Viscous profiles for traveling waves of scalar balance laws: The canard case

Jörg Härterich
Freie Universität Berlin, Arnimallee 2-6, D-14195 Berlin, Germany

1 Introduction

In [5], we studied some viscous profiles of scalar hyperbolic balance laws

$$u_t + f(u)_x = g(u), \quad x \in \mathbb{R}, \quad u \in \mathbb{R}, \quad f \in C^3, \quad g \in C^2. \tag{H}$$

Equations of this type are often considered as an approximation for a viscous equation

$$u_t + f(u)_x = \varepsilon u_{xx} + g(u), \quad x \in \mathbb{R}, \quad u \in \mathbb{R} \tag{P}$$

where the viscosity parameter ε is very small. Applications of balance laws, although not scalar, can be found e.g. in combustion [10] or nozzle flow [1].
Recently, Mascia [8] has classified several types of traveling waves for the case of a convex flow function f. Here we will treat the question whether these traveling waves are admissible with respect to the viscosity criterion, i.e. whether they can be obtained as limits of traveling waves of the viscous balance law (P) when ε tends to zero. It turns out that not all solutions are viscosity solutions in this sense. In particular, all waves of the hyperbolic equation with more than one discontinuity do not admit viscous profiles.
We assume the following about f and g:

(F) f is convex: $f''(u) > 0$

(G) The zeroes of g are simple

We denote the zeroes of g with u_i where either $i \in \{i_0, i_0+1, \ldots, i_1\}$ for some i_0, $i_1 \in \mathbb{N}$, or $i \in \mathbb{Z}$. The set of all zeroes of g is called $\mathcal{Z}(g)$ and can be divided into two disjoint subsets:

$$\begin{aligned} \mathcal{R}(g) &:= \{u_i \in \mathcal{Z}(g) : g'(u_i) > 0\} \\ \mathcal{A}(g) &:= \{u_i \in \mathcal{Z}(g) : g'(u_i) < 0\} \end{aligned}$$

Here $\mathcal{R}$ and $\mathcal{A}$ stand for repelling and attracting, both with respect to the pure reaction dynamics $u_t = g(u)$.
Moreover, for simpler notation we assume that the indices are chosen in a way such that

$$i \text{ odd} \iff u_i \in \mathcal{A}(g).$$

Since the hyperbolic equation (1) does in general not possess global smooth solutions, and since on the other hand weak solutions are not unique, one considers as a solution usually a weak solution which satisfies an additional so-called entropy condition. However, since this paper is concerned with traveling waves only, we state for this special case of traveling waves what is meant by an entropy solution.

Definition 1.1 *An* **entropy traveling wave** *is a solution of (1) of the form $u(x,t) = u(x - st)$ for some velocity $s \in \mathbb{R}$ with the following properties:*

(i) u is piecewise C^1, i.e. $u \in C^1(\mathbb{R} \setminus J)$ and the set of accumulation points of J has only isolated points. At points where u is continuously differentiable it satisfies the ordinary differential equation

$$(f'(u(\xi)) - s)\ u'(\xi) = g(u(\xi)). \tag{1}$$

(ii) The jumps at the points of discontinuity satisfy the Rankine-Hugoniot condition

$$s\,(u(\xi+) - u(\xi-)) = f(u(\xi+)) - f(u(\xi-))$$

and the entropy condition

$$u(\xi+) \leq u(\xi-)$$

where $u(\xi+)$ and $u(\xi-)$ stand for the one-sided limits of u.

From the convexity assumption (F) it follows that for any $u \in \mathbb{R}$ and any speed s there is at most one other value $h(u,s)$ such that u and $h(u,s)$ satisfy the Rankine-Hugoniot condition

$$\frac{f(u) - f(h(u,s))}{u - h(u,s)} = s.$$

If there is no such $h(u,s)$ we set

$$h(u,s) := \begin{cases} -\infty & \text{for} \quad f'(u) - s > 0 \\ +\infty & \text{for} \quad f'(u) - s < 0 \end{cases}$$

Definition 1.2 *A traveling wave u is said to be a* **heteroclinic wave** *connecting equilibria u_i and u_j if*

$$\begin{aligned} \lim_{\xi \to -\infty} u(\xi) &= u_i \\ \lim_{\xi \to +\infty} u(\xi) &= u_j \end{aligned}$$

for some $u_i, u_j \in \mathbb{R}$.
It is called a **homoclinic wave** *if $u_i = u_j$.*

1.1 Heteroclinic waves of the hyperbolic equation

There are several types of heteroclinic waves as shown in [8]. We collect here the results of Mascias theorems 2.3-2.5 but reorder them in three groups:

(A) Heteroclinic waves which exist for a whole interval of wave speeds s

(B) waves which can be found only if the speed s takes the discrete value $f'(u_i)$ for some i and

(C) undercompressive waves which do also show up only for particular shock speeds.

