BV Estimates for Multicomponent Chromatography with Relaxation

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Abstract. We consider the Cauchy problem for a system of $2n$ balance laws which arises from the modelling of multi-component chromatography:

$$\begin{cases} u_t + u_x = -\dfrac{1}{\varepsilon}\big(F(u) - v\big), \\ \qquad v_t = \dfrac{1}{\varepsilon}\big(F(u) - v\big), \end{cases} \tag{1}$$

This model describes a liquid flowing with unit speed over a solid bed. Several chemical substances are partly dissolved in the liquid, partly deposited on the solid bed. Their concentrations are represented respectively by the vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$.

We show that, if the initial data have small total variation, then the solution of (1) remains with small variation for all times $t \geq 0$. Moreover, using the $\mathbf{L}^1$ distance, this solution depends Lipschitz continuously on the initial data, with a Lipschitz constant uniform w.r.t. ε. Finally we prove that as $\varepsilon \to 0$, the solutions of (1) converge to a limit described by the system

$$\big(u + F(u)\big)_t + u_x = 0, \qquad\qquad v = F(u). \tag{2}$$

The proof of the uniform BV estimates relies on the application of probabilistic techniques. It is shown that the components of the gradients v_x, u_x can be interpreted as densities of random particles travelling with speed 0 or 1. The amount of coupling between different components is estimated in terms of the expected number of crossing of these random particles. This provides a first example where BV estimates are proved for general solutions to a class of $2n \times 2n$ systems with relaxation.

1 - Introduction

Consider the system of $2n$ balance laws

$$\begin{cases} u_t + u_x = -\dfrac{1}{\varepsilon}\big(F(u) - v\big), \\ \qquad v_t = \dfrac{1}{\varepsilon}\big(F(u) - v\big), \end{cases} \tag{1.1}$$

with initial data

$$u(0,x) = u_0(x), \qquad\qquad v(0,x) = v_0(x). \tag{1.2}$$

This system arises from the modelling of multi-component chromatography: A liquid flows with unit speed over a solid bed. Several chemical substances are partialy dissolved in the liquid, partialy deposited on the solid bed. Their concentrations, in the fluid and in the solid phase respectively, are described by the vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$. The balance equations (1.1) describe two facts:

- Particles dissolved in the fluid travel forward with unit speed, while those deposited on the solid bed do not move.

- It is possible for a particle to pass from the fluid to the solid phase, and viceversa. The parameter ε represents the *relaxation time*, determining how quickly the equilibrium configuration $v = F(u)$ is reached.

The equilibrium relation can be modelled in several ways. In our paper, we consider the Langmuir isotherm, i.e., the function $F = (F_1, \ldots, F_n)$ here takes the form (cf. [RAA1])

$$F_i(u) \doteq \frac{k_i u_i}{1 + k_1 u_1 + \cdots + k_n u_n}, \tag{1.3}$$

for some constants $0 < k_1 < \cdots < k_n$. Letting the relaxation parameter $\varepsilon \to 0$, one expects that the solution of (1.1) converges to a limit described by the system

$$\big(u + F(u)\big)_t + u_x = 0, \tag{1.4}$$

$$v = F(u). \tag{1.5}$$

Aim of this paper is to provide a rigorous proof of this convergence, valid for all solutions with small total variation. Our results apply more generally to the case of functions $F : \Omega \mapsto \mathbb{R}^n$, defined on an open set $\Omega \subset \mathbb{R}^n$, which satisfy the following assumptions:

(A1) At each point $u \in \Omega$, the $n \times n$ Jacobian matrix $A(u) \doteq DF(u)$ has n real, positive, distinct eigenvalues $0 < \eta_1(u) < \cdots < \eta_n(u)$.

(A2) Every characteristic field is either linearly degenerate or genuinely nonlinear.

(A3) All the integral curves of the eigenvectors of A are straight lines.

Observe that, by (A1), we can choose bases of right and left eigenvectors $r_i(u)$, $l_i(u)$, normalized according to

$$\big|r_i(u)\big| = 1, \qquad\qquad l_i(u) \cdot r_j(u) = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{1.6}$$

Throughout the following, the directional derivative of a function $u \mapsto \varphi(u)$ in the direction of a vector $\mathbf{v} \in I\!R^n$ will be written as

$$\mathbf{v} \bullet \varphi(u) \doteq \lim_{h \to 0} \frac{\varphi(u + h\mathbf{v}) - \varphi(u)}{h}.$$

With the normalization (1.6), the assumption (A3) can thus be written in the form

$$r_i \bullet r_i \equiv 0 \qquad\qquad i = 1, \dots, n. \tag{1.7}$$

A direct computation shows that the assumptions (A1)-(A3) are indeed satisfied by the function F in (1.3).

Our first main result shows that, if the initial data (u_0, v_0) are close to equilibrium and have small total variation, then the solution of (1.1) remains with small variation for all times $t \geq 0$. Moreover, this solution depends Lipschitz continuously on the initial data, with a Lipschitz constant uniform w.r.t. both t and ε.

Theorem 1. *For every compact set $K \subset \Omega$ and $\varepsilon > 0$, there exist constants $L, \delta_0 > 0$, a closed domain $\mathcal{D} \subset \mathbf{L}^1_{\mathrm{loc}}(I\!R; I\!R^{2n})$ and a continuous semigroup $S^\varepsilon : \mathcal{D} \times [0, \infty[\mapsto \mathcal{D}$ with the following properties.*

(i) The domain $\mathcal{D}$ contains every initial data (u_0, v_0) such that

$$\begin{aligned} &\mathit{Tot.Var.}\{u_0\} + \mathit{Tot.Var.}\{v_0\} \leq \delta_0, \\ &u_0(\bar{x}) \in K, \qquad \Big| F\big(u_0(\bar{x})\big) - v_0(\bar{x}) \Big| \leq \delta_0 \qquad \textit{for some } \bar{x} \in I\!R. \end{aligned} \tag{1.8}$$

(ii) Each trajectory of the semigroup: $t \mapsto \big(u(t), v(t)\big) = S^\varepsilon_t(u_0, v_0)$ is the solution of the corresponding Cauchy problem (1.1)-(1.2).

(iii) For every pair of initial data $(u_0, v_0), (\tilde{u}_0, \tilde{v}_0) \in \mathcal{D}$ and every $t \geq 0$ one has

$$\big\| S^\varepsilon_t(u_0, v_0) - S^\varepsilon_t(\tilde{u}_0, \tilde{v}_0) \big\|_{\mathbf{L}^1(I\!R; I\!R^{2n})} \leq L \cdot \big\| (u_0, v_0) - (\tilde{u}_0, \tilde{v}_0) \big\|_{\mathbf{L}^1(I\!R; I\!R^{2n})}. \tag{1.9}$$

The constants L, δ_0 and the domain $\mathcal{D}$ do not depend on ε.

Our second theorem is concerned with the limit of trajectories of (1.1), as $\varepsilon \to 0$.

Theorem 2. *In the same setting as Theorem 1, for every $(u_0, v_0) \in \mathcal{D}$ and $\varepsilon > 0$, call $t \mapsto \big(u^\varepsilon(t), v^\varepsilon(t)\big)$ the corresponding solution of (1.1)-(1.2). Then, as $\varepsilon \to 0$, for each $t > 0$ one has the convergence*

$$u^\varepsilon(t) \to u(t), \qquad v^\varepsilon(t) \to v(t) \qquad \textit{in } \mathbf{L}^1_{\mathrm{loc}}(I\!R; I\!R^n) \tag{1.10}$$

The limit function u can be characterized as the unique entropic solution of the system of conservation law (1.4) with initial data $u(0, \cdot)$ implicitly defined by

$$F\big(u(0, x)\big) = v(0, x), \qquad u(0, x) + v(0, x) = u_0(x) + v_0(x). \tag{1.11}$$

Moreover, for each $t > 0$, one has $v(t, x) = F\big(u(t, x)\big)$ for a.e. $x \in I\!R$.

The above results can be regarded as a counterpart of those proved in [BB] for vanishing viscosity limits. The various steps in the proof are also quite similar. A priori estimates on the total variation of u, v are obtained from uniform $\mathbf{L}^1$ bounds on the derivatives u_x, v_x. The evolution equations for the derivatives take the form of n coupled 2×2 systems of balance laws. The key property of these systems is that the $\mathbf{L}^1$ norm of the components u_x^i, v_x^i can increase only as a result of the coupling terms, and these terms are uniformly integrable in space-time. The basic argument in the proof, here as in [BB], can be described as follows: Calling $\Psi(t)$ the $\mathbf{L}^1$ norm of the derivatives of the solution at time t, we have a relation of the form

$$\Psi(t) = \Psi(0) + \int_0^t \mathcal{Q}(s)\, ds, \tag{1.12}$$

where $\mathcal{Q}$ describes the contribution of interaction terms. Since these terms are quadratic, a key transversality property of our system yields an estimate of the form

$$\int_0^t \mathcal{Q}(s)\, ds \leq \kappa \cdot \left(\Psi(0) + \int_0^t \mathcal{Q}(s)\, ds \right)^2, \tag{1.13}$$

for some (possibly large) constant κ. An elementary argument now shows that, if the total variation of the initial data is small enough so that $\Psi(0) < 1/4\kappa$, then $\Psi(t) < 1/2\kappa$ for all $t \geq 0$. This will yield the uniform BV estimates on solutions of (1.1).

