Pointwise Error Estimates For Relaxation Approximations to Conservation Laws

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Abstract

We obtain sharp *pointwise* error estimates for relaxation approximation to scalar conservation laws with piecewise smooth solutions. We first prove that the first order partial derivatives for the perturbation solutions are uniformly upper bounded (the so-called Lip^+ stability). An one-sided interpolation inequality between classical L^1 error estimates and Lip^+ stability bounds enables us to convert a global L^1 result into a (non-optimal) local estimate. Optimal error bounds on the weighted error then follow from the maximum principle for weakly coupled hyperbolic systems. The main difficulties in obtaining the Lip^+ stability and the optimal pointwise errors are how to construct appropriate "difference functions" so that the maximum principle can be applied.

Contents

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1 Introduction

Consider the following stiff relaxation system:

$$\left\{\begin{array}{l} u_t^\epsilon + v_x^\epsilon = 0 \\ \\ v_t^\epsilon + \alpha u_x^\epsilon = -\frac{1}{\epsilon}\Big(v^\epsilon - f(u^\epsilon)\Big), \qquad \epsilon > 0\,, \end{array}\right. \tag{1.1}$$

for $(x,t) \in \mathbf{R} \times (0,\infty)$. The initial conditions associated with the above system are

$$u^\epsilon(x,0) = u_0(x), \qquad v^\epsilon(x,0) = f(u_0(x)). \tag{1.2}$$

The system (1.1) can be regarded as a singular perturbation problem, and the solutions are expected to converge, as ϵ tends to zero, to the entropy solutions of the equilibrium equation

$$\left\{\begin{array}{ll} u_t + f(u)_x = 0, & v = f(u) \\ \\ u(x,0) = u_0(x), & v(x,0) = f(u_0(x))\,. \end{array}\right. \tag{1.3}$$

The relaxation limit for 2×2 nonlinear systems of conservation laws was first studied by Liu [6], who justified some nonlinear stability criteria for diffusion waves, expansion waves and traveling waves. A general mathematical framework was analyzed for the nonlinear systems by Chen, Levermore and Liu [1]. Consult [12] for a bird's eye view of recent results in this direction.

The presence of relaxation mechanisms is widespread in both the continuum mechanics as well as the kinetic theory contexts. Relaxation is known to provide a subtle dissipative mechanism for discontinuities against the destabilizing effect of nonlinear response [6]. The relaxation models can be loosely interpreted as discrete velocity kinetic equations. The relaxation parameter, ϵ, plays the role of the mean free path and the system models the macroscopic conservation law. In that sense they are a discrete velocity analogue of the kinetic equations introduced by Perthame and Tadmor [15] and Lions et al. [11].

The relaxation approximation can also be used to construct numerical approximations to the equilibrium conservation laws. In [7], Jin and Xin developed a class of first- and second-order nonoscillatory numerical schemes for the conservation law (1.3), based on the relaxation approximation (1.1). Since the relaxation approximation (1.1) is formally an $\mathcal{O}(\epsilon)$ perturbation to (1.3), they can compute (1.1) without resolving the computational grid to $\mathcal{O}(\epsilon)$. Indeed, in their final form, it is seen that the relaxation parameter in these relaxation schemes plays no role. In particular, their $\epsilon = 0$-limit in the first order case coincide with the central Lax-Friedrichs scheme, and their $\epsilon = 0$-limit in the second-order version corresponds to the central scheme of Nessyahu & Tadmor [13]. The nonoscillatory central schemes introduced in [13] is based on staggered evolution of the reconstructed averages – a high-order sequel to the celebrated first-order Lax-Friedrichs (staggered) scheme. An extension of the high-resolution central scheme to multidimensional problems can be found in [5]. The central schemes are simple, efficient, stable and enjoy the main advantage of avoiding costly (upwind) Riemann solvers. In this context, relaxation schemes offer yet another way to derive a whole class of high-resolution Riemann-solvers-free central schemes. The key is how to discretize the relaxation, as outlined in [7].

There have been many recent studies concerning the asymptotic convergence of the relaxation systems to the corresponding equilibrium conservation laws as the rate of the relaxation tends to zero. Most of these results deal with either large-time, nonlinear asymptotic stability or the zero relaxation limit for Cauchy problems. Tveito and Winther [18, 26] provided an $\mathcal{O}(\epsilon^{1/3})$-rate of convergence for some relaxation systems with nonlinear convection arising in chromatography. Katsoulakis and Tzavaras [8] introduced a class of relaxation systems, the contractive relaxation systems, and established an $\mathcal{O}(\sqrt{\epsilon})$ error bound in the case that the equilibrium equation is a scalar multi-dimensional one. The approaches in [8, 18, 26] are based on the extensions of Kruzhkov and Kuznentzov-type error estimates [10]. Kurganov and Tadmor [9] studied convergence and error estimates for a class of relaxation systems, including (1.1) and the one arising in chromatography, and concluded an $\mathcal{O}(\epsilon)$ order of convergence for scalar convex conversation laws. The novelty of their approach is the use of a weak Lip'-measure of the error, which allow them to obtain sharp error estimates[1]. For the relaxation system (1.1), Natalini [12] proved that the solutions to the relaxation system converges strongly to the unique entropy solution of (1.2) as $\epsilon \to 0$. Based on a general framework developed in [23, 25], the first-order rate of convergence for (1.1) is established in the case when its equilibrium solutions are piecewise smooth [24], which is an improvement on the $\mathcal{O}(\sqrt{\epsilon})$ error bounds [8, 9]. The boundary layer effect in the small relaxation limit to the equilibrium scalar conservation laws was investigated in [27]. The existence and uniqueness for the initial-boundary value problems are established.

The convergence and the rate of convergence mentioned above are mostly in the L^1 sense. It is understood that the L^1 error estimate is a global one, while in many practical cases we are interested in the *local* behavior of $u(x,t)$. Consequently, when the error is measured by the L^1-norm, there is a loss of information due to the poor resolution of shock waves in $u(x,t)$. Several authors have investigated pointwise error estimates: For a system of conservation laws, Goodman and Xin [4] proved that the viscosity methods approximating piecewise smooth solutions with finitely many *noninteracting* shocks have a local $\mathcal{O}(\epsilon)$ error bound away from the shocks. A general convergence theory for 1D scalar convex conservation laws was developed by Tadmor and co-authors, see e.g. [13, 19]. They proved that when measured in the weak Lip'-topology, the convergence rate of the viscous solution is of order $\mathcal{O}(\epsilon)$ in case of rarefaction-free initial data and is of order $\mathcal{O}(\epsilon|\ln\epsilon|)$ in the general case. These weak Lip'-estimates are then converted into the usual L^1 error bounds of order one-half, and moreover, pointwise error estimates of order one-third, $\mathcal{O}(\epsilon^{1/3})$, are derived. Pointwise error analysis for finite difference methods to scalar and system of conservation laws is given recently by Engquist and Yu [3], Engquist and Sjogreen [2]. In [20], the authors provided the *optimal* pointwise convergence rate for the viscosity approximation. They used an innovative idea which enables them to convert a *global* L^1-error estimate into a *local* error estimate. Using this local error estimate and a bootstrap argument they proved that the viscosity approximation satisfies a *pointwise* error estimate of order $\mathcal{O}(\epsilon)$ for all but finitely many neighborhoods of shock discontinuities, each of width $\mathcal{O}(\epsilon)$. The previous results for the optimal order one convergence rates, in both L^1 and L^∞ spaces, are all based on a matching method and traveling wave solutions, see e.g. [3, 4, 23]. The approach introduced in [20] does not follow the characteristics but instead makes use of the energy method, and hence can be extended to other types of approximate solutions, e.g. [21].

