Canadian Mathematical Society
Conference Proceedings

Operator Splitting Methods for Degenerate Convection–Diffusion Equations I: Convergence and Entropy Estimates

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Dedicated to Sergio Albeverio on the occasion of his 60th birthday

Abstract. We present and analyze a numerical method for the solution of a class of scalar, multi-dimensional, nonlinear degenerate convection-diffusion equations. The method is based on operator splitting to separate the convective and the diffusive terms in the governing equation. The nonlinear, convective part is solved using front tracking and dimensional splitting, while the nonlinear diffusion equation is solved by a suitable difference scheme. We verify L^1 compactness of the corresponding set of approximate solutions and derive precise entropy estimates. In particular, these results allow us to pass to the limit in our approximations and recover an entropy solution of the problem in question. The theory presented covers a large class of equations. Important subclasses are hyperbolic conservation laws, porous medium type equations, two-phase reservoir flow equations, and strongly degenerate equations coming from the recent theory of sedimentation-consolidation processes. A thorough numerical investigation of the method analyzed in this paper (and similar methods) is presented in a companion paper.

1. Introduction

We address the important issue of constructing and analyzing numerical methods for a class of scalar, nonlinear, degenerate convection–diffusion problems of the form

$$u_t + \sum_{j=1}^{m} V_j(x) f_j(u)_{x_j} = \varepsilon \sum_{j=1}^{m} \big(K_j(x) A_j(u)_{x_j}\big)_{x_j}, \quad u(x,0) = u_0(x), \tag{1}$$

for $(x,t) \in Q_T = \mathbb{R}^m \times (0,T)$. Here $u = u(x,t)$ is the unknown function; f_j, K_j, A_j, and u_0 are given functions; and $\varepsilon > 0$ is a (small) scaling constant. In applications related to fluid flow (see, e.g, [**27**]), $f = (f_1, \dots, f_m)$ is often referred to as the flux function, $V = (V_1, \dots, V_m)$ as the velocity field, $K = (K_1, \dots, K_m)$ as the permeability field, and $A = (A_1, \dots, A_m)$ as the function of diffusivity.

Key words and phrases. Degenerate convection–diffusion equations, entropy solutions, operator splitting, front tracking, convergence, entropy estimates.

Throughout this paper, we will assume that the data of our problem satisfy the following conditions:

$$
\begin{cases}
\text{(i) } f, A \in C^2(\mathbb{R};\mathbb{R}^m) \text{ and } A' \geq 0. \\
\text{(ii) } V \in C^2(\mathbb{R};\mathbb{R}^m) \text{ and } \sum_{j=1}^m V(x)_{x_j} = 0 \text{ (i.e., } V \text{ is divergence free).} \\
\text{(iii) } K \in C^2(\mathbb{R};\mathbb{R}^m) \text{ and } K > 0. \\
\text{(iv) } u_0 \in L^1(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m) \cap BV(\mathbb{R}^m).
\end{cases} \tag{2}
$$

We mention that the "maximal regularity" conditions (i)–(iii) can be significantly weakened and they are assumed here only to make our presentation as short and as transparent as possible. On the other hand, condition (iv) concerning the initial function u_0 is fairly weak. Notice that the convective part of (1) is written on transport, or non-conservative, form, which is reasonable since V is assumed to be divergence free.

We only require that $A_j' \geq 0$ for each j and hence the parabolic term is allowed to degenerate. In fact, each A_j' may be zero on a set of positive measure, in which case we call the equation *strongly degenerate*, and thus the well known hyperbolic conservation law

$$
u_t + \sum_{j=1}^m V_j(x) f_j(u)_{x_j} = 0 \tag{3}
$$

is a special case of (1). Other subclasses of (1) include one-point degenerate porous medium type equations [**40**]; two-point degenerate two-phase reservoir flow equations [**17**]; and strongly degenerate equations coming from the recent theory of sedimentation-consolidation processes [**5, 6, 15**].

A characteristic feature of nonlinear partial differential equations of hyperbolic-parabolic type such as (1) is that solutions may exhibit quite complex behavior, like singularities and sharp transitions, in a small region of space (and time) and this makes them particularly hard to solve numerically. The aim of the paper is to construct and analyze a numerical method for nonlinear convection-diffusion equations of degenerate parabolic type (1) that works "uniformly" in the diffusion coefficient $A_j' \geq 0$, and as such is able to resolve the issues concerning singularities and sharp transitions in the solutions of (1); more details are given towards the end of this section.

If (1) is allowed to degenerate at certain points, that is, $A_j'(u) = 0$ for some values of u, solutions are not necessarily smooth, but typically continuous, and *weak solutions* must be sought. On the other hand, if $A_j(u)$ is zero on a set of positive measure, weak solutions may be discontinuous and are not necessarily uniquely determined by their initial data, as can be easily deduced from the maximum principle and what is known about the hyperbolic conservation law. Consequently, additional admissibility criteria — so-called *entropy conditions* — must be imposed to single out the physically correct solution. We use the following definition of an entropy or generalized solution:

Definition 1.1 (Entropy Solution). *We call a function $u \in L^1(Q_T) \cap L^\infty(Q_T)$ an entropy solution of the Cauchy problem* (1) *provided*

$$
\sqrt{\varepsilon K_j(x)} A(u)_{x_j} \in L^2_{loc}(Q_T), \quad j = 1, \dots, m, \tag{4}
$$

and the following entropy inequality holds

$$
\begin{aligned}
(5)\qquad \iint_{Q_T} \Big(S(u,k)\phi_t + \sum_{j=1}^{m} F_j(u,k)(\phi V_j)_{x_j} + \varepsilon \sum_{j=1}^{m} Q_j(u,k)\big(K_j\phi_{x_j}\big)_{x_j} \Big)\, dt\, dx \\
+ \int |u_0 - k|\phi(x,0)\, dx - \int |u(x,T) - k|\phi(x,T)\, dx \geq 0,
\end{aligned}
$$

for every constant $k \in \mathbb{R}$ *and test function* $\phi \in C_0^\infty(\mathbb{R}^m \times [0,\infty))$, $\phi \geq 0$. *Here we have used the short-hand notations:* $S(u,k) = |u-k|$ *for the entropy,* $F_j(u,k) = \operatorname{sgn}(u-k)(f_j(u) - f_j(k))$ *for the convective entropy flux, and* $Q_j(u,k) = |A_j(u) - A_j(k)|$ *for the diffusive entropy flux.*

We denote the left-hand side of (5) by $\mathcal{L}(u,\phi,k; f, V, A, u_0, T)$ and sometimes $\mathcal{L}(u,\phi,k)$ or even simply $\mathcal{L}_\phi(u)$. Furthermore, $\mathcal{L}(u,\phi,k; f, V, 0, u_0, T)$ denotes the left-hand side of the entropy inequality corresponding to the purely hyperbolic case. Observe that (5) implies that (1) holds in the sense distributions, i.e., the equality

$$
\begin{aligned}
(6)\qquad \iint_{Q_T} \Big(u\phi_t + \sum_{j=1}^{m} V_j(x) f_j(u)\phi_{x_j} + \varepsilon \sum_{j=1}^{m} A_j(u)\big(K_j(x)\phi_{x_j}\big)_{x_j} \Big)\, dt\, dx \\
+ \int u_0(x)\phi(x,0)\, dx - \int u(x,T)\phi(x,T)\, dx = 0,
\end{aligned}
$$

holds for each test function $\phi \in C_0^\infty(\mathbb{R}^m \times [0,\infty))$.

When $A_j \equiv 0$ for all j, the entropy condition (5) reduces to the entropy condition introduced independently by Vol'pert [**42**] and Kružkov [**35**] for hyperbolic problems and thus Definition 1.1 contains the hyperbolic problem as a special case [**3**]. Definition 1.1 goes back to Vol'pert and Hudjaev [**43**], who were the first to consider strongly degenerate parabolic equations of nonlinear (or quasilinear) type. They also showed the existence of a BV entropy solution, provided u_0 is sufficiently smooth, by passing to the limit in a parabolic regularization and obtained some partial uniqueness results for BV entropy solutions, i.e., entropy solutions whose first order partial derivatives are finite measures on Q_T. In the one-dimensional case, Wu and Yin [**44**] later provided the complete uniqueness proof for BV entropy solutions. See Bénilan and Touré [**1, 2**] for further results — via nonlinear semigroup theory — on entropy solutions in the one-dimensional case (without variable coefficients). In this context, let us also mention that Cockburn and Gripenberg [**13**] have established continuous dependence on the nonlinearities of semigroup solutions of the Cauchy problem for multi-dimensional equations of the type (with $f_j = f$ for all j)

$$
(7)\qquad u_t + \sum_{j=1}^{m} f_j(u)_{x_j} = \Delta_x A(u).
$$

In the multi-dimensional case, Brézis and Crandall [**4**] established uniqueness of weak solutions in $L^1(Q_T) \cap L^\infty(Q_T)$ of the Cauchy problem for (7) with no lower order terms ($f_j = 0$ for all j). Later, under the assumption that $A(u)$ is strictly increasing, which does not rule out the possibility of $A'(u)$ having infinite number of zero points, Yin [**45**] showed uniqueness of weak solutions in $L^\infty(Q_T) \cap BV(Q_T)$ of the Cauchy problem for (7). Note that (7) is a special case of (1). The assumption that u_t should be a finite measure was removed in [**47**]. The initial-boundary value problem for (7) with variable coefficients was treated in [**46**].

An important step forward in the general case of $A(u)$ being merely non-decreasing (i.e., the strongly degenerate case) was made recently by Carrillo [**12**], who showed uniqueness of entropy solutions for a particular boundary value problem for (7). His method of proof, which is an elegant extension of Kružkov's "doubling of the variables" device [**35**], also applies to the Cauchy problem for (7) as well weakly coupled systems of equations such as (7), see [**28**]. In [**7**], the uniqueness proof of Carrillo was adopted to several initial-boundary value problems arising the theory of sedimentation-consolidation processes [**5, 6, 15**], which in some cases call for the notion of entropy boundary condition (see also [**10, 11**] for the BV approach).

