Global weak solutions to the Navier-Stokes equations for a 1D viscous polytropic ideal gas

Song Jiang *
Institute of Applied Physics and Computational Mathematics
P.O. Box 8009, Beijing 100088, China
E-Mail: jiang@mail.iapcm.ac.cn

Ping Zhang
Institute of Mathematics, Academia Sinica, Beijing 100080, China
E-Mail: zp@math03.math.ac.cn

Abstract

We prove the existence of global weak solutions to the Navier-Stokes equations for a one-dimensional viscous polytropic ideal gas. We require only that the initial density is in $L^\infty \cap L^2_{loc}$ with positive infimum, the initial velocity is in L^2 and the initial temperature is in a larger space than L^2 with positive infimum. The initial density and the initial velocity may have differing constant states at $x = \pm\infty$. In particular, piesewise constant data with arbitrary large jump discontinuities are included. Our results show that neither vacuum states nor concentration states can form and the temperature remains positive in finite time.

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1 Introduction

We study the global existence of weak solutions to the Cauchy problem for the Navier–Stokes equations for a one-dimensional viscous polytropic ideal gas:

$$\rho_t + (\rho v)_x = 0, \tag{1.1}$$

$$(\rho v)_t + (\rho v^2)_x + P_x = \mu v_{xx}, \tag{1.2}$$

$$\left[\rho\left(c_V\theta + \frac{v^2}{2}\right)\right]_t + \left[v\left(\rho\frac{v^2}{2} + c_V\rho\theta + P\right)\right]_x = \kappa\theta_{xx} + \mu(vv_x)_x \tag{1.3}$$

with initial conditions

$$(\rho(x,0), v(x,0), \theta(x,0)) = (\rho_0(x), v_0(x), \theta_0(x)), \qquad x \in \mathbb{R}. \tag{1.4}$$

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Here ρ, v and θ denote the density, the velocity and the absolute temperature, respectively, $P = P(\rho, \theta)$ is the pressure having the form (ideal gases)

$$P = R\rho\theta, \tag{1.5}$$

and μ, c_V, κ, R are positive constants, and at infinity the initial data ρ_0, v_0, θ_0 satisfy

$$\lim_{x\to\pm\infty} (\rho_0(x), v_0(x), \theta_0(x)) = (\rho^\pm, v^\pm, \overline{\theta}) \tag{1.6}$$

where, $\rho^\pm, \overline{\theta}$ are positive constants and $v^\pm$ are constants satisfying $v^- \le v^+$.

To describe the assumptions on the initial data, we define smooth functions $0 < \overline{\rho}(x)$, $\overline{v}(x) \in C^\infty(\mathbb{R})$ satisfying

$$\overline{\rho}(x) = \begin{cases} \rho^+, & x \ge 1, \\ \rho^-, & x \le -1, \end{cases} \qquad \overline{v}(x) = \begin{cases} v^+, & x \ge 1, \\ v^-, & x \le -1. \end{cases} \tag{1.7}$$

Then our hypotheses upon the initial data are that

$$\begin{cases} \rho_0 \in L^\infty(\mathbb{R}), \ \inf_{\mathbb{R}} \rho_0(\cdot) > 0, \ \rho_0 - \overline{\rho}, \ v_0 - \overline{v} \in L^2(\mathbb{R}), \\ \dfrac{\theta_0}{\overline{\theta}} - \log \dfrac{\theta_0}{\overline{\theta}} - 1 \in L^1(\mathbb{R}), \ \inf_{\mathbb{R}} \theta_0(\cdot) > 0. \end{cases} \tag{1.8}$$

Since the first work of Kazhikhov and Shelukhin [13] on the global existence in the dynamics of a one-dimensional compressible viscous heat-conducting fluid for large initial data, significant progress has been made on the mathematical aspect of the initial and initial boundary value problems for (1.1)–(1.3). For initial data in H^1, the existence and uniqueness of global generalized (strong) solutions to the Cauchy problem and to different initial boundary value problems have been known. Moreover, the global solution is asymptotically stable as time tends to infinity; see for example [12, 1, 15, 16, 17, 5, 9, 10] and among others, where $\rho^+ = \rho^-$ and $v^\pm = 0$ are assumed. In [8] global generalized solutions to the Cauchy problem was studied for the initial data in H^1 where differing end states $\rho^\pm, v^\pm$ with $v^- \le v^+$ are allowed.

Concerning weak solutions, Serre in 1986 [19] proved the global existence of weak solutions to the Cauchy problem under the conditions that $\rho_0 \in BV$, $\rho^+ = \rho^-$ and $v^\pm = 0$. For the initial data satisfying $(\rho_0, v_0, \theta_0) \in L^\infty \times L^2 \times L^1$ and $\inf \rho_0, \inf \theta_0 > 0$, Amosov and Zlotnik [2, 3, 20] obtained global weak solutions of initial boundary value problems in bounded intervals, also see [5] for non-negative initial density and [11, 6] for small initial data. Recently, Hoff [7] studied the Navier-Stokes equations of compressible isentropic flow and proved the existence of global weak solutions for $\rho_0 \in L^\infty$ with positive infimum, $v_0 \in L^2$ with differing end states. To our best knowledge, the problem concerning the global existence of weak solutions to the Cauchy problem (1.1)–(1.4) under (1.6) and (1.8) still remains open.

Our aim in this paper is to prove a global existence theorem on weak solutions to (1.1)–(1.4). In particular, in our result piecewise constant data with arbitrary large jump discontinuities are allowed. An important physical consequence of our result is that neither vacuum states ($\rho = 0$) nor concentration states ($\rho = \infty$) can occur in the solution and the temperature remains positive, no matter how large the oscillation of initial data.