Proposition 1.3 *(A1) Continuous monotone waves that connect adjacent equilibria*

(A2) Discontinuous heteroclinic waves (see [5] for details)

(B1) Continuous, monotone waves with increasing profile $j = i + 2$, $u_i, u_j \in \mathcal{A}(g)$ *and* $s = f'(u_{i+1})$

(B2) Continuous, monotone waves with decreasing profile $j = i - 2$, $u_i, u_j \in \mathcal{R}(g)$ *and* $s = f'(u_{i-1})$

(B3) *(i)* $i \geq j$, $u_i \in \mathcal{A}(g)$, $u_j \in \mathcal{A}(g)$, $s = f'(u_{i+1})$ *and* $h(u_j, s) < u_{i+2}$

(ii) $i > j$, $u_i \in \mathcal{A}(g)$, $u_j \in \mathcal{R}(g)$, $s = f'(u_{i+1})$ *and* $h(u_{j-1}, s) < u_{i+2}$.

(iii) $i > j$, $u_i \in \mathcal{R}(g)$, $u_j \in \mathcal{A}(g)$, $s = f'(u_{j-1})$ *and* $h(u_{j-2}, s) < u_{i-1}$.

(iv) $i \geq j$, $u_i \in \mathcal{A}(g)$, $u_j \in \mathcal{A}(g)$, $s = f'(u_{j-1})$ *and* $h(u_i, s) < u_{j-2}$.

(B4) *(i) Nonmonotone waves that connect* u_i *to* u_{i+2} *with speed* $s = f'(u_{i+1})$ *as in (B1) but possess some discontinuity.*

(ii) Discontinuous nonmonotone waves that connect u_i *to* u_{i+1} *with speed* $s = f'(u_{i+1})$ *and pass at least once continuously through* $u = u_{i+1}$ *before converging monotonously to* u_{i+1}.

(C) Undercompressive shocks (saddle-saddle connections): $i > j$, u_i, $u_j \in \mathcal{A}(g)$, $s = \dfrac{f(u_i) - f(u_j)}{u_i - u_j}$.

Remark 1.4 *(i) Homoclinic waves appear as special cases of type (B3)(i) and (B3)(iv) waves with* $i = j$.

(ii) Type (B1) comprises a whole one-parameter family of entropy traveling waves depending on a parameter $\lambda > 0$, *see below*

1.2 Viscous Profiles

Looking for traveling waves of (P), i.e. solutions of the form $u(x,t) = u(x - st)$ leads to the equation

$$\varepsilon u'' = (f'(u) - s)u' - g(u) \tag{2}$$

where the prime denotes differentiation with respect to a new coordinate $\xi := x - st$. We can now make precise what we mean by a viscosity traveling wave solution.

Definition 1.5 *A traveling wave solution of (1) with wave speed s_0 is said to be* **admissible** *or to* **admit a viscous profile** *if there is a sequence of solutions (u_{ε_n}) of (2) with $\varepsilon_n \searrow 0$, $s_n \to s_0$ such that $\|u_{\varepsilon_n} - u\|_{L^1(\mathbb{R})} \to 0$.*

1.3 The main results

In another article [5] we were able to show by classical singular perturbation theory the following statement:

Proposition 1.6 *All heteroclinic waves of type (A1), (A2) and (C) admit a viscous profile.*

In two cases we had to use different wave speeds $s_n \neq s_0$. The need to have the wave speed as an additional parameter will also become very important when we discuss which of the waves of type (B1) possess a viscous profile. It will turn out that a heteroclinic orbit of (2) connecting the equilibria u_i and u_{i+2} exists for exactly one wave speed $s(\varepsilon)$ near $s_0 = f'(u_{i+1})$. However, for the hyperbolic equation we can find a one-parameter family of entropy traveling waves with speed s_0 connecting u_i and u_{i+2} in the following way: Fixing a parameter $\lambda \geq 0$ a profile $u_{0,\lambda}(\xi)$ can be defined by the requirements that

- $u_{0,\lambda}(\xi) \equiv u_{i+1}$ for $|\xi| \leq \lambda$,
- $u_{0,\lambda}(\xi)$ solves $(f'(u(\xi)) - s)\, u'(\xi) = g(u(\xi))$ for $\xi \in (-\infty, -\lambda]$ and for $\xi \in [\lambda, +\infty)$,
- $u_{0,\lambda}(\xi)$ is continuous.

It is easily verified that $u_{0,\lambda}(\xi)$ is an entropy traveling wave for all $\lambda \geq 0$. Now the result about admissibility can be stated.

Theorem 1.7 *Consider the family $u_{0,\lambda}$ of entropy traveling waves of type (B1) depending on the parameter λ. The differentiable heteroclinic wave of type (B1) with $\lambda = 0$ admits a viscous profile, while all other waves $u_{0,\lambda}$ with $\lambda > 0$ are not admissible.*

We also present another result which shows that not all heteroclinic waves with jumps according to the entropy criterion satisfy the admissibility condition, although both restriction are derived from the same viscous approximation.

Theorem 1.8 *The heteroclinic waves of type (B4)(i) admit no viscous profile.*

This will be an easy consequence of the Jordan curve theorem and thus is probably a low-dimensional effect which appears only in *scalar* balance laws.
Although we do not give a proof here we remark that the type (B3) waves can be handled by combining the results of theorem 1.7 with [5]. Loosely speaking, those (B3) waves are admissible which are discontinuous at every point where they are not differentiable.