Later on we will introduce our numerical approximations and prove that they converge, based on a suitable compactness argument. We recall here that such convergence proofs merely ensure convergence along subsequences. However, when we are equipped with a uniqueness result, as in the cases mentioned above, we automatically get convergence of the whole sequence in question and not just some subsequence. To the best of our knowledge, uniqueness of entropy solutions for the general problem (1) that we consider here is still open when $A(u)$ is merely non-decreasing, so in the general case we have to resort to convergence along subsequences.

Although there seems to be an increasing interest in the (analysis of) numerical approximation of entropy solutions of degenerate convection-diffusion equations, the amount of literature on the subject is as the moment extremely modest, at least compared with the purely hyperbolic case which has long traditions. Recent literature include papers by Evje and Karlsen [**23, 19, 21, 22, 18**], Cockburn and Shu [**14**], Kurganov and Tadmor [**36**], and Bouchut, Guarguaglini, and Natalini [**3**].

In [**23**], an operator splitting method for one-dimensional degenerate equations is proposed and analyzed. In particular, convergence results and precise entropy estimates are given for the proposed approximations which allow the authors to pass to the limit and recover the unique entropy solution of the equation in question. Related papers include [**20, 24, 8, 9, 17**].

In [**19**], the authors build a convergence theory for explicit monotone difference approximations of degenerate convection-diffusion equations. This theory parallels, and includes, the classical theory of Harten, Hyman, and Lax [**25**] and Crandall and Majda [**16**] for conservation laws. Implicit monotone difference approximations are treated via nonlinear semigroup theory in [**21, 18**]. High order MUSCL type methods are analyzed in [**22**].

In [**14**], the authors present so-called local discontinuous Galerkin method for (degenerate) convection-diffusion problems. This method is an extension of the Runge-Kutta discontinuous Galerkin method for hyperbolic problems. A convergence analysis within the entropy solution framework is, however, not provided.

In [**36**], the authors introduce and test a new family of central schemes for conservation laws and (degenerate) convection-diffusion equations. The main feature of the schemes is that they possess a much smaller numerical viscosity than the original central schemes [**39**] and as such apply to convection-diffusion equations. In particular, they admit a semi-discrete formulation. A convergence analysis of their convection-diffusion schemes within the entropy solution framework is not presented.

In [**3**], the authors introduce and analyze a class of discrete velocity BGK type methods for degenerate convection-diffusion equations, which in turn are extensions of similar methods for hyperbolic conservation laws. Convergence results and entropy estimates are proved, which imply that the BGK approximations converge to the unique entropy solution of the governing equation.

We conclude this section by giving more details about the contents of the present paper, which deals with a class of numerical methods called operator splitting methods. The underlying design principle behind any splitting method for an evolutionary partial differential equation is to split the time evolution into several partial steps in order to separate out the different (physical) effects present in the governing equation. In our context, the idea is to split the original problem (1) into a first order convection (hyperbolic) problem and a second order degenerate diffusion (parabolic) problem. The split problems are then solved sequentially to approximate the exact solution of the original problem. The main attraction of splitting methods lies, of course, in the fact that one can employ optimal existing methods for each split problem. A detailed description of the operator splitting method analyzed in this paper is given in §2. For an introduction to operator splitting methods in general, we refer to Espedal and Karlsen [**17**].

The main purpose of the present paper is to extend the analysis (and the splitting method) of Evje and Karlsen [**23**] (which in turn borrowed from Karlsen and Risebro [**34**]) to more general problems arising in applications. To be more precise, we extend the analysis of [**23**] as follows:

- In [**23**], the authors proved convergence to a limit satisfying (4) and (5) only for the semi-discrete method. For their fully discrete method, however, they did not establish (4). In the present paper, we establish at least that the limit of *one* of our fully discrete methods also satisfies (4).
- In [**23**], the authors proposed and analyzed operator splitting method for one-dimensional equations without variable coefficients. Here we propose and analyze an operator splitting method that works for a general class of multi-dimensional equations with variable coefficients, thereby containing [**23**] as a special case.

The convergence (or compactness) part of the analysis, together with the proof that the limit of one of the methods satisfies (4), is presented in §3. Precise entropy estimates are established in §4. These estimates imply in particular that the limit of a converging sequence of numerical approximations satisfies the entropy condition (5). Finally, we mention that a variety of (convincing) numerical applications are presented in our companion paper [**27**], including applications to oil reservoir simulation and sedimentation-consolidation processes.

Acknowledgement. KHK would like to thank Raimund Bürger and Steinar Evje for discussions and collaboration on the subject of degenerate parabolic equations and their applications to sedimentation-consolidation processes.

2. Operator Splitting Methods

To construct approximate solutions of (1) it is often favourable to use a splitting method. There are several motivations and possibilities for splitting the equation. First, one could use dimensional splitting directly on (1), i.e., split it into m one-dimensional equations of the form

$$v_t + V(x)f(v)_x = \varepsilon\big(K(x)A(v)_x\big)_x, \tag{8}$$

and use ones favourite one-dimensional method to approximate the solution operator $\mathcal{P}_t$ of (8). In this case the splitting reads

$$u(x, n\Delta t) \approx \left[\mathcal{P}^m_{\Delta t} \circ \cdots \circ \mathcal{P}^1_{\Delta t}\right]^n u_0(x), \tag{9}$$

where $\Delta t > 0$ denotes the splitting step. To approximate $\mathcal{P}_j t$, one could for instance use a monotone difference method [**19, 21**] or a higher-order method based on MUSCL techniques [**22**]. Another alternative is the *corrected operator splitting method* (COS) introduced by Karlsen and Risebro [**33**], which is a large-time-step method. The method splits (8) into a hyperbolic step $v_t + V(x)f(v)_x = 0$ and a degenerate parabolic step $w_t = \varepsilon(K(x)A(u)_x)_x$. Once a shock is formed in the hyperbolic step, there will be an entropy loss due to Oleinik's convexification. The source of the entropy loss can be identified as a residual flux (defined locally as the flux f minus its convexification) which can be taken into account either in a separate correction step or included in the parabolic step. This way, COS resolves the correct balance between convection and diffusion in regions with sharp gradients for an (almost) arbitrarily large time step and can be used in combination with (9) to yield a very efficient numerical method (if the dynamics of (1) allows for large splitting steps). This method has been investigated in other studies [**24, 31, 30, 32**], see also the companion paper [**27**].

As an alternative to dimensional splitting, one can split with respect to physical mechanisms, which is what we will pursue in this paper. To this end, split (1) into a hyperbolic part

$$v_t + \sum_{j=1}^m V_j(x) f_j(v)_{x_j} = 0, \qquad v(x,0) = v_0(x) \tag{10}$$

and a degenerate parabolic part

$$w_t = \varepsilon \sum_{j=1}^m \big(K_j(x) A_j(w)_{x_j}\big)_{x_j}, \qquad w(x,0) = w_0(x). \tag{11}$$

In what follows, we choose a time step $\Delta t > 0$ and an integer N such that $N\Delta t = T$. We also use the notation $t^n = n\Delta t$ for $n = 0, \dots, N$. The corresponding (semi-discrete) splitting method then reads

$$u_{\Delta t}(t) = \left[\mathcal{H}_{\Delta t} \circ \mathcal{S}_{\Delta t}\right]^n u_0, \qquad \text{for } t \in (t^{n-1}, t^n], \quad n = 1, \dots, N, \tag{12}$$

where $\mathcal{S}_t$ and $\mathcal{H}_t$ denote the solution operators of (10) and (11), respectively.

A slightly different method is obtained if we apply dimensional splitting to the hyperbolic part, i.e., (10) is further split into one-dimensional equations

$$v_t + V_j(x) f_j(v)_{x_j} = 0, \qquad v(x,0) = v_0(x). \tag{13}$$

The corresponding semi-discrete splitting method takes the form

$$u_{\Delta t}(t) = \left[\mathcal{H}_{\Delta t} \circ \mathcal{S}^m_{\Delta t} \circ \cdots \circ \mathcal{S}^1_{\Delta t}\right]^n u_0, \qquad \text{for } t \in (t^{n-1}, t^n], \quad n = 1, \dots, N, \tag{14}$$

where $\mathcal{S}^j_t$ is the exact solution operator associated with (13).

Concerning the semi-discrete methods (12) and (14), we have the following main theorem:

THEOREM 2.1 (Semi-discrete methods). *Suppose that conditions* (i)-(iv) *in* (2) *hold. Let $\{u_{\Delta t}\}$ be the semi-discrete splitting sequence given by* (12) *or* (14). *Then $\{u_{\Delta t}\}$ converges along a subsequence in $L^1_{loc}(Q_T)$ to a limit u that satisfies $u(\cdot, t) \in L^1(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m) \cap BV(\mathbb{R}^m)$ uniformly in t and $u(\cdot, t)$ is uniformly L^1 Hölder* *continuous in time with exponent* $1/2$. *Finally, the limit* u *satisfies the regularity requirement* (4) *and the entropy inequality* (5).

This theorem shows that the semi-discrete splitting sequences given by (12) and (14) both converge along subsequences to entropy solutions of (1) in the sense of Definition 1.1. Moreover, when it is known that the entropy solution is unique (see §1), the whole sequence $\{u_{\Delta t}\}$ converges to this solution.