Before stating the main result of the present paper we introduce the notation used throughout this paper: Let s be a real number and let $1 \le p \le \infty$. By $W^{s,p}$ we denote the usual Sobolev space defined over $\mathbb{R}$ with norm $\|\cdot\|_{W^{s,p}}$; $H^s \equiv W^{s,2}$ with norm $\|\cdot\|_{H^s}$, $L^p \equiv W^{0,p}$ with norm $\|\cdot\|_{L^p}$. $W^{s,p}_{loc}$, H^s_{loc} and L^p_{loc} denote $W^{s,p}_{loc}(\mathbb{R})$, $H^s_{loc}(\mathbb{R})$ and $L^p_{loc}(\mathbb{R})$, respectively. $L^p(I,B)$ resp. $\|\cdot\|_{L^p(I,B)}$ denotes the space of all strongly measurable, pth-power integrable (essentially bounded if $p=\infty$) functions from I to B resp. its norm, $I \subset \mathbb{R}$ an interval, B a Banach space. We also use $\mathbb{R}_+ := [0,\infty)$. The same letter C (sometimes used as $C(X,\cdots)$ to emphasize the dependence of C on $X,\cdots$) will denote various positive constants which may depend on T but not on ϵ.

Now we introduce the definition of weak solutions.

Definition. *We say that $(\rho(x,t), v(x,t), \theta(x,t))$ is a global weak solution of (1.1)–(1.4), if for any $T>0$, $\rho(x,t) \ge 0$ and $\theta(x,t) > 0$ on $[0,T]\times\mathbb{R}$, and*

$$\rho \in L^\infty_{loc}([0,\infty), L^\infty), \quad \rho - \overline{\rho},\ v - \overline{v} \in L^\infty_{loc}([0,\infty), L^2), \quad v_x \in L^{4/3}_{loc}([0,\infty)\times\mathbb{R}),$$
$$\frac{\theta}{\overline{\theta}} - \log\frac{\theta}{\overline{\theta}} - 1 \in L^\infty_{loc}([0,\infty), L^1), \quad \theta \in L^\infty_{loc}([0,\infty), L^1_{loc}), \quad \theta_x \in L^1_{loc}([0,\infty)\times\mathbb{R}),$$

and the following equations hold:

$$\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\mathbb{R}} \{\rho\phi_t + \rho v\phi_x\}\, dxdt = \int_{\mathbb{R}} \rho_0\phi(x,0)dx,\\
&\int_{\mathbb{R}_+}\int_{\mathbb{R}} \{\rho v\phi_t + \rho v^2\phi_x + P\phi_x - \mu v_x\phi_x\}\, dxdt = \int_{\mathbb{R}} \rho_0 v_0\phi(x,0)dx,\\
&\int_{\mathbb{R}_+}\int_{\mathbb{R}} \left\{\rho\left(c_V\theta + \frac{v^2}{2}\right)\phi_t + v\left(\rho\frac{v^2}{2} + c_V\rho\theta + P\right)\phi_x - \kappa\theta_x\phi_x + \mu\frac{v^2}{2}\phi_{xx}\right\} dxdt\\
&\qquad = \int_{\mathbb{R}} \rho_0\left(c_V\theta_0 + \frac{v_0}{2}\right)\phi(x,0)dx
\end{aligned} \tag{1.9}$$

for any $\phi \in C_0^\infty([0,\infty)\times\mathbb{R})$.

Thus, the main result of this paper reads:

Theorem 1.1 *Assume that ρ_0, v_0, θ_0 satisfy (1.8). Then there exists a global weak solution (ρ, v, θ) of (1.1)–(1.4), such that for any $T>0$,*

$$\begin{aligned}
&C_1 \le \rho(x,t) \le C_2, \quad C_1 \le \theta(x,t) \qquad \text{for a.e. } x\in\mathbb{R},\ t\in[0,T],\\
&\int_{\mathbb{R}} \left\{\left(\frac{\overline{\rho}}{\rho} - \log\frac{\overline{\rho}}{\rho} - 1\right) + \left(\frac{\theta}{\overline{\theta}} - \log\frac{\theta}{\overline{\theta}} - 1\right) + (v-\overline{v})^2\right\}(x,t)dx\\
&\qquad \le C \qquad \forall\, t\in[0,T],
\end{aligned} \tag{1.10}$$

where C_1, C_2, C are positive constants which may depend on T.

The proof of Theorem 1.1 is based on passing to the limit for the smooth approximate solutions and a basic energy estimate which is deduced by using some relations associated with the second law of thermodynamics. Due to the differing end states some difficulties arise in the derivation of the energy estimate. We can overcome these difficulties by exploiting the facts that $v^- \leq v^+$ and that the end states for θ_0 are the same. The boundedness of ρ from below and above can be obtained by using arguments similar to those in [12, 1] while lower bounds of θ can be achieved by applying the maximum principle for linear parabolic equations in unbounded domains. In order to pass to the limit for the smooth approximate solutions we apply Lemma 5.1 of Lions [14, page 12] on weak compactness. This needs the estimates of some derivatives of (ρ, v, θ) in some L^p spaces, and we use the basic energy estimate and the interpolation theory to derive the estimates of the derivatives. In Section 2 we derive uniform a priori estimates for the approximate solutions and the proof of Theorem 1.1 is given in Section 3.

2 Uniform a priori estimates for the approximate solutions

In this section we derive some uniform a priori estimates for the approximate solutions which are obtained by mollifying the initial data.

Let $j_\epsilon(x)$ denote the standard Friedrichs' mollifier, let $\rho_0^\varepsilon(x) := \rho_0 * j_\varepsilon$, $v_0^\varepsilon(x) := v_0 * j_\varepsilon$, $\theta_0^\varepsilon(x) := \theta_0 * j_\varepsilon$. Then under the conditions of Theorem 1.1, the Cauchy problem (1.1)–(1.4) with (ρ_0, v_0, θ_0) replaced by $(\rho_0^\varepsilon, v_0^\varepsilon, \theta_0^\varepsilon)$ possesses a unique global smooth solution $(\rho^\varepsilon(x,t), v^\varepsilon(x,t), \theta^\varepsilon(x,t))$ on $\mathbb{R} \times [0,T]$ for any $T > 0$. Moreover, ρ^ϵ satisfies

$$C_1^\epsilon(T) \leq \rho^\epsilon(x,t) \leq C_2^\epsilon(T), \quad \theta^\epsilon(x,t) > 0, \qquad x \in \mathbb{R},\ t \in [0,T], \tag{2.1}$$

where $C_1^\epsilon(T)$, $C_2^\epsilon(T)$ are positive constants depending on ϵ and T.