2 Singular perturbation theory

We return now to the viscous traveling wave equation. The second-order equation (2) that arises from plugging the traveling wave ansatz into the viscous PDE (P) can be written as a first-order system in two different ways. Apart from the "phase plane" coordinates

$$\left.\begin{array}{rcl} \varepsilon u' &=& w \\ w' &=& \dfrac{f(u)-su}{\varepsilon} - g(u) \end{array}\right\} \tag{3}$$

one can use the "Liénard coordinates"

$$\left.\begin{array}{rcl} \varepsilon u' &=& v + f(u) - su \\ v' &=& -g(u). \end{array}\right\} \tag{4}$$

We will distinguish these two possibilities by a consequent use of the variable w for phase plane considerations and v for Liénard plane arguments. From the "slow-fast"-system (4) two limiting systems can be derived which both capture a part of the behavior that is observed for $\varepsilon > 0$.
One is the "slow" system obtained by simply setting $\varepsilon = 0$:

$$\left.\begin{array}{rcl} 0 &=& v + f(u) - su \\ v' &=& -g(u). \end{array}\right\} \tag{5}$$

The flow is confined to the **singular curve**

$$\mathcal{C}_s := \{(u,v) : v + f(u) - su = 0\}.$$

Rescaling the variable ξ we arrive at

$$\left.\begin{array}{rcl} \dot{u} &=& v + f(u) - su \\ \dot{v} &=& -\varepsilon g(u). \end{array}\right\}$$

In the limit $\varepsilon = 0$, this equation leads to the "fast" (or "layer") system

$$\left.\begin{array}{rcl} \dot{u} &=& v + f(u) - su \\ \dot{v} &=& 0. \end{array}\right\} \tag{6}$$

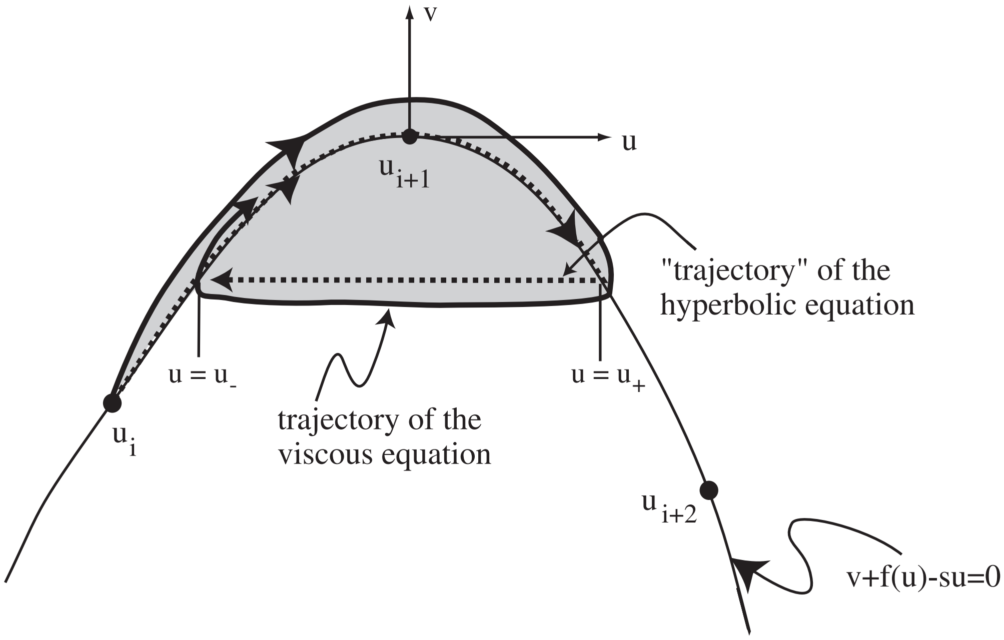

Figure 1: Why a type (B4)(i) entropy wave cannot possess a viscous profile

Equation (6) defines a vector field for which the singular curve $\mathcal{C}_s$ consists of equilibrium points only. It points to the left below the curve $\mathcal{C}_s$ and to the right above. Trajectories of the fast system connect only points for which $v + f(u) - su$ has the same values. This is exactly the Rankine-Hugoniot condition for waves propagating with speed s. Moreover the direction of the fast vector field is in accordance with the entropy condition.
Geometric singular perturbation theory in the spirit of Fenichel [4] is a strong tool in regions where the singular curve is normally hyperbolic, i.e. where the points on $\mathcal{C}_s$ are hyperbolic with respect to the fast field.
Unfortunately, trajectories that pass near the top of the curve $\mathcal{C}_s$ are not captured by this classical theory. However, in recent years, methods have been developped by Dumortier and Roussarie [3] and by Milik and Szmolyan [9] to describe those trajectories.

3 Nonexistence of viscous profiles

We begin our sequence of proofs with the apparently most complicated heteroclinic waves and show that all discontinuous heteroclinic waves from u_i to u_{i+2} do not admit a viscous profile.

