On the regularity of approximate solutions to conservation laws with piecewise smooth solutions

Tao Tang*
Department of Mathematics, Hong Kong Baptist University, Kowloon Tong, Hong Kong
E-mail: `ttang@math.hkbu.edu.hk`

Zhen–huan Teng†
Department of Mathematics, Peking University, Beijing 100871, China
E-mail: `tengzh@sxx0.math.pku.edu.cn`

Abstract

In this paper we address the questions of the convergence rate for approximate solutions to conservation laws with piecewise smooth solutions in a weighted $W^{1,1}$ space. Convergence rate for the *derivative* of the approximate solutions is established under the assumption that a weak *pointwise-error* estimate is given. In other words, we are able to convert weak pointwise-error estimates to optimal error bounds in a weighted $W^{1,1}$ space.

For convex conservation laws, the assumption of a weak pointwise-error estimate is verified by Tadmor [*SIAM J. Numer. Anal.*, 28 (1991), pp. 891-906]. Therefore, one immediate application of our $W^{1,1}$-convergence theory is that for convex conservation laws we indeed have $W^{1,1}$-error bounds for the approximate solutions to conservation laws. Furthermore, the $\mathcal{O}(\epsilon)$-pointwise error estimates of Tadmor and Tang [*SIAM J. Numer. Anal.*, 36 (1999), pp. 1739-1758] is recovered by the use of our $W^{1,1}$-convergence theory.

AMS(MOS) subject classification. 35L65, 65M10, 65M15
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Contents

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1 Introduction

We study the convergence of vanishing viscosity solutions governed by the single conservation law

$$u_t^\epsilon + f(u^\epsilon)_x = \epsilon u_{xx}^\epsilon, \qquad x \in \mathbf{R},\, t > 0, \epsilon > 0 \tag{1.1}$$

and subject to the initial condition prescribed at $t = 0$,

$$u^\epsilon(x, 0) = u_0(x). \tag{1.2}$$

We are interested in the convergence rate of u_x^ϵ towards the derivative of the inviscid solution, u_x, of the corresponding inviscid conservation law

$$u_t + f(u)_x = 0, \qquad x \in \mathbf{R},\, t > 0, \tag{1.3}$$

which is subject to the same initial conditions

$$u(x, 0) = u_0(x). \tag{1.4}$$

There has been an enormous amount of papers related to the error estimates for the viscosity or more general approximations to scalar conservation laws. The methods of analysis include matching method and traveling wave solutions, see e.g. Goodman & Xin [5]; matching the Green function of the linearized problem, see e.g. Liu [10]; weak $W^{-1,1}$ convergence theory, see e.g. Tadmor [17]; the Kruzkov-functional method, see e.g. Kuznetsov [7]; and energy-like methods, see e.g. Tadmor & Tang [18]. The results on error estimates include

- For BV entropy solutions to (1.3), an $\mathcal{O}(\sqrt{\epsilon})$ convergence rate in L^1 obtained by Kuznetsov [7], Lucier [11], Sanders [14], Cockburn-Gremaud-Yang [3] etc;

- For BV entropy solutions, an $\mathcal{O}(\epsilon)$ convergence rate in $W^{-1,1}$ obtained by Tadmor [17], Nessyahu & Tadmor [12], Nessyahu-Tadmor-Tassa [13], Liu-Wang-Warnecke [9] etc;

- For piecewise smooth solutions for (1.3), an $\mathcal{O}(\epsilon)$ convergence rate in L^1 obtained by Bakhvalov [1], Harabetian [6], Teng & Zhang [25], Fan [2], Tang & Teng [23], Teng [24] etc;

- For piecewise smooth solutions, an $\mathcal{O}(\epsilon)$ convergence rate in smooth region of the entropy solution obtained by Goodman & Xin [5], Engquist & Sjogreen [4], Tadmor & Tang [18, 21] etc.