For the sake of simplicity we will drop the subscript ϵ throughout this section.

To derive the a priori estimates for the solution $(\rho^\varepsilon, v^\varepsilon, \theta^\varepsilon)$, it is convenient to use Lagrangian coordinates instead of Eulerian coordinates (x,t). We introduce the coordinate transformation:

$$y = \int_0^x \rho(\xi, t) dy \equiv r(x,t), \qquad \tau = t, \tag{2.2}$$

and by virtue of (2.1), for each $t \geq 0$ the mapping (the inverse mapping of $r(x,t)$) $x \equiv x(y,t) : \mathbb{R} \to \mathbb{R}$ is surjective. Hence, we can reformulate the porblem (1.1)–(1.4) in Lagrangian cordinates (y, τ), that is, if we denote $u := 1/\rho$, the specific volume, and use t instead of τ in the case without confusion, then the problem (1.1)–(1.4) with (1.6) is transformed into

$$u_t = v_y, \tag{2.3}$$

$$v_t = \sigma_y, \qquad \left(\sigma := \mu \frac{v_y}{u} - R\frac{\theta}{u} \right) \tag{2.4}$$

$$c_V \theta_t = \sigma v_y + \kappa \left[\frac{\theta_y}{u} \right]_y \tag{2.5}$$

with the initial data

$$(u(y,0), v(y,0), \theta(y,0)) = (u_0(y), v_0(y), \theta_0(y)), \quad y \in \mathbb{R} \tag{2.6}$$

and the conditions at infinity

$$\lim_{y \to \pm\infty} (u(y,t), v(y,t), \theta(y,t)) = (u^\pm, v^\pm, \overline{\theta}) \quad \text{with } u^\pm := 1/\rho^\pm. \tag{2.7}$$

Now we take functions $\overline{u}(y), \overline{v}(y) \in C^\infty(\mathbb{R})$ with $\overline{u} > 0$, $\overline{v}_y \geq 0$ and

$$\overline{u}(y) = \begin{cases} u^+, & y \geq 1, \\ u^-, & y \leq -1, \end{cases} \qquad \overline{v}(y) = \begin{cases} v^+, & y \geq 1, \\ v^-, & y \leq -1. \end{cases} \tag{2.8}$$

Then, by (1.8), similar conditions to (1.8) still hold for $(u_0(y), v_0(y), \theta_0(y))$. It should be pointed out here that because of $u^\pm > 0$ and $v^- \leq v^+$ we can easily construct functions $\overline{u}$ and $\overline{v}$ which satisfy $\overline{u} > 0$, $\overline{v}_y \geq 0$.

We have the following basic energy estimate which is obtained by using some relations associated with the second law of thermodynamics and which embodies the dissipation induced by viscosity and thermal diffusion.

Lemma 2.1 *Let*

$$W(y,t) := R\overline{\theta}\left(\frac{u}{\overline{u}} - \log\frac{u}{\overline{u}} - 1\right) + c_V\overline{\theta}\left(\frac{\theta}{\overline{\theta}} - \log\frac{\theta}{\overline{\theta}} - 1\right).$$

Then we have

$$\int_{\mathbb{R}} \left[W(y,t) + \frac{1}{2}(v(y,t) - \overline{v}(y))^2\right] dy + \overline{\theta}\int_0^t \int_{\mathbb{R}} \left[\mu\frac{v_y^2}{u\theta} + \kappa\frac{\theta_y^2}{u\theta^2}\right] dyds \leq C \tag{2.9}$$

for all $t \in [0,T]$.

Proof. Using Eqs. (2.3)–(2.5), we obtain after a straightforward calculation that

$$\begin{aligned} &\left[W + \frac{(v-\overline{v})^2}{2}\right]_t + \overline{\theta}\left[\mu\frac{v_y^2}{u\theta} + \kappa\frac{\theta_y^2}{u\theta^2}\right] \\ &\qquad = R\frac{\overline{\theta}}{\overline{u}}v_y + [\sigma(v-\overline{v})]_y + \sigma\overline{v}_y + \kappa\left[\frac{\theta_y}{u}\left(1 - \frac{\overline{\theta}}{\theta}\right)\right]_y. \end{aligned}$$

Recalling the definition of $\overline{u}$ and $\overline{v}$, we integrate the above equality over $\mathbb{R} \times (0,t)$ to infer that

$$\begin{aligned} &\int_{\mathbb{R}} \left(W + \frac{(v-\overline{v})^2}{2}\right)(y,t)dy + \overline{\theta}\int_0^t\int_{\mathbb{R}}\left(\mu\frac{v_y^2}{u\theta} + \kappa\frac{\theta_y^2}{u\theta^2}\right)dyds \\ &\quad \leq \int_{\mathbb{R}}\left\{W(y,0) + \left(\frac{v_0 - \overline{v}}{2}\right)^2\right\}dy + \int_0^t\int_{\mathbb{R}}\left(R\frac{\overline{\theta}}{\overline{u}}v_y + \sigma\overline{v}_y\right)dyds \\ &\quad = E_0 + \int_0^t\int_{\mathbb{R}}\left\{R\frac{\overline{\theta}}{\overline{u}}\overline{v}_y + R\frac{\overline{\theta}}{\overline{u}}(v-\overline{v})_y + \sigma\overline{v}_y\right\}dyds \\ &\quad \leq E_0 + C - R\int_0^t\int_{\mathbb{R}}\left[\frac{\overline{\theta}}{\overline{u}}\right]_y(v-\overline{v})dyds + \int_0^t\int_{\mathbb{R}}\sigma\overline{v}_y dyds, \end{aligned} \tag{2.10}$$

where, by Taylor's expansion, (1.8) and the coordinate transformation (2.2),

$$\begin{aligned} E_0 :=& \int_{\mathbb{R}} \left\{ W(y,0) + \frac{(v_0-\overline{v})^2}{2} \right\}(y)dy \le C \int_{\mathbb{R}} \{(u_0-\overline{u})^2 + (v_0-\overline{v})^2\}(y)dy \\ &\le C \int_{\mathbb{R}} \{(\rho_0-\overline{\rho})^2 + (v_0-\overline{v})^2\}(x)dx \le C. \end{aligned}$$