Lemma 3.1 *All heteroclinic traveling waves of type (B4)(i) are not admissible.*

Proof: We consider a heteroclinic traveling wave u_0 with speed s_0 parametrized in a way that its first discontinuity is at $\xi = 0$. At this first shock u_0 jumps from $u_+ \in (u_{i+1}, u_{i+2})$ to $u_- \in (u_i, u_{i+1})$, see figure 1. In this figure we have drawn the

hyperbolic wave profile as a dashed line projected on the singular curve $\mathcal{C}_s$. The shock discontinuity is shown as a vertical line connecting two points on $\mathcal{C}_s$.
For the viscous equation (4) with ε small and wave speed s near s_0, the equilibrium u_i is of saddle type with a one-dimensional unstable manifold. Hence, if there was a heteroclinic orbit u_ε from u_i to u_{i+2}, then it would have to be a branch of the unstable manifold of u_i. We follow thus the unstable manifold of u_i and show that it does not converges to u_{i+2}. If the (smooth) trajectory corresponding to the unstable manifold of u_i was L^1-close to u_0, then it would have to intersect the curve $\mathcal{C}_s$ near $u = u_+$ because u' can only change sign by crossing $\mathcal{C}_s$. For ε small the trajectory would then intersect the curve $\mathcal{C}_s$ again near $u = u_-$ and enter a positively invariant region whose boundary is formed by the trajectory and a part of $\mathcal{C}_s$. This region is painted in grey in figure 1. Since the grey region is positively invariant and u_{i+2} lies in the exterior of this region, there can be no heteroclinic connection from u_i to u_{i+2} that could provide a viscous profile for the entropy traveling wave u_0.

$\square$

4 Admissibility of monotone heteroclinic waves

In this chapter we show that the only monotone heteroclinic wave of type (B1) which admits a viscous profile is the one corresponding to $\lambda = 0$. In a first step we prove, that for fixed ε small there is a unique wave speed near s_0 that allows for a heteroclinic connection from u_i to u_{i+2} in (4).

4.1 Rotated Vector Fields

A useful concept for the study of some special planar systems are the so-called *rotated vector fields* introduced by Duff [2] and later improved by Perko [11, 12].

Definition 4.1 *Consider a family of planar vector field*

$$\begin{aligned} \dot{x} &= P(x, y, \nu) \\ \dot{y} &= Q(x, y, \nu) \end{aligned} \tag{7}$$

depending on a scalar parameter ν. Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be an analytic function such that $G(x, y) = 0$ defines a curve which is not a trajectory of (7). The family is called a family of **rotated vector fields (mod G =0)** *if*

$$\begin{vmatrix} P(x, y, \nu) & Q(x, y, \nu) \\ P_\nu(x, y, \nu) & Q_\nu(x, y, \nu) \end{vmatrix} \geq 0 \tag{8}$$

for all x, y and ν and if the inequality is strict except on the set $\{(x, y);\ G(x, y) = 0\}$.

Geometrically, this means that varying the parameter ν, the vector field is "rotated" in the same direction at every point with the possible exception of some points where it may keep its direction. We will use the following

Proposition 4.2 (Duff, Perko) *Consider a family of rotated vector fields. Suppose there is an equilibrium, which for all values of ν possesses a one-dimensional unstable manifold. Then this unstable manifold moves either clockwise or anticlockwise as the parameter ν is increased. The stable manifold moves in the same direction. Moreover, these directions are the same for all saddle equilibria of the system.*

Lemma 4.3 *Consider (4) written in phase plane coordinates*

$$\left.\begin{aligned} \dot{u} &= w/\varepsilon \\ \dot{w} &= \frac{f'(u)-s}{\varepsilon}w - g(u) \end{aligned}\right\} \qquad (9)$$

depending on the parameters ε and s. Then this is a family of rotated vector fields mod $w = 0$ with respect to the parameter s.

Proof: We only have to evaluate (8), i.e.

$$\begin{vmatrix} 0 & w/\varepsilon \\ -w/\varepsilon & \dfrac{f'(u)-s}{\varepsilon}w - g(u) \end{vmatrix} = \left(\frac{w}{\varepsilon}\right)^2 \geq 0$$

and check that this expression only vanishes on the lines $w = 0$ that are not trajectories of the system.

□

Now we may use proposition 4.2 to show the existence of a heteroclinic connection from u_i to u_{i+2} for the viscous traveling wave equation.