From a computational point of view, we need to replace the exact solution operators $\mathcal{S}_t$ and $\mathcal{H}_t$ in (12) by suitable numerical methods. In this paper we use an explicit-implicit finite difference method to approximate $\mathcal{H}_t$ and a large-step, front tracking method to approximate $\mathcal{S}_t$. The large-step method is based on dimensional splitting, which means that we shall here use (14) as the basis for constructing the fully discrete splitting method. We let $\mathcal{H}_{\Delta x,t}$ denote the finite difference solution operator associated with (11) at time t and $\mathcal{S}^j_{\delta,\Delta x,t}$ the front tracking solution operator associated with (13) at time t. To simplify the dimensional splitting process, each one-dimensional hyperbolic solution is projected onto a Cartesian grid with mesh parameter Δx by a projection operator

$$\pi v(x) = \frac{1}{\Delta x^m} \int_{\Omega_j} v(y)\, dy, \quad \text{for } x \in \Omega_j,$$

where Ω_j denotes

$$\Omega_j = [j_1\Delta x, (j_1+1)\Delta x) \times \cdots \times [j_m\Delta x, (j_m+1)\Delta x), \qquad j = (j_1,\dots,j_m) \in \mathbb{Z}^m.$$

Then our fully discrete splitting method reads

$$u_\eta(t) = \left[\mathcal{H}_{\Delta x,\Delta t} \circ \pi \circ \mathcal{S}^m_{\delta,\Delta x,\Delta t} \circ \cdots \circ \pi \circ \mathcal{S}^1_{\delta,\Delta x,\Delta t}\right]^n u^0, \tag{15}$$

for $t \in (t^{n-1}, t^n]$ and $n = 1,\dots,N$. Here η signifies the discretization parameters $(\Delta x, \Delta t, \delta)$ and $u^0 = \pi u_0$. For simplicity of notation, we sometimes use the shorthand

$$\mathcal{S}_{\delta,\Delta x,t} = \pi \circ \mathcal{S}^m_{\delta,\Delta x,t} \circ \cdots \circ \pi \circ \mathcal{S}^1_{\delta,\Delta x,t}, \qquad t > 0. \tag{16}$$

i.e., $\mathcal{S}_{\delta,\Delta x,t}$ denotes the front tracking-dimensional splitting solution operator associated with (10), see Holden and Risebro [**29**] and Lie [**37**] for more details about this approximate solution operator.

Concerning the fully discrete method (15), we have the following similar main theorem:

THEOREM 2.2 (Fully discrete method). *Suppose that conditions* (i)-(iv) *in* (2) *hold. Let* $\{u_\eta\}$ *be the fully discrete splitting sequence given by* (15). *Then* $\{u_\eta\}$ *converges along a subsequence in* $L^1_{loc}(Q_T)$ *to a limit* u *that satisfies* $u(\cdot,t) \in L^1(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m) \cap BV(\mathbb{R}^m)$ *uniformly in* t *and* $u(\cdot,t)$ *is uniformly* L^1 *Hölder continuous in time with exponent* $1/2$. *The precise entropy estimate for* u_η *reads* $\mathcal{L}_\phi(u_\eta) \geq -C\big(\sqrt{\Delta t} + \Delta x + \delta^2\big)$, *for some constant* $C > 0$ *independent of* η. *In particular, this implies that the limit* u *satisfies* (5). *In the case of a fully implicit diffusion solver, the limit* u *also satisfies the regularity requirement* (4).

Note that in the case of an explicit-implicit (or fully explicit) diffusion solver $\mathcal{H}_{\Delta x,\Delta t}$, we have not yet managed to establish the regularity requirement (4).

Theorem 2.2 follows easily from the results stated and proved in §3 and §4. On the other hand, the proof of Theorem 2.1 will not be written out explicitly in this paper. It is however easy to modify the proof of Theorem 2.2 so that it applies to the semi-discrete methods (12) and (14). We leave the details to the interested reader as an exercise.

Next, we describe the front tracking and the finite difference method in more detail.

The Hyperbolic Step. In each hyperbolic step we solve an equation of the form

$$v_t + V(x)f(v)_x = 0, \qquad v(x,0) = v_0(x), \qquad x \in \mathbb{R},\ t > 0. \tag{17}$$

To this end we will use front tracking, which is an unconditionally stable method without an intrinsic time step. The method seeks approximations within the class of step functions. First, the initial data v_0 is approximated by a step function $v_{0,\Delta x}$. This is achieved by the projection in (15), i.e., $v_{0,\Delta x} = \pi v_0$. Thus the initial value problem is decomposed into a series of simple Riemann problems that can be solved analytically one by one. If the flux function is approximated by a continuous, piecewise linear function $f^\delta(v)$, the solution of each Riemann problem will again be a step function. The global solution, until the first wave interaction occurs, thus consists of constant states separated by discontinuities propagating along paths $x(t)$. Each path $x(t)$ is given by the differential equation $\dot{x} = V(x)s$, where s is the Rankine–Hugoniot speed $(f^\delta(v_L) - f^\delta(v_R))/(v_L - v_R)$. By approximating the velocity V by either a piecewise linear or a piecewise constant function $V_{\Delta x}$, the differential equations can be solved explicitly and the paths $x(t)$ are given in closed form. Since every wave interaction leads to a new Riemann problem, this construction can be continued up to any desired time. In fact, it can be shown [**26, 38**] that there is a finite number of wave interactions, even in infinite time, if the initial data is bounded and has finite total variation. We stress that we replace the flux functions f by f^δ in most of the following convergence analysis and derivation of entropy estimates. Only at the final step we consider the limit as $f^\delta \to f$. One can prove that the limit (as $\delta \to 0$) is indeed the entropy solution of (17), see Lie [**38**].

By construction, front tracking solutions are not increasing in L^∞ norm and have bounded total variation. Since all waves have finite speed of propagation, the solution is Lipschitz continuous in time with respect to the L^1 norm. Each solution satisfies an entropy condition for the perturbed equation with f^δ and $V_{\Delta x}$ and is thus an entropy solution. However, the solution operators are not L^1 contractions because of the velocity field and the non-conservative form. These claims can be verified by using a Kružkov type analysis, see Lie [**38**]. We summarize properties of solutions of (17) in the following lemma:

Lemma 2.1. *Let $v(x,t)$ be a solution of* (17). *Then v satisfies the following estimates*

$$\|v\|_\infty \le \|v_0\|_\infty, \quad |v(\,\cdot\,,t)|_{BV} \le |v_0|_{BV}, \quad \|v(\,\cdot\,,t) - v_0\|_1 \le Ct,$$

where C is a constant depending on the data. Moreover, let $u(x,t)$ be a solution of (17) *with flux function g, velocity field U, and initial data u_0. Then we have the stability estimate*

$$\begin{aligned}\|v(\,\cdot\,,t) - u(\,\cdot\,,t)\|_1 \le e^{Dt}\big[&\|v_0 - u_0\|_1 \\ &+ t\big(E_1\|U - V\|_\infty + E_2\|f - g\|_{Lip}\big)\min(|u_0|_{BV}, |v_0|_{BV})\big],\end{aligned}$$

where the constants D, E_1, and E_2 depend on the data (f, g, V, U, v_0, u_0).

Note that each front tracking solution is an exact solution of (17) for special functions f, V, and v_0, so that Lemma 2.1 also gives the properties of the front tracking solutions used for the hyperbolic steps.

The Parabolic Step. In each parabolic step we solve a possibly degenerate equation of the form

$$w_t = \varepsilon \sum_{j=1}^{m} \big(K_j(x) A_j(w)_{x_j}\big)_{x_j}, \qquad w(x,0) = w_0(x), \tag{18}$$

where we assume, without loss of generality, that w_0 has compact support. We will use an implicit–explicit finite difference method to solve (18). Difference methods for general degenerate parabolic equations are treated in [**19, 21, 22, 18**]. We assume a mesh with a uniform spacing Δx in each spatial direction and a time step τ. Let $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}^m$ and e_j the index with the jth component equal unity and all other components equal zero. Let W_α^n denote the approximate solution in grid cell α at time $t = n\tau$. Furthermore, let $\beta_\alpha^j = \varepsilon\tau K^j_{\alpha+e_j/2}/\Delta x^2$ and $A_\alpha^{j,n} = A_j(W_\alpha^n)$. Then the scheme reads

$$\begin{aligned} W_\alpha^{n+1} =& W_\alpha^n + \theta \sum_{j=1}^{m} \Big[\beta_\alpha^j \big(A_{\alpha+e_j}^{j,n} - A_\alpha^{j,n}\big) - \beta_{\alpha-e_j}^j \big(A_\alpha^{j,n} - A_{\alpha-e_j}^{j,n}\big)\Big] \\ &+ (1-\theta) \sum_{j=1}^{m} \Big[\beta_\alpha^j \big(A_{\alpha+e_j}^{j,n+1} - A_\alpha^{j,n+1}\big) - \beta_{\alpha-e_j}^j \big(A_\alpha^{j,n+1} - A_{\alpha-e_j}^{j,n+1}\big)\Big], \end{aligned} \tag{19}$$

where the parameter θ is in the interval $[0,1]$; $\theta = 0$ giving a fully implicit scheme and $\theta = 1$ a fully explicit scheme. We make the following simplifying assumption in this paper.

ASSUMPTION 2.1. *The scheme* (19) *admits a unique solution* $\{W_\alpha^{n+1}\}$.

The existence of a unique solution of (19) can be established by suitable fixed point argument. For $n = 0, 1 \dots$, we let $W^n = W^n(x)$ denote the piecewise constant function

$$W^n(x) = W_\alpha^n, \qquad \text{for } x \in \Omega_\alpha \tag{20}$$

and some $\alpha \in \mathbb{Z}^m$. Moreover, we let $\mathcal{H}_{\Delta x,t}$ denote the (finite difference) solution operator defined by

$$\mathcal{H}_{\Delta x,t} w_0(x) = W^n(x), \qquad x \in \mathbb{R}^m,\ t \in ((n-1)\tau, n\tau],\ n = 1, 2, \dots, \tag{21}$$

for any initial function $w_0 = w_0(x)$ that is piecewise constant with respect to the grid.