Observing that $\overline{v}_y \ge 0$, $\theta/u > 0$ and $\operatorname{supp}\overline{v}_y \subset \{y \mid |y| \le 1\}$, we obtain by Eq. (2.3) that

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} \sigma \overline{v}_y dyds &= -R \int_0^t \int_{\mathbb{R}} \frac{\theta}{u} \overline{v}_y dyds + \mu \int_0^t \int_{\mathbb{R}} \frac{v_y}{u} \overline{v}_y dyds \\ &\le \mu \int_0^t \int_{\mathbb{R}} (\log u)_t \overline{v}_y dyds \\ &= \mu \int_{\mathbb{R}} \overline{v}_y \log \frac{u}{\overline{u}} dy - \mu \int_{\mathbb{R}} \overline{v}_y \log \frac{u_0}{\overline{u}} dy \\ &= \mu \int_{\{y\ |u(y,t)\le \overline{u}(y)\}} \overline{v}_y \log \frac{u}{\overline{u}} dy + \mu \int_{\{y\ |u(y,t)> \overline{u}(y)\}} \overline{v}_y \log \frac{u}{\overline{u}} dy + C \\ &\le C + \mu \int_{\{y\ |u(y,t)> \overline{u}(y)\}} \overline{v}_y \log \frac{u}{\overline{u}} dy \\ &\le C + C \int_{\{y\ |u(y,t)> \overline{u}(y), |y|\le 1\}} \log \frac{u}{\overline{u}} dy. \end{aligned}$$

Note that for $x \ge 1$, $\log x \le \sqrt{2}(x - \log x - 1)^{1/2}$, we get from the above inequality that

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} \sigma \overline{v}_y dyds &\le C \int_{\{y\ |u(y,t)\ge \overline{u}(y), |y|\le 1\}} \left(\frac{u}{\overline{u}} - \log \frac{u}{\overline{u}} - 1 \right)^{1/2} dy + C \\ &\le \varepsilon \int_{\mathbb{R}} \left(\frac{u}{\overline{u}} - \log \frac{u}{\overline{u}} - 1 \right) dy + C\varepsilon^{-1} \qquad (0 \le \varepsilon < 1). \end{aligned} \tag{2.11}$$

Substituting (2.11) into (2.10), taking ε appropriately small, taking into account $\operatorname{supp}\overline{u}_y \subset \{y \mid |y| \le 1\}$ and applying Cauchy-Schwarz's inequality, we conclude that

$$\int_{\mathbb{R}} \left(W + (v-\overline{v})^2 \right)(y,t)dy + \int_0^t \int_{\mathbb{R}} \left(\frac{v_y^2}{u\theta} + \frac{\theta_y^2}{u\theta^2} \right) dyds \le C + C \int_0^t \int_{\mathbb{R}} (v-\overline{v})^2 dyds,$$

which, by applying Gronwall's inequality, immediately gives the lemma. The proof is complete. $\square$

With the above basic energy estimate, we are able to bound the specific volume $u(y,t)$ and the temperature $\theta(y,t)$.

Lemma 2.2 *There are positive constants C_1, C_2 which may depend on T, such that*

$$C_1 \le u(y,t) \le C_2, \quad C_1 \le \theta(x,t) \qquad \forall\, x \in \mathbb{R},\ \ t \in [0,T]. \tag{2.12}$$

Proof. The proof of the boundedness of ρ is parallel to that for generalized solutions (strong solutions), e.g. see [1, 12]. For the sake of completeness we present the proof here.

It is easy to see that Lemma 2.1 implies

$$\int_i^{i+1} \left(\frac{u}{\overline{u}} - \log \frac{u}{\overline{u}} - 1 \right)(y,t)dy, \quad \int_i^{i+1} \left(\frac{\theta}{\overline{\theta}} - \log \frac{\theta}{\overline{\theta}} - 1 \right)(y,t)dy \le C, \quad t \in [0,T]. \tag{2.13}$$

Since $x-\log x-1$ is convex, one sees that

$$\int_i^{i+1}\frac{u}{\overline{u}}dy-\log\int_i^{i+1}\frac{u}{\overline{u}}dy-1,\quad \int_i^{i+1}\frac{\theta}{\overline{\theta}}dy-\log\int_i^{i+1}\frac{\theta}{\overline{\theta}}dy-1\le C,\quad t\in[0,T]. \tag{2.14}$$

The estimates (2.13), (2.14) show that

$$\alpha_1\le\int_i^{i+1}u(y,t)dy,\ \int_i^{i+1}\theta(y,t)dy\le\beta_1,\quad i=0,\pm1,\pm2,\cdots,\ \ t\in[0,T], \tag{2.15}$$

and that by virtue of the mean value theorem, for each $t\in[0,T]$ there are $a_i(t),b_i(t)\in[i,i+1]$, such that

$$\alpha_1\le u(a_i(t),t),\ \ \theta(b_i(t),t)\le\beta_1,\quad i=0,\pm1,\pm2,\cdots, \tag{2.16}$$

where α_1,β_1 are the positive roots of the equation $x-\log x-1=C$.