The question that we address in this paper is concerned with the rate of pointwise convergence for the relaxation approximation (1.1). The main purpose is to establish the *optimal* pointwise convergence. The proof of our results is based on two ingredients:

- An one-sided interpolation inequality between the L^1 error estimates and Lip^+ stability bounds; and
- A comparison theorem (the maximum principle) for weakly coupled hyperbolic systems.

In Section 2, we will review some existed results and theory useful for obtaining our error bounds. As mentioned earlier, the L^1 error bounds for the relaxation approximation have been established by several authors. A rigorous Lip^+ stability bounds for the relaxation approximation will be established in Section 3. In Section 4 we first consider the case when there is only one shock for

[1]Here and below, Lip' stands for the dual of Lip topology

the solutions to the equilibrium equation (1.3), i.e. the set of shock S consists only one smooth curve. In this case, we show that

$$\mathrm{dist}(x,S)|u(x,t)-u^\epsilon(x,t)| \leq C\epsilon. \tag{1.4}$$

It implies that $|u(x,t)-u^\epsilon(x,t)| \leq C(h)\epsilon$ for (x,t) with $\mathcal{O}(h)$ distance away from the set of shocks. The result (1.4) can be generalized to finitely many shocks with *possible collisions*. In the final section, we discuss the possible extensions of the results obtained in this work.

2 Preliminaries

Several useful results for the relaxation approximation will be reviewed in this section. We begin by introducing the sub-characteristic condition.

2.1 Sub-characteristic condition

The main stability criterion can be (formally) derived by using the Chapman-Enskog expansion for the stiff relaxation system (1.1)

$$u_t^\epsilon + f(u^\epsilon)_x = \epsilon\Big((\alpha - f'(u^\epsilon)^2)u_x^\epsilon\Big)_x + O(\epsilon^2). \tag{2.1}$$

The above equation will be of parabolic type under the following stability condition, i.e. the sub-characteristic condition [28]:

$$\alpha > f'(u^\epsilon)^2\,. \tag{2.2}$$

In a recent paper, Natalini [12] provided a rigorous analysis for (1.1) that leads to the sub-characteristic condition (2.2) under some assumptions on α and the initial data u_0. More precisely, we state his results as follows.

Lemma 2.1 *If α in the relaxation equation (1.1) and the initial data u_0 in (1.2) satisfy*

$$\sqrt{\alpha} > M(N_0), \tag{2.3}$$

where N_0 and M_0 are defined by

$$\begin{cases} N_0 := \max\left(\|u_0\|_{L^\infty},\, \|f(u_0)\|_{L^\infty}\right), & M(N_0) := \sup_{|\zeta|\leq B(N_0)} |f'(\zeta)|, \quad \textit{with} \\ B(N_0) := 2N_0 + F(2N_0), & F(N_0) := \sup_{|\zeta|\leq N_0} |f(\zeta)|\,, \end{cases} \tag{2.4}$$

then the relaxation system (1.1) with initial condition (1.2) satisfies the sub-characteristic inequality (2.2). Moreover, the solution (u^ϵ, v^ϵ) for (1.1) is uniformly bounded with respect to ϵ:

$$|u^\epsilon(x,t)| \leq B(N_0), \qquad |v^\epsilon(x,t)| \leq \sqrt{\alpha}B(N_0) \qquad \textit{for} \quad (x,t)\in \mathbf{R}\times(0,\infty). \qquad \blacksquare \tag{2.5}$$

Throughout of this paper we will assume that the condition (2.3) is satisfied. Under this assumption, the sub-characteristic inequality is guaranteed and will be used to establish the Lip^+ stability and the pointwise error bounds.

2.2 Global L^1 error bounds

The L^1-error analysis for the relaxation approximation method has been presented by several authors. For general data, an optimal L^1-rate can be found in [8],[9], for example. This optimal $\mathcal{O}(\sqrt{\epsilon})$ L^1-rate is overviewed in §3.2, based on the *Lip'* approach taken in [9] (for a more general class of relaxation models). For piecewise-smooth data, the *optimal* $\sim \mathcal{O}(\epsilon)$ L^1-convergence rate was recently obtained by Teng [24]. We state his results as follows.

Lemma 2.2 *Assume α in the relaxation equation (1.1) and the initial data u_0 in (1.2) satisfy the conditions stated in Lemma 2.1. Assume that the solutions to the scalar convex conservation law (1.3) are piecewise smooth. Let (u^ϵ, v^ϵ) be the solutions of the relaxation problems (1.1)-(1.2). Then the following error estimate holds:*

$$\sup_{0\le t\le T}\Big(\|u^\epsilon(\cdot,t)-u(\cdot,t)\|_{L^1(\mathbf{R})}+\|v^\epsilon(\cdot,t)-v(\cdot,t)\|_{L^1(\mathbf{R})}\Big)\le C(T)\epsilon\,|\ln\epsilon|, \tag{2.6}$$

where $v = f(u)$. If there is no initial central rarefaction wave and no new generated shocks, then the error bound is improved to

$$\sup_{0\le t\le T}\Big(\|u^\epsilon(\cdot,t)-u(\cdot,t)\|_{L^1(\mathbf{R})}+\|v^\epsilon(\cdot,t)-v(\cdot,t)\|_{L^1(\mathbf{R})}\Big)\le C(T)\epsilon. \qquad \blacksquare \tag{2.7}$$

We shall utilize these L^1 global error bounds to derive the pointwise error estimate (1.4). The order of the global L^1 error bounds will not affect the general $\mathcal{O}(\epsilon)$-pointwise result (1.4), but it will affect the choice of the distance function d, see [20] for detail. Thus, improved L^1-error bounds lead to sharper description of the shock layer, with an optimal shock layer of size $\sim \mathcal{O}(\epsilon)$ corresponding to the piecewise-smooth cases (2.6) and (2.7).

2.3 An interpolation inequality

We let $\|\bullet\|_{Lip^+}$ denote the Lip^+-seminorm

$$\|w\|_{Lip^+} := \operatorname{ess\,sup}_{x\neq y}\left[\frac{w(x)-w(y)}{x-y}\right]^+$$

where $[w]^+ = H(w)w$, with $H(\bullet)$ the Heaviside function. The following lemma is due to Nessyahu & Tadmor [14, §2]; its proof can be found in [20].