For later use, recall that the L^∞ norm and the L^1 norm of a grid function $W = \{W_\alpha\}$ are defined respectively as follows:

$$\|W\|_\infty = \sup_{\alpha\in\mathbb{Z}^m} \|W_\alpha\|, \quad \|W\|_1 = \sum_{\alpha\in\mathbb{Z}^m} |W_\alpha| \Delta x^m,$$

and the BV semi-norm as

$$TV(W) = \sum_{j,\alpha} |W_\alpha^n - W_{\alpha-e_j}^n| \Delta x^{m-1}.$$

For the scheme (19) we have the following properties.

LEMMA 2.2. *Let $W^n = \{W^n_\alpha\}$ and $V^n = \{V^n_\alpha\}$ be approximate solutions of* (18) *generated by* (19) *with a time step τ satisfying the stability condition*

$$\theta \max_{\alpha,j,n} \beta^j_\alpha (A'_j)^n_\alpha \le \theta \frac{\varepsilon\tau}{\Delta x^2} \max_j \Big(\sup_x K_j(x) \sup_u A'_j(u) \Big) \le \frac{1}{2m}. \tag{22}$$

Then

$$\|W^n\|_\infty \le \|W^0\|_\infty, \quad TV(W^n) \le TV(W^0), \quad \|W^n - V^n\|_1 \le \|W^0 - V^0\|_1.$$

PROOF. The estimates are straightforward and we present the details only for $m = 1$ (with $K_1 = K$, $A_1 = A$, and $\alpha = i \in \mathbb{Z}$). First, we consider the L^∞ estimate for the approximation $\{W^n_i\}$. Using Taylor expansions of $A(W)$ we can rewrite the scheme as

$$\begin{aligned}
\big[1 + (1-\theta)(\beta_i A'(\xi_i) + \beta_{i-1} A'(\xi_{i-1}))\big] W^{n+1}_i = \\
\big[1 - \theta(\beta_i A'(\zeta_i) + \beta_{i-1} A'(\zeta_{i-1}))\big] W^n_i \\
+ \theta\big[\beta_i A'(\zeta_i) W^n_{i+1} + \beta_{i-1} A'(\zeta_{i-1}) W^n_{i-1}\big] \\
+ (1-\theta)\big[\beta_i A'(\xi_i) W^{n+1}_{i+1} + \beta_{i-1} A'(\xi_{i-1}) W^{n+1}_{i-1}\big],
\end{aligned}$$

for suitable ξ_i, ζ_i. If we now choose $i = \ell$ such that $W^{n+1}_\ell = \max_i W^{n+1}_i$ and use the stability condition (22) we derive

$$\begin{aligned}
\big[1 + (1-\theta)(\beta_\ell A'(\xi_\ell) + \beta_{\ell-1} A'(\xi_{\ell-1}))\big] W^{n+1}_\ell \le \\
\big[1 - \theta(\beta_i A'(\zeta_i) + \beta_{i-1} A'(\zeta_{i-1}))\big] \max_i W^n_i \\
+ \theta\big[\beta_i A'(\zeta_i) \max_i W^n_{i+1} + \beta_{i-1} A'(\zeta_{i-1}) \max_i W^n_{i-1}\big] \\
+ (1-\theta)\big[\beta_\ell A'(\xi_\ell) W^{n+1}_{\ell+1} + \beta_{i-1} A'(\xi_{\ell-1}) W^{n+1}_{\ell-1}\big].
\end{aligned}$$

Since $W^{n+1}_{\ell\pm1} \le W^{n+1}_\ell$, the inequality simplifies to $\max_i W^{n+1}_i \le \max_i W^n_i$. Similarly, by picking $i = k$ such that $W^{n+1}_k = \min_i W^{n+1}_i$ we can bound W^{n+1}_i from below. Hence, $\max_i |W^{n+1}_i| \le \max_i |W^n_i|$ and the L^∞ estimate follows by induction on n.

To prove estimate on the total variation, we use almost the same argument. Introduce $Z^n_i = W^n_{i+1} - W^n_i$. From the difference scheme (19) evaluated at i and $i+1$, we then get

$$\begin{aligned}
Z^{n+1}_i = & Z^n_i + \theta\big[\beta_{i+1} A'(\zeta_{i+1}) Z^n_{i+1} - 2\beta_i A'(\zeta_i) Z^n_i + \beta_{i-1} A'(\zeta_{i-1}) Z^n_{i-1}\big] \\
& + (1-\theta)\big[\beta_{i+1} A'(\xi_{i+1}) Z^{n+1}_{i+1} - 2\beta_i A'(\xi_i) Z^{n+1}_i + \beta_{i-1} A'(\xi_{i-1}) Z^{n+1}_{i-1}\big],
\end{aligned}$$

for suitable ξ_i, ζ_i. Using once more the stability condition, we get

$$\begin{aligned}
\sum_i \big(1 + 2(1-\theta)\beta_i A'(\xi_i)\big) |Z^{n+1}_i| \le \sum_i (1 - 2\theta)\beta_i A'(\zeta_i) |Z^n_i| \\
+ \theta\Big(\sum_i \beta_{i+1} A'(\zeta_{i+1}) |Z^n_{i+1}| + \sum_i \beta_{i-1} A'(\zeta_{i-1}) |Z^n_{i-1}|\Big) \\
+ (1-\theta)\Big(\sum_i \beta_{i+1} A'(\xi_{i+1}) |Z^{n+1}_{i+1}| + \sum_i \beta_{i-1} A'(\xi_{i-1}) |Z^{n+1}_{i-1}|\Big).
\end{aligned}$$

Hence $\sum_i |Z^{n+1}_i| \le \sum_i |Z^n_i|$, from which the TV estimate follows. The L^1 stability estimate is derived by a similar argument. $\square$

3. Convergence Results

In this section we will prove that the fully discrete splitting method (15) converges to a limit $u(x,t)$. The proof is based on a standard L^1 compactness argument, where we first use Lemmas 2.1 and 2.2 to bound the approximate solutions u_η in suitable norms

LEMMA 3.1. *Let u_η denote the approximate solution defined by* (15). *Then u_η satisfies, for $n, \ell = 1, 2, \dots$,*

$$\|u_\eta(\cdot, n\Delta t)\|_\infty \le \|u_0\|_\infty, \qquad |u_\eta(\cdot, n\Delta t)|_{BV} \le e^{C_1 n\Delta t}|u_0|_{BV},$$
$$\|u_\eta(\cdot, \ell\Delta t) - u_\eta(\cdot, n\Delta t)\|_1 \le C_2\sqrt{|n-\ell|\,\Delta t},$$

where C_1 and C_2 are constants independent of $\delta, \Delta x, \Delta t$.

PROOF. The first inequality follows since neither of the operators $\mathcal{H}_{\Delta x, \Delta t}$, $\mathcal{S}^j_{\delta,\Delta x,\Delta t}$, or π introduce new extrema into the solution (see Lemmas 2.1 and 2.2).

To prove the second inequality, we need to establish a bound on the total variation of the composite hyperbolic operator $\mathcal{S}_{\delta,\Delta x,\Delta t}$ defined by (16). This result is well-known and relies on the total variation non-increasing property and the L^1 stability of each one-dimensional hyperbolic operator given in Lemma 2.1, see Lie [**37**] and Holden and Risebro [**29**]. Combining this result with the fact that the total variation is non-increasing in the parabolic step (see Lemma 2.2), the second inequality follows.

It remains to prove the third inequality, which will be done by a technique used first by Karlsen and Risebro [**34**]. For the product of the hyperbolic operators, i.e., for the composite operator $\mathcal{S}_{\delta,\Delta x,\Delta t}$ defined in (16), the result is valid with exponent 1 (i.e., Lipschitz continuity in time), see e.g., Lie [**37**]. Thus we focus on the parabolic operator here. The key point is to establish weak Lipschitz continuity in time for the finite difference approximation W^n defined in (20). To this end, let D^j_- and D^j_+ denote the backward and forward differences in direction j. Then:

$$\begin{aligned}
&\Big|\int \phi(W^{n+1} - W^n)\,dx\Big| \\
&= \varepsilon\tau\left|\sum_{j,\alpha}\left[\theta D^j_-\left(K^j_{\alpha+e_j/2}D^j_+A^{j,n}_\alpha\right) + (1-\theta)D^j_-\left(K^j_{\alpha+e_j/2}D^j_+A^{j,n+1}_\alpha\right)\right]\phi_\alpha\Delta x^m\right| \\
&= \varepsilon\tau\left|\sum_{j,\alpha}\Big(\theta K^j_{\alpha-e_j/2}A'_j(\zeta_\alpha)D^j_-W^n_\alpha + (1-\theta)K^j_{\alpha-e_j/2}A'_j(\xi_\alpha)D^j_-W^{n+1}_\alpha\Big)D^j_-\phi_\alpha\Delta x^m\right| \\
&\le \varepsilon\tau\max_j\{\|K_j\|_\infty\,\|A'_j\|_\infty\}\,\|\nabla\phi\|_\infty\sum_{j,\alpha}\Big(\theta|D^j_-W^n_\alpha| + (1-\theta)|D^j_-W^{n+1}_\alpha|\Big)\Delta x^{m-1} \\
&\le \mathrm{Const}\,\tau\,\|\nabla\phi\|_\infty,
\end{aligned}$$

where summation by parts (once in space) was used to derive the second equality from the first one. Moreover, we have used the averaged test function $\phi_\alpha = \pi\phi(\alpha\Delta x)$, and the Taylor expansions $A^{j,n}_{\alpha+e_j} - A^{j,n}_\alpha = A'_j(\theta_\alpha)[W^n_{\alpha+e_j} - W^n_\alpha]$ for suitable θ_α and $|\phi_{\alpha+e_j} - \phi_\alpha| \le \|\partial_{x_j}\phi\|_\infty\Delta x$. By repeating the above argument, we easily derive

$$\Big|\int \phi(W^\ell - W^n)\,dx\Big| \le \mathrm{Const}\,\|\nabla\phi\|_\infty\,|\ell - n|\,\tau, \tag{23}$$

which implies that $|\int \phi(\mathcal{H}_{\Delta x,\Delta t}w_0 - w_0)\,dx| \le \text{Const}\,\|\nabla\phi\|_\infty \Delta t$. Combining this with the strong Lipschitz continuity for the hyperbolic operator $\mathcal{S}_{\delta,\Delta x,\Delta t}$ we derive

$$\left|\int \phi(u_\eta(x,\ell\Delta t) - u_\eta(x,n\Delta t))\,dx\right| \le \text{Const}\left(\|\phi\|_\infty + \|\nabla\phi\|_\infty\right)|\ell - n|\,\Delta t. \tag{24}$$