We may write Eq. (2.4) as $v_t=-R[\theta/u]_y+\mu[\log u]_{ty}$. Integrating this equation first over $(0,t)$ with respect to t, then over $(a_i(t),y)$ ($y\in[i,i+1]$) with respect to y, and taking exponential on both sides of the resulting equation, we obtain

$$\frac{1}{u(y,t)}\exp\left\{\frac{R}{\mu}\int_0^t\frac{\theta(y,s)}{u(y,s)}ds\right\}=B_i(y,t)Y_i(t),\qquad y\in[i,i+1], \tag{2.17}$$

where

$$\begin{aligned}B_i(y,t)&:=\frac{u_0(a_i(t))}{u(a_i(t),t)u_0(y)}\exp\left\{-\frac{1}{\mu}\int_{a_i(t)}^y(v-v_0)d\xi ds\right\},\\ Y_i(t)&:=\exp\left\{\frac{R}{\mu}\int_0^t\frac{\theta(a_i(t),s)}{u(a_i(t),s)}ds\right\}.\end{aligned}$$

Multiplying (2.17) by $R\theta/\mu$ and integrating over $[0,t]$, we arrive at

$$\exp\left\{\int_0^t\frac{\theta(y,s)}{u(y,s)}ds\right\}=1+\frac{R}{\mu}\int_0^tB_i(y,s)Y_i(s)\theta(y,s)ds.$$

Inserting the above identity into (2.17), we immediately get the following representation for $u(y,t)$:

$$u(y,t)=\frac{1}{B_i(y,t)Y_i(t)}\left\{1+\frac{R}{\mu}\int_0^tB_i(y,s)Y_i(s)\theta(y,s)ds\right\} \tag{2.18}$$

for $y\in[i,i+1]$, $t\ge0$ and $i=0,\pm1,\pm2,\cdots$. Recalling the definition of B_i and Y_i, using Cauchy-Schwarz's inequality, Lemma 2.1 and (2.16), we see that

$$0<C^{-1}\le B_i(y,t)\le C,\ \ Y_i(t)\ge1,\quad y\in[i,i+1],\,t\in[0,T],\ \ i=0,\pm1,\pm2,\cdots. \tag{2.19}$$

Now we integrate (2.18) over $(i,i+1)$ with respect to y and make use of (2.15) and (2.19) to obtain $Y_i(t)\le C+C\int_0^tY_i(s)ds$, which together with Gronwall's inequality gives

$$Y_i(t)\le C\quad\text{for all } t\in[0,T],\ \ i=0,\pm1,\pm2,\cdots. \tag{2.20}$$

It follows from (2.16) that

$$\begin{aligned}\theta(y,t) &\le C\left(\theta^{1/2}(y,t)-\theta^{1/2}(b_i(t),t)\right)^2+C\\ &\le C+C\left(\int_i^{i+1}\frac{|\theta_y|}{\theta^{1/2}}dy\right)^2\\ &\le C+C\int_i^{i+1}\frac{\theta_y^2}{u\theta^2}dy\max_{[i,i+1]}u(\cdot,t),\qquad y\in[i,i+1].\end{aligned}\tag{2.21}$$

Inserting (2.19)–(2.21) into (2.18), we find that

$$u(y,t)\le C+C\int_0^t\int_{\mathbb{R}}\frac{\theta_y^2}{u\theta^2}dy\max_{[i,i+1]}u(\cdot,s)ds\qquad\forall y\in[i,i+1],\ t\in[0,T],$$

from which, Gronwall's inequality and Lemma 2.1 we obtain $u(y,t)\le C_2$ for all $y\in[i,i+1], t\in[0,T]$ and $i=0,\pm1,\pm2,\cdots$. On the other hand, from (2.18)–(2.20) one easily gets $u(y,t)\ge C_1$ for all $y\in[i,i+1]$, $t\in[0,T]$ and $i=0,\pm1,\pm2,\cdots$. Thus, we have proved the lower and upper boundedness of ρ.

To show that the temperature is bounded below away from zero, we apply the maximum principle. To this end we write Eq. (2.5), using the estimate $C_1\le u(y,t)\le C_2$, as follows

$$\begin{aligned}c_V\left[\frac{1}{\theta}\right]_t &= -\mu\frac{v_y^2}{u\theta^2}+R\frac{v_y}{u\theta}+\kappa\left[\frac{1}{u}\left(\frac{1}{\theta}\right)_y\right]_y-2\kappa\frac{\theta_y^2}{u\theta^3}\\ &\le C+\kappa\left[\frac{1}{u}\left(\frac{1}{\theta}\right)_y\right]_y,\qquad y\in\mathbb{R},\ t\in[0,T],\end{aligned}\tag{2.22}$$

where the positive constant C is independent of ϵ. Applying the maximum principle for parabolic equations in unbounded domains [18, Chapter 3], we obtain $1/\theta(y,t)\le C$ for all $x\in\mathbb{R}$ and $t\in[0,T]$, which gives the lower boundedness of θ. This completes the proof of the lemma. $\square$

Based on the results established in Lemmas 2.1–2.2, we show the following estimates which will be important in the limit procedure in the next section.

Lemma 2.3 *Let $M\in\mathbb{N}$. Then we have*

$$\begin{aligned}&\int_0^T\sup_{\mathbb{R}}\theta(t,\cdot)dt\le C,\\ &\int_0^T\int_{-M}^M|\theta_y|dydt\le C(M),\\ &\int_0^T\int_{-M}^M|v_y|^{4/3}dydt\le C(M).\end{aligned}\tag{2.23}$$

Proof. By virtue of (2.15)–(2.16), (2.12) and Lemma 2.1, one finds that

$$\int_0^T\max_{[i,i+1]}\theta(\cdot,t)dt\le C+\int_0^T\max_{[i,i+1]}|\theta(\cdot,t)-\theta(b_i(t),t)|dt$$

$$
\begin{aligned}
&\leq C+\int_0^T\int_i^{i+1}|\theta_y|dydt\\
&\leq C+\left(\int_0^T\int_i^{i+1}\frac{\theta_y^2}{\theta^2}dydt\right)^{1/2}\left(\int_0^T\int_i^{i+1}\theta^2dydt\right)^{1/2}\\
&\leq C\left\{1+\left(\int_0^T\max_{[i,i+1]}\theta(\cdot,t)\int_i^{i+1}\theta(\cdot,t)dydt\right)^{1/2}\right\}\\
&\leq C\left\{1+\left(\int_0^T\max_{[i,i+1]}\theta(\cdot,t)dt\right)^{1/2}\right\},\qquad i=0,\pm1,\pm2,\cdots,
\end{aligned}
$$

which imples $(2.23)_1$.