Lemma 4.4 *For ε sufficiently small there exists a unique value $s = s(\varepsilon)$ such that there is a monotone heteroclinic connection $u_\varepsilon(\xi)$ from u_i to u_{i+2}. Moreover, $|s(\varepsilon) - s_0| = \mathcal{O}(\varepsilon)$.*

Proof: We fix ε and consider a branch of the unstable manifold of u_i and denote with $w_-(s)$ the value of its first intersection with the line $u = u_{i+1}$. In case $W^u(u_i)$ is a heteroclinic connection between u_i and u_{i+1} we set $w_-(s) := 0$. Similarly the first intersection of $W^s(u_{i+2})$ determines a number $w_+(s)$. Both numbers w_- and w_+ depend continuously on s by standard invariant manifold theory.
For $s - s_0 > 2\sqrt{\varepsilon g'(u_{i+1})}$ the equilibrium u_{i+1} is a sink with two real eigenvalues. In this case the unstable manifold of u_i connects to u_{i+1}, so $w_-(s) = 0$. However

$w_+(s) > 0$ since u_{i+1} is a stable equilibrium and $v' < 0$ along the line $w = 0$ as long as $u > u_{i+1}$.
For $s - s_0 < -2\sqrt{\varepsilon g'(u_{i+1})}$ the situation is vice versa: We have $w_+(s) = 0$ and $w_-(s) > 0$. The intermediate value theorem yields now immediately the existence of a number $s(\varepsilon)$ for which $w_-(s) = w_+(s)$. Due to proposition 4.2 we know that the function w_- is monotone increasing while w_+ is monotone decreasing. From this we can conclude that $s(\varepsilon)$ is unique.
The $\mathcal{O}(\varepsilon)$ estimate can be obtained by calculating the vector field along a curve $v + f(u) - su = \varepsilon \frac{g(u)}{f'(u)-s_0}$ as in [5]. One can show that the invariant manifolds miss each other if $|s - s_0|$ is too large. We do not give the details here since we will not use this estimate later. However, it indicates which might be the correct scaling of s in the blow-up construction that is carried out below.

□

4.2 A Blow-up construction

The next step is a blow-up construction in the spirit of [3] and [6, 7, 9]. In fact, our analysis is close to the one of Krupa and Szmolyan so we only sketch the method and refer to [6] for more details. A similar study in a three-dimensional setting is carried out in [9].
From the viewpoint of geometrical singular perturbation theory, parts of the singular curve $\mathcal{C}_s$ that are normally hyperbolic, survive for small $\varepsilon > 0$. Deleting a small neighborhood of the fold point from $\mathcal{C}_s$ leaves two branches: One branch A_0 which is attracting for the fast dynamics and one branch R_0 which is repelling. By Fenichels theory [4], there will be two invariant curves A_ε and R_ε for $\varepsilon > 0$ small. The dynamics on this branches is close to the slow dynamics on A_0 and R_0, in particular, the equilibria u_i and u_{i+2}, which survive for $\varepsilon > 0$, will lie on A_ε and R_ε. Moreover, A_ε must contain the unstable manifold of u_i and R_ε must contain the one-dimensional stable manifold of u_{i+2}, at least up to a vicinity of the fold point. Of course, we may continue A_ε and R_ε with the flow in forward, resp. backward direction.
A heteroclinic orbit from u_i to u_{i+2} exist if and only if the forward continuation of A_ε intersects the backward continuation of R_ε. This chapter is concerned with the question how to determine the value of $s(\varepsilon)$ for which such a heteroclinic connection occurs.
The key is a good understanding of the dynamics near the fold point when ε is small and s is varied near s_0.
To examine this dynamics, we rescale the variables according to

$$\begin{aligned} u &= u_{i+1} + \mu \bar{u} \\ v &= -f(u_{i+1}) + s u_{i+1} + \mu^2 \bar{v} \\ s &= s_0 + \mu^2 \bar{s} \end{aligned}$$

$$\varepsilon = \mu^2\bar{\varepsilon}.$$

The rescaled equation is studied in two different charts, one with $\bar{\varepsilon} \equiv 1$, and another with $\bar{v} \equiv -1$. To distinguish these two charts we write u_1, v_1, s_1 and μ_1 for the variables in the first chart and u_2, μ_2, s_2 and ε_2 for the variables in the second chart. The new variables are hence

$$\begin{aligned} u - u_{i+1} &= \mu_1 u_1 &&= \mu_2 u_2 \\ v + f(u_{i+1}) - s u_{i+1} &= \mu_1^2 v_1 &&= \mu_2^2 \\ s - s_0 &= \mu_1^2 s_1 &&= \mu_2^2 s_2 \\ \varepsilon &= \mu_1^2 &&= \mu_2^2 \varepsilon_2 \end{aligned}$$

and the change of variables between the two charts is given by

$$\left.\begin{aligned} \frac{\mu_1^2}{\mu_2^2} &= \varepsilon_2 \\ u_1 &= \frac{1}{\sqrt{\varepsilon_2}} u_2 \\ v_1 &= -\frac{1}{\varepsilon_2} \end{aligned}\right\} \tag{10}$$