Let now ω_h be a smooth mollifier with support in $[-h,h]^m$ and define ψ to be equal $\operatorname{sgn}(u_\eta^\ell - u_\eta^n)$ for $|x| \le r - h$ (r is some fixed positive number) and zero otherwise, and finally $\psi_h = \omega_h * \psi$. Standard approximating arguments (see, e.g., [**34**]) show that

$$\int_{[-r,r]^m} |u_\eta(x,\ell\Delta t) - u_\eta(x,n\Delta t)|\;dx \le C_1 h + C_2\,|\ell - n|\,\Delta t/h.$$

By choosing $h = \sqrt{|\ell - n|\,\Delta t}$, and finally letting $r \to \infty$ we conclude that

$$\int |u_\eta(x,\ell\Delta t) - u_\eta(x,n\Delta t)|\;dx \le \text{Const}\,\sqrt{|\ell - n|\,\Delta t}.$$

□

Using essentially the above lemma, Helly's theorem, and several diagonal arguments, we can prove that the sequence $\{u_\eta\}$ converges to a function u.

Lemma 3.2. *Fix $T > 0$. Let u_η denote the approximate solution defined by* (15). *Then for any sequences $\Delta t_i \to 0$ and $\Delta x_i \to 0$ with $\Delta x_i/\Delta t_i$ fixed, there exists a subsequence of $\{u_\eta\}$, denoted $\{u_{\eta_i}\}$, and a function u such that*

$$u_{\eta_i} \to u \text{ in } L^1_{loc}(\mathbb{R}^m \times [0,T]).$$

Moreover, the limit $u(\cdot,t)$ belongs to $BV(\mathbb{R}^m)$ uniformly in t and is uniformly L^1 Hölder continuous in time with exponent $1/2$.

Proof. The convergence proof is fairly standard, see, e.g., Smoller [**41**] or Karlsen and Risebro [**34**]. From Lemma 3.1 it follows that the sequence $\{u_\eta\}$ is uniformly bounded and has uniformly bounded total variation. Therefore, Helly's theorem ensures convergences of a subsequence $\{u_{\eta_i}\}$ in L^1 on bounded boxes $[-r,r]^m$ for each fixed t. Since r is arbitrary, the argument can be applied a countable number of times to form a further subsequence, still denoted by $\{u_{\eta_i}\}$, such that $\{u_{\eta_i}(\cdot,t)\}$ converges in $L^1_{loc}(\mathbb{R}^m)$ for each fixed t. Yet another diagonalization argument gives convergence for a dense countable subset $\{t_\ell\}$ in $[0,T]$. For $t \notin \{t_\ell\}$, there exists a sequence $\{t_k\} \subset \{t_\ell\}$ such that $t_k \to t$. By a triangle inequality

$$\begin{aligned}\int_{[-r,r]^m} |u_{\eta_i}(x,t) - u_{\eta_j}(x,t)|\,dx &\le \int_{[-r,r]^m} |u_{\eta_i}(x,t) - u_{\eta_i}(x,t_k)|\,dx \\ &\quad + \int_{[-r,r]^m} |u_{\eta_i}(x,t_k) - u_{\eta_j}(x,t_k)|\,dx + \int_{[-r,r]^m} |u_{\eta_j}(x,t_k) - u_{\eta_j}(x,t)|\,dx.\end{aligned}$$

The first and third term can be made arbitrary small by making k large and using the Hölder continuity in time (see Lemma 3.1). The second term can be made arbitrary small by making i,j large and using that $\{u_{\eta_i}(\cdot,t_k)\}$ is a Cauchy sequence for each t_k. Hence, $\{u_{\eta_i}(\cdot,t)\}$ is a Cauchy sequence in $L^1_{loc}(\mathbb{R}^m)$ for all $t \in [0,T]$. We denote the limit by u. Finally, in view of Lemma 3.1, it is clear that the limit u possesses the regularity claimed in the lemma. □

It remains to prove that u is an entropy solution, i.e., that it satisfies (4) and (5). We start with (4). However, as already mentioned, we will only manage to establish this estimate for the implicit diffusion solver (and, of course, the case of exact solution operators). In the next section we derive an upper bound on how far each approximate solution u_η is from satisfying (5).

LEMMA 3.3. *Suppose $\theta = 0$ in* (19). *Then the limit u constructed in Lemma 3.2 satisfies* (4).

PROOF. The proof is an adaption of the argument used in [**22**] (see also [**18**]). For simplicity of presentation, we only treat the two-dimensional equation

$$w_t = \varepsilon(K(x,y)A(w)_x)_x + \varepsilon(K(x,y)A(w)_y)_y \tag{25}$$

and the corresponding implicit difference scheme

$$\frac{W_{i,j}^{l+1} - W_{i,j}^{l}}{\tau} = \varepsilon D_-^x(K_{i+1/2,j}D_+^x A(W_{i,j}^{l+1})) + \varepsilon D_-^y(K_{i,j+1/2}D_+^y A(W_{i,j}^{l+1})), \tag{26}$$

where $W_{i,j}^l$ denotes the approximate solution of (25) at $(i\Delta x, j\Delta x, l\tau)$ and D_-^ℓ, D_+^ℓ are the backward and forward differences in direction $\ell = x, y$. Then extension of the proof to the more general problem (11) is straightforward and is left to the reader. Let us introduce the function

$$A_\eta(x,y,t) = \begin{cases} A(W_{i,j}^{l+1}) + D_+^x A(W_{i,j}^{l+1})(x - x_i) + D_+^y A(W_{i+1,j}^{l+1})(y - y_j), \\ \quad \text{for } (x,y) \in T_{i,j}^L,\ t \in (t^{n-1} + l\tau, t^{n-1} + (l+1)\tau], \\ A(W_{i,j}^{l+1}) + D_+^x A(W_{i,j+1}^{l+1})(x - x_i) + D_+^y A(W_{i,j}^{l+1})(y - y_j), \\ \quad \text{for } (x,y) \in T_{i,j}^U,\ t \in (t^{n-1} + l\tau, t^{n-1} + (l+1)\tau], \end{cases}$$

for some $i, j \in \mathbb{Z}$, $l = 0, \dots, N_\tau - 1$, and $n = 1, \dots, N$ with $N_\tau \tau = \Delta t$ and $N\Delta t = T$. Here $T_{i,j}^L$ denotes the triangle with vertices (x_i, y_j), (x_{i+1}, y_j), and (x_{i+1}, y_{j+1}), while $T_{i,j}^U$ denotes the triangle with vertices (x_i, y_j), (x_i, y_{j+1}), and (x_{i+1}, y_{j+1}). Let

$$R_{i,j} = [x_i, x_{i+1}] \times [y_i, y_{i+1}]$$

and note that $R_{i,j} = T_{i,j}^L \cup T_{i,j}^U$. For later use, observe that

$$\partial_x A_\eta \equiv D_+^x A(W_{i,j}^{l+1}) \text{ on } P_{i,j} = T_{i,j-1}^L \cup T_{i,j}^U,$$
$$\partial_y A_\eta \equiv D_+^y A(W_{i,j}^{l+1}) \text{ on } Q_{i,j} = T_{i-1,j}^L \cup T_{i,j}^U.$$

Observe next that, independently of η,

$$\|\sqrt{\varepsilon K} A_\eta\|_{L^2(Q_T)} \le \mathrm{Const}(T),$$

which is due to the fact that K, $\{W_i^l\}$, and $\sum_i |W_i^l|$ are uniformly bounded. We next claim that, independently of η,

$$\sqrt{\varepsilon K(x,y)}\,\partial_z A_\eta \in L^2(Q_T), \quad z = x, y. \tag{27}$$

To see this, multiply the difference equation (26) by $\Delta x^2 A(W_{i,j}^{l+1})$, sum over i, j, and then do summation by parts. The result reads

$$\Delta x^2 \sum_{i,j} \frac{W_{i,j}^{l+1} - W_{i,j}^{l}}{\tau} A(W_{i,j}^{l+1}) + \Delta x^2 \sum_{i,j} \varepsilon K_{i+1/2,j} \big(D_+^x A(W_{i,j}^{l+1})\big)^2 + \Delta x^2 \sum_{i,j} \varepsilon K_{i,j+1/2} \big(D_+^y A(W_{i,j}^{l+1})\big)^2 = 0.$$

Since $\mathcal{A}(s)=\int_0^s A(\xi)\,d\xi$ is convex, it follows that

$$\mathcal{A}(W_{i,j}^{l+1})-\mathcal{A}(W_{i,j}^{l})\le (W_{i,j}^{l+1}-W_{i,j}^{l})A(W_{i,j}^{l+1}).$$

We thus get

$$\Delta x^2\sum_{i,j}\frac{\mathcal{A}(W_{i,j}^{l+1})-\mathcal{A}(W_{i,j}^{l})}{\tau}+\Delta x^2\sum_{i,j}\varepsilon K_{i+1/2,j}\big(D_+^x A(W_{i,j}^{l+1})\big)^2 + \Delta x^2\sum_{i,j}\varepsilon K_{i,j+1/2}\big(D_+^y A(W_{i,j}^{l+1})\big)^2\le 0.$$