On the other hand, it follows from $(2.23)_1$, Lemma 2.1, (2.12) and (2.15) that

$$
\begin{aligned}
\int_0^T\int_i^{i+1}|\theta_y|dydt &\leq\left(\int_0^T\int_i^{i+1}\frac{\theta_y^2}{\theta^2}dydt\right)^{1/2}\left(\int_0^T\int_i^{i+1}\theta^2dydt\right)^{1/2}\\
&\leq C\left(\int_0^T\max_{[i,i+1]}\theta(\cdot,t)\int_i^{i+1}\theta(y,t)dydt\right)^{1/2}\\
&\leq C,\qquad i=0,\pm1,\pm2,\cdots,
\end{aligned}
$$

which gives $(2.23)_2$. In the same manner, we have

$$
\begin{aligned}
\int_0^T\int_i^{i+1}|v_y|^{4/3}dydt &\leq\left(\int_0^T\int_i^{i+1}\frac{v_y^2}{\theta}dydt\right)^{2/3}\left(\int_0^T\int_i^{i+1}\theta^2dydt\right)^{1/3}\\
&\leq C,\qquad i=0,\pm1,\pm2,\cdots.
\end{aligned}
$$

This completes the proof of the lemma. $\square$

3 Proof of Theorem 1.1

In this section, using the a priori estimates established in Section 1 and the method of weak convergence, we pass to the limit and consequently prove the existence of global weak solutions to (1.1)–(1.4).

Let $(\rho^\epsilon(x,t),v^\epsilon(x,t),\theta^\epsilon(x,t))$ denote the approximate solutions of (1.1)–(1.4) given at the begining of Section 1 by mollifying the initial data. First we observe that in Eulerian coordinates (x,t) Lemma 2.2 implies

$$
C_1\leq\rho^\epsilon(x,t)\leq C_2,\ \ C_1\leq\theta^\epsilon(x,t)\qquad\forall x\in\mathbb{R},\ t\in[0,T],\tag{3.1}
$$

and (3.1), (2.15), Lemma 2.3, Lemma 2.1 and Sobolev's imbedding theorem imply that for any $M\in\mathbb{N}$,

$$
\int_0^T\sup_{\mathbb{R}}\theta^\epsilon(\cdot,t)dt\leq C,
$$

$$\sup_{[0,T]}\int_{-M}^{M}\theta^\epsilon(x,\cdot)dx,\ \int_0^T\int_{-M}^{M}|\theta_x^\epsilon|dxdt\le C(M), \tag{3.2}$$
$$\int_0^T\int_{-M}^{M}|v_x^\epsilon|^{4/3}dxdt,\ \int_0^T\sup_{[-M,M]}|v^\epsilon(\cdot,t)|^{4/3}dt\le C(M),$$

and

$$\begin{aligned}\int_{\mathbb{R}}\Big\{\Big(\frac{\tilde\rho(x,t)}{\rho^\epsilon(x,t)}-\log\frac{\tilde\rho(x,t)}{\rho^\epsilon(x,t)}-1\Big)+\Big(\frac{\theta^\epsilon(x,t)}{\overline\theta}-\log\frac{\theta^\epsilon(x,t)}{\overline\theta}-1\Big)\\ +(v^\epsilon(x,t)-\tilde v(x,t))^2\Big\}\,dx\le C\qquad\forall\, t\in[0,T],\end{aligned} \tag{3.3}$$

where

$$\tilde\rho(x,t):=\frac{1}{\overline u(r(x,t))}\equiv\frac{1}{\overline u(y)},\quad \tilde v(x,t):=\frac{1}{\overline v(r(x,t))}\equiv\frac{1}{\overline v(y)},$$

and $r(x,t)$ is defined by (2.2) with ρ replaced by ρ^ϵ.

Recalling the definition of $r(x,t)$ and $(\overline\rho(x),\overline v(x))$ in the introduction, using (3.1), one finds that $(\tilde\rho(x,t),\tilde v(x,t))=(\overline\rho(x),\overline v(x))=(\rho^\pm,v^\pm)$ for all $\pm x\ge K$ and $t\in[0,T]$ uniformly in ϵ, where K is some large constant. Hence, from (3.3) and (3.1), it follows that

$$\begin{aligned}\int_{\mathbb{R}}\Big\{\Big(\frac{\overline\rho(x)}{\rho^\epsilon(x,t)}-\log\frac{\overline\rho(x)}{\rho^\epsilon(x,t)}-1\Big)+\Big(\frac{\theta^\epsilon(x,t)}{\overline\theta}-\log\frac{\theta^\epsilon(x,t)}{\overline\theta}-1\Big)\\ +(v^\epsilon(x,t)-\overline v(x))^2\Big\}\,dx\le C\qquad\forall\, t\in[0,T].\end{aligned} \tag{3.4}$$