The mathematic modelling of chromatography was studied by several authors. The books of Rhee, Aris and Ammundson [RAA2, RAA3] provide a good introduction to this field. For the special case of Langmuir isotherm, the equilibrium system was studied in great detail in [RAA1], where the authors developed an exact Riemann solver and analysed wave interactions. Hyperbolic conservation laws with relaxation effects were first considered by Liu in a fundamental paper [Li]. Chen & Liu [CL] studied the zero relaxation and dissipation limit, for two models of viscoelasticity and phase transition. In the case where the limit is a single equation, many results are known for various models. Among them, Natalini [N1] studied convergence towards equilibrium. See also his recent survey paper [N2] and the references therein. For the model of single component chromatography, convergence towards equilibrium and error estimates, in the framework of BV solutions, were studied by Tveito & Winther in [TW1, TW2] for one-dimensional cases, and Shen, Tveito & Winther [STW] for the two-dimensional cases. However, for multicomponent chromatography with relaxation few results are known. To the authors' knowledge, this paper would be the first one that gives a rigorous proof of convergence, for general BV solutions.

2 - A Basic Relaxation Problem

Toward the analysis of more general systems, a basic first step is the study of 2×2 relaxation problems. In this section we shall consider two solutions of two distinct 2×2 linear systems, with strictly different average speeds. Relyng on a probabilistic interpretation, we will prove an a priori bound on the integral of their product. We also derive some corollaries which will be useful in the proof of the BV bound in Section 3.

Consider the 2×2 system

$$\begin{cases} u_t + u_x = -\big(f(u) - v\big), \\ \qquad v_t = \big(f(u) - v\big). \end{cases} \tag{2.1}$$

We assume that $f : I\!R \mapsto I\!R$ is smooth, with $f'(u) > 0$ for all $u \in I\!R$. Since the system (2.1) is semilinear, for smooth initial data, the global existence and uniqueness of smooth solutions is well known.

The spatial derivatives u_x, v_x satisfy the equations

$$\begin{cases} u_{xt} + u_{xx} = -f'(u)u_x + v_x, \\ \qquad\quad v_{xt} = f'(u)u_x - v_x. \end{cases} \tag{2.2}$$

Since the solutions of (2.1) remain smooth, a priori bounds on the total variation of u, v can be obtained from estimates on the $\mathbf{L}^1$ norm of u_x, v_x. Toward this goal, we first study the more general linear system

$$\begin{cases} \xi_t + \xi_x = -a(t,x)\xi + b(t,x)\zeta, \\ \qquad\quad \zeta_t = \ \ a(t,x)\xi - b(t,x)\zeta, \end{cases} \tag{2.3}$$

with

$$0 < a_- \leq a(t,x) \leq a_+, \qquad\qquad 0 < b_- \leq b(t,x) \leq b_+. \tag{2.4}$$

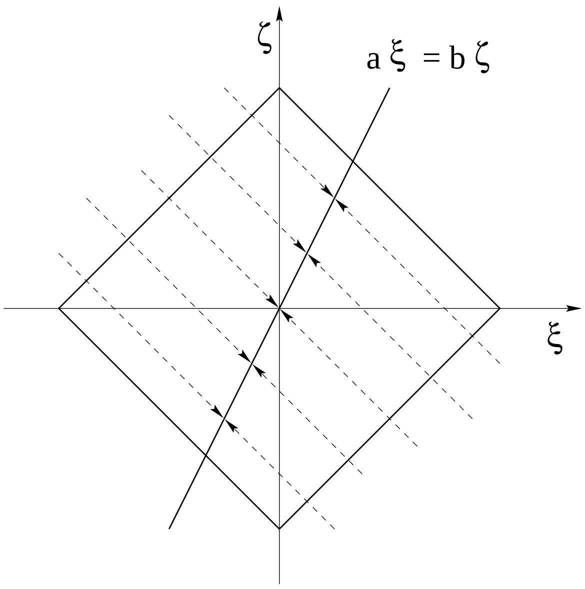

figure 1

Observe that, on $I\!R^2$, the diamonds

$$B_\rho \doteq \{(\xi,\zeta)\,; \quad |\xi| + |\zeta| \leq \rho\}$$

are positively invariant for a system of the form (fig.1)

$$\begin{cases} \dot{\xi} = -a\xi + b\zeta, \\ \dot{\zeta} = \ \ a\xi - b\zeta, \end{cases}$$

with $a, b > 0$. For every solution of (2.3), it is thus clear that the quantity

$$\Phi(t) \doteq \int_{-\infty}^{\infty} \big|\xi(t,x)\big| + \big|\zeta(t,x)\big| \; dx \tag{2.5}$$

is non-increasing in time. Applying this to the case

$$\xi = u_x, \quad \zeta = v_x, \qquad a = f'(u), \quad b = 1,$$

we obtain the uniform bound

$$\int_{-\infty}^{\infty} \big|v_x(t,x)\big| + \big|u_x(t,x)\big| \, dx \leq \int_{-\infty}^{\infty} \big|v_x(0,x)\big| + \big|u_x(0,x)\big| \, dx, \tag{2.6}$$

valid for every solution of (2.1).

We observe that the equation (2.3) is associated to the following stochastic process (see [Za], pp.15-17). A medium is composed by particles that can move with random speed 0 or 1. Let $P(t)$ denote the position of a particle at time t. The speed of a particle can switch from 1 to 0 with rate a and from 0 to 1 with rate b. In other words, if at a given time t we have $\dot P(t) = 0$, the probability that $\dot P(t+\Delta t) = 1$ is $a \cdot \Delta t + o(\Delta t)$. Similarly, if $\dot P(t) = 1$, then the probability that $\dot P(t+\Delta t) = 0$ is $b \cdot \Delta t + o(\Delta t)$. We use here the Landau symbol $o(\Delta t)$ to denote a higher order infinitesimal, satisfying $o(\Delta t)/\Delta t \to 0$ as $\Delta t \to 0+$. At any given time t, the functions $\zeta(t,\cdot)$ and $\xi(t,\cdot)$ then yield the density of particles travelling with speed 0 and 1 respectively.

It is convenient to represent the general solution of (2.3) in terms of fundamental solutions. We call $\Gamma_{ji}(t,x;t_0,x_0)$, $i,j \in \{0,1\}$ the density of probability that a particle, which initially is at x_0 at time t_0 with speed i, reaches the point x at time $t > t_0$ with speed j. Since the only possible speeds are 0 and 1, one clearly has

$$\Gamma_{ji}(t,x;t_0,x_0) = 0 \qquad \text{if } \ x \notin \big[x_0, \ x_0 + (t-t_0)\big].$$

Given an initial data

$$\xi(t_0,x) = \xi_0(x), \qquad \zeta(t_0,x) = \zeta_0(x), \tag{2.7}$$

for $t > t_0$ the corresponding solution of (2.3) can thus be written in the form

$$\begin{aligned} \xi(t,x) &= \int_{-\infty}^{\infty} \big[\Gamma_{10}(t,x;t_0,y)\zeta_0(y) + \Gamma_{11}(t,x;t_0,y)\xi_0(y)\big]\, dy, \\ \zeta(t,x) &= \int_{-\infty}^{\infty} \big[\Gamma_{00}(t,x;t_0,y)\zeta_0(y) + \Gamma_{01}(t,x;t_0,y)\xi_0(y)\big]\, dy. \end{aligned} \tag{2.8}$$

Observe that (2.8) contains a slight abuse of notation. Indeed, while Γ_{10} and Γ_{01} are absolutely continuous w.r.t. Lebesgue measure, the probability measure $\Gamma_{00}(t,\cdot\,;t_0,x_0)$ contains an atom at the point x_0 (whose mass is exponentially decreasing), while the measure $\Gamma_{11}(t,\cdot\,;t_0,x_0)$ contains an atom at the point $x_0 + (t-t_0)$. In spite of this, we shall continue to write the probability measures $\Gamma_{ji}(t,\cdot;t_0,x_0)$ in the form (2.8), since this will not generate confusion.