Multiplying this inequality by τ and summing over l, we get

$$\Delta x^2\sum_{i,j}\big(\mathcal{A}(W_{i,j}^{N_\tau})-\mathcal{A}(W_{i,j}^{0})\big)+\Delta x^2\tau\sum_{i,j}\sum_{l}\varepsilon K_{i+1/2,j}\big(D_+^x A(W_{i,j}^{l+1})\big)^2 + \Delta x^2\tau\sum_{i,j}\sum_{l}\varepsilon K_{i,j+1/2}\big(D_+^x A(W_{i,j}^{l+1})\big)^2\le 0. \tag{28}$$

Introduce the splitting solution

$$\tilde u_\eta(t)=\big[\mathcal{H}_{\Delta x,t-t^{n-1}}\circ\mathcal{S}_{\delta,\Delta x,\Delta t}\big]u_\eta^{n-1},\quad \text{for } t\in(t^{n-1},t^n],$$

where $n=1,\dots,N$ and $u_\eta^{n-1}=u_\eta(t^{n-1})$. Note that

$$\tilde u_\eta(x,y,t)=W_{i,j}^{l+1},\quad \text{for } (x,y)\in R_{i,j},\ t\in(t^{n-1}+l\tau,t^{n-1}+(l+1)\tau],$$

where $l=0,\dots,N_\tau-1$. Moreover, we have that,

$$\begin{cases}\tilde u_\eta(t^n)=u_\eta(t^n)\text{ for all } n=1,\dots,N,\\ \|\tilde u_\eta(t)-u_\eta(t)\|_1=\mathcal{O}(\sqrt{\Delta t})\text{ whenever } t\ne t^n \text{ for some } n=1,\dots,N,\end{cases}$$

uniformly in $t\in[0,T]$, so that $\tilde u_\eta\to u$, with u being the limit constructed in Lemma 3.2. We can thus replace (28) by

$$\int\big(\mathcal{A}(\tilde u_\eta)\big|_{t=t^n}-\mathcal{A}(\tilde u_\eta)\big|_{t=t^{n-1}+}\big)\,dx\,dy+\iint_{t^{n-1}}^{t^n}\varepsilon K(x,y)\big(\partial_x A_\eta\big)^2\,dt\,dx\,dy + \iint_{t^{n-1}}^{t^n}\varepsilon K(x,y)\big(\partial_y A_\eta\big)^2\,dt\,dx\,dy=\mathcal{O}(\Delta x).$$

Observe that

$$\Big|\int\big(\mathcal{A}(\tilde u_\eta)\big|_{t=t^{n-1}+}-\mathcal{A}(\tilde u_\eta)\big|_{t=t^{n-1}}\big)\,dx\,dy\Big|=\mathcal{O}(\Delta x)+\mathcal{O}(\Delta t),$$

which yields

$$\int\big(\mathcal{A}(\tilde u_\eta)\big|_{t=t^n}-\mathcal{A}(\tilde u_\eta)\big|_{t=t^{n-1}}\big)\,dx\,dy+\iint_{t^{n-1}}^{t^n}\varepsilon K(x,y)\big(\partial_x A_\eta\big)^2\,dt\,dx\,dy + \iint_{t^{n-1}}^{t^n}\varepsilon K(x,y)\big(\partial_y A_\eta\big)^2\,dt\,dx\,dy=\mathcal{O}(\Delta x)+\mathcal{O}(\Delta t).$$

Summing over n, we get

$$\int\big(\mathcal{A}(\tilde u_\eta)\big|_{t=T}-\mathcal{A}(\tilde u_\eta)\big|_{t=0}\big)\,dx\,dy+\iint_0^T\varepsilon K(x,y)\big(\partial_x A_\eta\big)^2\,dt\,dx\,dy + \iint_0^T\varepsilon K(x,y)\big(\partial_y A_\eta\big)^2\,dt\,dx\,dy=\mathcal{O}(T).$$

Since, independently of η, $\mathcal{A}(\tilde{u}_\eta)\big|_{t=0,T} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, (27) follows. Therefore, passing if necessary to a subsequence,

$$A_\eta \to \overline{A} \text{ weakly in } H^1_{loc}(Q_T).$$

Since obviously

$$\iint_0^T \left|A_\eta(x,y,t) - A(\tilde{u}_\eta(x,y,t))\right| dt\, dx = \mathcal{O}(\Delta x)$$

and, thanks to Lemma 3.2, $A(\tilde{u}_\eta) \to A(u)$ a.e., we conclude that $\overline{A} = A(u)$ a.e., and thus (27) holds. This concludes the proof of the lemma. □

4. Entropy Estimates

In this section we derive a precise entropy estimate for u_η, i.e., an estimate saying how far the approximate solution u_η is from fulfilling the entropy condition (5). However, to accomplish this we need to introduce a new time interpolant $\hat{u}_\eta$ and first establish an entropy estimate for $\hat{u}_\eta$. To this end, we define intermediate solutions

$$u_\eta^{n,j} = \pi \circ \mathcal{S}^j_{\delta,\Delta x,\Delta t} \circ \cdots \circ \pi \circ \mathcal{S}^1_{\delta,\Delta x,\Delta t} u_\eta^{n-1}, \qquad u_\eta^{n,0} = u_\eta^{n-1} = u_\eta(t^{n-1}),$$

for $j = 1,\dots,m$, and $n = 1,\dots,N$. Next we split each time interval $[t^{n-1},t^n]$ into subintervals $[t^n_j, t^n_{j+1}]$, $j = 0,\dots,m$, where

$$t^n_j = \Big((n-1) + \frac{j}{m+1}\Big)\Delta t, \quad j = 0,\dots,m+1,$$

so that $t^n_0 = t^{n-1}$ and $t^n_{m+1} = t^n$. We then define (see [**34, 23**])

$$\hat{u}_\eta(t) = \begin{cases} \mathcal{S}^j_{\delta,\Delta x,(m+1)(t-t^n_{j-1})}\, u_\eta^{n,j-1}, & t \in (t^n_{j-1}, t^n_j), \\ u_\eta^{n,j}, & t = t^n_j, \\ \mathcal{H}_{\Delta x,(m+1)(t-t^n_m)}\, u_\eta^{n,m} & t \in (t^n_m, t^n_{m+1}], \end{cases} \tag{29}$$

for $j = 1,\dots,m$ and $n = 1,\dots,N$. Observe that the sequence $\{\hat{u}_\eta\}$ converges to the limit function u constructed in Lemma 3.2, since obviously

$$\begin{cases} \hat{u}_\eta(t^n) = u_\eta(t^n) \text{ for all } n = 1,\dots,N, \\ \|\hat{u}_\eta(t) - u_\eta(t)\|_1 = \mathcal{O}(\sqrt{\Delta t}) \text{ whenever } t \neq t^n \text{ for some } n = 1,\dots,N. \end{cases} \tag{30}$$

4.1. Hyperbolic step. Since each one-dimensional hyperbolic step in (14) is the exact solution of a perturbed version of (13), the corresponding solution operator satisfies the entropy inequality

$$\mathcal{L}(v_{\Delta x,\delta}, \varphi, k; f^\delta, V^{\Delta x}, 0, v_{0,\Delta x}, T) \geq 0 \text{ for } m = 1,$$

where $v_{\Delta x,\delta}$ denotes the front tracking solution of (17). For sufficiently smooth V, f and initial data v_0 of bounded variation, a straightforward calculation gives (see, e.g., [**38, 23, 33**])

$$\begin{aligned} &\left|\mathcal{L}(v_{\Delta x,\delta}, \phi, k; f, V, 0, v_0, T) - \mathcal{L}(v_{\Delta x,\delta}, \phi, k; f^\delta, V^{\Delta x}, 0, v_{0,\Delta x}, T)\right| \\ &\qquad \leq \operatorname{Const} T(\Delta x + \delta^2), \end{aligned}$$

In particular, it follows from this that the one-dimensional front tracking solution $v_{\Delta x,\delta}$ satisfies

$$
\begin{aligned}
&\iint_{\mathbb{R}}\int_0^{\Delta t} \Big\{ |v_{\Delta x,\delta} - k|\phi_t + F(v_{\Delta x,\delta}, k)\big(\phi V_j(x,t)\big)_x \Big\}\, dt dx \\
&\geq \int |v_{\Delta x,\delta}(x,\Delta t) - k|\phi(x,\Delta t)\, dx - \int |v_{\Delta x,\delta}(x,0) - k|\phi(x,0)\, dx \\
&\quad - \mathrm{Const}\, \Delta t(\Delta x + \delta^2),
\end{aligned}
\tag{31}
$$

for all test functions $\phi \geq 0$. In (31), $v_{\Delta x,\delta}(x,0)$ coincides, of course, with the initial function $v_{0,\Delta x}$. Exploiting the particular structure of $\hat{u}_\eta$ together with a suitable change of (time) variable, it is fairly easy to translate (31) into (see [**34, 23**] for details)

$$
\begin{aligned}
&\int_{\mathbb{R}^m}\int_{t_{j-1}^n}^{t_j^n} \Big\{ |\hat{u}_\eta - k|\phi_t + (m+1)F_j(\hat{u}_\eta, k)\big(\phi V_j(x,t)\big)_{x_j} \Big\}\, dt dx \\
&\quad \geq \int |\hat{u}_\eta(t_j^n -) - k|\phi(x,t_j^n)\, dx - \int |\hat{u}_\eta(t_{j-1}^n) - k|\phi(x,t_{j-1}^n)\, dx \\
&\qquad\qquad - \mathrm{Const}\, \Delta t(\Delta x + \delta^2),
\end{aligned}
\tag{32}
$$

for $j = 1,\dots,m$ and $n = 1,\dots,N$. One should notice the factor $m+1$ coming from the particular time scaling in (29). In (32), $\hat{u}_\eta(x,t_j^n -)$ denotes the function before the operator π is applied.