By $(3.2)_1$ and $(3.2)_2$, one sees that $\theta^\epsilon\in L^1_{loc}(\mathbb{R}_+,L^\infty_{loc})\cap L^\infty_{loc}(\mathbb{R}_+,L^1_{loc})$. So the interpolation theory gives $\theta^\epsilon\in L^{p_1}_{loc}(\mathbb{R}_+,L^{p_2}_{loc})$ and $\|\theta^\epsilon\|_{L^{p_1}_{loc}(\mathbb{R}_+,L^{p_2}_{loc})}\le C$, where p_1,p_2 satisfy $1/p_1=1-\Theta$, $1/p_2=\Theta$ for any $\Theta\in[0,1]$. This, together with (3.1)–(3.2) and (3.4) shows that we can extract a subsequence of $(\rho^\epsilon,v^\epsilon,\theta^\epsilon)$, still denoted by $(\rho^\epsilon,v^\epsilon,\theta^\epsilon)$, such that as $\epsilon\downarrow 0$, we have the following weak or weak-$*$ convergences:

$$\begin{aligned}&\rho^\epsilon\rightharpoonup\rho\ \text{ in } L^\infty_{loc}(\mathbb{R}_+,L^\infty),\qquad \theta^\epsilon\rightharpoonup\theta\ \text{ in } L^{p_1}_{loc}(\mathbb{R}_+,L^{p_2}_{loc}),\\ &v^\epsilon\rightharpoonup v\quad\text{ in } L^\infty(\mathbb{R}_+,L^2_{loc})\text{ and in } L^{4/3}_{loc}(\mathbb{R}_+,L^\infty_{loc}),\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&\rho^\epsilon v^\epsilon\rightharpoonup\overline{\rho v}\text{ in } L^\infty_{loc}(\mathbb{R}_+,L^2_{loc}),\qquad \rho^\epsilon(v^\epsilon)^2\rightharpoonup\overline{\rho v^2}\text{ in } L^{4/3}_{loc}(\mathbb{R}_+,L^2_{loc}),\\ &\rho^\epsilon\theta^\epsilon\rightharpoonup\overline{\rho\theta}\text{ in } L^2_{loc}(\mathbb{R}_+,L^2_{loc}),\qquad \rho^\epsilon v^\epsilon\theta^\epsilon\rightharpoonup\overline{\rho v\theta}\text{ in } L^{4/3}_{loc}(\mathbb{R}_+,L^{4/3}_{loc}),\\ &\rho^\epsilon(v^\epsilon)^3\rightharpoonup\overline{\rho v^3}\text{ in } L^{10/9}_{loc}(\mathbb{R}_+,L^{10/9}_{loc}),\end{aligned} \tag{3.6}$$

where $\overline{\rho v}$, $\overline{\rho v^2}$, $\overline{\rho\theta}$, $\overline{\rho v\theta}$ and $\overline{\rho v^3}$ denote the weak or weak-$*$ limit of $\rho^\epsilon v^\epsilon$, $\rho^\epsilon(v^\epsilon)^2$, $\rho^\epsilon\theta^\epsilon$, $\rho^\epsilon v^\epsilon\theta^\epsilon$ and $\rho^\epsilon(v^\epsilon)^3$, respectively. We should point out here that (3.6) easily follows from (3.1)–(3.2) and (3.4), for example,

$$\int_0^T\int_{-M}^{M}|\rho^\epsilon v^\epsilon\theta^\epsilon|^{4/3}dxdt\le C\int_0^T\max_{[-M,M]}\theta^\epsilon\int_{-M}^{M}\{(v^\epsilon)^2+\theta^\epsilon\}dxdt\le C$$

uniformly in ϵ, and the other terms can uniformly be bounded in the same way.

By the convexity of the function $x - \log x - 1$ and (3.4), the inequality (3.4) still holds for the limit functions, that is

$$\int_{\mathbb{R}} \left\{ \left(\frac{\overline{\rho}}{\rho} - \log \frac{\overline{\rho}}{\rho} - 1 \right) + \left(\frac{\theta}{\overline{\overline{\theta}}} - \log \frac{\theta}{\overline{\overline{\theta}}} - 1 \right) + (v - \overline{v})^2 \right\} (x,t) dx \le C \tag{3.7}$$

for $t \in [0,T]$, which combined with (3.1) and Taylor's expansion gives

$$\int_{\mathbb{R}} (\rho - \overline{\rho})^2 (x,t) dx \le C \qquad \forall\, t \in [0,T].$$

Now in order to justify that (ρ, v, θ) is indeed a weak solution of (1.1)–(1.4), in view of the definition of the weak solutions we have to show that

$$\begin{aligned} &\overline{\rho v} = \rho v, \quad \overline{\rho v^2} = \rho v^2, \quad \overline{\rho\theta} = \rho\theta, \quad \overline{\rho v\theta} = \rho v\theta, \quad \overline{\rho v^3} = \rho v^3, \\ &(v^\epsilon)^2 \to v^2 \quad \text{strongly in } L^2_{loc}(\mathbb{R}_+ \times \mathbb{R}) \end{aligned} \tag{3.8}$$

hold. In the following we apply Lemma 5.1 in [14, page 12] to justify (3.8).

First, by Eq. (1.1), (3.1) and (3.4) we find that

$$\partial_t \rho^\epsilon = \partial_x(\rho^\epsilon v^\epsilon) \text{ uniformly bounded in } L^\infty(\mathbb{R}_+, H^{-1}_{loc}), \tag{3.9}$$

and by $(3.2)_3$, we have

$$\|v^\epsilon - v^\epsilon(\cdot + \xi, t)\|_{L^{4/3}([0,T], L^\infty_{loc})} \to 0 \quad \text{as } |\xi| \to 0, \text{ uniformly in } \epsilon. \tag{3.10}$$

Thus by (3.5), (3.9), (3.10) and Lemma 5.1 in [14], we conclude that $\rho^\epsilon v^\epsilon \rightharpoonup \rho v$ in the sense of distributions, and therefore $\overline{\rho v} = \rho v$. Since by virtue of (3.1), (3.2) and (3.4), $\{\rho^\epsilon v^\epsilon\}$ is uniformly bounded in $L^\infty_{loc}(\mathbb{R}_+, L^2_{loc})$ and

$$\partial_t(\rho^\epsilon v^\epsilon) = -\partial_x[\rho^\epsilon (v^\epsilon)^2] - R\partial_x(\rho^\epsilon\theta^\epsilon) + \mu\partial_x^2 v^\epsilon \tag{3.11}$$

is uniformly bounded in $L^\infty_{loc}(\mathbb{R}_+, W^{-1,1}_{loc}) + L^{4/3}_{loc}(\mathbb{R}_+, W^{-1,4/3}_{loc})$, we obtain in the same manner that $\overline{\rho v^2} = \rho v^2$.