The results list above are concerned with the convergence of the approximate solution itself, essentially nothing is obtained for its derivative. In this work, we will investigate the convergence of the first derivative of the approximate solutions. We will assume that the inviscid solution u has finitely many discontinuities, which is the generic situation, [15, 22]. By properly choosing a weighted function, we will obtain an $\mathcal{O}(\epsilon)$-bound for $u^\epsilon - u$ in a weighted $W^{1,1}$ space. More precisely, we will show that the following estimate holds:

$$\int_{\mathbf{R}} \rho(x,t)\Big(|u_x^\epsilon - u_x| + |u^\epsilon - u|\Big)dx \le C\epsilon\,,$$

where ρ is a *distance* function to the singular support of $u(x,t)$. In case that there is only one shock discontinuity $x = X(t)$, the above result implies that

$$\|u_x^\epsilon(\cdot,t) - u_x(\cdot,t)\|_{L^1(\mathbf{R}\backslash[X(t)-h,X(t)+h])} \le C(h)\epsilon\,,$$

for any given $h > 0$. If there are finitely many shock curves $S(t) = \{(x,t)|x = X_k(t)\}_{k=1}^K$, then we have

$$\|u_x^\epsilon(\cdot,t) - u_x(\cdot,t)\|_{L^1(\mathbf{R}\backslash\cup_k[X_k(t)-h,X_k(t)+h])} \le C(h)\epsilon.$$

In this work, the above estimates are established under the assumption that a (non-optimal) pointwise-error estimate holds away from the singular support of the entropy solution $u(x,t)$. In other words, we are able to convert weak pointwise-error estimates to optimal error bounds in a weighted $W^{1,1}$ space. It is noticed that such a weak pointwise-error estimate is verified for convex conservation laws by Tadmor [17]. Furthermore, our $W^{1,1}$-estimate recovers the $\mathcal{O}(\epsilon)$-pointwise error bound obtained by Tang & Tadmor [18], i.e.

$$|(u^\epsilon - u)(x,t)| \le C(h)\epsilon\,, \qquad dist(x,S(t)) \ge h,$$

for any given $h > 0$, where $S(t) = \{(x,t)|x = X_k(t)\}_{k=1}^K$. Since our result is obtained through a completely different way, this work gives an independent check of the result in [18].

We close this introductory section by emphasizing two important points of this work: (1) Unlike most of previous work on error estimates, this work gives the error bounds for the *derivative* of the approximate solutions; (2) The present results suggest that if a weak pointwise-error estimate can be established for the *nonconvex* conservation laws, then the optimal pointwise-error estimate can be obtained for the *nonconvex* case. So far, almost no pointwise-error estimates have been obtained for nonconvex conservation laws.

2 Preliminaries

For ease of exposition we shall make the following assumptions in this section:

- (A1): The initial data u_0 is piecewisely C^3-smooth and is compactly supported;
- (A2): We assume that there exists a smooth curve, $x = X(t)$, such that $u(x,t)$ is smooth at any point away from $x = X(t)$;
- (A3): There exists a constant $0 < \gamma \le 1$ and a constant $C_T > 0$ such that

$$|u^\epsilon(x,t) - u(x,t)| \le C_T\epsilon^\gamma\,, \qquad \text{for} \quad |x - X(t)| > C_T\epsilon^\gamma\,. \tag{2.1}$$

Remark 1 We make the following remarks and observations:

- (a): The assumption (A1) implies that $u_{xxx}(\bullet, t) \in L^1(\mathbf{R})$ which will be used in the proofs of next section, see the estimate (3.8). Also, if we are interested in pointwise error estimates in a finite domian (not too far from the shock curve), then it is reasonable to assume that u_0 is compactly supported;

- (b): Although we only consider the case with one shock, the extension to finitely many shocks/rarefaction waves can be carried out by following [18], see Section 4 for more detail discussion;

- (c): The assumption (A3) is satisfied for convex conservation laws with Lip^+-bounded initial data, with $\gamma = 1/3$, see Tadmor [17]. It can be improved to $\gamma = 1/2$ as discussed in Tadmor & Tang [18]. However, it is still unclear if (A3) holds for nonconvex conservation laws;

- (d): There has no convexity assumption for the flux function f in this section. ■

At a point on the shock curve $x = X(t)$, we have the Rankine-Hugoniot condition:

$$X'(t) = \frac{f(u(X(t)+,0)) - f(u(X(t)-,0))}{u(X(t)+,0) - u(X(t)-,0)}. \tag{2.2}$$

Also the Lax geometrical entropy condition is satisfied [8]:

$$f'(u(X(t)-,t)) \geq X'(t) \geq f'(u(X(t)+,t)). \tag{2.3}$$

Following [18], we introduce a function $\phi(x) \in C^2(\mathbf{R})$ which satisfies

- (i): $\phi(x) \sim |x|^\alpha, \qquad |x| \ll 1,$
- (ii): $x\phi'(x) > 0, \qquad x \neq 0,$
- (iii): $\phi(x) \to 1, \qquad |x| \to \infty$,

where $\alpha \geq 1$ is a finite constant. The second requirement above implies that ϕ is monotonely decreasing for $x < 0$ and increasing for $x > 0$. More precisely, the distance function ϕ is required to satisfy

$$\begin{cases} \phi(0) = 0, & 0 \leq x\phi'(x) \leq \alpha\phi(x), \\ \\ |\phi(x)| \leq x^\alpha, & |\phi^{(k)}(x)| \leq C, \quad k = 0, 1, 2, \end{cases} \tag{2.4}$$

where the constants C are independent of x. The functions satisfy the above requirement include $\phi(x) = (1 - e^{-x^2})^{\alpha/2}$.