A more detailed analysis is possible in the special case where the switching rates a, b are constants. For the system

$$\begin{cases} \xi_t + \xi_x = -\alpha\xi + \beta\zeta, \\ \qquad \zeta_t = \ \alpha\xi - \beta\zeta, \end{cases} \tag{2.9}$$

the fundamental solution is invariant under time and space translation. Hence

$$\Gamma_{ji}(t,x;t_0,x_0) = G_{ji}(t-t_0, \ x-x_0) \qquad i,j \in \{0,1\}.$$

The kernels G_{ji} could be explicitly computed in terms of a Fourier transform. However, such a detailed analysis will not be needed for our purposes. The main information we shall use is an asymptotic property of particle trajectories, related to the law of large numbers. Consider a particle starting at the origin. Regardless of its initial velocity, on a large time interval $[0,T]$ this particle will have speed 1 during a fraction of time approximately proportional to $\beta/(\alpha+\beta)$. More precisely,

$$\lim_{t\to\infty}\frac{P(t)}{t}=\lambda\doteq\frac{\beta}{\alpha+\beta}\qquad\text{with probability 1.}\tag{2.10}$$

We now consider a second linear system with constant coefficients

$$\begin{cases}\xi_t+\xi_x=-\alpha^*\xi+\beta^*\zeta,\\ \qquad\zeta_t=\ \alpha^*\xi-\beta^*\zeta,\end{cases}\tag{2.11}$$

and call G^*_{ji} the corresponding kernels. Throughout the following, we assume that the average speed λ^* of the particles P^* described by (2.11) is strictly smaller than the average speed λ of the particles P described by (2.9), namely:

$$0\ <\ \lambda^*\ \doteq\ \frac{\beta^*}{\alpha^*+\beta^*}\ <\ \lambda\ \doteq\ \frac{\beta}{\alpha+\beta}\ <\ 1.\tag{2.12}$$

Lemma 1. *Consider the double integrals*

$$E_0\doteq\int_0^\infty\int_{-\infty}^\infty G_{01}(t,x)\cdot G^*_{10}(t,x)\ dxdt,\qquad E_1\doteq\int_0^\infty\int_{-\infty}^\infty G_{10}(t,x)\cdot G^*_{01}(t,x).\tag{2.13}$$

If (2.12) holds, then

$$E_1=1+E_0<\infty.\tag{2.14}$$

Proof. Call P the "fast" particles and P^* the "slow" particles, whose densities are described by the solutions of (2.9), (2.11) respectively. We say that P *overtakes* P^* at (t,x) if the two particles cross at time t and moreover

$$P(t)=P^*(t)=x,\qquad \dot P(t)=1,\quad \dot P^*(t)=0.$$

Observe that the first integral E_0 in (2.13) yields the expected number of times where P^* overtakes P, assuming that the slow particle P^* starts at the origin with speed 0 and that the fast particle P starts at the origin with speed 1. Indeed,

$$\begin{aligned}E_0&=\int_0^\infty\int_{-\infty}^\infty\lim_{\varepsilon\to0+}\frac{1}{\varepsilon^2}\left\{\int_x^{x+\varepsilon}G_{01}(t,y)\int_{y-\varepsilon}^y G^*_{10}(t,y')\,dy'\,dy\right\}\,dxdt\\&=\int_0^\infty\lim_{\varepsilon\to0+}\frac{1}{\varepsilon}\cdot\text{Prob.}\Big\{P^*\text{ overtakes }P\text{ during the time interval }[t,\ t+\varepsilon]\Big\}.\end{aligned}$$

Similarly, the second integral E_1 in (2.13) yields the expected number of times where P overtakes P^*, assuming that the slow particle P^* starts at the origin with speed 1 and that the fast particle P starts at the origin with speed 0. By (2.10) and (2.12) we have

$$\lim_{t\to\infty}\big[P(t)-P^*(t)\big]=\infty\qquad\text{with probability 1.}\tag{2.15}$$

As a consequence, if the fast particle initially lies behind the slow particle, then by (2.15) P overtakes P^* with probability 1. Let this first crossing occur at (t_1, x_1). At time t_1 we then have

$$P(t_1) = P^*(t_1) = x_1, \qquad \dot P(t_1) = 1, \qquad \dot P^*(t_1) = 0.$$

Since the speed of particles is a Markov process, invariant under translation of coordinates, the expected number of subsequent times where P^* overtakes P is precisely E_0. Because of (2.15), this coincides with the number of further crossings where P overtakes P^*. This yields the first equality in (2.14).

We now show that $E_0 < \infty$. Call p_0 the probability that a slow particle P^* starting at the origin with speed 0 never crosses a fast particle starting from the origin with speed 1. If we can show that $p_0 > 0$ we are done. Indeed, in this case there holds

$$E_0 = 0 \cdot p_0 + (1 + E_0)(1 - p_0),$$

$$E_0 = \frac{1 - p_0}{p_0} < \infty.$$

Choose a speed μ such that $\lambda^* < \mu < \lambda$. We claim that there exists a positive probability $p > 0$ such that $P(t) \geq \mu t$ for all $t \geq 0$. Indeed, call $0 = t_0 < t_1 < \cdots$ the random times where the particle changes speed, so that

$$\dot P(t) = \begin{cases} 1 & \text{if} \quad t \in]t_{2i},\ t_{2i+1}[\,, \\ 0 & \text{if} \quad t \in]t_{2i+1},\ t_{2i+2}[\,, \end{cases} \qquad i = 0, 1, 2, \ldots \tag{2.16}$$

Consider the random variables

$$X_i \doteq P(t_{2i}) - P(t_{2i-2}) - \mu(t_{2i} - t_{2i-2}).$$

By the assumptions, these random variables are independent and identically distributed. Moreover, their expected value is strictly positive. Indeed, since $\mu < \lambda$ one has

$$\begin{aligned} E[X_i] &= (1-\mu) \int_0^\infty \alpha t\, e^{-\alpha t}\, dt - \mu \int_0^\infty \beta t\, e^{-\beta t}\, dt \\ &= \frac{1-\mu}{\alpha} - \frac{\mu}{\beta} > \frac{1}{\alpha} - \frac{\lambda}{\alpha} - \frac{\lambda}{\beta} = 0. \end{aligned}$$

A well known probability theorem (see [L], p.399) now states that, with positive probability $p > 0$, all of the partial sums

$$S_n \doteq X_1 + \cdots + X_n, \qquad n \geq 1,$$

are non-negative. By construction, this is the case if and only if $P(t) \geq \mu t$ for all $t \geq 0$.

An entirely similar argument shows that, with some positive probability $p^* > 0$ one has $P^*(t) \leq \mu t$ for all $t \geq 0$. We conclude that with probability $p_0 \geq p \cdot p^* > 0$ the two points P, P^* never cross. This proves the lemma.

Corollary 1. *In the same setting of Lemma 1, for every $y \neq 0$ and every $i, j, h, k \in \{0, 1\}$ with $j \neq k$, one has*

$$\int_0^\infty \int_{-\infty}^\infty G^*_{ji}(t, x - y) \cdot G_{kh}(t, x)\ dxdt \leq E_1. \tag{2.17}$$

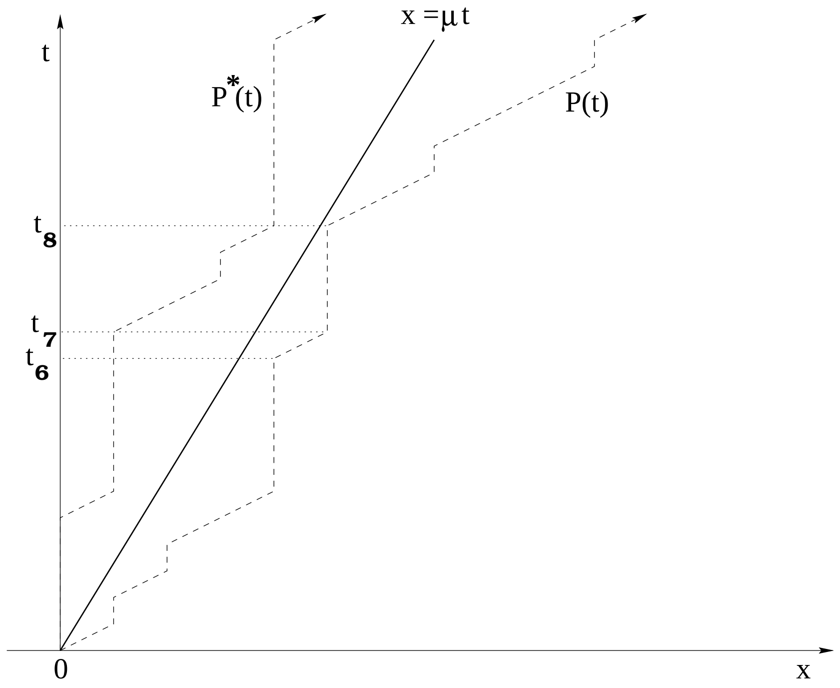

figure 2

Moreover,

$$\lim_{y\to-\infty}\int_0^\infty\int_{-\infty}^\infty G^*_{10}(t,x-y)\cdot G_{01}(t,x)\ dxdt=0 \tag{2.18}$$

Proof. The various cases of (2.17) are all easy consequences of Lemma 1. To prove (2.18) we observe that, if $\lambda^*<\mu<\lambda$, by (2.10) it follows

$$\lim_{y\to-\infty}\ \text{Prob.}\left\{\inf_{t\geq 0}\ \big(P(t)-\mu t\big)<\frac{y}{2}\right\}=0,$$

where P is a fast particle starting at the origin. Similarly,

$$\lim_{y\to\infty}\ \text{Prob.}\left\{\sup_{t\geq 0}\ \big(P^*(t)-\mu t\big)>\frac{y}{2}\right\}=0,$$

where P^* is a slow particle starting at the origin. We now observe that the double integral in (2.18) is the expected number of times where a slow particle starting at y overtakes a fast particle starting at the origin. The above inequalities show that the probability that the two particles ever cross approaches zero as $y\to-\infty$. Hence the result.