Lemma 2.3 *Assume that $z \in L^1 \cap Lip^+(\mathcal{I})$, and $w \in C^1_{loc}(x-\delta, x+\delta)$ for an interior x such that $(x-\delta, x+\delta) \subset \mathcal{I}$. Then the following estimate holds:*

$$|z(x)-w(x)| \le Const\cdot\left[\frac{1}{\delta}\|z-w\|_{L^1}+\delta\{\|z\|_{Lip^+(x-\delta,x+\delta)}+|w|_{C^1_{loc}(x-\delta,x+\delta)}\}\right]. \tag{2.8}$$

In particular, if the size of the smoothness neighborhood for w can be chosen so that

$$\delta \sim \|z-w\|^{1/2}_{L^1(\mathcal{I})}\cdot(\|z\|_{Lip^+}+|w|_{C^1_{loc}})^{-1/2}\le\frac{1}{2}|\mathcal{I}| \tag{2.9}$$

then the following estimate holds:

$$|z(x)-w(x)|\le Const\cdot\|z-w\|^{1/2}_{L^1(\mathcal{I})}\cdot\Big[\|z\|_{Lip^+}+|w|_{C^1_{loc}(x-\delta,x+\delta)}\Big]^{1/2}. \qquad \blacksquare \tag{2.10}$$

Thus, (2.10) tells us that if the global L^1-error $\|z-w\|_{L^1}$ is small, then the pointwise error $|z(x)-w(x)|$ is also small wherever w_x is bounded. This does not require the C^1-boundedness of z – the weaker one-sided Lip^+ bound of z will suffice.

2.4 A comparison lemma

The following maximum principle for weakly coupled hyperbolic systems plays an important role in this work. Consider the following system

$$\left\{\begin{array}{l}\partial_t u_1 + \lambda_1(x,t)\partial_x u_1 = \alpha_{11}(x,t)u_1 + \alpha_{12}(x,t)u_2 + \beta_1(x,t)\,, \\[2ex] \partial_t u_2 + \lambda_2(x,t)\partial_x u_2 = \alpha_{21}(x,t)u_1 + \alpha_{22}(x,t)u_2 + \beta_2(x,t)\,,\end{array}\right. \tag{2.11}$$

where α_{ij}, $1 \leq i, j \leq 2$, are also of functions of x and t. The initial conditions $u_j(x,0)$ are given, $j = 1, 2$. The following lemma can be found in [16].

Lemma 2.4 *Consider the Cauchy problem for the weakly coupled hyperbolic systems (2.11) in a domain $E := \mathbf{D} \times (0,T)$. If the coefficient functions and the initial conditions for (2.11) satisfy*

$$\left\{\begin{array}{lll}\alpha_{12}(x,t) \geq 0, & \alpha_{21}(x,t) \geq 0, & (x,t) \in E \\[2ex] \beta_1(x,t) \leq 0, & \beta_2(x,t) \leq 0, & (x,t) \in E \\[2ex] u_1(x,0) \leq 0, & u_2(x,0) \leq 0, & x \in \mathbf{D} \\[2ex] u_1(x,t) \leq 0, & u_2(x,t) \leq 0, & (x,t) \in \partial\mathbf{D} \times (0,T)\end{array}\right. \tag{2.12}$$

then the solution of (2.11) satisfies the following estimates:

$$u_1(x,t) \leq 0, \qquad u_2(x,t) \leq 0, \qquad \textit{for} \quad (x,t) \in E. \qquad \blacksquare \tag{2.13}$$

The two important results, the Lip^+ stability and the optimal pointwise error bounds are all based on the above lemma. The main difficulty is how to construct appropriate object functions u_1, u_2 so that above lemma can be suitablly applied.

3 Lip^+ stability and local error bounds

In this section, we assume that f is strictly convex, i.e.

$$f''(u) \geq \beta > 0, \qquad \text{for} \quad u \in \mathbf{R}, \tag{3.1}$$

and that u_0 is Lip^+-bounded,

$$\|u_0\|_{Lip^+} < \infty. \tag{3.2}$$

3.1 Lip^+ stability

We will show that the family $\{u^\epsilon(x,t)\}_{\epsilon>0}$ is Lip^+-stable. Assume first that $u_0 \in C_0^1(\mathbf{R})$. This implies, by the standard regularity theory for the semilinear hyperbolic problems, that $(u^\epsilon, v^\epsilon) \in C^1(\mathbf{R} \times (0,T))$ for some $T > 0$. Differentiating the equations (1.1) with respect to x gives

$$(u_x^\epsilon)_t + (v_x^\epsilon)_x = 0, \tag{3.3}$$

$$(v_x^\epsilon)_t + \alpha(u_x^\epsilon)_x = -\frac{1}{\epsilon}\Big(v_x^\epsilon - f'(u^\epsilon)u_x^\epsilon\Big). \tag{3.4}$$

By doing $\sqrt{\alpha}\times(3.3)+(3.4)$ and $\sqrt{\alpha}\times(3.3)-(3.4)$, the above system can be put in the following diagonal form:

$$(\sqrt{\alpha}u_x^\epsilon + v_x^\epsilon)_t + \sqrt{\alpha}(\sqrt{\alpha}u_x^\epsilon + v_x^\epsilon)_x = -\frac{1}{\epsilon}\Big(v_x^\epsilon - f'(u^\epsilon)u_x^\epsilon\Big),$$
$$(\sqrt{\alpha}u_x^\epsilon - v_x^\epsilon)_t - \sqrt{\alpha}(\sqrt{\alpha}u_x^\epsilon - v_x^\epsilon)_x = \frac{1}{\epsilon}\Big(v_x^\epsilon - f'(u^\epsilon)u_x^\epsilon\Big).$$

Letting

$$\bar{p} = \sqrt{\alpha}u_x^\epsilon + v_x^\epsilon, \qquad \bar{q} = \sqrt{\alpha}u_x^\epsilon - v_x^\epsilon$$

and by using the above results yield

$$\left\{\begin{array}{l} \bar{p}_t + \sqrt{\alpha}\bar{p}_x = \frac{1}{2\epsilon}\left(\frac{f'(u^\epsilon)}{\sqrt{\alpha}} - 1\right)\bar{p} + \frac{1}{2\epsilon}\left(\frac{f'(u^\epsilon)}{\sqrt{\alpha}} + 1\right)\bar{q}, \\ \bar{q}_t - \sqrt{\alpha}\bar{q}_x = -\frac{1}{2\epsilon}\left(\frac{f'(u^\epsilon)}{\sqrt{\alpha}} - 1\right)\bar{p} - \frac{1}{2\epsilon}\left(\frac{f'(u^\epsilon)}{\sqrt{\alpha}} + 1\right)\bar{q}. \end{array}\right. \tag{3.5}$$

We further introduce the transformations:

$$\left\{\begin{array}{l} p = \bar{p} - \Big(\sqrt{\alpha} + f'(u^\epsilon)\Big)\|u_0\|_{Lip^+}, \\ q = \bar{q} - \Big(\sqrt{\alpha} - f'(u^\epsilon)\Big)\|u_0\|_{Lip^+}. \end{array}\right. \tag{3.6}$$

Applying the above transformations to (3.5) gives

$$\begin{aligned} p_t + \sqrt{\alpha}p_x &= \frac{1}{2\epsilon}\left(\frac{f'(u^\epsilon)}{\sqrt{\alpha}} - 1\right)p + \frac{1}{2\epsilon}\left(\frac{f'(u^\epsilon)}{\sqrt{\alpha}} + 1\right)q \\ &\quad - f''(u^\epsilon)\Big(u_t^\epsilon + \sqrt{\alpha}u_x^\epsilon\Big)\|u_0\|_{Lip^+}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} q_t - \sqrt{\alpha}q_x &= -\frac{1}{2\epsilon}\left(\frac{f'(u^\epsilon)}{\sqrt{\alpha}} - 1\right)p - \frac{1}{2\epsilon}\left(\frac{f'(u^\epsilon)}{\sqrt{\alpha}} + 1\right)q \\ &\quad + f''(u^\epsilon)\Big(u_t^\epsilon - \sqrt{\alpha}u_x^\epsilon\Big)\|u_0\|_{Lip^+}. \end{aligned} \tag{3.8}$$

