1, 1 - \alpha_n\} & \text{if } -1 = \alpha_1 < \beta_n = 1. \end{cases}$$

Fix $\delta \in (0, \delta_0)$, let x_1 be the unique value such that $-1+\delta \in [\phi(x_1-), \phi(x_1+)]$ and let x_2 be the unique value such that $1 - \delta \in [\phi(x_2-), \phi(x_2+)]$. The construction of the subsolution w is slightly different in the case $\beta_n < 1$ and $\beta_n = 1$.

Case 1. If $\beta_n < 1$ or ϕ with no jumps, then ϕ is continuous in $(x_2, +\infty)$. Then we define w as follows:

$$w(x) := \begin{cases} -1 & \text{if } x < x_1, \\ \phi(x) & \text{if } x_1 \leq x \leq x_2, \\ 1 - \delta & \text{if } x > x_2. \end{cases}$$

In order to prove that w is a subsolution we apply Proposition 5.4. It is immediate to check that -1, ϕ and $1 - \delta$ are separetely subsolution of the problem, therefore we have to check that at x_1 and at x_2 condition (13) is satisfied.

Since $\beta_n < 1$ and $\delta < 1 - \beta_n$, then ϕ is continuous at x_2 and condition (13) holds at that point.

Let us consider the point x_1. If $\alpha_1 = -1$, then x_1 is a jump point for ϕ and $\phi(x) = -1$ for any $x < x_1$. So $w(x) = \phi(x)$ for any $x \leq x_2$ and, by previous analysis at x_2, it is a subsolution. If $\alpha_1 > -1$ then ϕ is continuous at x_1. Then $f(s) \geq f(-1+\delta)$ for any $s \in [-1, -1+\delta)$. Indeed since ϕ is continuous then $f(s) = f_l(s; 0) = \min\{f(y) : y \in [-1, s]\}$ in $[-1, -1+\delta)$. This conclude Case 1.

Case 2. If $\beta_n = 1$, the construction is more tricky. By entropy condition, $f(s) > f(\alpha_n) = f(1)$ and $f'(s) > 0$ for any $s \in (\alpha_n, \alpha_n + \eta)$ for some $\eta > 0$. Then there exist $\psi = \psi(x)$, $x \in [x_2, x_3)$ (for some $x_3 > x_2$) solution of

$$f'(\psi)\psi' = g(\psi), \qquad \psi(x_2) = \alpha_n,$$

and ψ is a strictly increasing function in (x_2, x_3). Then the function w is defined as follows:

$$w(x) := \begin{cases} -1 & \text{if } x < x_1, \\ \phi(x) & \text{if } x_1 \leq x \leq x_2, \\ \psi(x) & \text{if } x_2 < x \leq x_3, \\ 1-\delta & \text{if } x > x_3. \end{cases}$$

Again, applying Proposition 5.4, it can be proved that w is a subsolution. Let us check condition (13) only at x_2 and x_3 (the other points being similar to Case 1). At x_2 it is readily seen that, since $\beta_n = 1$, then $\phi(x_2-) = \alpha_n$, hence w is continuous at x_2. At point x_3, since ψ is increasing, then $\psi(x_3) \in (\alpha_n, \beta_n)$. Therefore condition (13) is satisfied.

This concludes the proof.

Proof of Theorem 1.1. Firstly we prove that assuming also hypothesis (H), then we have uniform convergence on Σ_τ for any positive τ.

Fix $0 < \delta < \delta_0$ with δ_0 as in the proof of Lemma 5.7. Following the idea sketched in The Remark of Lemma 5.6, let $0 < \sigma < \liminf\limits_{x \to +\infty} u_0(x)$ be such that $x_\sigma < \delta$ (x_σ defined in Lemma 5.5). Applying Lemma 5.6, $\underline{u}^\varepsilon = \underline{u}^\varepsilon(\,\cdot\,, \,\cdot\,; \sigma, x_\sigma, \sup f')$ is a subsolution of problem (1) and there exists $T_1 > 0$, independent of ε such that $\underline{u}^\varepsilon(\,\cdot\,, \varepsilon T_1) \in I_\delta(u^R)$ for $\varepsilon < \varepsilon_0$ for some ε_0 (for simplicity we have dropped the dependence on $\sigma, x_\sigma,\ sup f'$). Similarly a supersolution $\overline{u}^\varepsilon$ can be built such that $\overline{u}^\varepsilon(\,\cdot\,, \varepsilon T_2) \in I_\delta(u^R)$ for some $T_2 > 0$, independent on ε. Hence we conclude that

$$u^\varepsilon(\,\cdot\,, \varepsilon T) \in I_\delta(u^R) \qquad \text{for some } T > 0 \text{ independent on } \varepsilon \text{ small enough.}$$

Claim. There exists $\varepsilon_1 > 0$ such that for any $\varepsilon < \varepsilon_1$ and $t \geq \varepsilon T$, the entropy solution u^ε is such that $u^\varepsilon(\,\cdot\,,t) \in I_{2\delta}(u^R)$.