Expanding f and g near $u_1 = 0$, the viscous traveling wave equation (4) written in the first chart reads

$$\left.\begin{aligned} \dot{u}_1 &= v_1 + \alpha_2 u_1^2 + \mu_1 u_1(\alpha_3 u_1^2 - s_1) + \mathcal{O}(\mu_1^2) \\ \dot{v}_1 &= -\beta_1 u_1 + \mu_1 \beta_2 u_1^2 + \mathcal{O}(\mu_1^2) \end{aligned}\right\} \tag{11}$$

with $\alpha_2 = \frac{1}{2} f''(u_{i+1})$, $\alpha_3 = \frac{1}{6} f'''(u_{i+1})$, $\beta_1 = g'(u_{i+1})$ and $\beta_2 = \frac{1}{2} g''(u_{i+1})$.
For $\mu_1 = 0$ this is a well-known equation in the theory of singular perturbations. It is integrable, more precisely

$$H(u_1, v_1) = (v_1 + \alpha_2 u_1^2 - \frac{\beta_1}{2\alpha_2}) e^{2\alpha_2 v_1/\beta_1}$$

is a conserved quantity. Equation (11) possesses a family of periodic orbits which accumulate onto a special unbounded solution γ_1 corresponding to $H = 0$. This special solution can be parametrized as

$$\left.\begin{aligned} u_1(\tau) &= \frac{\beta_1}{2\alpha_2}\tau \\ v_1(\tau) &= \frac{\beta_1}{2\alpha_2} - \frac{\beta_1^2}{4\alpha_2}\tau^2 \end{aligned}\right\} \tag{12}$$

A perturbation of this special solution will later provide the connection between the unstable manifold of u_i and the stable manifold of u_{i+2}. By (10), in the (u_2, v_2)-variables this special solution corresponds to

$$\alpha_2 u_2^2 - 1 = \frac{\varepsilon_2 \beta_1}{2\alpha_2}.$$

In this second chart $\bar{v} = -1$ the vector field can after a rescaling of time be written as

$$\left.\begin{array}{rcl} \dot{u}_2 &=& 2 - 2\alpha_2 u_2^2 - 2\mu_2\alpha_3 u_2^3 + 2\mu_2 s_2 u_2 \\ && +\varepsilon_2 u_2(\beta_1 u_2 + \beta_2\mu_2 u_2^2) + \mathcal{O}(\mu_2^2) \\ \dot{\mu}_2 &=& -\varepsilon_2\mu_2(\beta_1 u_2 + \beta_2\mu_2 u_2^2 + \mathcal{O}(\mu_2^2)) \\ \dot{s}_2 &=& 2\varepsilon_2 s_2(\beta_1 u_2 + \beta_2\mu_2 u_2^2 + \mathcal{O}(\mu_2^2)) \\ \dot{\varepsilon}_2 &=& 2\varepsilon_2^2(\beta_1 u_2 + \beta_2\mu_2 u_2^2 + \mathcal{O}(\mu_2^2)) \end{array}\right\} \tag{13}$$

The three-dimensional subspace $\{\varepsilon_2 = 0\}$ is invariant, and in this invariant space the equations simplify to

$$\begin{array}{rcl} \dot{u}_2 &=& 2 - 2\alpha_2 u_2^2 - 2\mu_2\alpha_3 u_2^3 + 2\mu_2 s_2 u_2 + \mathcal{O}(\mu_2^2) \\ \dot{\mu}_2 &=& 0 \\ \dot{s}_2 &=& 0 \end{array}$$

There exist two manifolds of equilibria with $u_2^\pm = \pm\frac{1}{\sqrt{\alpha_2}} + \mathcal{O}(\mu_2)$ and arbitrary values of μ_2 and s_2. These two-dimensional manifolds of equilibria will turn out to be connected to the two surviving normally hyperbolic parts A_ε and R_ε of the slow manifold. The advantage from our blow-up construction is that we have now normal hyperbolicity of the equilibria up to $\mu_2 = 0$. In fact, the linearization in $u_2^\pm$ has eigenvalues $\mp 4\sqrt{\alpha_2} + \mathcal{O}(\mu_2)$. Note that $\alpha_2 = \frac{1}{2}f''(u_{i+1}) \neq 0$ by our assumptions on f.

The following proposition is an adaption of theorem 3.1 in [9] to our lower-dimensional situation, so we refer the reader to [9] for the proof:

Proposition 4.5 *Consider the equilibrium $p_a = (\frac{1}{\sqrt{\alpha_2}}, 0, 0, 0)$ of (13). There exists a 3-dimensional center manifold $\mathcal{M}_a$ at p_a with a strong stable foliation. The intersection of $\mathcal{M}_a$ with the invariant subspace $\mu_2 = s_2 = 0$, $\varepsilon_2 > 0$ is a single solution γ_a converging to p_a. Moreover, this solution γ_a corresponds to a part of the special solution γ_1 in the other chart.*
An analogous center manifold $\mathcal{M}_r$ with similar properties exists near the equilibrium $p_r = (-\frac{1}{\sqrt{\alpha_2}}, 0, 0, 0)$ of (13).