It follows from (1.1), $u_t^\epsilon + v_x^\epsilon = 0$, that

$$u_t^\epsilon + \sqrt{\alpha}u_x^\epsilon = -v_x^\epsilon + \sqrt{\alpha}u_x^\epsilon = q + \Big(\sqrt{\alpha} - f'(u^\epsilon)\Big)\|u_0\|_{Lip^+},$$
$$u_t^\epsilon - \sqrt{\alpha}u_x^\epsilon = -v_x^\epsilon - \sqrt{\alpha}u_x^\epsilon = -p - \Big(\sqrt{\alpha} + f'(u^\epsilon)\Big)\|u_0\|_{Lip^+}.$$

The above observation, together with (3.7) and (3.8), lead to

$$\begin{aligned} p_t + \sqrt{\alpha}p_x &= \frac{1}{2\epsilon}\left(\frac{f'(u^\epsilon)}{\sqrt{\alpha}} - 1\right)p + \frac{1}{2\epsilon}\left(\frac{f'(u^\epsilon)}{\sqrt{\alpha}} + 1 - 2\epsilon f''(u^\epsilon)\|u_0\|_{Lip^+}\right)q \\ &\quad - f''(u^\epsilon)\Big(\sqrt{\alpha} - f'(u^\epsilon)\Big)\|u_0\|_{Lip^+}^2, \end{aligned} \tag{3.9}$$

$$\begin{aligned} q_t - \sqrt{\alpha}q_x &= -\frac{1}{2\epsilon}\left(\frac{f'(u^\epsilon)}{\sqrt{\alpha}} - 1 + 2\epsilon f''(u^\epsilon)\|u_0\|_{Lip^+}\right)p - \frac{1}{2\epsilon}\left(\frac{f'(u^\epsilon)}{\sqrt{\alpha}} + 1\right)q \\ &\quad - f''(u^\epsilon)\Big(\sqrt{\alpha} + f'(u^\epsilon)\Big)\|u_0\|_{Lip^+}^2. \end{aligned} \tag{3.10}$$

It follows from the sub-characteristic condition (2.2) that (3.9)-(3.10) is a weakly coupled hyperbolic system and its coefficients satisfy the requirements in (2.12) provided that ϵ is sufficiently small. We now check the initial conditions. First checking $p(x,0)$:

$$\begin{aligned} p(x,0) &= \sqrt{\alpha}u_0' + f'(u_0)u_0' - (\sqrt{\alpha} + f'(u_0))\|u_0\|_{Lip^+} \\ &= (\sqrt{\alpha} + f'(u_0))(u_0' - \|u_0\|_{Lip^+}) \le 0. \end{aligned}$$

Similarly, we have

$$q(x,0) = \left(\sqrt{\alpha} - f'(u_0)\right)(u_0' - \|u_0\|_{Lip^+}) \leq 0.$$

Using Lemma 2.4, we obtain

$$p(x,t) \leq 0, \qquad q(x,t) \leq 0, \qquad \text{for} \quad (x,t) \in \mathbf{R} \times (0,T). \tag{3.11}$$

It follows from (3.6) that

$$u_x^\epsilon = \frac{1}{2\sqrt{\alpha}}(p+q) + \|u_0\|_{Lip^+}\,.$$

This identity, together with (3.11), yields

$$u_x^\epsilon \leq \|u_0\|_{Lip^+}\,,$$

which is the Lip^+ stability (**??**) for u^ϵ when it is smooth. Finally, we extend our result to general initial data by the following standard procedure:

$$u_0^\delta(x) := \int \psi_\delta(x-y)u_0(y)dy,$$

where ψ_δ is a compactly supported nonnegative unit mass mollifier,

$$\psi_\delta(x) = \frac{1}{\delta}\psi\left(\frac{x}{\delta}\right), \qquad \int_{-\infty}^{\infty} \psi_\delta(x)dx = 1.$$

It is obvious that if $\|u_0\|_{Lip^+} < \infty$ then $\|u_0^\delta\|_{Lip^+}$ is also bounded. Consider the 2×2 stiff relaxation system (1.1) with the smooth initial data:

$$u^\epsilon(x,0) = u_0^\delta(x)\,, \qquad v^\epsilon(x,0) = f(u_0^\delta(x)). \tag{3.12}$$

Using the above proof we know that there exists a $T > 0$, such that

$$\|u^{\epsilon,\delta}(\bullet,t)\|_{Lip^+} \leq \|u_0^\delta\|_{Lip^+}, \qquad \text{for} \quad t \in (0,T), \tag{3.13}$$

where $u^{\epsilon,\delta}$ is one component of the solution to (1.1) and (3.12). Letting $\delta \to 0+$ in (3.13) gives

$$\|u^\epsilon(\bullet,t)\|_{Lip^+} \leq \|u_0\|_{Lip^+}, \qquad \text{for} \quad t \in (0,T).$$

By standard continuation arguments for time, we can extend the desired Lip^+ stability result (**??**) for u^ϵ to any finite time interval.

We summarize what we have shown by stating the following:

Theorem 3.1 *Assume α in the relaxation equation* (1.1) *and the initial data u_0 in* (1.2) *satisfy the conditions stated in Lemma 2.1. Assume $f'' > 0$. Then the family of solutions $\{u^\epsilon(x,t)\}_{\epsilon>0}$, given by the relaxation system* (1.1) *and initial data* (1.2), *are Lip^+-stable. Moreover, the functions $\{\sqrt{\alpha}u^\epsilon + v^\epsilon\}_{\epsilon>0}$ and $\{\sqrt{\alpha}u^\epsilon - v^\epsilon\}_{\epsilon>0}$ are also Lip^+-stable.* ■

3.2 Error estimates based on Lip' theory

Equipped with the Lip^+-stability, one can derive $O(\sqrt{\epsilon})$ L^1- and local error bounds using the Lip' theory presented in [19]. The case for a general family of relaxation models was outlined in [9]; here is a brief overview for the particular case of the relaxation model (1.1).