Remark 2 Unlike in [18], the distance function ϕ is now extended to the whole $\mathbf{R}$. This enables us to make a uniform treatment to the (weighted) error function. ■

In this work, we choose $\alpha = \gamma^{-1} + 1$, where γ is given in the assumption (A3). In other words,

$$\phi(x) \sim \begin{cases} x^{1/\gamma+1}, & |x| \ll 1, \\ \\ 1, & |x| \gg 1. \end{cases}$$

3 Main results

We define the error function between the viscosity solution and the entropy solution as $e(x,t) := u^\epsilon(x,t) - u(x,t)$. The main results of this paper are given in the following theorem.

Theorem 1 *Assume that the assumptions* (A1)-(A3) *are satisfied. Then there exists a constant C independent of ϵ such that*

$$\int_{\mathbf{R}} \phi(x - X(t))\Big(|e(x,t)| + |e_x(x,t)|\Big)dx \le C\epsilon\,, \tag{3.1}$$

where $\phi(x - X(t))$ represents the distance from a point (x,t) to the shock curve $x = X(t)$. In particular, for any given $h > 0$,

$$\|u^\epsilon_x(\cdot,t) - u_x(\cdot,t)\|_{L^1(\mathbf{R}\backslash[X(t)-h,X(t)+h])} \le C(h)\epsilon. \tag{3.2}$$

An immediate application of the above theorem is to recover the (optimal) pointwise-error estimate of Tadmor & Tang [18].

Corollary 1 *Assume that the assumptions* (A1)-(A3) *are satisfied. Then for a weighted distance function ϕ, $\phi \sim \min(|x|^{1/\gamma+1}, 1)$,*

$$|(u^\epsilon - u)(x,t)\,\phi(x - X(t))| = \mathcal{O}(\epsilon). \tag{3.3}$$

In particular, if (x,t) is away from the shock discontinuity $S(t) = \{(x,t)|\, x = X(t)\}$, then

$$|(u^\epsilon - u)(x,t)| \le C(h)\epsilon, \qquad dist(x, S(t)) \ge h. \tag{3.4}$$

3.1 Some lemmas

In order to establish the results in Theorem 1, we need following three lemmas which will lead to a Gronwall inequality for the left hand side function of (3.1). Moreover, in the remaining of this section, we always denote $\phi = \phi(x - X(t)), \phi' = \phi'(x - X(t))$.

Lemma 1 *For any $F \in L^1(\mathbf{R})$, we have*

$$\int_{\mathbf{R}} \Big(f'(u^\epsilon) - \dot{X}(t)\Big)\phi'|F|dx \le C\int_{\mathbf{R}} \phi|F|dx + C\epsilon\,. \tag{3.5}$$

Proof. We split the left hand side of (3.5) into two parts: $I_1 + I_2$, where

$$I_1 = \int_{|x-X(t)|\ge\epsilon^\gamma} \Big(f'(u^\epsilon) - \dot{X}(t)\Big)\phi'(x - X(t))|F|dx,$$
$$I_2 = \int_{|x-X(t)|\le\epsilon^\gamma} \Big(f'(u^\epsilon) - \dot{X}(t)\Big)\phi'(x - X(t))|F|dx\,,$$

where γ is the constant given in (A3). It follows from (2.4) that $|\phi'(x)| \le C|x|^{1/\gamma}$. This result gives that

$$I_2 \le C\epsilon. \tag{3.6}$$

Now for $x - X(t) \geq \epsilon^\gamma$, we use the facts that $\phi' \geq 0, \dot{X}(t) > f'(u_+)$, where $u_+ = u(X(t)+0,t)$, to obtain

$$
\begin{aligned}
& \Big(f'(u^\epsilon) - \dot{X}(t)\Big)\phi' \\
\leq\ & \Big(f'(u^\epsilon) - f'(u)\Big)\phi' + \Big(f'(u) - f'(u_+)\Big)\phi' \\
\leq\ & Cf''(\bullet)\epsilon^\gamma \phi' + f''(\bullet)u_x(\bullet)(x - X(t))\phi' \qquad \text{(using (A3))} \\
\leq\ & Cf''(\bullet)(x - X(t))\phi' + f''(\bullet)u_x(\bullet)(x - X(t))\phi' \\
\leq\ & C(x - X(t))\phi' \\
\leq\ & C\phi\,.
\end{aligned}
$$