Using a comparison argument, we now derive similar estimates in the case where the switching rates a,b possibly depend on t,x. Together with (2.3), consider a second system

$$\begin{cases}\xi_t+\xi_x=-a^*(t,x)\xi+b^*(t,x)\zeta,\\ \qquad\ \zeta_t=a^*(t,x)\xi-b^*(t,x)\zeta,\end{cases} \tag{2.19}$$

with

$$0 < a_-^* \le a^*(t,x) \le a_+^*, \qquad\qquad 0 < b_-^* \le b^*(t,x) \le b_+^*.$$

Assume that the coefficients of the systems (2.3), (2.9) and (2.19), (2.11) can be compared as follows:

$$a(t,x) \le \alpha, \qquad b(t,x) \ge \beta, \qquad\qquad a^*(t,x) \ge \alpha^*, \qquad b^*(t,x) \le \beta^* \qquad \text{for all } \ t,x. \tag{2.20}$$

Intuitively, this means that the particles Q whose density is described by (2.3) are even faster (on average) than those of (2.9), because their speed is more likely to switch from 0 to 1 and less likely to switch from 1 to 0. On the other hand, the particles Q^* described by (2.19) are even slower than those of (2.11). As a consequence, the particles of the two systems (2.3), (2.19) cross each other a smaller number of times than the particles of (2.9), (2.11). We thus expect that the corresponding double integrals of $\Gamma\Gamma^*$ will be smaller than those of GG^*. This is indeed the content of the following lemma.

Lemma 2. *Let E_0, E_1 be the double integrals defined at (2.13). Under the assumptions (2.20), for every τ_0, x_0, one then has*

$$E^\sharp(\tau_0,x_0) \doteq \int_{\tau_0}^\infty \int_{-\infty}^\infty \Gamma_{01}(t,x;\tau_0,x_0)\cdot\Gamma_{10}^*(t,x;\tau_0,x_0)\ dxdt \le E_0, \tag{2.21}$$

Proof. Consider any trajectory $t\mapsto P(t)$ of a particle described by (2.9), with $P(0)=0$, $\dot P(0)=1$. Call $0=t_0<t_1<\cdots$ the switching times of the derivative $\dot P$, so that (2.16) holds. For any given starting point (τ_0,x_0), to this trajectory $P(\cdot)$ we now associate a second trajectory $Q(\cdot)$, with

$$Q(\tau_0)=x_0, \qquad\qquad \dot Q(t) = \begin{cases} 1 & \text{if} \quad t\in\,]\tau_{2i},\ \tau_{2i+1}[\,, \\ 0 & \text{if} \quad t\in\,]\tau_{2i+1},\ \tau_{2i+2}[\,, \end{cases} \qquad\qquad i=0,1,2,\ldots \tag{2.22}$$

where the switching times are inductively defined as follows. Assume that $\tau_0,\ldots,\tau_{2i}$ have already been constructed. By (2.22) this implies

$$Q(\tau_{2i}) = x_0 + \sum_{j=0}^{i-1}(\tau_{2j+1}-\tau_{2j}).$$

We then define the further switching times τ_{2i+1}, τ_{2i+2} implicitly by the relations

$$\int_{\tau_{2i}}^{\tau_{2i+1}} a\big(s,\ Q(\tau_{2i})+(s-\tau_{2i})\big)\,ds = \alpha\cdot(t_{2i+1}-t_{2i}), \tag{2.23}$$

$$\int_{\tau_{2i+1}}^{\tau_{2i+2}} b\big(s,\ Q(\tau_{2i+1})\big)\,ds = \beta\cdot(t_{2i+2}-t_{2i+1}). \tag{2.24}$$

This construction guarantees that, if P is a random particle whose distribution is a fundamental solution of (2.9), then the distribution of Q provides a fundamental solution of (2.3). More precisely, assume that for every Borel set J one has

$$\text{Prob.}\Big\{P(t)\in J,\ \ \dot P(t)=0\Big\} = \int_J G_{01}(t,x)\,dx.$$

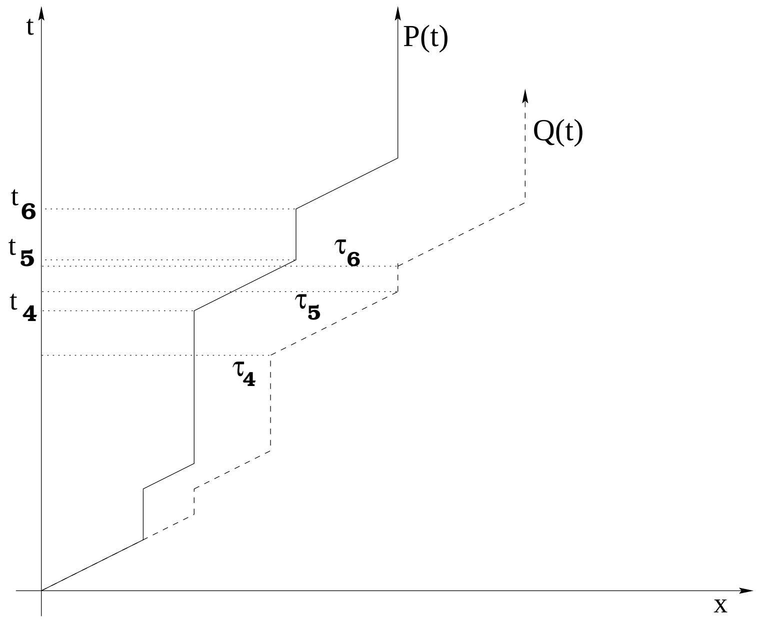

figure 3

Then for every $t > \tau_0$ and every set J there holds

$$\text{Prob.}\Big\{Q(t) \in J, \quad \dot{Q}(t) = 0\Big\} = \int_J \Gamma_{01}(t, x; t_0, x_0)\, dx.$$

Because of the assumptions (2.20), from the definitions (2.23)-(2.24) it follows

$$\tau_{2i+1} - \tau_{2i} \geq t_{2i+1} - t_{2i}, \qquad\qquad \tau_{2i+2} - \tau_{2i+1} \leq t_{2i+2} - t_{2i+1}.$$

In other words, each interval where $\dot{Q} = 1$ is longer than the corresponding time interval where $\dot{P} = 1$, while each interval where $\dot{Q} = 0$ is shorter than the corresponding time interval where $\dot{P} = 0$.

We have thus constructed a map $P(\cdot) \mapsto Q(\cdot)$, consistent with the probability measures, such that

$$Q(t) \geq x_0 + P(t - t_0) \qquad\qquad \text{for every } t \geq t_0. \tag{2.25}$$

In an entirely similar way, in connection with the systems (2.11), (2.17) we can construct a map $P^*(\cdot) \mapsto Q^*(\cdot)$, consistent with the probability measures, such that

$$Q^*(t) \leq x_0 + P^*(t - t_0) \qquad\qquad \text{for every } t \geq t_0. \tag{2.26}$$

As in the proof of Lemma 1, in connection with the systems (2.9), (2.11), we now call p_0 the probability that a slow particle P^* starting at the origin with speed 0 never crosses a fast particle P starting from the origin with speed 1. In connection with the systems (2.3), (2.19), given (τ, y), call $\hat{p}_0(\tau, y)$ the probability that a slow particle Q^* starting from x_0 at time τ_0 with speed 0 never crosses a fast particle Q starting from y at time τ with speed 1. By (2.25)-(2.26), we have

$$\hat{p}_0(\tau, y) \geq p_0 \tag{2.27}$$

Observing that the integral $E^\sharp(t_0,x_0)$ in (2.21) represents the expected number of crossings between particles Q, Q^*, setting

$$p_0^* \doteq \min_{\tau,y} \hat p_0(\tau,y) > p_0$$

we have

$$E^\sharp(t_0,x_0) \leq \frac{1-p_0^*}{p_0^*} < \frac{1-p_0}{p_0} = E_0,$$

proving the lemma.

Corollary 2. *For every $k \neq j$ one has*

$$\int_{\max\{t_1,t_2\}}^{\infty} \int_{-\infty}^{\infty} \Gamma_{ji}(t,x;t_1,x_1)\cdot\Gamma^*_{kh}(t,x;t_2,x_2)\,dxdt \leq E_1. \tag{2.28}$$

Proof. For every $s\in[t_0,t]$ one has the identity

$$\Gamma_{ji}(t,x;t_0,x_0) = \sum_{\ell=0,1}\int_{-\infty}^{\infty} \Gamma_{j\ell}(t,x;s,y)\cdot\Gamma_{\ell i}(s,y;t_0,x_0)\,dy. \tag{2.29}$$

Using (2.29) with $s \doteq \max\{t_1,t_2\}$, it is not restrictive to assume that $t_1 = t_2$. As before, we can interpret the integral in (2.28) as the expected number of times where P overtakes P^* (if $j=1, k=0$) or where P^* overtakes P (if $j=0, k=1$).

To fix the ideas, assume $j = 0$, the other case being similar. Let now $\tau > 0$ be the first time where the fast particle overtakes the slow one. Since $P(\tau) = P^*(\tau)$, $\dot P(\tau) = 1$, $\dot P^*(\tau) = 0$, by Lemma 2 the expected number of crossings with $\dot P^* = 1$ occurring after time τ is $\leq E_0$. By construction, there is at most one such crossing within the time interval $[0,\tau]$. Recalling that $E_1 = E_0 + 1$, this yields (2.28).