To begin with, we derive the modified equation satisfied by u^ϵ. Consider the second equation in (1.1),

$$u_t^\epsilon + \alpha v_x^\epsilon = -\frac{1}{\epsilon}(v^\epsilon - f(u^\epsilon)). \tag{3.14}$$

We differentiation with respect to x, expressing $f(u^\epsilon)_x$ in terms of u^ϵ and v^ϵ – consult (3.3), $f(u^\epsilon)_x = v_x^\epsilon + \epsilon(v_t^\epsilon + \alpha u_x^\epsilon)_x$. Inserted into the first equation of (1.1) we find

$$u_t^\epsilon + f(u^\epsilon)_x = \epsilon(v_t^\epsilon + \alpha u_x^\epsilon)_x. \tag{3.15}$$

The term on the right is the *truncation error*. The main result in [19, 14] shows that when measured in the Lip'-norm, the global error, $u^\epsilon - u$, is governed by the truncation+initial errors

$$\|u^\epsilon - u\|_{Lip'} \leq Const\left[\epsilon\|(v_t^\epsilon + \alpha u_x^\epsilon)_x\|_{Lip'} + \|u_0^\epsilon - u_0\|_{Lip'}\right]. \tag{3.16}$$

In our case of (1.2), there is no initial error. To measure the Lip'-size of the truncation error, we proceed along the lines of [9, Example 3]: we differentiate (1.1) with respect to t, obtaining

$$(u_t^\epsilon)_t + (v_t^\epsilon)_x = 0, \tag{3.17}$$

$$(v_t^\epsilon)_t + \alpha(u_t^\epsilon)_x = -\frac{1}{\epsilon}\left(v_t^\epsilon - f'(u^\epsilon)u_t^\epsilon\right). \tag{3.18}$$

Performing $\sqrt{\alpha}\times(3.17)+(3.18)$ and $\sqrt{\alpha}\times(3.17)-(3.18)$, then the above system can be put in the following diagonal form in terms of the characteristic variables, $\bar{r} := \sqrt{\alpha}u_t^\epsilon + v_t^\epsilon$ and $\bar{s} := \sqrt{\alpha}u_t^\epsilon - v_t^\epsilon$,

$$\begin{cases} \bar{r}_t + \sqrt{\alpha}\bar{r}_x = \frac{1}{2\epsilon}\left(\frac{f'(u^\epsilon)}{\sqrt{\alpha}} - 1\right)\bar{r} + \frac{1}{2\epsilon}\left(\frac{f'(u^\epsilon)}{\sqrt{\alpha}} + 1\right)\bar{s}, \\ \\ \bar{s}_t - \sqrt{\alpha}\bar{s}_x = -\frac{1}{2\epsilon}\left(\frac{f'(u^\epsilon)}{\sqrt{\alpha}} - 1\right)\bar{r} - \frac{1}{2\epsilon}\left(\frac{f'(u^\epsilon)}{\sqrt{\alpha}} + 1\right)\bar{s}. \end{cases} \tag{3.19}$$

Integrate the first equation against $sgn(\bar{r})$, the second against $sgn(\bar{s})$, and add; in view of the sub-characteristic condition (2.2) we find (compare [9, equation 4.10])

$$\|\bar{r}\|_{L^1} + \|\bar{s}\|_{L^1} \leq \|\bar{r}_0\|_{L^1} + \|\bar{s}_0\|_{L^1} \tag{3.20}$$

If the initial are prepared in the sense that $\|v_0^\epsilon - f(u_0^\epsilon)\|_{L^1} = \mathcal{O}(\epsilon)$ (– and in fact, in our case we ignore initial errors by restricting attention to (1.2)), then initial time derivatives

$$\|v^\epsilon(\cdot, t=0)_t\|_{L^1} + \|(u^\epsilon(\cdot, t=0)_t\|_{L^1}$$

are bounded, and by (3.20), they remain bounded in later time. In particular, $\|v_t^\epsilon(\cdot,t)\|_{L^1} \leq Const.$ This, together with the BV bound of u^ϵ (– which follows from the Lip^+ stability), imply that the Lip'-size of the local truncation error is of order ϵ

$$\|\epsilon(v_t^\epsilon + \alpha u_x^\epsilon)_x\|_{Lip'} \leq \epsilon(\|v_t^\epsilon\|_{L^1} + \|u_x^\epsilon\|_{L^1}) \leq \mathcal{O}(\epsilon), \tag{3.21}$$

and consequently, (3.16) implies that the Lip' size of the global error is of the same order of $\mathcal{O}(\epsilon)$. If we interpolate between this Lip' bound and the BV boundedness of $u^\epsilon - u$, we arrive at an L^1 convergence rate estimate of order $\mathcal{O}(\sqrt{\epsilon})$,

$$\|u^\epsilon - u\|_{L^1} \leq Const\|u^\epsilon - u\|_{Lip'}^{1/2} \cdot \|u^\epsilon - u\|_{BV}^{1/2} \leq Const.\sqrt{\epsilon}.$$

The Lip^+ stability of u^ϵ enables us to convert this global estimate into a local one: using Lemma 2.3 with $(z,w)=(u^\epsilon,u)$ we find, consult (2.10)

$$|u^\epsilon(x,t)-u(x,t)| \le Const.\delta \cdot |u|_{C^1_{loc}(x-\delta,x+\delta)}, \quad \delta \sim \epsilon^{1/4}. \tag{3.22}$$

There are several possible improvements.

- If one utilize the $\mathcal{O}(\epsilon)$-Lip' error estimate (– instead of the L^1 estimate of order $\mathcal{O}(\sqrt{\epsilon})$), then this pointwise error estimate can be further improved outside a smaller shock region of size $\delta \sim \epsilon^{1/3}$, consult [14, Corollary 2.4].
- Moreover, for piecewise smooth data one has an L^1 error estimate of order ϵ, [24], and the above arguments yield pointwise error estimate of order $\delta \sim \|u^\epsilon - u\|_{L^1}^{1/2} = \mathcal{O}(\sqrt{\epsilon})$; this will be outlined in §3.3.
- Finally, in the §4 we will present a bootstrap argument for a further improvement of this pointwise error estimate; we prove an pointwise error of order δ outside the a shock zone of optimal size $\delta \sim \epsilon$.

Remark. In (1.3) we restrict attention to initial data which are *excatly* matched with their assumed limit, $v_0^\epsilon = f(u_0^\epsilon)$. It is clear from the above discussion that Lip' error bound of order $\mathcal{O}(\epsilon)$ holds for more general initial data, which are only required to be prepared so that $\|u_0^\epsilon - u_0\|_{Lip'} + \|v_0^\epsilon - f(u_0^\epsilon)\|_{L^1} = \mathcal{O}(\epsilon)$.

3.3 A non-optimal pointwise error estimate

In the following section, we will consider the case that the entropy solution for (1.3) is piecewise smooth, with finitely many shock discontinuities. Thus, if we let $S(t)$ denote the singular support of $u(\bullet,t)$, then it consists of finitely many shocks, $S(t) := \{(x,t)\,|\,x = X_k(t)\}$, each of which satisfies the Rankine-Hugoniot and the Lax conditions:

$$X_k' = \frac{[f(u(X_k,t)]}{[u(X_k,t)]}, \tag{3.23}$$

$$f'(u(X_k(t)-,t)) > X_k'(t) > f'(u(X_k(t)+,t))\,. \tag{3.24}$$

We note in passing that many practical initial data lead to finite number of shocks (see, e.g. [17, 22]), and in this case one has a global L^1-error bound of order ϵ, (2.7). Next we consider the characteristic variables, $\sqrt{\alpha}u^\epsilon \pm v^\epsilon$: It follows that their L^1 convergence rate from their limiting value $\sqrt{\alpha}u \pm v$ with $v = f(u)$ is also or der (ϵ); Moreover, Theorem 3.1 implies the Lip^+ boundedness

$$\sqrt{\alpha}u_x^\epsilon + v_x^\epsilon \le C, \qquad \sqrt{\alpha}u_x^\epsilon - v_x^\epsilon \le C.$$

We can now apply the interpolation inequality (2.10), with $(z,w) = (\sqrt{\alpha}u^\epsilon \pm v^\epsilon, \sqrt{\alpha}u \pm f(u))$. We obtain the following pointwise error bound (see also [20]):

$$\begin{cases} |\sqrt{\alpha}u^\epsilon + v^\epsilon - (\sqrt{\alpha}u + f(u))| \le C\sqrt{\epsilon}\,, \\ \\ |\sqrt{\alpha}u^\epsilon - v^\epsilon - (\sqrt{\alpha}u - f(u))| \le C\sqrt{\epsilon}\,, \quad \text{for} \quad dist(x,S(t)) \ge \sqrt{\epsilon}. \end{cases} \tag{3.25}$$

It follows from the above results that

$$\begin{cases} |u^\epsilon(x,t) - u(x,t)| \le C\sqrt{\epsilon}\,, \\ \\ |v^\epsilon(x,t) - f(u(x,t))| \le C\sqrt{\epsilon}\,, \quad \text{for} \quad dist(x,S(t)) \ge \sqrt{\epsilon}. \end{cases} \tag{3.26}$$

Although the above pointwise local estimate is not optimal, it will suffice to derive the optimal error bound by a bootstrap argument which employs the comparison Lemma 2.4.