Similarly, by noting that $\phi' \leq 0$ for $x \leq X(t)$ we can also prove that

$$\Big(f'(u^\epsilon) - \dot{X}(t)\Big)\phi' \leq C\phi\,, \qquad \text{for} \quad x - X(t) \leq -\epsilon^\gamma\,.$$

The above results lead to $I_1 \leq C\|\phi F\|_{L^1(\mathbf{R})}$. This, together with (3.6), yield the inequality (3.5). The proof of this lemma is complete. ∎

Lemma 2 *There exists constant C independent of ϵ such that*

$$\int_{\mathbf{R}} \phi \mathrm{sgn}(e_x)\partial_t e_x dx \leq \int_{\mathbf{R}} \phi' f'(u^\epsilon)|e_x|dx + C\int_{\mathbf{R}} \phi|e_x|dx + C\int_{\mathbf{R}} \phi|e|dx\ + C\epsilon\,. \tag{3.7}$$

Proof. Direct calculation from the viscosity equation (1.1) gives

$$\partial_t e_x + (f(u^\epsilon) - f(u))_{xx} = \epsilon(e_x)_{xx} + \epsilon u_{xxx}\,.$$

Applying the above result gives

$$\int_{\mathbf{R}} \phi \mathrm{sgn}(e_x)\partial_t e_x dx = J_1 + J_2 + J_3$$

where

$$
\begin{aligned}
J_1 &= -\int_{\mathbf{R}} \phi \mathrm{sgn}(e_x)(f(u^\epsilon) - f(u))_{xx} dx\,, \\
J_2 &= \epsilon \int_{\mathbf{R}} \phi \mathrm{sgn}(e_x)(e_x)_{xx} dx\,, \\
J_3 &= \epsilon \int_{\mathbf{R}} \phi \mathrm{sgn}(e_x) u_{xxx} dx\,.
\end{aligned}
$$

It is easy to verify that

$$J_3 = \mathcal{O}(\epsilon). \tag{3.8}$$

We now estimate J_2 by integration by parts. It is noted that $e_x(\cdot,t) \in C((X(t),\infty))$. As in Lax [8], we divide the interval $[X(t),\infty)$ into intervals, $[X(t),\infty) = \cup_m I_m(t)$, $I_m(t) = [p_m(t), p_{m+1}(t))$, with $p_0(t) = X(t)$ and e_x changing signs accross points p_m, $m \geq 1$. Assuming that

$$(-1)^s = \mathrm{sgn}(e_x)\,\Big|_{x \in I_0(t)}\,,$$

we have

$$\phi \text{sgn}(e_x) e_{xx} \Big|_{x=p_0(t)} = 0\,,$$

$$(-1)^{s+m} e_{xx} \Big|_{x=p_{m+1}(t)} \leq 0\,, \qquad (-1)^{s+m} e_{xx} \Big|_{x=p_m(t)} \geq 0\,, \quad \text{for} \quad m \geq 1\,. \tag{3.9}$$

It follows from the above results that

$$\begin{aligned}
\int_{X(t)}^{\infty} \phi \text{sgn}(e_x)(e_x)_{xx} dx &= \sum_{m=0}^{\infty} (-1)^{s+m} \int_{p_m(t)}^{p_{m+1}(t)} \phi (e_x)_{xx} dx \\
&= \sum_m (-1)^{m+s} \phi e_{xx} \Big|_{p_m(t)}^{p_{m+1}(t)} - \sum_m (-1)^{m+s} \int_{p_m(t)}^{p_{m+1}(t)} \phi' e_{xx} dx \\
&\leq 0 - \sum_m (-1)^{m+s} \int_{p_m(t)}^{p_{m+1}(t)} \phi' e_{xx} dx\,,
\end{aligned} \tag{3.10}$$

where in the last step we have used (3.9). Note that $\phi' = 0$ when $x = p_0(t) = X(t)$ and $e_x = 0$ when $x = p_m(t), m \geq 1$. Using integration by parts for (3.10) leads to