Corollary 3. *For every $k\neq j$, $j,k\in\{0,1\}$, one has*

$$\int_{\max\{t_1,t_2\}}^{\infty} \int_{-\infty}^{\infty} \left(\int_{t_1}^{t} e^{s-t}\,\Gamma_{ji}(s,x;t_1,x_1)\,ds\right)\Gamma^*_{kh}(t,x;t_2,x_2)\,dxdt \leq E_1. \tag{2.30}$$

Proof. This follows from Corollary 2, by a change in the order of integration.

Corollary 4. *For every $t_1,t_2\geq 0$ and every $x_1,x_2\in I\!R$, one has*

$$\int_{\max\{t_1,t_2\}}^{\infty} \int_{-\infty}^{\infty} \left(\int_{t_1}^{t} e^{s-t}\,\Gamma^*_{1i}(s,x;t_1,x_1)\,ds\right)\Gamma_{1h}(t,x;t_2,x_2)\,dxdt \leq \frac{2e^{b_+}}{a_-}\,E_1. \tag{2.31}$$

$$\int_{\max\{t_1,t_2\}}^{\infty} \int_{-\infty}^{\infty} \left(\int_{t_1}^{t} e^{s-t}\,\Gamma_{1i}(s,x;t_1,x_1)\,ds\right)\Gamma^*_{1h}(t,x;t_2,x_2)\,dxdt \leq \frac{2e^{b_+}}{a_-}\,E_1. \tag{2.32}$$

Proof. To fix the ideas, assume $t_1 \geq t_2$, the other case being similar. Performing a change of variables, the left hand side of (2.31) can be estimated as

$$\begin{aligned}
&\int_{t_1}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} e^{-\sigma}\Gamma^*_{1i}(s,x;t_1,x_1)\cdot\Gamma_{1h}(s+\sigma,x;t_2,x_2)\,d\sigma dx dt\\
&\qquad\leq \int_{t_1}^{\infty}\int_{-\infty}^{\infty}\left(\sum_{m\geq 0} e^{-m}\int_m^{m+1}\Gamma_{1h}(s+\sigma,x;t_2,x_2)\,d\sigma\right)\Gamma^*_{1i}(s,x;t_1,x_1)\,dxdt\\
&\qquad\leq \sum_{m\geq 0} e^{-m}\frac{e^{b_+}}{a_-}\int_{t_1}^{\infty}\int_{-\infty}^{\infty}\Gamma^*_{1i}(s,x;t_1,x_1)\cdot\Gamma_{0h}(s+m+1,x;t_2,x_2)\,dxdt\\
&\qquad\leq \frac{2e^{b_+}}{a_-}\,E_1.
\end{aligned}$$

Indeed, the function $Z(t) \doteq \Gamma_{0h}(t,x;t_2,x_2)$ satisfies

$$\dot Z(t) = a(t,x)\Gamma_{1h}(t,x;t_2,x_2) - b(t,x)Z(t) \geq a_-\Gamma_{1h}(t,x;t_2,x_2) - b_+Z(t), \qquad Z(s+m)\geq 0,$$

hence

$$Z(s+m+1) = \Gamma_{0h}(s+m+1,x;t_2,x_2) \geq e^{-b_+}a_-\cdot\int_{s+m}^{s+m+1}\Gamma_{1h}(t,x;t_2,x_2)\,dt.$$

3 - Proof of the BV bounds

Consider the system of $2n$ balance laws (1.1). By the linear rescaling of the independent variables

$$t' = t/\varepsilon, \qquad x' = x/\varepsilon,$$

it is not restrictive to assume that $\varepsilon = 1$. We shall henceforth consider the system

$$\begin{cases} u_t + u_x = -F(u) + v,\\ \qquad v_t = F(u) - v. \end{cases} \tag{3.1}$$

Observe that the assumptions (1.8) imply

$$\operatorname{dist}\big(u_0(x),\,K\big) \leq \delta_0, \qquad \Big|F\big(u_0(x)\big) - v_0(x)\Big| = \mathcal{O}(1)\cdot\delta_0$$

for all $x \in I\!R$. Here and in the sequel we use the Landau symbol $\mathcal{O}(1)$ to denote a function whose absolute value remains uniformly bounded.

As in (1.6), we call $l_i(u), r_i(u)$ the left and right eigenvectors of the Jacobian matrix $A(u) = DF(u)$. For a given smooth solution (u,v) of (3.1), call

$$u_x^i \doteq l_i(u)\cdot u_x, \qquad v_x^i \doteq l_i(u)\cdot v_x, \qquad i = 1,\ldots,n,$$

the components of the gradients u_x, v_x w.r.t. the basis of eigenvectors $r_1(u), \dots, r_n(u)$, so that

$$u_x = \sum_i u_x^i r_i(u), \qquad v_x = \sum_i v_x^i r_i(u). \tag{3.2}$$

Differentiating (3.1) we obtain

$$\begin{cases} u_{xt} + u_{xx} = -A(u)u_x + v_x, \\ \qquad v_{xt} = A(u)u_x - v_x. \end{cases} \tag{3.3}$$

In components, we have

$$\begin{aligned} (r_i)_t &= -\sum_j (r_j \bullet r_i) u_x^j - \big(F(u) - v\big) \bullet r_i, \\ (r_i)_x &= \sum_j (r_j \bullet r_i) u_x^j, \end{aligned} \tag{3.4}$$

$$\begin{aligned} u_{xt} &= \sum_i (u_x^i)_t r_i + \sum_i u_x^i (r_i)_t, \qquad & u_{xx} &= \sum_i (u_x^i)_x r_i + \sum_i u_x^i (r_i)_x, \\ v_{xt} &= \sum_i (v_x^i)_t r_i + \sum_i v_x^i (r_i)_t. \end{aligned} \tag{3.5}$$

Hence

$$\begin{aligned} \sum_i (u_x^i)_t r_i + \sum_i (u_x^i)_x r_i &= \sum_i u_x^i \cdot \big(F(u) - v\big) \bullet r_i - \sum_i \big(\eta_i u_x^i - v_x^i\big) r_i, \\ \sum_i (v_x^i)_t r_i - \sum_{i,j} (r_j \bullet r_i) v_x^i u_x^j &= -\sum_i v_x^i \cdot \big(F(u) - v\big) \bullet r_i + \sum_i \big(\eta_i u_x^i - v_x^i\big) r_i. \end{aligned} \tag{3.6}$$

We recall that $\eta_i = \eta_i(u)$ denotes the i-th eigenvalue of the matrix $A(u)$. Taking the product of (3.6) with $l_i(u)$ one obtains a system of $2n$ scalar equations of the form

$$\begin{cases} (u_x^i)_t + (u_x^i)_x = -\eta_i u_x^i + v_x^i - \sum_j G_{ij} u_x^j \\ \qquad (v_x^i)_t = \eta_i u_x^i - v_x^i + \sum_j G_{ij} v_x^j + \sum_{j \neq k} H_{ijk} v_x^j u_x^k \end{cases} \tag{3.7}$$

where

$$G_{ij}(u,v) \doteq l_i \cdot \Big(\big(F(u) - v\big) \bullet r_j \Big), \qquad H_{ijk}(u,v) \doteq l_i \cdot (r_j \bullet r_k).$$

Note that if the terms involving G_{ij} and H_{ijk} are integrable, then the integrals of u_x and v_x are bounded, and we achieve the BV bound of u and v. We thus seek a priori bounds on these terms.

Concerning the term involving H_{ijk}, we notice that the product $v_x^j u_x^k$ only appears for $j \neq k$. Since the wave speeds from different families are strictly different, Corollary 2 will show that the integral of this term is bounded. Therefore, the simple estimate

$$\big| H_{ijk}(u,v) \big| = \mathcal{O}(1) \tag{3.8}$$

will do. Bounds on the terms involving G_{ij} require more work. The analysis given below shows that G_{ij} includes only terms u_x^k with $k \neq j$. The contribution due to these terms can be bounded using Corollaries 2–4.