4 Pointwise error estimate

The key tool in obtaining the optimal pointwise error estimate is to use Lemma 2.4. In order to use it, we need to construct appropriate functions u_1 and u_2, in this section they are error functions, such that they satisfy (2.11) and those conditions listed in the lemma. To illustrate the main idea of our proof, we first concentrate on the case that there is only one shock curve.

4.1 The case of a single shock

We assume that there is a smooth curve, $S(t) := \{(x,t)\,|\,xX(t)\}$, so that $u(x,t)$ is C^2-smooth at any point $x \neq X(t)$. There are two smooth regions $x > X(t)$ and $x < X(t)$. We consider the pointwise error estimate in the region $x > X(t)$; the results for $x < X(t)$ can be obtained in a similar way. The function $\phi(x) \in C^2([0,\infty))$ satisfies

$$\phi(x) \sim \begin{cases} x, & \text{if} \quad 0 \le x \ll 1 \\ 1, & \text{if} \quad x \gg 1. \end{cases}$$

More precisely, the function ϕ satisfies

$$\begin{cases} \phi(0) = 0, \qquad \phi'(x) > 0, \quad \phi(x) \le x, \quad \text{for} \quad x > 0; \\ x\phi'(x) \le \phi(x), \quad \text{for} \quad x \ge 0; \\ |\phi^{(k)}(x)| \le 1, \quad x \ge 0, \end{cases} \tag{4.1}$$

e.g., $\phi(x) = 1 - e^{-x}$. Roughly speaking, the weighted function behaves like $\phi(x) \sim \min(|x|, 1)$.

We define two functions, which roughly speaking are the errors for u^ϵ and v^ϵ, in the following form

$$U = u^\epsilon - u - \sigma(x,t), \qquad V = v^\epsilon - f(u+\sigma) + \epsilon\Psi(x,t) \tag{4.2}$$

where

$$\begin{cases} \sigma = \epsilon d e^{\gamma t}/\phi(x - X(t)), \\ \Psi = \Big(\alpha - f'(u+\sigma)f'(u)\Big)u_x - \Big(\sqrt{\alpha} + f'(u+\sigma)\Big)\Big(\sqrt{\alpha} - \dot{X}(t)\Big)\sigma\phi'/\phi \end{cases} \tag{4.3}$$

In the above definitions, $\phi = \phi(x - X(t))$ is the so-called weighted distance to the shock set [2]. Also, in the above definitions, d and γ are two positive numbers to be determined.

Remark. It is seen from (4.2) and (4.3) that U is the error function for u^ϵ with first order correction $O(\epsilon)$, while V is the error function for v^ϵ with first- and second-order corrections.

4.1.1: The basic idea. In order to put the error functions U and V to the framework of Lemma 2.4, we further let

$$p = \sqrt{\alpha}U + V, \qquad q = \sqrt{\alpha}U - V \tag{4.4}$$

and will verify the following estimates:

- (C1): for $x \ge X(0) + \sqrt{\epsilon}$,

$$p(x,0) \le 0, \qquad q(x,0) \le 0.$$

[2] In the case $x < X(t)$, the weighted distance is $\phi(X(t) - x)$. In other words, the weighted distance for any choice of x is $\phi(|x - X(t)|)$ in the single shock case.

- (C2): for all $t \ge 0$,

$$p(X(t)+\sqrt{\epsilon},t) \le 0, \qquad q(X(t)+\sqrt{\epsilon},t) \le 0.$$

- (C3): The functions p and q satisfies the following equations:

$$\left\{ \begin{array}{l} p_t + \sqrt{\alpha} p_x = \alpha_{11} p + \alpha_{12} q + \beta_1(x,t), \\ \\ q_t - \sqrt{\alpha} q_x = \alpha_{21} p + \alpha_{22} q + \beta_2(x,t), \end{array} \right. \tag{4.5}$$

  where for $x \ge X(t)+\sqrt{\epsilon}$ the coefficients α_{12} and α_{21} are nonnegative, and the source terms β_1 and β_2 are non-positive.

The idea is to choose d and γ sufficiently large so that Lemma 2.4 can be applied. The estimates (C1) and (C2) are satisfied by choosing sufficiently large d. Then for the time interval $0 < t \le T_1 := \gamma^{-1}$, i.e.

$$e^{\gamma t} \le e \tag{4.6}$$

we show that (C3) is satisfied by choosing sufficiently large γ.

After showing that (C1)-(C3) are verified for $t \in [0, T_1]$, we know that the error bounds for u^ϵ and v^ϵ can be established for $0 \le t \le T_1$. We can then use $u^\epsilon(x,T_1)$ and $u(x,T_1)$ as new initial data and repeat the same procedure to obtain the local error bounds for $T_1 < t \le T_2$. By this standard continuation arguments, we can obtain the error bounds up to $t = T$.

4.1.2: The verification for (C1). Observe that for $x \ge X(t)+\sqrt{\epsilon}$

$$|\Psi(x,t)| \le C + C\epsilon^{-1/2}\sigma, \qquad \sigma \ge \epsilon d\,. \tag{4.7}$$

Since $u^\epsilon(x,0) = u(x,0) = u_0(x)$ and $v^\epsilon(x,0) = f(u_0)$, we have, for $x \ge X(0)+ \ge \sqrt{\epsilon}$,

$$\begin{aligned} p(x,0) &= -\sqrt{\alpha}\sigma(x,0) + f(u_0) - f(u_0+\sigma) + \epsilon\Psi \\ &= \Big(- \sqrt{\alpha} + f'(\bullet)\Big)\sigma(x,0) + \epsilon\,\Psi \\ &\le -C_1\sigma + C\epsilon + C\sqrt{\epsilon}\sigma \\ &\le \Big(- C_1 + C/d + C\sqrt{\epsilon}\Big)\,\sigma \le 0 \end{aligned}$$

provided that d is sufficiently large and ϵ sufficiently small. Similarly, we can show that $q(x,0) < 0$ for $x > X(0)+\sqrt{\epsilon}$ with sufficiently large d and small ϵ.