Any section $\varepsilon = const.$ of $\mathcal{M}_a$ contains a part of A_ε while $\mathcal{M}_r$ contains a part of R_ε. For $\varepsilon_2 = 0$ there is a heteroclinic connection between the two manifolds A_0 and R_0 by the special orbit. For $\varepsilon_2 > 0$ we have to determine by a Melnikov calculation the value of s_2 for which such a heteroclinic connection is preserved. We work in the first chart again and use a distance function $d(\mu_1, s_1)$ measuring the distance of the unstable manifold of p_r and the stable manifold of p_a at $u_1 = 0$. For the derivative of d with respect to μ_1, an integral representation as a Melnikov integral is derived in [13] and [14]. To perform the computation one needs explicitly all bounded solutions of the adjoint linearized equation. Linearizing (11) around the solution γ_1 with $\mu_1 = 0$ yields the non-autonomous linear system

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \beta_1\tau & 1 \\ -\beta_1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

The adjoint equation

$$\dot{\psi} = \begin{pmatrix} -\beta_1\tau & \beta_1 \\ -1 & 0 \end{pmatrix} \psi$$

has the (up to a constant factor) unique bounded solution

$$\psi(\tau) = \begin{pmatrix} \beta_1 \tau e^{-\frac{\beta_1}{2}\tau^2} \\ e^{-\frac{\beta_1}{2}\tau^2} \end{pmatrix}. \tag{14}$$

The Melnikov integral is then computed by integrating the scalar product of ψ with the μ_1-derivative of (11) evaluated along ψ:

$$\begin{aligned}
\frac{\partial d}{\partial \mu_1} &= \int_{-\infty}^{+\infty} \begin{pmatrix} \beta_1\tau \\ 1 \end{pmatrix}^T \begin{pmatrix} -\frac{\alpha_3\beta_1^3}{8\alpha_2^3}\tau^3 + \frac{\beta_1 s_1}{2\alpha_2}\tau \\ \frac{\beta_1^2\beta_2}{4\alpha_2^2}\tau^2 \end{pmatrix} e^{-\frac{\beta_1}{2}\tau^2}\, d\tau \\
&= \int_{-\infty}^{+\infty} \left(-\frac{\alpha_3\beta_1^4}{8\alpha_2^3}\tau^4 + \frac{\beta_1^2 s_1}{2\alpha_2}\tau^2 + \frac{\beta_1^2\beta_2}{4\alpha_2^2}\tau^2 \right) e^{-\frac{\beta_1}{2}\tau^2}\, d\tau \\
&= \int_{-\infty}^{+\infty} \left(-\frac{\alpha_3\beta_1^{3/2}}{\sqrt{2}\alpha_2^3}\nu^4 + \frac{\sqrt{2}\beta_1^{1/2} s_1}{\alpha_2}\nu^2 + \frac{\beta_1^{1/2}\beta_2}{\sqrt{2}\alpha_2^2}\nu^2 \right) e^{-\nu^2}\, d\nu \\
&= \sqrt{\frac{\pi\beta_1}{2}} \left(-\frac{3\alpha_3\beta_1}{4\alpha_2^3} + \frac{s_1}{\alpha_2} + \frac{\beta_2}{2\alpha_2^2} \right)
\end{aligned}$$

This expression has a simple zero for the critical value of s_1

$$s_1^c = \frac{3\alpha_3\beta_1}{4\alpha_2^2} - \frac{\beta_2}{2\alpha_2}.$$

Hence the heteroclinic orbit persists at a value

$$s_1 = s_1^c + \mathcal{O}(\mu_1).$$

Restating this result of the Melnikov calculation in the original coordinates, we have:

Lemma 4.6 *The heteroclinic connection from u_i to u_{i+2} occurs at*

$$s(\varepsilon) = s_0 + \left(\frac{f'''(u_{i+1})g'(u_{i+1}) - g''(u_{i+1})f''(u_{i+1})}{2f''(u_{i+1})^2} \right) \varepsilon + h.o.t.$$

It remains to show that the heteroclinic orbits u_ε provide a viscous profile for the entropy traveling wave $u_{0,\lambda}$ with $\lambda = 0$. To this end, we localize the heteroclinic orbit in the Liénard plane.

Lemma 4.7 *Given $d > 0$ small, there exist positive constants k, σ and ε_1 such that for all $0 < \varepsilon \le \varepsilon_1$ and $|s - s_0| \le \sigma$ trajectories of (4) can leave the region*

$$U_+ := \left\{ u_i \le u \le u_{i+1} - d,\ \left| v + f(u) - su + \varepsilon \frac{g(u)}{f'(u) - s} \right| \le k\varepsilon^{3/2}|g(u)| \right\}$$

only at the right boundary $u = u_{i+1} - d$. Moreover, U_+ contains a branch of the unstable manifold of u_i. Similarly trajectories may enter

$$U_- := \left\{ u_{i+1} + d \leq u_{i+2}, \ \left| v + f(u) - su + \varepsilon \frac{g(u)}{f'(u) - s} \right| \leq k\varepsilon^{3/2} |g(u)| \right\}$$

only at $u = u_{i+1} + d$.