By virtue of $(3.2)_2$ and the mean value theorem, $\|\theta^\epsilon - \theta^\epsilon(\cdot + \xi, t)\|_{L^1((0,T), L^1_{loc})} \to 0$ as $|\xi| \to 0$, uniformly in ϵ. Therefore by the interpolation theory and $(3.2)_1$ we have

$$\|\theta^\epsilon - \theta^\epsilon(\cdot + \xi, t)\|_{L^1([0,T], L^p_{loc})} \to 0 \text{ as } |\xi| \to 0, \text{ uniformly in } \epsilon \tag{3.12}$$

for any $p < \infty$. Thus again applying (3.5), (3.11) and Lemma 5.1 of [14], one obtains $\rho^\epsilon v^\epsilon \theta^\epsilon \rightharpoonup \rho v\theta$ in the sense of distributions. Hence $\overline{\rho v\theta} = \rho v\theta$. In the same way we can show that $\overline{\rho\theta} = \rho\theta$.

By (3.1), (3.2) and Eq. (1.3), one sees that $\partial_t \left\{ \rho^\epsilon \left(\frac{(v^\epsilon)^2}{2} + c_V \theta^\epsilon \right) \right\}$ is uniformly bounded in

$$L^1_{loc}(\mathbb{R}_+, W^{-1,1}_{loc}) + L^\infty_{loc}(\mathbb{R}_+, W^{-2,1}_{loc}),$$

and because

$$\int_0^T \left\{ \int_{-M}^M (|v^\epsilon|^{12/5} + |\theta^\epsilon|^{6/5}) dx \right\}^{10/3} dt$$
$$\le C \int_0^T \max_{[-M,M]} (|v^\epsilon|^{4/3} + |\theta^\epsilon|^{2/3}) \left\{ \int_{-M}^M [(v^\epsilon)^2 + \theta^\epsilon] dx \right\}^{10/3} dt$$
$$\le C,$$

$\rho^\epsilon \left(\frac{(v^\epsilon)^2}{2} + c_V \theta^\epsilon \right)$ is uniformly bounded in $L^4_{loc}(\mathbb{R}_+, L^{6/5}_{loc})$, thus by (3.10) and Lemma 5.1 in [14],

$$\rho^\epsilon \left(\frac{(v^\epsilon)^2}{2} + c_V \theta^\epsilon \right) \cdot v^\epsilon \rightharpoonup \overline{\rho \left(\frac{v^2}{2} + c_V \theta \right)} \cdot v = \rho \left(\frac{v^2}{2} + \theta \right) \cdot v,$$

which implies $\overline{\rho v^3} = \rho v^3$, where we have used the fact $\rho^\epsilon v^\epsilon \theta^\epsilon \rightharpoonup \rho v \theta$, and $\overline{\rho \left(\frac{v^2}{2} + c_V \theta \right)}$ denotes the weak limit of $\rho^\epsilon \left(\frac{(v^\epsilon)^2}{2} + \theta^\epsilon \right)$.

Finally, we show $(v^\epsilon)^2 \rightharpoonup v^2$. It follows from (3.1), (3.2) and (3.4) that $(v^\epsilon)^2$ is uniformly bounded in $L^{4/3}_{loc}(\mathbb{R}_+, L^2_{loc})$ (cf. $(3.6)_1$). Thus, $(v^\epsilon)^2 \rightharpoonup \overline{v^2}$ weakly in $L^{4/3}_{loc}(\mathbb{R}_+, L^2_{loc})$. On the other hand, by (3.10) and (3.4),

$$\|(v^\epsilon)^2 - (v^\epsilon)^2(\cdot + \xi, t)\|_{L^{4/3}([0,T], L^{4/3}_{loc})} \le 2 \|v^\epsilon - v^\epsilon(\cdot + \xi, t)\|_{L^{4/3}([0,T], L^\infty_{loc})} \|v^\epsilon\|_{L^\infty([0,T], L^{4/3}_{loc})}$$
$$\le C \|v^\epsilon - v^\epsilon(\cdot + \xi, t)\|_{L^{4/3}([0,T], L^\infty_{loc})} \to 0 \text{ as } |\xi| \to 0, \text{ uniformly in } \epsilon. \tag{3.13}$$

From (3.13), (3.9), Lemma 5.1 in [14] and the fact that $\rho^\epsilon (v^\epsilon)^2 \rightharpoonup \rho v^2$ and $\rho > 0$, one gets

$$\rho v^2 \leftharpoonup \rho^\epsilon (v^\epsilon)^2 \rightharpoonup \rho \overline{v^2},$$

which gives $\overline{v^2} = v^2$. Hence, by Theorem 1 of [4], $v^\epsilon(x,t) \to v(x,t)$ stronly in $L^2_{loc}(\mathbb{R}_+ \times \mathbb{R})$. We have proved (3.8).

Now we multiply the system (1.1)–(1.3) with (ρ, v, θ) replaced by $(\rho^\epsilon, v^\epsilon, \theta^\epsilon)$ by $\phi \in C_0^\infty([0,\infty) \times \mathbb{R})$, integrate over $(0,\infty) \times \mathbb{R}$ and integrate by parts. Letting $\epsilon \downarrow 0$, applying (3.5), (3.6) and (3.8), summing up the results for (ρ, v, θ) obtained above, we see that (ρ, v, θ) obtained in (3.5) is indeed a weak solution of (1.1)–(1.4). Finally, the estimate (1.10) follows from (3.1) and (3.7). This completes the proof of Theorem 1.1. $\square$

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