4.1.3: The verification for (C2). It follows from the non-optimized local error estimates (3.26) that

$$u^\epsilon - u = O(\sqrt{\epsilon}), \qquad v^\epsilon - f(u) = O(\sqrt{\epsilon}), \quad \text{for} \quad x \ge X(t)+\sqrt{\epsilon}.$$

It is also observed that $\sigma(x,t) \ge Cd\sqrt{\epsilon}$ for $x = X(t)+\sqrt{en}$. From the definition of p we obtain

$$\begin{aligned} p(X(t)+\sqrt{\epsilon},t) &= \sqrt{\alpha}O(\sqrt{\epsilon}) - \sqrt{\alpha}\sigma(x,t) + O(\sqrt{\epsilon}) + f(u) - f(u+\sigma) + O(\epsilon) + O(\epsilon\sigma) \\ &= O(\sqrt{\epsilon}) + \Big(- \sqrt{\alpha} - f'(\bullet)\Big)\sigma(X(t)+\sqrt{\epsilon},t) + O(\epsilon) + O(\epsilon\sigma) \\ &\le C\sqrt{\epsilon} - C_1\,C\sqrt{\epsilon}\,d + C\epsilon \le 0 \end{aligned}$$

provided that d is sufficiently large. Similarly, we can show that $q(X(t)+\sqrt{\epsilon},t) \le 0$.

4.1.4: The verification for (C3). By the definitions of U and V, as well as the relaxation equations (1.1) and its equilibrium equation (1.3), we have

$$
\begin{aligned}
U_t + V_x &= \underbrace{u_t^\epsilon + v_x^\epsilon}_{=0} - \underbrace{(u_t + f(u)_x)}_{=0} + (f(u)_x - f(u+\sigma)_x) - \sigma_t + \epsilon \Psi_x \\
&= \Big(f'(u) - f'(u+\sigma)\Big) u_x - f'(u+\sigma)\sigma_x - \sigma_t + \epsilon \Psi_x \\
&= -f''(\bullet) u_x \sigma - f'(u+\sigma)\sigma_x - \sigma_t + \epsilon \Psi_x
\end{aligned}
\tag{4.8}
$$

Similarly, we calculate $V_t + \alpha U_x$ and obtain:

$$
\begin{aligned}
V_t + \alpha U_x &= v_t^\epsilon + \alpha u_x^\epsilon - f(u+\sigma)_t - \alpha u_x + \epsilon \Psi_t - \alpha \sigma_x \\
&= \frac{1}{\epsilon}\Big(f(u^\epsilon) - v^\epsilon\Big) - f'(u+\sigma)(u_t + \sigma_t) - \alpha u_x + \epsilon \Psi_t - \alpha \sigma_x \\
&= \frac{1}{\epsilon}\Big(f(u^\epsilon) - f(u+\sigma)\Big) - \frac{1}{\epsilon}\Big(v^\epsilon - f(u+\sigma) + \epsilon \Psi\Big) \\
&\qquad - f'(u+\sigma)\sigma_t + \epsilon \Psi_t - \alpha \sigma_x + \Psi + \Big(f'(u+\sigma) f'(u) - \alpha\Big) u_x \\
&= \frac{1}{\epsilon} f'(\bullet) U - \frac{1}{\epsilon} V - f'(u+\sigma)\sigma_t + \epsilon \Psi_t - \alpha \sigma_x \\
&\qquad - \Big(\sqrt{\alpha} + f'(u+\sigma)\Big)\Big(\sqrt{\alpha} - \dot{X}(t)\Big)\frac{\sigma \phi'}{\phi}.
\end{aligned}
\tag{4.9}
$$

By the definition of p, $p = \sqrt{\alpha} U + V$, we obtain from (4.8) and (4.9) that p satisfies the first equation in (4.5) with

$$
\begin{cases}
\alpha_{11} = \frac{1}{2\epsilon}\Big(\frac{f'(\bullet)}{\sqrt{\alpha}} - 1\Big), \qquad \alpha_{12} = \frac{1}{2\epsilon}\Big(\frac{f'(\bullet)}{\sqrt{\alpha}} + 1\Big) \\
\beta_1(x,t) = -\sqrt{\alpha} f''(\bullet) u_x \sigma - \Big(\sqrt{\alpha} + f'(u+\sigma)\Big)(\sqrt{\alpha}\sigma_x + \sigma_t) \\
\qquad\qquad + \epsilon\Big(\sqrt{\alpha}\Psi_x + \Psi_t\Big) - \Big(\sqrt{\alpha} + f'(u+\sigma)\Big)(\sqrt{\alpha} - \dot{X}(t))\frac{\sigma\phi'}{\phi}.
\end{cases}
\tag{4.10}
$$

We observe that $\alpha_{12} \geq 0$. Now we need to verify that $\beta_1(x,t) \leq 0$ for $x \geq X(t) + \sqrt{\epsilon}$ provided that γ is sufficiently large. It follows from the definitions of σ that

$$
\sigma_x = -\frac{\sigma \phi'}{\phi}, \qquad \sigma_t = \gamma \sigma + \frac{\sigma \phi'}{\phi}\dot{X}.
\tag{4.11}
$$

Using the above results and the definition of Ψ gives

$$
\begin{aligned}
\Psi_x &= O(1) + O\left(\frac{\sigma^2}{\phi^2}\right) + O\left(\frac{\sigma}{\phi^2}\right) \\
&\leq C + C\epsilon^{-1}\sigma^2 + C\epsilon^{-1}\sigma \leq C + C\epsilon^{-1}\sigma \qquad \Big(\text{using (4.6)}\Big); \\
\Psi_t &= O(1) + O\left(\frac{\sigma^2}{\phi^2}\right) + O\left(\frac{\sigma}{\phi^2}\right) + O\left(\frac{\sigma^2}{\phi}\right) \\
&\leq C + C\epsilon^{-1}\sigma + C\sigma.
\end{aligned}
$$

The above estimates, together with (4.11) and the definition of $\beta_1(x,t)$, yield

$$
\begin{aligned}
\beta_1(x,t) &= -\sqrt{\alpha} f''(\bullet) u_x \sigma - \Big(\sqrt{\alpha} + f'(u+\sigma)\Big)\gamma\sigma + \epsilon\Big(\sqrt{\alpha}\Psi_x + \Psi_t\Big) \\
&\leq C\sigma - C_1\gamma\sigma + C\epsilon.
\end{aligned}
\tag{4.12}
$$

From the definition of ψ we have $\epsilon = \phi e^{-\gamma t}\sigma/d \le C\sigma/d$. This, together with (4.12), gives

$$\beta_1(x,t) \le \Big(C - C_1\gamma + C/d\Big)\sigma \le 0, \qquad \text{for} \quad x \ge X(t) + \sqrt{\epsilon} \tag{4.13}$$

provided that γ is sufficiently large.