$$\begin{aligned}
\int_{X(t)}^{\infty} \phi \text{sgn}(e_x)(e_x)_{xx} dx &\leq \sum_m (-1)^{m+s} \int_{p_m(t)}^{p_{m+1}(t)} \phi'' e_x dx \\
&= \int_{X(t)}^{\infty} \phi'' |e_x| dx \leq C \|u_0\|_{BV(\mathbf{R})} = \mathcal{O}(1),
\end{aligned}$$

where in the second last step we have used the facts that u^ϵ and u are BV-bounded by $||u_0||_{BV(R)}$. Similarly, we can show that

$$\int_{-\infty}^{X(t)} \phi \text{sgn}(e_x)(e_x)_{xx} dx \leq \mathcal{O}(1).$$

The above two results yield

$$J_2 \leq \mathcal{O}(\epsilon). \tag{3.11}$$

Finally, we need to estimate J_1. Observe

$$\begin{aligned}
&- \int_{X(t)}^{\infty} \phi \, \text{sgn}(e_x) \Big(f(u^\epsilon) - f(u) \Big)_{xx} dx \\
= \; &- \sum_m (-1)^{m+s} \phi \Big(f(u^\epsilon) - f(u) \Big)_x \Big|_{p_m(t)}^{p_{m+1}(t)} + \sum_m (-1)^{m+s} \int_{p_m(t)}^{p_{m+1}(t)} \Big(f(u^\epsilon) - f(u) \Big)_x \phi' dx
\end{aligned} \tag{3.12}$$

Using the following observation

$$\Big(f(u^\epsilon) - f(u) \Big)_x = \Big(f'(u^\epsilon) - f'(u) \Big) u_x\,, \qquad \text{when } x = p_m(t), m \geq 1\,,$$

we obtain from (3.12) that

$$\begin{aligned}
&- \int_{X(t)}^{\infty} \phi \, \text{sgn}(e_x) \Big(f(u^\epsilon) - f(u) \Big)_{xx} dx \\
= \; &- \sum_m (-1)^{m+s} \phi \Big(f'(u^\epsilon) - f'(u) \Big) u_x \Big|_{p_m}^{p_{m+1}}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_m (-1)^{m+s} \int_{p_m}^{p_{m+1}} \phi' \Big[f'(u^\epsilon) e_x + (f'(u^\epsilon) - f'(u)) u_x \Big] dx \\
= \;& \underbrace{-\sum_m (-1)^{m+s} \int_{p_m}^{p_{m+1}} \Big(\phi (f'(u^\epsilon) - f'(u)) u_x \Big)_x dx}_{K_1} + \int_{X(t)}^{\infty} \phi' f'(u^\epsilon) |e_x| dx \\
&\quad \underbrace{+\sum_m (-1)^{m+s} \int_{p_m}^{p_{m+1}} \phi' (f'(u^\epsilon) - f'(u)) u_x dx}_{K_2} \; . \qquad (3.13)
\end{aligned}
$$

Using product rule for the integrand of K_1 gives

$$
\begin{aligned}
K_1 \;=\; & -K_2 - \sum_m (-1)^{m+s} \int_{p_m}^{p_{m+1}} \phi (f'(u^\epsilon) - f'(u))_x u_x dx \\
& - \sum_m (-1)^{m+s} \int_{p_m}^{p_{m+1}} \phi (f'(u^\epsilon) - f'(u)) u_{xx} dx \\
=\; & -K_2 - \int_{X(t)}^{\infty} \phi \, \mathrm{sgn}(e_x) \Big[f''(u^\epsilon) e_x + (f''(u^\epsilon) - f''(u)) u_x \Big] u_x dx \\
& - \int_{X(t)}^{\infty} \phi \mathrm{sgn}(e_x) (f'(u^\epsilon) - f'(u)) u_{xx} dx \\
\leq\; & -K_2 + C \int_{X(t)}^{\infty} \phi |e_x| dx + C \int_{X(t)}^{\infty} \phi |e| dx \, . \qquad (3.14)
\end{aligned}
$$

Combining the above results, (3.13) and (3.14), gives

$$
\begin{aligned}
& - \int_{X(t)}^{\infty} \phi \, \mathrm{sgn}(e_x) \Big(f(u^\epsilon) - f(u) \Big)_{xx} dx \\
\leq \;& \int_{X(t)}^{\infty} \phi' f'(u^\epsilon) |e_x| dx + C \int_{X(t)}^{\infty} \phi |e_x| dx + C \int_{X(t)}^{\infty} \phi |e| dx \, .
\end{aligned}
$$