By the key assumption (1.7) it follows the estimate

$$\big|G_{ij}(u,v)\big| = \mathcal{O}(1) \cdot \Big|\pi^{\perp}_{r_j(u)}\big(F(u) - v\big)\Big|, \tag{3.9}$$

where

$$\pi^{\perp}_{r_j(u)}\mathbf{v} \doteq \mathbf{v} - \big\langle r_j(u),\, \mathbf{v}\big\rangle\, r_j(u)$$

denotes the perpendicular projection of a vector $\mathbf{v} \in I\!R^n$ on the $(n-1)$-dimensional subspace perpendicular to $r_j(u)$. By (3.1) it follows

$$\begin{aligned}
\frac{d}{dt}\Big[\pi^{\perp}_{r_j(u)}\big(F(u) - v\big)\Big] &= \pi^{\perp}_{r_j(u)}\big(A(u)u_t - v_t\big) + \mathcal{O}(1) \cdot \big|F(u) - v\big| \cdot \big|(r_j)_t\big| \\
&= -\sum_{k \neq j} \eta_k u_x^k \pi^{\perp}_{r_j(u)} r_k - \sum_{k \neq j} (1 + \eta_k) \cdot \Big(l_k \cdot \big(F(u) - v\big)\Big) \cdot \pi^{\perp}_{r_j(u)} r_k \\
&\quad + \mathcal{O}(1) \cdot \bigg\{ \sum_{k \neq j} |u_x^k| + \Big|\pi^{\perp}_{r_j(u)}\big(F(u) - v\big)\Big| \bigg\} \cdot \big|F(u) - v\big|.
\end{aligned} \tag{3.10}$$

Fix a value $\bar{u}$ and consider the eigenvalues and the right and left eigenvectors of the Jacobian matrix $A(\bar{u}) = DF(\bar{u})$:

$$\bar{\eta}_k \doteq \eta_k(\bar{u}), \qquad \bar{r}_k \doteq r_k(\bar{u}), \qquad \bar{l}_k \doteq l_k(\bar{u}).$$

For u in a neighborhood of $\bar{u}$, on the $(n-1)$-dimensional space $r_j(u)^{\perp}$ we can put the norm

$$\|z\|_{\dagger} \doteq \sum_{k \neq j} \big|\bar{l}_k \cdot z\big|. \tag{3.11}$$

Let now $z = z(t)$ be a vector moving in the hyperplane $\bar{r}_j^{\perp}$ according to the linear system

$$\dot{z} = -\sum_{h \neq j} (1 + \bar{\eta}_h)(\bar{l}_h \cdot z)\pi^{\perp}_{\bar{r}_j} \bar{r}_h.$$

Recalling (3.2), for $h, k \neq j$ we have

$$\bar{l}_k \cdot \pi^{\perp}_{\bar{r}_j} \bar{r}_h = \bar{l}_k \cdot \bar{r}_h = \begin{cases} 1 & \text{if} \quad h = k, \\ 0 & \text{if} \quad h \neq k. \end{cases}$$

By the assumption (A1) all eigenvalues η_k are strictly positive. We thus have

$$\begin{aligned}
\frac{d}{dt}\big\|z(t)\big\|_{\dagger} &= \sum_{k \neq j} -(1 + \bar{\eta}_k) \cdot \big|\bar{l}_k \cdot z\big| \\
&\leq -\bar{\kappa}\|z\|_{\dagger},
\end{aligned}$$

where

$$\bar{\kappa} \doteq \min_k (1 + \bar{\eta}_k) \;>\; 1.$$

By linearity and continuity, for all u in a neighborhood of $\bar u$ the following holds. If $z = z(t)$ is a vector in $r_j(u)^\perp$ which evolves according to the linear equation

$$\dot z = -\sum_{k\neq j} \big(1+\eta_k(u)\big)\big(\bar l_k(u)\cdot z\big)\pi^\perp_{r_j(u)} r_h(u),$$

then its norm (3.11) satisfies

$$\frac{d}{dt}\big\|z(t)\big\|_\dagger \le -\kappa\|z\|_\dagger \tag{3.12}$$

for some constant $\kappa \in]1,\bar\kappa[$. We now observe that, in (3.10), one has

$$l_k\cdot\big(F(u)-v\big) = l_k\cdot\pi^\perp_{r_j(u)}\big(F(u)-v\big) \qquad k\neq j,$$

simply because $l_k\cdot r_j\equiv 0$ when $k\neq j$. Moreover, by choosing δ_0 small in (1.8), we can assume that the term $\big|F(u)-v\big|$ remains as small as we like, say $<\kappa-1$. From (3.9) and (3.10), using (3.12) with $\pi^\perp_{r_j(u)}\big(F(u)-v\big)$ in the role of z, we thus obtain the estimate

$$\begin{aligned}\big|G_{ij}(t,x)\big| &= \mathcal{O}(1)\cdot\Big|\pi^\perp_{r_j(u)}\big(F(u)-v\big)\Big|\\ &= \mathcal{O}(1)\cdot\Big\|\pi^\perp_{r_j(u)}\big(F(u)-v\big)\Big\|_\dagger\\ &= \mathcal{O}(1)\cdot e^{-t}\Big|\big(F(u)-v\big)(0,x)\Big| + \mathcal{O}(1)\cdot\int_0^t e^{s-t}\sum_{k\neq j}\big|u_x^k(s,x)\big|\,ds,\end{aligned} \tag{3.13}$$

for all $t\ge 0$, $x\in I\!R$.

Motivated by (3.7) and (3.13), we now consider a semilinear system of $2n$ equations for the variables ξ^i,ζ^i, $i=1,\dots,n$:

$$\begin{aligned}\xi^i_t+\xi^i_x+\eta_i(t,x)\xi^i-\zeta^i &= C\cdot\sum_j\left(\rho_0e^{-t}+\sum_{k\neq j}\int_0^t e^{s-t}\xi^k(s,x)\,ds\right)\xi^j,\\ \zeta^i_t-\eta_i(t,x)\xi^i+\zeta^i &= C\cdot\sum_j\left(\rho_0e^{-t}+\sum_{k\neq j}\int_0^t e^{s-t}\xi^k(s,x)\,ds\right)\zeta^j + C\cdot\sum_{j\neq k}\xi^j\zeta^k.\end{aligned} \tag{3.14}$$

By choosing the constant C sufficiently large, a comparison argument yields the following: If u,v provide a smooth solution of (3.1), setting

$$\xi^i(0,x)=\xi^i_0(x)\doteq\big|u_x(0,x)\big|, \qquad \zeta^i(0,x)=\zeta^i_0(x)\doteq\big|v_x(0,x)\big|, \tag{3.15}$$

$$\eta_i(t,x)\doteq\eta_i\big(u(t,x)\big), \qquad \rho_0\doteq\sup_x\Big|F\big(u(0,x)\big)-v(0,x)\Big|, \tag{3.16}$$

the corresponding solution of the Cauchy problem (3.14)-(3.15) provides a pointwise upper bound for derivative components u^i_x, v^i_x, namely

$$\big|u^i_x(t,x)\big|\le\xi^i(t,x), \qquad \big|v^i_x(t,x)\big|\le\zeta^i(t,x) \qquad \text{for all } t,x. \tag{3.17}$$

In the remaining part of the proof we thus seek an a priori bound on the $\mathbf{L}^1$ norm of solutions of (3.14)-(3.15).

Using Duhamel's principle, the solution of the system (3.14) can be written in the form

$$\begin{aligned}\xi^i(t,x) = &\int \Big\{\Gamma^i_{10}(t,x;0,y)\zeta^i_0(y) + \Gamma^i_{11}(t,x;0,y)\xi^i_0(y)\Big\}\,dy \\ &+ \int_0^t \int \Big\{\Gamma^i_{10}(t,x;s,y)\mathcal{Q}^i_0(s,y) + \Gamma^i_{11}(t,x;s,y)\mathcal{Q}^i_1(s,y)\Big\}\,dyds,\end{aligned} \tag{3.18}$$

$$\begin{aligned}\zeta^i(t,x) = &\int \Big\{\Gamma^i_{00}(t,x;0,y)\zeta^i_0(y) + \Gamma^i_{01}(t,x;0,y)\xi^i_0(y)\Big\}\,dy \\ &+ \int_0^t \int \Big\{\Gamma^i_{00}(t,x;s,y)\mathcal{Q}^i_0(s,y) + \Gamma^i_{01}(t,x;s,y)\mathcal{Q}^i_1(s,y)\Big\}\,dyds,\end{aligned} \tag{3.19}$$

where we denoted by $\mathcal{Q}^i_1, \mathcal{Q}^i_0$ the right hand sides of (3.14). Define

$$\Psi(t) \doteq \sum_{i=1}^n \Big(\big\|\xi^i(t)\big\|_{\mathbf{L}^1} + \big\|\zeta^i(t)\big\|_{\mathbf{L}^1}\Big),$$

$$\mathcal{Q}(t) \doteq \sum_{i=1}^n \Big(\big\|\mathcal{Q}^i_1(t)\big\|_{\mathbf{L}^1} + \big\|\mathcal{Q}^i_0(t)\big\|_{\mathbf{L}^1}\Big),$$

From (3.18)-(3.19) it follows

$$\Psi(t) \le \Psi(0) + \int_0^t \mathcal{Q}(s)\,ds, \tag{3.20}$$

To give an estimate on $\int_0^t \mathcal{Q}(s)\,ds$, we notice that it can be written as the sum of three terms:

$$\int_0^t \mathcal{Q}(s)\,ds = J_1 + J_2 + J_3$$

where

$$J_1 \doteq C\int_0^t \sum_{i=1}^n \int_R \sum_j \rho_0 e^{-s}\Big(\xi^j(s,x) + \zeta^j(s,x)\Big)\,dx\,ds,$$

$$J_2 \doteq C\int_0^t \sum_{i=1}^n \int_R \sum_j \sum_{k\neq j}\Big(\int_0^s e^{\tau-s}\xi^k(\tau,x)\,d\tau\Big)\Big(\xi^j(s,x) + \zeta^j(s,x)\Big)\,dx\,ds,$$

$$J_3 \doteq C\int_0^t \sum_{i=1}^n \int_R \sum_{j\neq k} \xi^j(s,x)\zeta^k(s,x)\,dx\,ds.$$

Using (3.18)-(3.19) and Corollaries 2, 3 and 4, we obtain the following estimates.

$$J_1 \le C\int_0^t \rho_0 e^{-s}\Psi(s)\,ds,$$

$$J_2 \le C\Big(\Psi(0) + \int_0^t \mathcal{Q}(s)\,ds\Big)^2,$$

$$J_3 \le C\Big(\Psi(0) + \int_0^t \mathcal{Q}(s)\,ds\Big)^2,$$

for some positive constant C. In turn, this yields

$$\int_0^t \mathcal{Q}(s)\,ds \le C' \cdot \int_0^t \rho_0 e^{-s}\Psi(s)\,ds + C' \cdot \left(\Psi(0) + \int_0^t \mathcal{Q}(s)\,ds\right)^2, \tag{3.21}$$

for a suitable constant C'. For any given $\varepsilon > 0$, if ρ_0 and $\Psi(0)$ are suffciently small, by (3.20)-(3.21) it follows $\Psi(t) \le \varepsilon$ for all $t \ge 0$. This proves the uniform BV bounds on the solutions of (3.1).