In order to prove that u^ε cannot escape $I_{2\delta}(u^R)$ from below, we use the subsolution given by Lemma 5.7. Analogous construction of supersolution guarantees that the claim holds. Since $w = w(x)$ is a subsolution of equation (9), then for any $\varepsilon > 0$ and any $y \in \mathbb{R}$ the function $w^\varepsilon_y = w^\varepsilon_y := w((x-y)/\varepsilon)$ is a subsolution of equation (9). Set $\varepsilon_1 := \min\{\delta/2x_1, \varepsilon_0\}$ with x_1 given in Lemma 5.7. Then $w^\varepsilon_{3\delta/2}(x) \leq \text{sgn }(x-\delta)$ for any $x \in \mathbb{R}$. Indeed the assertion is trivially true for $x \geq \delta$. If $x < \delta$, then $w^\varepsilon_{3\delta/2}(x) \leq w^\varepsilon_{3\delta/2}(\delta) = w(-\delta/2\varepsilon) \leq w(-x_1) = -1$ since $x_1 < \delta/2\varepsilon$ for any $\varepsilon < \varepsilon_1$. Since $w^\varepsilon_{3\delta/2} \geq -1$ in $(-\infty, 2\delta)$ and $w^\varepsilon_{3\delta/2} \geq 1-\delta$ in $(2\delta, +\infty)$ the same hold for u^ε. By constructing suitable supersolution we get the estimates from above, completing the proof of the claim.

This completes the proof of the second part of the Theorem. In order to prove also the first part, assume that assumption (H) does not hold. Hence $f(s) \geq f(0) + f'(0)s$. This implies that $\gamma = f'(0)$. Again, withou loss of generality, we can assume $f'(0) = 0$.

For any $s_1 < 0 < s_2$ there exist two functions g_1 and g_2 in $C^1([-1,1])$ such that

$$g_i(\pm 1) = g_i(s_i) = 0, \quad (s - s_i)\cdot g(s) > 0 \text{ for any } s \in (-1,1)\backslash\{s_i\}, \quad g_i'(s_i) > 0,$$

and

$$g_2(s) \leq g(s) \leq g_1(s) \qquad \text{for any } s \in [-1,1].$$

Next, let us denote by u^ε_i the solution of the Cauchy problem (1) with g replaced by g_i, $i = 1,2$, Then it can be proved that $u^\varepsilon_1 \leq u^\varepsilon \leq u^\varepsilon_2$ in $\mathbb{R}\times(0,\infty)$ for any $\varepsilon > 0$. Applying Theorem 1.1, we deduce that u^ε_i converges to $\text{sgn }(x - \gamma_i t)$ uniformly in Σ_τ for any $\tau > 0$ where γ_i is characterized by

$$\min_{y\in[-1,s_i]} (f(y) - \gamma_i y) = \min_{y\in[s_i,1]} (f(y) - \gamma_i y)\,,$$

for $i = 1,2$. Then γ_i is a continuous function of s_i. Fix $0 < \tau < T$, $\delta > 0$ and choose s_1, s_2 sufficiently close to zero in such a way that $|\gamma_i| < \delta/(2T)$ ($i = 1,2$). Hence $\text{sgn }(x - \gamma_i t) \in I_{\delta/2}(\text{sgn } x)$ in $\mathbb{R}\times[\tau,T]$ ($i = 1,2$). By Theorem 1.1, there exists ε_i such that if $\varepsilon < \varepsilon_i$ then $u^\varepsilon_i(\cdot,t) \in I_{\delta/2}(\text{sgn }(x - \gamma_i t))$ for any $t \geq \tau$ ($i = 1,2$). Setting $\varepsilon_0 := \min\{\varepsilon_1, \varepsilon_2\}$ and using the fact that $u^\varepsilon_1 \leq u^\varepsilon \leq u^\varepsilon_2$ we get that $u^\varepsilon \in I_\delta(\text{sgn } x)$ in $\mathbb{R}\times[\tau,T]$ whenever $\varepsilon < \varepsilon_0$. This concludes the proof.

Proof of Corollary 1.2. By assumptions, it is possible to find two continuous functions v_0, w_0 such that $v_0, w_0 \geq u_0$, $v_0(\mathbb{R}), w_0(\mathbb{R}) \subset [-1,1]$,

$\liminf_{x\to+\infty} |v_0(x)| > 0$, $\liminf_{x\to\pm\infty} |w_0(x)| > 0$ and $(x-x_1)\cdot v_0(x) > 0$, $(x-x_2)\cdot w_0(x) < 0$ for any $x \neq 0$ for some $x_1, x_2 \in \mathbb{R}$ $(x_1 < x_2)$.