4.2. Parabolic step. We now derive a corresponding entropy inequality for the (degenerate) parabolic step in (14). This will be established by first proving a discrete cell entropy inequality for the difference scheme (19). Similar (discrete) entropy inequalities have been derived in Evje and Karlsen [**23, 19, 21, 18**].

We derive the cell entropy inequality only for $m = 1$ (with $K_1 = K$, $A_1 = A$, and $\alpha = i \in \mathbb{Z}$) to avoid complicated notation. Let

$$
S_k(u) = S(u,k) = |u - k|, \qquad Q_k(u) = Q(u,k) = |A(u) - A(k)|,
$$

define the discrete difference operator

$$
D_0\, W_i^n = \frac{W_{i+1/2}^n - W_{i-1/2}^n}{\Delta x},
$$

and let $a \vee k = \max(a,k)$ and $a \wedge k = \min(a,k)$.

We now set out to derive the following cell entropy condition

$$
S_k(W_i^{n+1}) - S_k(W_i^n) - \varepsilon\tau D_0\Big[K_i D_0\Big(\theta Q_k(W_i^n) + (1-\theta)Q_k(W_i^{n+1})\Big)\Big] \leq 0.
\tag{33}
$$

Using the scheme and the fact that A is an increasing function, we can write (here A_i^n denotes $A(W_i^n)$, etc.)

$$
\begin{aligned}
S_k(W_i^n) + \varepsilon\tau\theta D_0\big(K_i D_0 Q_k(W_i^n)\big) &= W_i^n \vee k - W_i^n \wedge k \\
&+ \theta\beta_i\Big[\big(A_{i+1}^n \vee A(k) - A_{i+1}^n \wedge A(k)\big) - \big(A_i^n \vee A(k) - A_i^n \wedge A(k)\big)\Big] \\
&+ \theta\beta_{i-1}\Big[\big(A_i^n \vee A(k) - A_i^n \wedge A(k)\big) - \big(A_{i-1}^n \vee A(k) - A_{i-1}^n \wedge A(k)\big)\Big] \\
=&H_i\big(W_{i+1}^n \vee k, W_i^n \vee k, W_{i-1}^n \vee k\big) - H_i\big(W_{i+1}^n \wedge k, W_i^n \wedge k, W_{i-1}^n \wedge k\big),
\end{aligned}
$$

where we have introduced the function

$$H_i(u,v,w) = v + \theta\beta_i\big(A(u) - A(v)\big) - \theta\beta_{i-1}\big(A(v) - A(w)\big).$$

The function H_i is increasing in all three arguments due to the stability condition (22), so that

$$\begin{aligned}
&S_k(W_i^n) + \varepsilon\tau\theta D_0\big(K_i D_0 Q_k(W_i^n)\big)\\
&\geq H_i\big(W_{i+1}^n, W_i^n, W_{i-1}^n\big) \vee H_i(k,k,k) - H_i\big(W_{i+1}^n, W_i^n, W_{i-1}^n\big) \wedge H_i(k,k,k)\\
&= \big|H_i\big(W_{i+1}^n, W_i^n, W_{i-1}^n\big) - H_i(k,k,k)\big|\\
&= \big|W_i^{n+1} - k - (1-\theta)\varepsilon\tau D_0\big(K_i D_0 Q_k(W_i^{n+1})\big)\big|,
\end{aligned} \tag{34}$$

where we have used that $H_i(k,k,k) = k$. Since A is a nondecreasing function, a case analysis will reveal that

$$|A(v) - A(k)| - |A(w) - A(k)| \leq \operatorname{sgn}(v-k)\big(A(v) - A(w)\big).$$

Now, we estimate as follows

$$\begin{aligned}
S_k&(W_i^{n+1}) - \varepsilon\tau(1-\theta)D_0\big(K_i D_0 Q_k(W_i^{n+1})\big)\\
&= W_i^{n+1} \vee k - W_i^{n+1} \wedge k + (1-\theta)\beta_i\Big[|A_i^{n+1} - A(k)| - |A_{i+1}^{n+1} - A(k)|\Big]\\
&\quad + (1-\theta)\beta_{i-1}\Big[|A_i^{n+1} - A(k)| - |A_{i-1}^{n+1} - A(k)|\Big]\\
&\leq \operatorname{sgn}(W_i^{n+1} - k)(W_i^{n+1} - k)\\
&\quad + (1-\theta)\operatorname{sgn}(W_i^{n+1} - k)\big(\beta_i\big(A_i^{n+1} - A_{i+1}^{n+1}\big) + \beta_{i-1}\big(A_i^{n+1} - A_{i-1}^{n+1}\big)\big)\\
&\leq \big|W_i^{n+1} - k - (1-\theta)\varepsilon\tau D_0\big(K_i D_0 Q_k(W_i^{n+1})\big)\big|.
\end{aligned}$$

Combining this inequality with (34), we arrive at the cell entropy condition (33). To derive a global entropy condition, let $\phi \geq 0$ be a test function, multiply (33) with $\tau\Delta x\phi_i^n$ ($\phi_i^n = \phi(i\Delta x, n\tau)$) and sum over all $i \in \mathbb{Z}$ and $n = 0, \dots, N_\tau - 1$, where $N_\tau\tau = \Delta t$. Then, using summation by parts twice in space, we get

$$\begin{aligned}
0 &\geq \sum_i \sum_{n=0}^{N_\tau - 1}\Big(S_k(W_i^{n+1}) - S_k(W_i^n)\Big)\phi_i^n\Delta x\\
&\quad - \tau\varepsilon\sum_i\sum_{n=0}^{N_\tau-1}\Big[D_0\Big(K_i D_0\big[\theta Q_k(W_i^n) + (1-\theta)Q_k(W_i^{n+1})\big]\Big)\Big]\phi_i^n\Delta x\\
&= \sum_i S_k(W_i^{N_\tau})\phi_i^{N_\tau - 1}\Delta x - \sum_i S_k(W_i^0)\phi_i^0\Delta x\\
&\quad - \sum_i\sum_{n=0}^{N_\tau-1}\Big[S_k(W_i^{n+1})\frac{\phi_i^{n+1} - \phi_i^n}{\tau}\Big]\tau\Delta x\\
&\quad + \varepsilon\sum_i\sum_{n=0}^{N_\tau-1}\Big[\big(\theta Q_k(W_i^n) + (1-\theta)Q_k(W_i^{n+1})\big)D_0(K_i D_0\phi_i^n)\Big]\tau\Delta x.
\end{aligned}$$

Using the L^1 Hölder continuity in time of $\{W_i^n\}$ (see Lemma 3.1 and its proof), we can write

$$\begin{aligned}\sum_i \sum_{n=0}^{N_\tau-1} &\Big(\theta Q_k(W_i^n) + (1-\theta)Q_k(W_i^{n+1})\Big)\tau\Delta x \\ &= \sum_i \sum_{n=0}^{N_\tau-1} Q_k(W_i^{n+1})\tau\Delta x + \mathcal{O}\Big(\Delta t\sqrt{\tau}\Big),\end{aligned}$$

so that we end up with

$$\begin{aligned}\sum_i \sum_{n=0}^{N_\tau-1} &\Big[S_k(W_i^{n+1})\frac{\phi_i^{n+1}-\phi_i^n}{\tau} + \varepsilon Q_k(W_i^{n+1})D_0(K_i D_0\phi_i^n)\Big]\tau\Delta x \\ &\geq \sum_i S_k(W_i^{N_\tau})\phi_i^{N_\tau}\Delta x - \sum_i S_k(W_i^0)\phi_i^0\Delta x - \text{Const}\,\Delta t\sqrt{\tau}.\end{aligned} \tag{35}$$

Letting $w(t) = \mathcal{H}_{\Delta x,t}w_0$ denote the finite difference solution (as defined in (21)), we can obviously replace the discrete inequality (35) by its continuous counterpart

$$\begin{aligned}&\int_{\mathbb{R}^m}\int_0^{\Delta t} \Big\{|w-k|\phi_t + \varepsilon Q_k(w)\big(K(x)\phi_x\big)_x\Big\}\,dtdx \\ &\geq \int |w(x,\Delta t)-k|\phi(x,\Delta t)\,dx - \int |w(x,0)-k|\phi(x,0)\,dx \\ &\qquad - \text{Const}\,\Delta t(\sqrt{\Delta t}+\Delta x),\end{aligned}$$

where $w(x,0)$ coincides with w_0. A similar inequality can be obtained (with the same argument) in the multi-dimensional case. Similar to the hyperbolic step, we can translate this multi-dimensional inequality into the following entropy inequality for the degenerate parabolic step (see [**34, 23**] for further details)

$$\begin{aligned}&\int_{\mathbb{R}^m}\int_{t_m^n}^{t_{m+1}^n} \Big\{|\hat{u}_\eta-k|\phi_t + (m+1)\varepsilon\sum_{j=1}^m |Q_j(\hat{u}_\eta,k)|\big(K_j(x)\phi_{x_j}\big)_{x_j}\Big\}\,dtdx \\ &\geq \int |\hat{u}_\eta(t_{m+1}^n)-k|\phi(x,t_{m+1}^n)\,dx - \int |\hat{u}_\eta(t_m^n)-k|\phi(x,t_m^n)\,dx \\ &\qquad - \text{Const}\,\Delta t(\sqrt{\Delta t}+\Delta x).\end{aligned} \tag{36}$$