It follows from (4.8)-(4.9) and the definition of q, $q\sqrt{\alpha}U - V$, that q satisfies the second equation in (4.5) with

$$\begin{cases} \alpha_{21} = \frac{1}{2\epsilon}\left(1 - \frac{f'(\bullet)}{\sqrt{\alpha}}\right), \qquad \alpha_{22} = -\frac{1}{2\epsilon}\left(1 + \frac{f'(\bullet)}{\sqrt{\alpha}}\right) \\ \beta_2(x,t) = -\sqrt{\alpha}f''(\bullet)u_x\sigma + \Big(\sqrt{\alpha} - f'(u+\sigma)\Big)\Big(\sqrt{\alpha}\sigma_x - \sigma_t\Big) + \epsilon\Big(\sqrt{\alpha}\Psi_x - \Psi_t\Big) \\ \qquad +\Big(\sqrt{\alpha} + f'(u+\sigma)\Big)(\sqrt{\alpha} - \dot{X})\frac{\sigma\phi'}{\phi}. \end{cases} \tag{4.14}$$

It is seen that $\alpha_{21} \ge 0$. Moreover, it follows from (4.11) that

$$\beta_2(x,t) = -\sqrt{\alpha}f''(\bullet)u_x\sigma - \Big(\sqrt{\alpha} - f'(u+\sigma)\Big)\gamma\sigma + \epsilon\Big(\sqrt{\alpha}\Psi_x + \Psi_t\Big) + J \tag{4.15}$$

where the last term J is defined by

$$J = 2\sqrt{\alpha}\Big(f'(u+\sigma) - \dot{X}\Big)\frac{\sigma\phi'}{\phi}. \tag{4.16}$$

Let $u_+ := u(X(t)+0,t)$. Using Lax geometrical entropy condition, $\dot{X}(t) \ge f'(u_+)$, gives

$$\begin{aligned} & f'(u+\sigma) - \dot{X}(t) \le f'(u+\sigma) - f'(u_+) \\ = \;& f''(\bullet)\Big(u(x,t) - u(X(t)+0,t)\Big) + f''(\bullet)\sigma \\ = \;& f''(\bullet)u_x(\bullet,t)(x - X(t)) + f''(\bullet)\sigma \end{aligned}$$

Using the fact that $x\phi'(x) \le \phi(x)$ gives

$$(x - X(t))\phi'(x - X(t)) \le C\,\phi(x - X(t)).$$

By the definition of J and the above observations, we have

$$J \le C\sigma + C\frac{\sigma^2}{\phi} \le C\sigma$$

where in the last step we have used the fact $\sigma/\phi \le Cd$. It follows from the above results and the equation for β_2, (4.15), that $\beta_2 \le 0$ provided that γ is sufficiently large.

In summary, if d and $\gamma = \gamma(d)$ are sufficiently large, then the comparison lemma, Lemma 2.4, gives

$$p(x,t) \le 0, \qquad q(x,t) \le 0, \qquad \text{for} \quad x \ge X(t) + \sqrt{\epsilon}. \tag{4.17}$$

Similarly, by changing $\phi(x - X(t))$ in (4.3) to $\phi(X(t) - x)$ will handle the case for $x \le X(t) - \sqrt{\epsilon}$. We will then obtain the following results

$$p(x,t) \le 0, \qquad q(x,t) \le 0, \qquad \text{for} \quad x \le X(t) - \sqrt{\epsilon}. \tag{4.18}$$

Since

$$u^\epsilon - u - \epsilon d e^{\gamma t}/\phi = \frac{1}{2\sqrt{\alpha}}(p+q),$$

the estimates (4.17) and (4.18) yield

$$u^\epsilon - u \le \epsilon\, d e^{\gamma t}/\phi\,, \qquad \text{for} \quad |x - X(t)| \ge \sqrt{\epsilon} \tag{4.19}$$

By letting $U = u - u^\epsilon - \sigma$ and $V = f(u-\sigma) - v^\epsilon + \epsilon\tilde{\Psi}$, where $\tilde{\Psi}$ is of a similar form for Ψ, we can again using the comparison lemma to obtain

$$u - u^\epsilon \le \epsilon\, d e^{\gamma t}/\phi\,, \qquad \text{for} \quad |x - X(t)| \ge \sqrt{\epsilon}. \tag{4.20}$$

We summarize what we have shown by stating the following.

Assertion 4.1 *Let $u^\epsilon(x,t)$ be the relaxation solutions of* (1.1)-(1.2) *and $u(x,t)$ be the entropy solution of* (1.3)*. If the entropy solution has* only one *shock discontinuity $S(t) = \{(x,t)|x = X(t)\}$, then the following error estimate holds:*

- *For a weighted distance function ϕ, $\phi(x) \sim \min(|x|, 1)$,*

$$|(u^\epsilon - u)(x,t)|\phi(|x - X(t)|) = \mathcal{O}(\epsilon)\,, \quad |x - X(t)| \ge \sqrt{\epsilon}. \tag{4.21}$$

- *In particular, if (x,t) is away from the singular support, then*

$$|(u^\epsilon - u)(x,t)| \le C(h)\epsilon\,, \qquad dist(x, S(t)) \ge h. \tag{4.22}$$ ■

4.2 Finitely many shocks

In the case that the entropy solutions for the conservation law (1.3) have two shocks, our analysis in Section 4.1 can be extended to cover this case easily. The main difference is to change the weighted distance function ϕ to the product of two weighted distance functions, i.e. $\phi(|x - X_1(t)|) \cdot \phi(|x - X_2(t)|)$. This idea was used in [20].

In more general case when there are finitely many shocks, we replace the weighted distance function by

$$\rho(x,t) = \prod_{k=1}^{K} \phi\Big(|x - X_k(t)|\Big). \tag{4.23}$$

Then we consider the error functions similar to (4.3). We can apply the same techniques as used in the last subsection to obtain the optimal error bounds. We omit the detail procedure, but state our main result as following.

Theorem 4.1 *Let $u^\epsilon(x,t)$ be the relaxation solutions of* (1.1)-(1.2) *and $u(x,t)$ be the entropy solution of* (1.3)*. If the entropy solution has* finitely *many shock discontinuities, $S(t) = \{(x,t)|x = X_k(t)\}_{k=1}^{K}$, then the following error estimates hold:*

- *For a weighted distance function ϕ, $\phi(x) \sim \min(|x|, 1)$,*

$$|(u^\epsilon - u)(x,t)| \prod_{k=1}^{K} \phi\Big(|x - X_k(t)|\Big) = \mathcal{O}(\epsilon)\,, \quad |x - X(t)| \ge \sqrt{\epsilon}. \tag{4.24}$$

- *In particular, if (x,t) is away from the singular support, then*

$$|(u^\epsilon - u)(x,t)| \le C(h)\epsilon\,, \qquad dist(x, S(t)) \ge h. \tag{4.25}$$ ■

5 Concluding remarks

In this work, we have obtained the pointwise error bounds for relaxation approximations to scalar conservation laws with piecewise smooth solutions. The proof of our results is based on two ingredients: an one-sided interpolation inequality between the L^1 error estimates and Lip^+ stability bounds and a comparison theorem for weakly coupled hyperbolic systems. For simplicity, we only investigate the case that the entropy solution to the equilibrium equation (1.3) consists of finitely many shocks. However, the techniques used in this paper can be extended to the following situations:

- 1. `The equilibrium equation (1.3) consists of finitely many rarefaction waves.` The sharp pointwise error bounds can be obtained by combining the proof techniques used in this paper and in [20].
- 2. `Finite difference approximations to the relaxation system (1.1).` In [7], Jin and Xin developed a class of first- and second-order nonoscillatory numerical schemes for the conservation law (1.3), based on the relaxation approximation (1.1). It is also noted that the first-order Jin-Xin relaxation approximation is the Lax-Friedrichs scheme.
- 3. `The monotone finite difference scheme for the conservation laws (1.3).` As a special case, the sharp pointwise error bounds have been obtained for the Lax-Friedrichs scheme [21].

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