Denote by v^ε (by w^ε, respectively) the entropy solution of the Cauchy problem (1) with u_0 replaced by v_0 (by w_0, resp.). Then, by applying Theorem 1.1, we get that

$$\begin{aligned} &\lim_{\varepsilon\to 0} v^\varepsilon(x,t) = \text{sgn }(x - x_1 - \gamma t), \\ &\lim_{\varepsilon\to 0} w^\varepsilon(x,t) = -\text{sgn }(x - x_2 - \gamma' t). \end{aligned}$$

Moreover, by comparison principle, $-1 \leq u^\varepsilon(x,t) \leq \min\{v^\varepsilon(x,t), w^\varepsilon(x,t)\}$. Hence, setting $T := (x_2 - x_1)/(\gamma - \gamma')$, the conclusion follows.

6 A case of a 1-dimensional rest manifold

In this final Section we consider a slightly different form for the reaction term g. Instead of assumption (G), we assume hereafter the following

Hypotheses G ': $g \in C^1([-1,1])$ such that $g(s) = 0$ if and only if $s \in \{-1,1\} \cup [\alpha,\beta]$ for some $-1 < \alpha < 0 < \beta < 1$, $s \cdot g(s) > 0$ for any $s \in (-1,1) \setminus [\alpha,\beta]$.

Under this assumption the rest manifold $\{g(s) = 0\}$ has the 1-dimensional connected component $J := [\alpha,\beta]$, instead of being 0-dimensional as in the previous part of the paper. Again, the problem is to understand the limit of solution to (1) as $\varepsilon \to 0^+$. In this case any state in the interior of J is neutrally stable, while -1 and $+1$ are asymptotically stable and the points on the boundary of $\partial J = \{\alpha,\beta\}$ are unstable. The presence of a neutral rest manifold cause the possible appearence in the limit of pieces of solution to the scalar conservation law

$$\partial_t u + \partial_x f(u) = 0.$$

The size of such pieces of solution to the homogeneus equation depends on the structure of the flux function f and on the location of the interval J.

In what follows, we just consider the case of the Riemann problem given by the initial data $u_0(x) := \text{sgn } x$.

Theorem 6.1 *Assume (F), (G '). Let $u_0(x) = sgn\ x$ for any $x \in \mathbb{R}$ and let u^ε be the unique entropy solution to Cauchy problem (1).*

Then there exist $\alpha_* \leq \beta_*$ *(*$\alpha_* \geq \alpha$ *and* $\beta_* \leq \beta$*) such that, defining* $\gamma_1 := f'(\alpha_*)$ *and* $\gamma_2 := f'(\beta_*)$*,* $\gamma_1 \leq \gamma_2$ *and it holds, for almost all* $(x,t) \in \mathbb{R} \times (0,\infty)$*,*

$$\lim_{\varepsilon \to 0} u^\varepsilon(x,t) = u^0(x,t) := \begin{cases} -1 & \textit{if } x \leq \gamma_1 t, \\ v(x,t) & \textit{if } \gamma_1 t < x < \gamma_2 t \\ 1 & \textit{if } x \geq \gamma_2 t, \end{cases}$$

where $v = v(x,t)$ *is the solution of the Riemann problem*

$$\partial_t v + \partial_x f(v) = 0, \qquad v(x,0) = \begin{cases} \alpha_* & x < 0, \\ \beta_* & x \geq 0. \end{cases}$$

The convergence is uniform in $\Omega_{\tau,T} := \{(x,t) \in \mathbb{R} \times [\tau,T] \, : \, |x - \gamma_1 t| \geq \tau$ *and* $|x - \gamma_2 t| \geq \tau\}$ *for any* $0 < \tau < T$*.*

As in the previous case, the proof is based on the construction of an opportune "effective flux function" f_{eff}. In this case the structure of such function is more complicate since it has to take in account the presence of pieces of solution of the Riemann problem for the homogeneous equation. As an example, in the case of a convex flux f, then $\alpha_* = \alpha$ and $\beta_* = \beta$, while in the case of a concave f, then $\alpha_* = \beta_*$ arbitrarily chosen in $[\alpha, \beta]$.

Given f in the interval I, we set f_I the convex envelope of f in I.

Proof of Theorem 6.1. Proceeding as in Section 3, it can be proved that there exists a unique γ_0 such that $f_l(\alpha; \gamma_0) = f_r(\beta; \gamma_0)$

If $f(s) \geq f_l(\alpha; \gamma_0) + \gamma_0(s - \alpha)$ for any $s \in (\alpha, \beta)$ then ther exists a traveling wave with speed γ_0 connecting -1 with $+1$. In this case we can proceed exactly as in the case considered in the previous Section.