Similarly, we can show that

$$
\begin{aligned}
& - \int_{-\infty}^{X(t)} \phi \, \mathrm{sgn}(e_x) \Big(f(u^\epsilon) - f(u) \Big)_{xx} dx \\
\leq \;& \int_{-\infty}^{X(t)} \phi' f'(u^\epsilon) |e_x| dx + C \int_{-\infty}^{X(t)} \phi |e_x| dx + C \int_{-\infty}^{X(t)} \phi |e| dx \, .
\end{aligned}
$$

The above two results lead to an estimate for J_1:

$$
J_1 \leq \int_{\mathbf{R}} \phi' f'(u^\epsilon) |e_x| dx + C \int_{\infty} \phi |e_x| dx + C \int_{\infty} \phi |e| dx \, . \qquad (3.15)
$$

The desired inequality (3.7) follows from the above estimates for $J_1, J2$ and J_3. ■

Lemma 3 *There exists constant C independent of ϵ such that*

$$
\frac{d}{dt} \int_{\mathbf{R}} \phi |e| dx \leq C \int_{\mathbf{R}} \phi |e_x| dx + C \int_{\mathbf{R}} \phi |e| dx \; + C\epsilon \, . \qquad (3.16)
$$

Proof. It follows from the viscous equation (1.1) and the conservation law (1.3) that

$$e_t = -\Big(f(u^\epsilon) - f(u)\Big)_x + \epsilon e_{xx} + \epsilon u_{xx} \,.$$

Using the above equation gives

$$\begin{aligned} & \int_{\mathbf{R}} \phi \mathrm{sgn}(e) e_t dx \\ = & -\int_{\mathbf{R}} \phi \mathrm{sgn}(e) \Big(f(u^\epsilon) - f(u)\Big)_x dx + \epsilon \int_{\mathbf{R}} \phi \mathrm{sgn}(e) e_{xx} dx + \epsilon \int_{\mathbf{R}} \phi \mathrm{sgn}(e) u_{xx} dx \,. \end{aligned} \tag{3.17}$$

It can be verified that the last term above is of order $\mathcal{O}(\epsilon)$, and the second last term is bounded by

$$\epsilon \int_{\mathbf{R}} \phi \mathrm{sgn}(e) e_{xx} dx \le -\epsilon \int_{\mathbf{R}} \phi' \mathrm{sgn}(e) e_x dx = \mathcal{O}(\epsilon) \,,$$

where in the last step we have used the fact that u and u^ϵ are BV-bounded. Using the following observation

$$\begin{aligned} \Big(f(u^\epsilon) - f(u)\Big)_x &= f'(u^\epsilon) e_x + \Big(f'(u^\epsilon) - f'(u)\Big) u_x \\ &= f'(u^\epsilon) e_x + f''(\bullet)\, e\, u_x \,, \end{aligned}$$

and the fact $u_x = \mathcal{O}(1)$ for (x,t) away from the shock curve, we can bound the first term on the RHS of (3.17):

$$-\int_{\mathbf{R}} \phi \mathrm{sgn}(e) \Big(f(u^\epsilon) - f(u)\Big)_x dx \le C \int_{\mathbf{R}} \phi |e_x| dx + C \int_{\mathbf{R}} \phi |e| dx \,.$$

Therefore, we have proved that

$$\int_{\mathbf{R}} \phi \mathrm{sgn}(e) e_t dx \le C \int_{\mathbf{R}} \phi |e_x| dx + C \int_{\mathbf{R}} \phi |e| dx + C\epsilon \,. \tag{3.18}$$

Using integration by parts we obtain

$$\begin{aligned} \int_{\mathbf{R}} -\dot{X}(t) \phi' |e| dx &= \dot{X}(t) \int_{\mathbf{R}} \phi \mathrm{sgn}(e) e_x dx \\ &\le C \int_{\mathbf{R}} \phi |e_x| dx \,. \end{aligned} \tag{3.19}$$

Combining (3.18) and (3.19) we obtain the desired estimate (3.16). ■

3.2 Proof of Theorem 1

Having the above three lemmas, we are ready to prove Theorem 1. Observe that

$$\frac{d}{dt} \int_{\mathbf{R}} \phi |e_x| dx = \int_{\mathbf{R}} -\dot{X}(t) \phi' |e_x| dx + \int_{\mathbf{R}} \phi \partial_t |e_x| dx \,.$$