4 - Stability estimates

The goal of this section is to prove that the solutions of (3.1) depend continuously on the initial data, with a Lipschitz constant uniform in time. For this purpose, we first derive uniform estimates on solutions of the corresponding variational equation. Let u, v be a solution of (3.1) with small total variation. Consider a perturbation of the form $u + \varepsilon U$, $v + \varepsilon V$. Substituting this in (3.1) and letting $\varepsilon \to 0$ we obtain a linearized equation for the infinitesimal perturbations U, V:

$$\begin{cases} U_t + U_x = -A(u)U + V, \\ \qquad V_t = A(u)U - V. \end{cases} \tag{4.1}$$

As usual, $A(u) = DF(u)$ denotes the Jacobian of F at u. Call

$$U^i \doteq l_i(u) \cdot U, \qquad V^i \doteq l_i(u) \cdot V, \qquad i = 1, \dots, n,$$

the components of U, V w.r.t. the basis of eigenvectors $r_1(u), \dots, r_n(u)$, so that

$$U = \sum_i U^i r_i(u), \qquad V = \sum_i V^i r_i(u).$$

In terms of these variables, (4.1) can be written as

$$\begin{aligned} \sum_i U_t^i r_i + \sum_i U_x^i r_i &= \sum_i U^i \cdot \big(F(u) - v\big) \bullet r_i - \sum_i \big(\eta_i U_x^i - V_x^i\big) r_i, \\ \sum_i V_t^i r_i - \sum_{i,j} (r_j \bullet r_i) V^i u_x^j &= -\sum_i V^i \cdot \big(F(u) - v\big) \bullet r_i + \sum_i \big(\eta_i U^i - V^i\big) r_i. \end{aligned} \tag{4.2}$$

Taking the iner products of (4.2) with $l_1, \dots, l_n$ we obtain a system of $2n$ scalar equations in the variables U^i, V^i:

$$\begin{cases} U_t^i + U_x^i + \eta_i U^i - V^i = -\sum_j G_{ij} U^j \\ \qquad V_t^i - \eta_i U^i + V^i = -\sum_{j \ne k} H_{ijk} u_x^k V^j + \sum_j G_{ij} V^j \end{cases} \tag{4.3}$$

Using Duhamel's principle, the solution of the system (4.3) can be written in the form

$$\begin{aligned} U^i(t,x) = &\int \Big\{ \Gamma_{10}^i(t,x;0,y) V_0^i(y) + \Gamma_{11}^i(t,x;0,y) U_0^i(y) \Big\}\, dy \\ &+ \int_0^t \int \Big\{ \Gamma_{10}^i(t,x;s,y) \Lambda_0^i(s,y) + \Gamma_{11}^i(t,x;s,y) \Lambda_1^i(s,y) \Big\}\, dyds, \end{aligned} \tag{4.4}$$

$$
\begin{aligned}
V^i(t,x) = \int \Big\{ \Gamma^i_{00}(t,x;0,y) V^i_0(y) + \Gamma^i_{01}(t,x;0,y) U^i_0(y) \Big\}\, dy \\
+ \int_0^t \int \Big\{ \Gamma^i_{00}(t,x;s,y) \Lambda^i_0(s,y) + \Gamma^i_{01}(t,x;s,y) \Lambda^i_1(s,y) \Big\}\, dyds,
\end{aligned}
\tag{4.5}
$$

where we denoted by Λ^i_1, Λ^i_0 the right hand sides of (4.3). Following the same steps as for deriving (3.21), and using (4.3-4.5), (3.13), (3.15) and (3.18-3.19), we deduce an a priori estimate of the form

$$
\begin{aligned}
\|U(t)\|_{\mathbf{L}^1} + \|V(t)\|_{\mathbf{L}^1} \ \leq\ & \|U(0)\|_{\mathbf{L}^1} + \|V(0)\|_{\mathbf{L}^1} \\
& + C_0 \cdot \left(\rho_0 e^{-t} + \Psi(0) + \int_0^t \mathcal{Q}(s)\, ds \right) \cdot \max_{s\in[0,t]} \Big(\|U(s)\|_{\mathbf{L}^1} + \|V(s)\|_{\mathbf{L}^1} \Big)
\end{aligned}
\tag{4.6}
$$

Assuming that ρ_0 and the total variations of u and of v are sufficiently small, we have

$$
C_0 \cdot \left(\rho_0 + \Psi(0) + \int_0^\infty \mathcal{Q}(s)\, ds \right) < 1.
$$

This implies a uniform estimate on the size of tangent vectors:

$$
\|U(t)\|_{\mathbf{L}^1} + \|V(t)\|_{\mathbf{L}^1} \ \leq\ L \cdot \Big(\|U(0)\|_{\mathbf{L}^1} + \|V(0)\|_{\mathbf{L}^1} \Big). \tag{4.7}
$$

If now two initial data (u_0, v_0) and $(\tilde u_0, \tilde v_0)$ are given, following [BB] we construct the smooth path

$$
\theta \mapsto (u^\theta_0, v^\theta_0) \doteq \theta (u_0, v_0) + (1-\theta)(\tilde u_0, \tilde v_0) \qquad \theta \in [0,1]. \tag{4.8}
$$

Calling $t \mapsto (u^\theta, v^\theta)(t,\cdot)$ the solution of (3.1) with initial data (u^θ_0, v^θ_0), we can write

$$
\begin{aligned}
\|u(t) - \tilde u(t)\|_{\mathbf{L}^1} + \|v(t) - \tilde v(t)\|_{\mathbf{L}^1} &\leq \int_0^1 \left\| \frac{du^\theta(t)}{d\theta} \right\|_{\mathbf{L}^1} + \left\| \frac{dv^\theta(t)}{d\theta} \right\|_{\mathbf{L}^1} d\theta \\
&\leq L \cdot \int_0^1 \left\| \frac{du^\theta(0)}{d\theta} \right\|_{\mathbf{L}^1} + \left\| \frac{dv^\theta(0)}{d\theta} \right\|_{\mathbf{L}^1} d\theta \\
&= L \cdot \Big(\|u(0) - \tilde u(0)\|_{\mathbf{L}^1} + \|v(0) - \tilde v(0)\|_{\mathbf{L}^1} \Big).
\end{aligned}
\tag{4.9}
$$

Indeed, the tangent vector

$$
(U^\theta, V^\theta)(t,x) \doteq \left(\frac{du^\theta}{d\theta},\ \frac{dv^\theta}{d\theta} \right)(t,x)
$$

is a solution of the linearized Cauchy problem (4.1), hence it satisfies (4.7) for every $t \geq 0$ and every θ. This completes the proof of Theorem 1.

Remark. Since we are only assuming that the initial data are in $\mathbf{L}^1_{\mathrm{loc}}$, the right hand side of (1.9) may well be infinite, in which case the inequality does not provide any information. However, observing that all propagation speeds lie in the interval $[0,1]$, from the previous analysis we obtain the sharper estimate

$$
\int_a^b \Big\{ \big|u(t,x) - \tilde u(t,x)\big| + \big|v(t,x) - \tilde v(t,x)\big| \Big\}\, dx \ \leq\ L \cdot \int_{a-t}^b \Big\{ \big|u_0(x) - \tilde u_0(x)\big| + \big|v_0(x) - \tilde v_0(x)\big| \Big\}\, dx. \tag{4.10}
$$

This holds for every interval $[a,b]$ and every couple of solutions of (1.1) with small total variation. Performing the rescaling $t \mapsto t/\varepsilon$, $x \mapsto x/\varepsilon$, one checks that the constant L is independent of $\varepsilon > 0$.

5 - Proof of the relaxation limit

In this section we give a proof of Theorem 2.

For every fixed $\varepsilon > 0$, call $\big(u^\varepsilon(t), v^\varepsilon(t)\big) \doteq S^\varepsilon_t(u_0,v_0)$ the value at time t of the solution of (1.1)-(1.2). This defines a continuous semigroup $S^\varepsilon : \mathcal{D} \times [0,\infty[\mapsto \mathcal{D}$, satisfying the conditions (i)–(iii) stated in Theorem 1. Observe, however, that for general initial data the map $t \mapsto S^\varepsilon_t(u_0,v_0)$ may not be uniformly Lipschitz continuous as a function of time, as $\varepsilon \to 0$.