Otherwise there exists some s such that $f(s) < f_l(\alpha; \gamma_0) + \gamma_0(s - \alpha)$. Next, let us set

$$\begin{aligned} \gamma_1 &:= \inf\{\gamma < \gamma_0 \, : \, f_l(\cdot; \gamma) \text{ intersects } f \text{ in } [\alpha, \beta]\}, \\ \gamma_2 &:= \sup\{\gamma > \gamma_0 \, : \, f_r(\cdot; \gamma) \text{ intersects } f \text{ in } [\alpha, \beta]\}, \end{aligned}$$

and let α_* be such that $f_l(\alpha_*, \gamma_1) = f(\alpha_*)$, maximum with such property, and β_* be such that $f_r(\beta_*, \gamma_2) = f(\beta_*)$, minimum with such proprerty.

Then $\gamma_1 = f'(\alpha_*)$, $\gamma_2 = f'(\beta_*)$, and the function

$$f_{\text{eff}}(s) := \begin{cases} f_l(s; \gamma_1) & s \in [-1, \alpha_*], \\ f_{[\alpha_*, \beta_*]}(s) & s \in (\alpha_*, \beta_*), \\ f_r(s; \gamma_2) & s \in [\beta_*, 1], \end{cases}$$

has the following properties

$$f_{\text{eff}} \in C([-1,1]), \qquad f_{\text{eff}} \leq f, \qquad f_{\text{eff}}(\alpha_*) = f(\alpha_*), \qquad f_{\text{eff}}(\beta_*) = f(\beta_*).$$

The functions $f_l(\,\cdot\,;\gamma_1)$ and $f_r(\,\cdot\,;\gamma_2)$ are effective fluxes defining two traveling waves (ϕ_1,γ_1) and (ϕ_2,γ_2) connecting -1 to α_* and β_* to $+1$, respectively. Then it is possible to construct the following family $\underline{u}^\varepsilon$ of subsolutions to problem (1) with $u_0(x) = \operatorname{sgn} x$ as follows

$$\underline{u}^\varepsilon(x,t) := \begin{cases} -1 & \text{if } x \leq \gamma_1 t, \\ v(x,t) & \text{if } \gamma_1 t < x < \gamma_2 t, \\ \phi_2((x-\gamma_2 t)/\varepsilon) & \text{if } x \geq \gamma_2 t, \end{cases}$$

and the family of supersolutions $\overline{u}^\varepsilon$

$$\overline{u}^\varepsilon(x,t) := \begin{cases} \phi((x-\gamma_1 t)/\varepsilon) & \text{if } x \leq \gamma_1 t, \\ v(x,t) & \text{if } \gamma_1 t < x < \gamma_2 t, \\ 1 & \text{if } x \geq \gamma_2 t, \end{cases}$$

where $v = v(x,t)$ is defined as in the statement fo the Theorem. To check that $\overline{u}^\varepsilon$ and $\underline{u}^\varepsilon$ are sub- and supersolutions it is enough to proceed as in the proof of Theorem 1.1.

Since as $\varepsilon \to 0$ both $\underline{u}^\varepsilon$ and $\overline{u}^\varepsilon$ converges to u^0 in the sense of the extended graph (see Section 4), the conclusion follows.

Appendix

Here we show that if assumption (H) is not satisfied the convergence in Theorem 1.1 cannot be uniform in all Σ_τ (no matter how τ is chosen). Hereafter assume without restriction $f(0) = f'(0) = 0$ and that

$$f(s) \geq 0 \quad \forall\, s \in [-1,1] \qquad \text{and} \qquad f'/g \notin L^1(-\delta,\delta) \quad \forall\, \delta > 0.$$

In this case $\gamma = f'(0) = 0$.

Following the same line of Theorem 2.5 in [7], it can be proved that the solution u^ε of the Cauchy problem (1) with initial datum $u_0(x) = \operatorname{sgn} x$ is such that

$$\lim_{t\to+\infty} \sup_{x\in[-M,M]} |u^\varepsilon(x,t)| = 0 \qquad \text{for any } M > 0.$$

Moreover, since the initial datum u_0 is increasing, the solution u^ε is increasing with respect to x, and $u^\varepsilon(0,t) = 0$ for any $t > 0$. Therefore

$$\sup_{x\in[-M,M]} |u^\varepsilon(x,t)| = \max\{u^\varepsilon(-M,t), u^\varepsilon(M,t)\} \qquad \text{for any } t > 0.$$

Hence for any $\tau > 0$ and for any $\varepsilon > 0$

$$\sup_{(x,t)\in\Sigma_\tau} |u^\varepsilon(x,t) - \text{sgn } x| \geq \sup_{x,t\geq\tau} (1 - u^\varepsilon(x,t)) = 1 - \inf_{t\geq\tau} u^\varepsilon(\tau,t) = 1,$$

showing that the convergence is not uniform in Σ_τ.

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