The above result, together with Lemma 2, yield

$$\frac{d}{dt} \int_{\mathbf{R}} \phi |e_x| dx \le \int_{\mathbf{R}} \phi' \Big(f'(u^\epsilon) - \dot{X}(t)\Big) |e_x| dx + C \int_{\mathbf{R}} \phi |e_x| dx + C \int_{\mathbf{R}} \phi |e| dx + C\epsilon \,.$$

Since $e_x \in L^1(\mathbf{R})$, we apply Lemma 1 to obtain

$$\frac{d}{dt}\int_{\mathbf{R}} \phi|e_x|dx \le C\int_{\mathbf{R}} \phi|e_x|dx + C\int_{\mathbf{R}} \phi|e|dx + C\epsilon\,.$$

The above result, together with Lemma 3, lead to

$$\frac{d}{dt}\int_{\mathbf{R}} \phi\Big(|e| + |e_x|\Big)dx \le C\int_{\mathbf{R}} \phi\Big(|e| + |e_x|\Big)dx + C\epsilon\,. \tag{3.20}$$

The estimate (3.1) in 1 follows immediately from the above Gronwall inequality. It follows from (2.4) that for any $h > 0$ there exists a constant $c(h) > 0$ such that

$$|\phi(x)| \ge c(h), \qquad \text{as} \quad |x| \ge h.$$

The estimate (3.2) follows from the above result and (3.1). The proof of Theorem 1 is complete. ■

3.3 Proof of Corollary 1

Consider the weighted error function $\phi(x - X(t))e(x,t)$ with (x,t) on the right hand side of the shock curve, i.e. $x > X(t)$. In this case,

$$\phi(x - X(t))\, e(x,t) = -\int_x^\infty \phi e_x dx - \int_x^\infty \phi' e dx\,,$$

which leads to

$$|\phi(x - X(t))e(x,t)| \le \int_{X(t)}^\infty \phi|e_x|dx + \int_{X(t)}^\infty \phi'|e|dx\,.$$

Using integration by parts gives

$$\int_{X(t)}^\infty \phi'|e|dx = -\int_{X(t)}^\infty \phi \mathrm{sgn}(e)\, e_x dx \le \int_{X(t)}^\infty \phi|e_x|dx.$$

Combining the above two results we obtain

$$|\phi(x - X(t))e(x,t)| \le 2\int_{X(t)}^\infty \phi|e_x|dx \le 2\int_{\mathbf{R}} \phi|e_x|dx\,.$$

This, together with Theorem 1, yield

$$|\phi(x - X(t))e(x,t)| \le C\epsilon\,, \qquad x > X(t)\,.$$

Similar result holds for (x,t) on the left hand side of the shock curve. This completes the proof of Corollary 1. ■

4 Application to convex conservation laws

Assume that the flux function f in (1.3) is convex and that there exists only one shock discontinuity for the entropy solution of (1.3)-(1.4). As in [17], we let $\|\bullet\|_{Lip^+}$ denote the Lip^+-seminorm

$$\|w\|_{Lip^+} := \operatorname{ess}\sup_{x\ne y}\left[\frac{w(x) - w(y)}{x - y}\right]^+,$$

where $[w]^+ = H(w)w$, with $H(\bullet)$ the Heaviside function. Owing to the convexity of the flux f, the viscosity solutions of (1.1) satisfy a Lip^+-stability condition, similar to the familiar Oleinik's E-condition, which asserts an a priori upper bound for the Lip^+-seminorm of the viscosity solution

$$\frac{u^\epsilon(x,t) - u^\epsilon(y,t)}{x-y} \le \|u^\epsilon(\cdot,t)\|_{Lip^+} \le \frac{1}{\|u_0\|^{-1}_{Lip^+} + \beta t}, \tag{4.1}$$

where u^ϵ is the solution of (1.1)-(1.2), β is the convexity constant of the flux f, $f'' \ge \beta$, Consult e.g., [17]. The above result suggests that if the initial data do not contain non-Lipschitz increasing discontinuities then the viscosity solution of (1.1) will keep the same property. The same is true for entropy solution of (1.3)-(1.4).

It is shown in [17] that with the Lip^+ initial data, the following pointwise-error bound holds:

$$|u^\epsilon(x,t) - u(x,t)| \le C\sqrt[3]{\epsilon}, \quad \text{for} \quad dist(x, S(t)) \ge \sqrt[3]{\epsilon}, \tag{4.2}$$

where $S(t) = \{(x,t)|\, x = X(t)\}$. This result can be further improved by using the results in [18] and [23]. In other words, the assumption (A3) holds for convex conservation laws with Lip^+ initial data.