4.3. The splitting method. For $j=1,\dots,m+1$ and $n=1,\dots,N$, let $\chi_j^n(t)$ be the characteristic function of the subinterval $[t_{j-1}^n, t_j^n]$. To derive an entropy estimate for the splitting method, we add (32) and (36), sum over $j=1,\dots,m$ and $n=1,\dots,N$ and rearrange terms, yielding

$$\mathcal{L}(\hat{u}_\eta,\phi,k;f,V,A,u_0,T) \geq E^{\mathcal{S}} + E^{\pi} + E^{\mathcal{H}} - \text{Const}\,T\big(\sqrt{\Delta t}+\Delta x+\delta\big), \tag{37}$$

where

$$
\begin{aligned}
E^{\mathcal{S}} &= \sum_{j=1}^{m} \iint_{Q_T} \Big[1-(m+1)\sum_{n=1}^{N}\chi_j^n(t)\Big] F_j(\hat{u}_\eta,k)\partial_{x_j}\big(\phi V_j(x,t)\big)\,dt\,dx,\\
E^{\pi} &= \sum_{n=1}^{N}\sum_{j=1}^{m}\int_{\mathbb{R}^m}\Big(S_k(\hat{u}_\eta(x,t_j^n)) - S_k(\pi\hat{u}_\eta(x,t_j^n))\Big)\phi(x,t_j^n)\,dx,\\
E^{\mathcal{H}} &= \varepsilon\iint_{Q_T}\Big[1-(m+1)\sum_{n=1}^{N}\chi_{m+1}^n(t)\Big]\sum_{j=1}^{m} Q_j(\hat{u}_\eta,k)\partial_{x_j}\big(K_j\partial_{x_j}\phi\big)\,dt\,dx.
\end{aligned}
$$

The three terms can be identified with the following sources for entropy production: $E^{\mathcal{S}}$ comes from the hyperbolic steps, E^{π} from the projections in the hyperbolic steps, and $E^{\mathcal{H}}$ from the (degenerate) parabolic steps. We shall next bound the three terms $E^{\mathcal{S}}$, E^{π}, and $E^{\mathcal{H}}$, starting with $E^{\mathcal{H}}$. By first splitting the integration over $[0,T]$ into a sum over intervals $[t^{n-1},t^n]$, we see that the sum over n can be taken outside the double integral. Then writing

$$
Q_j(\hat{u}_\eta(t),k) = Q_j(\hat{u}_\eta(t^{n-1}),k) + \Big[Q_j(\hat{u}_\eta(t),k) - Q_j(\hat{u}_\eta(t^{n-1}),k)\Big],
$$

$E^{\mathcal{H}}$ becomes $E_1^{\mathcal{H}} + E_2^{\mathcal{H}}$, where

$$
\begin{aligned}
E_1^{\mathcal{H}} &= \varepsilon\sum_{j,n}\int_{\mathbb{R}^m}\int_{t^{n-1}}^{t^n}\Big[1-(m+1)\chi_{m+1}^n(t)\Big] Q_j(\hat{u}_\eta(t^{n-1}),k)\partial_{x_j}\big(K_j\partial_{x_j}\phi\big)\,dt\,dx,\\
E_2^{\mathcal{H}} &= \varepsilon\sum_{j,n}\int_{\mathbb{R}^m}\int_{t^{n-1}}^{t^n}\Big[1-(m+1)\chi_{m+1}^n(t)\Big]\\
&\qquad\qquad\times\Big[Q_j(\hat{u}_\eta(t),k) - Q_j(\hat{u}_\eta(t^{n-1}),k)\Big]\partial_{x_j}\big(K_j\partial_{x_j}\phi\big)\,dt\,dx.
\end{aligned}
$$

In $E_1^{\mathcal{H}}$ we can use the smoothness of $\phi(\,\cdot\,,t)$ to write

$$
\big(\partial_{x_j}(K_j\partial_{x_j}\phi)\big)\big|_{(x,t)} = \big(\partial_{x_j}(K_j\partial_{x_j}\phi)\big)\big|_{(x,t^{n-1})} + C_{j,n}(x,t)(t-t^{n-1}),
$$

for $t\in(t^{n-1},t^n)$ and some suitable $C_{j,n}(x,t)$ uniformly bounded (in j,n,x,t,η) by a constant $C>0$. Inserting this into $E_1^{\mathcal{H}}$ gives (where $\Omega=\operatorname{supp}(\phi)$)

$$
\begin{aligned}
|E_1^{\mathcal{H}}| \le \varepsilon\sum_{j,n}\int_\Omega\bigg(&\int_{t^{n-1}}^{t^n} Q_j(\hat{u}_\eta(t^{n-1}))\,\big(\partial_{x_j}(K_j\partial_{x_j}\phi)\big)\big|_{(x,t^{n-1})}\big[1-(m+1)\chi_{m+1}^n(t)\big]\,dt\\
&+\int_{t^{n-1}}^{t^n} C(t-t^{n-1})\Big[1-(m+1)\chi_{m+1}^n(t)\Big]\,dt\bigg)\,dx\\
\equiv \sum_{j,n}\int_\Omega\int_{t^{n-1}}^{t^n} &C(t-t^{n-1})\Big[1-(m+1)\chi_{m+1}^n(t)\Big]\,dt\,dx = \mathcal{O}(\Delta t).
\end{aligned}
$$

For the second term, we have

$$
\begin{aligned}
|E_2^{\mathcal{H}}| \leq & \sup_{|u|\leq\|u_0\|_\infty} |A'(u)|\, m\varepsilon \sum_{j,n} \sup_{Q_T} \big|\partial_{x_j}(K_j \partial_{x_j}\phi)\big| \\
& \times \int_{\mathbb{R}^m}\int_{t^{n-1}}^{t^n} |\hat{u}_\eta(x,t) - \hat{u}_\eta(x,t^{n-1})|\, dtdx \\
\leq & \mathrm{Const} \sum_{j,n} \int_0^{\Delta t} \sqrt{t}\, dt = \mathrm{Const} \sum_n \Delta t^{3/2} = \mathcal{O}(\sqrt{\Delta t}).
\end{aligned}
$$

For $E^{\mathcal{S}}$ we use a similar argument (to obtain a similar estimate), so that

$$
E^{\mathcal{S}} + E^{\mathcal{H}} \geq -\mathrm{Const}\, T\sqrt{\Delta t}.
$$

Now it remains to determine the entropy production due to the projection operator π. For simplicity we present the argument for $m = 1$; recall that each hyperbolic operator is one-dimensional, see also [**23, 33**]. Introduce the grid boundaries $x_i = i\Delta x$, let $v = v(x)$ be some function of bounded variation and let $\phi = \phi(x) \geq 0$ be a test function. Then

$$
\begin{aligned}
\int_{\mathbb{R}} \Big(S_k(v) - S_k(\pi v)\Big)\phi(x)\, dx = & I_1 + I_2 \\
:= & \sum_i \int_{x_i}^{x_{i+1}} \Big(S_k(v) - S_k(\pi v)\Big)\phi(x_i)\, dx \\
& + \sum_i \int_{x_i}^{x_{i+1}} \Big(S_k(v) - S_k(\pi v)\Big)\Big(\phi(x) - \phi(x_i)\Big)\, dx.
\end{aligned}
$$

The second term I_2 is straightforward estimated as follows:

$$
\begin{aligned}
|I_2| \leq & |\phi'|\Delta x \sum_i \int_{x_i}^{x_{i+1}} |v - \pi v|\, dx \\
\leq & |\phi'|\Delta x \sum_i \int_{x_i}^{x_{i+1}} \frac{1}{\Delta x}\Big| \int_{x_i}^{x_{i+1}} v(y) - v(x)\, dy\Big|\, dx \leq \mathrm{Const}\, \Delta x\, TV(v).
\end{aligned}
$$

To estimate the first term, let us first assume that $S_k(u)$ is a C^2 approximation of $|u-k|$. We can then obviously write

$$
S_k(v) = S_k(\pi v) + S_k'(\pi v)(v - \pi v) + C(x)(v-\pi v)^2 \text{ for some } C(x) \geq 0,
$$

and therefore

$$
I_1 = \sum_i \int_{x_i}^{x_{i+1}} \Big[S_k'(\pi v)(v - \pi v) + C(x)(v - \pi v)^2\Big]\, dx.
$$

The first term integrates to zero and the second is always positive, so that $I_1 \geq 0$. This latter inequality can also be made rigorous for $S_k(u) = |u-k|$ by suitably approximating $|u-k|$ with C^2 functions, see [**23**]. Hence, $I_1 + I_2 \geq -C\Delta x$. From this we conclude that also

$$
E^{\pi} \geq -\mathrm{Const}\, \Delta x.
$$

Summing up, we have derived the following entropy estimate for $\hat{u}_\eta$.

$$
\mathcal{L}(\hat{u}_\eta, \phi, k; f, V, A, u_0, T) \geq -\mathrm{Const}\, T\big(\sqrt{\Delta t} + \Delta x + \delta^2\big).
$$

In view of (30), this estimate translates into the following similar entropy estimate for u_η:

LEMMA 4.1. *The entropy production of the splitting method* (15) *is of order* $\mathcal{O}(\sqrt{\Delta t} + \Delta x + \delta)$, *or more precisely*

$$\mathcal{L}(u_\eta, \phi, k; f, V, A, u_0, T) \geq -\mathrm{Const}\, T\left(\sqrt{\Delta t} + \Delta x + \delta^2\right), \tag{38}$$

for some constant $\mathrm{C} > 0$ *independent of the discretization parameters* $\eta = (\Delta x, \Delta t, \delta)$.

From this lemma we can conclude that if the approximating sequence $\{u_\eta\}$ generated by (15) converges, then it does so to an entropy solution of (1).

5. Summary

In this paper we have presented operator splitting methods for degenerate convection-diffusion equations and shown that these approximations converge to entropy weak solutions. In a companion paper, [**27**], we apply these methods to examples from the simulation of two-phase flow in porous media and certain models for sedimentation-consolidation processes. We demonstrate that although the operator splittings may have certain potential pitfalls, such as splitting errors, lack of mass conservation and grid orientation effects, they perform very well on most cases and deliver more than the standard resolution with surprisingly high efficiency.

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