We shall first establish Theorem 2 restricted to the domain of equilibrium states

$$\mathcal{D}_0 \doteq \Big\{(u_0,v_0) \in \mathcal{D}\,; \quad v_0(x) = F\big(u_0(x)\big)\Big\}. \tag{5.1}$$

Lemma 3. *There exists a uniform Lipschitz constant L' such that, for every $(u_0,v_0) \in \mathcal{D}_0$, $\varepsilon > 0$ and $s,t \geq 0$, one has*

$$\big\|S^\varepsilon_t(u_0,v_0) - S^\varepsilon_s(u_0,v_0)\big\|_{\mathbf{L}^1(I\!\!R;I\!\!R^n)} \leq L'|t-s|. \tag{5.2}$$

Proof. The result follows by an easy adaptation of an argument by Natalini [N1]. Set $\bar u \doteq u_0(\bar x)$ for some $\bar x$ as in (1.8), and let $\bar l_i, \bar r_i$ be left and right eigenvectors of the matrix $DF(\bar u)$, normalized as in (1.6). Define

$$Z(t) \doteq \sum_{i=1}^n \int_{-\infty}^{\infty} \Big|\bar l_i \cdot \big(v(t,x) - F(u(t,x))\big)\Big|\,dx. \tag{5.3}$$

From (1.1), since u takes values in a small neighborhood of $\bar u$ and all of the eigenvalues of $DF(\bar u)$ are positive, we deduce

$$\dot Z(t) = \mathcal{O}(1) \cdot \Big\{ -\frac{Z(t)}{\varepsilon} + \text{Tot.Var.}\big(u(t)\big)\Big\}. \tag{5.4}$$

Since we are assuming that $Z(0) = 0$, the uniform boundedness of the total variation implies an estimate of the form

$$Z(t) \leq C'\varepsilon$$

for all $\varepsilon > 0$, $t \geq 0$. This clearly yields the uniform Lipschitz continuity of the solution of (1.1), as a function of time with values in $\mathbf{L}^1(I\!\!R;\, I\!\!R^{2n})$.

By the uniform Lipschitz estimates (5.2) and the uniform bounds on the total variation, for each given initial data $(u_0,v_0) \in \mathcal{D}_0$ we can now select a subsequence $\varepsilon_k \to 0$ such that the corresponding solutions $(u^{\varepsilon_k}, v^{\varepsilon_k})$ converge in $\mathbf{L}^1_{\text{loc}}$. Using a standard diagonal procedure we construct a further subsequence $\varepsilon'_k \to 0$ such that, for a countable dense set of initial data $(u_{0,\nu}, v_{0,\nu})$, the corresponding solutions $(u^{\varepsilon_k}_\nu, v^{\varepsilon_k}_\nu)$ converge in $\mathbf{L}^1_{\text{loc}}$ to some functions (u_ν, v_ν). By the uniform Lipschitz continuity w.r.t. the initial data, stated at (4.10), the limit

$$\big(\tilde u(t), \tilde v(t)\big) = \lim_{k\to\infty} S^{\varepsilon'_k}_t(u_0,v_0) \doteq S_t(u_0,v_0) \tag{5.5}$$

is well defined in $\mathbf{L}^1_{\mathrm{loc}}$ for every $(u_0, v_0) \in \mathcal{D}_0$, $t \geq 0$. This yields a semigroup which is uniformly Lipschitz continuous as a map from $\mathcal{D}_0 \times [0, \infty[$ into $\mathcal{D}_0$.

We claim that the limit functions $\tilde{u}, \tilde{v}$ can be characterized by (1.4)-(1.5) and (1.11). Indeed, consider the map

$$(u_0;\, t) \mapsto S'_t u_0 \doteq u(t), \tag{5.6}$$

where $\big(u(t),\, v(t)\big)$ denotes the limit in (5.5), with initial data $v_0 = F(u_0)$. By (4.10), S' is a uniformly Lipschitz continuous semigroup defined on the domain

$$\mathcal{D}' \doteq \Big\{ u_0 \in \mathbf{L}^1_{\mathrm{loc}}(I\!R;\, I\!R^n)\,; \quad \big(u_0,\, F(u_0)\big) \in \mathcal{D}_0 \Big\}.$$

To show that S' coincides with the unique Standard Riemann Semigroup generated by the system (1.4), it suffices to establish the following property (see Chapter 9 in [B]).

(P) Let $u_0 \in \mathcal{D}_0$ be a piecewise constant initial data. Moreover, assume that, at each point x_α where u_0 is discontinuous, the left and right states $u^-_\alpha = u_0(x_\alpha -)$, $u^+_\alpha = u_0(x_\alpha +)$ lie on a single rarefaction curve, say

$$u^+_\alpha = u^-_\alpha + \sigma_\alpha r_{k_\alpha}(u^-_\alpha) \tag{5.7}$$

for some $\sigma_\alpha \in I\!R$, $k_\alpha \in \{1, \ldots, n\}$. Then, for all $t \in [0, \tau]$ sufficiently small, the trajectory $S'_t u_0$ coincides with the solution of (1.4), (1.2) obtained by piecing together the solutions of the Riemann problems at the various points of jump.

The proof of the above property is straightforward. Indeed, let u_0 be piecewise constant with *simple jumps*, i.e. satisfying (5.7). Then, for $t \geq 0$ sufficiently small, for every $\varepsilon > 0$, in a neighborhood of each point of jump the component $u^\varepsilon = u^\varepsilon(t, x)$ of the solution of (1.1) takes values on the straight line

$$\gamma_\alpha \doteq \big\{ u^-_\alpha + \sigma r_{k_\alpha}(u^-_\alpha)\ ;\ \ \sigma \in I\!R \big\},$$

while $v^\varepsilon = v^\varepsilon(t, x)$ takes values on the straight line $F(\gamma_\alpha)$. In this case, the problem reduces to particular case of convergence for a 2×2 system, for which the result is well known [N1].

We conclude the proof of Theorem 2 by extending the convergence result to all initial data $(u_0, v_0) \in \mathcal{D}$. In this general case, let $t \mapsto \big(u(t), v(t)\big)$ the solution of (1.4)-(1.5) with initial data implicitly defined by (1.11). Moreover, call $\big(u^\varepsilon(t), v^\varepsilon(t)\big) \doteq S^\varepsilon_t(u_0, v_0)$ the solution of (1.1)-(1.2). Finally, define $\big(\tilde{u}^\varepsilon(t), \tilde{v}^\varepsilon(t)\big)$ as the solution of

$$\begin{cases} u_t = -\dfrac{1}{\varepsilon}\big(F(u) - v\big), \\[2mm] v_t = \dfrac{1}{\varepsilon}\big(F(u) - v\big), \end{cases} \tag{5.8}$$

We now have

$$\big|\tilde{u}^\varepsilon(t, x) - u(0, x)\big| + \big|\tilde{v}^\varepsilon(t, x) - v(0, x)\big| = \mathcal{O}(1) \cdot e^{-t/\varepsilon} \qquad\qquad x \in I\!R, \tag{5.9}$$

and, for an arbitrary interval $[a, b]$,

$$\int_a^b \Big\{ \big|\tilde{u}^\varepsilon(t, x) - u^\varepsilon(t, x)\big| + \big|\tilde{v}^\varepsilon(t, x) - v^\varepsilon(t, x)\big| \Big\}\, dx = \mathcal{O}(1) \cdot (b - a)t. \tag{5.10}$$

Choosing $t = \sqrt{\varepsilon}$, (5.9)-(5.10) yield

$$\int_a^b \Big(\big| u^\varepsilon(\sqrt{\varepsilon}, x) - u(0,x) \big| + \big| v^\varepsilon(\sqrt{\varepsilon}, x) - v(0,x) \big| \Big)\, dx = \mathcal{O}(1) \cdot \big((b-a) e^{-1/\sqrt{\varepsilon}} + \sqrt{\varepsilon} \big). \tag{5.11}$$

In other words, as $\varepsilon \to 0$ there holds

$$\big(u^\varepsilon(\sqrt{\varepsilon}), v^\varepsilon(\sqrt{\varepsilon}) \big) \to \big(u(0), v(0) \big) \qquad \text{in } \mathbf{L}^1_{\text{loc}}(I\!\!R;\, I\!\!R^{2n}). \tag{5.12}$$

Since the convergence result has already been established for initial data in $\mathcal{D}_0$, we can apply $S^\varepsilon_{t-\sqrt{\varepsilon}}$ to both sides of (5.12) and let $\varepsilon \to 0$. The uniform continuity of the semigroups S^ε w.r.t. the initial conditions implies

$$\Big\{ \big(u^\varepsilon(t),\ v^\varepsilon(t) \big) - \big(u(t - \sqrt{\varepsilon}),\ v(t - \sqrt{\varepsilon}) \big) \Big\} \to 0 \qquad \text{in } \mathbf{L}^1_{\text{loc}}(I\!\!R;\, I\!\!R^{2n})$$

for every $t > 0$. Recalling (5.2), we conclude

$$\big(u^\varepsilon(t), v^\varepsilon(t) \big) \to \big(u(t), v(t) \big) \qquad \text{in } \mathbf{L}^1_{\text{loc}}(I\!\!R;\, I\!\!R^{2n}),$$

completing the proof of Theorem 2.

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