4.1 One shock

It is noted that the assumption (A2), i.e. the entropy solution has *only one* shock discontinuity, implies that u_0 must be Lip^+-stable. Therefore, the theories developed in last section can be applied for convex conservation laws with one shock discontinuity. We summarize what we have shown by stating the following.

Assertion 1 *Let $u^\epsilon(x,t)$ be the viscosity solutions of* (1.1)-(1.2) *and $u(x,t)$ be the entropy solution of* (1.3)-(1.4). *If the flux function f in* (1.3) *is convex and that the entropy solution has* only one *shock discontinuity $S(t) = \{(x,t)|x = X(t)\}$, then the following error estimates hold:*

- *For a weighted distance function ϕ, $\phi(x) \sim \min(|x|^4, 1)$,*

$$\begin{aligned} &(\mathsf{a}): \quad \int_{\mathbf{R}} \phi(x - X(t))\Big(|(u^\epsilon - u)(x,t)| + |(u_x^\epsilon - u_x)(x,t)|\Big)dx \le C\epsilon, \\ &(\mathsf{b}) \quad |(u^\epsilon - u)(x,t)|\phi(x - X(t)) = \mathcal{O}(\epsilon). \end{aligned}$$

- *In particular, if (x,t) is away from the singular support, then for any given $h > 0$*

$$\begin{aligned} &(\mathsf{a}): \quad \|u_x^\epsilon(\cdot,t) - u_x(\cdot,t)\|_{L^1(\mathbf{R}\setminus[X(t)-h, X(t)+h])} \le C(h)\epsilon, \\ &(\mathsf{b}): \quad |(u^\epsilon - u)(x,t)| \le C(h)\epsilon, \qquad |x - X(t)| \ge h. \end{aligned}$$

4.2 Finitely many shocks

In this general case, we define the weighted distance function as

$$\rho(x,t) = \prod_{k=1}^{K} \phi\left(x - X_k(t)\right). \tag{4.3}$$

We can apply the same techniques as used in §3 for the weighted error functions $(u^\epsilon(x,t) - u(x,t))\rho(x,t)$ and $(u^\epsilon_x(x,t) - u_x(x,t))\rho(x,t)$. The results in Theorem 1 and Corollary 1 can be extended to these error functions. Following [18], we can extend Assertion 1 and conclude the following.

Assertion 2 *Let $u^\epsilon(x,t)$ be the viscosity solutions of* (1.1)-(1.2) *and $u(x,t)$ be the entropy solution of* (1.3)-(1.4). *If the flux function f in* (1.3) *is convex and that the entropy solution has* finitely many *shock discontinuities $S(t) = \{(x,t)|x = X_k(t)\}_{k=1}^K$, then the following error estimates hold:*

- *For a weighted distance function ϕ, $\phi(x) \sim \min(|x|^4, 1)$,*

$$\text{(a)}: \quad \int_{\mathbf{R}} \prod_{k=1}^{K} \phi\left(x - X_k(t)\right) \Big(|(u^\epsilon - u)(x,t)| + |(u^\epsilon_x - u_x)(x,t)|\Big) dx \le C\epsilon\,,$$

$$\text{(b)}: \quad |(u^\epsilon - u)(x,t)| \prod_{k=1}^{K} \phi\left(x - X_k(t)\right) = \mathcal{O}(\epsilon)\,.$$

- *In particular, if (x,t) is away from the singular support, then for any $h > 0$*

$$\text{(a)}: \quad \|u^\epsilon_x(\cdot,t) - u_x(\cdot,t)\|_{L^1(\mathbf{R}\setminus\cup_k[X_k(t)-h, X_k(t)+h])} \le C(h)\epsilon\,,$$

$$\text{(b)}: \quad |(u^\epsilon - u)(x,t)| \le C(h)\epsilon\,, \qquad dist(x, S(t)) \ge h.$$

4.3 Finite difference methods

Unlike previous work for studying the viscous conservation laws, the approach used in Tadmor & Tang [18] does not follow the characteristics but instead makes use of the energy method. Therefore their results for the viscosity methods can be extended to finite difference methods [19, 20]. It is seen that the work in this paper also makes use of the energy method and it is expected that the results in this work could be extended to other types of approximate solutions.

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