Proof: We consider the upper boundary $v(u) = -f(u) + su + \varepsilon \frac{g(u)}{f'(u) - s} - k\varepsilon^{3/2} g(u)$ of U_+. The slope of this boundary curve is

$$\frac{dv(u)}{du} = -(f'(u) - s) + \varepsilon \frac{(f'(u) - s)g'(u) - f''(u)g(u)}{(f'(u) - s)^2} - k\varepsilon^{3/2} g'(u).$$

The slope of the vector field (4) along this upper boundary of U_+ is

$$\begin{aligned} \frac{\dot{v}}{\dot{u}} &= \frac{-\varepsilon g(u)}{v(u) + f(u) - su} \\ &= \frac{-(f'(u) - s)}{1 - \varepsilon^{1/2} k (f'(u) - s)} \\ &= -(f'(u) - s) + \frac{1}{2} k \varepsilon^{1/2} (f'(u) - s)^2 + \mathcal{O}(\varepsilon). \end{aligned}$$

Since $f'(u_{i+1}) = s_0$ and f' is strictly monotone, we can find σ, $c_1 > 0$ such that

$$f'(u) - s \leq f'(u_{i+1} - d) - (s_0 - \sigma) < -c_1 < 0$$

holds for $|s - s_0| \leq \sigma$ and all $u \in [u_i, u_{i+1} - d]$. Hence, the $\mathcal{O}(\varepsilon^{1/2})$-term of the vector field dominates the $\mathcal{O}(\varepsilon)$-term of the boundary slope. So by making ε small we can make sure that the vector field on the upper boundary of U_+ points into the interior of U_+. In exactly the same way one can show that along the lower boundary the vector field points into the interior of U_+ as well. So the only way how a trajectory can leave U_+ is through the right boundary at $u = u_{i+1} - d$.
To show that U_+ contains a branch of the unstable manifold of u_i one has to calculate the linearization of (4) at the equilibrium u_i as in [5]. It turns out that the eigenvector of the positive eigenvalue has slope $-(f'(u_i) - s) + \mathcal{O}(\varepsilon)$, so by a suitable choice of k the tangent vector to the unstable manifold lies in U_+. This implies immediately that the unstable manifold itself passes through U_+.
The proof for the region U_- is completely analogous and is omitted for this reason.

□

Lemma 4.8 *Let $\delta := \frac{\sqrt{g'(u_{i+1})}}{f''(u_{i+1})} > 0$ and $d > 0$ from the previous lemma sufficiently small. Then there exists $\varepsilon_2 \in (0, \varepsilon_1]$ such that for $\varepsilon \leq \varepsilon_2$ and $|s - s_0| \leq \frac{1}{2}\sqrt{\varepsilon g'(u_{i+1})}$*

all trajectories leave the region

$$V_+ := \left\{ u_{i+1} - d \le u \le u_{i+1} - \delta\sqrt{\varepsilon}, -k_1\varepsilon^{3/2} \le v + f(u) - su + \varepsilon \frac{g(u)}{f'(u)-s} \le k_1\varepsilon^{3/2} \right\}$$

at the right boundary $u = u_{i+1} - \delta\sqrt{\varepsilon}$. *Here* $k_1 := k|g(u_{i+1} - d)|$.
Similarly, trajectories may enter

$$V_- := \left\{ u_{i+1} + \delta\sqrt{\varepsilon} \le u \le u_{i+1} + d, -k_1\varepsilon^{3/2} \le v + f(u) - su + \varepsilon \frac{g(u)}{f'(u)-s} \le k_1\varepsilon^{3/2} \right\}$$

only at $u = u_{i+1} + \delta\sqrt{\varepsilon}$.

Remarks :

1) The constant k_1 is chosen in a way such that every trajectory after leaving U_+ will automatically enter V_+.

2) For ε sufficiently small the wave speed $s(\varepsilon)$ from lemma 4.6 satisfies the condition $|s(\varepsilon) - s_0| \le \frac{1}{2}\sqrt{\varepsilon g'(u_{i+1})}$.

Proof: To prove the lemma, we fix $d > 0$ so small that we can find constants c_2, $c_3 > 0$ with

$$0 \ge g(u) \ge -c_2(u_{i+1} - u)$$

and

$$f'(u) - s \le -c_3(u_{i+1} - u)$$

for $u \in [u_{i+1} - d, u_{i+1} - \delta\sqrt{\varepsilon}]$ and all $|s - s_0| \le \frac{1}{2}\sqrt{\varepsilon g'(u_{i+1})}$. The latter is possible since

$$\begin{aligned} f'(u) - s &= f'(u_{i+1}) - s_0 + f''(u_{i+1})(u - u_{i+1}) + s_0 - s + \mathcal{O}(|u - u_{i+1}|^2) \\ &\le \frac{1}{2}f''(u_{i+1})(u - u_{i+1}) + \frac{1}{2}f''(u_{i+1})(u - u_{i+1}) + s_0 - s + \mathcal{O}(|u - u_{i+1}|^2) \\ &\le \frac{1}{2}f''(u_{i+1})(u - u_{i+1}) + \mathcal{O}(|u - u_{i+1}|^2) \end{aligned}$$

As in the previous lemma, we calculate the difference between the slope of the upper boundary of V_+ and the slope of the vector field along this curve, which is

$$\begin{aligned} \frac{dv(u)}{du} - \frac{\dot{v}}{\dot{u}} &= -(f'(u) - s) + \varepsilon \frac{(f'(u) - s)g'(u) - f''(u)g(u)}{(f'(u) - s)^2} \\ &\quad + (f'(u) - s) - \frac{k_1}{2}\varepsilon^{1/2}\frac{(f'(u) - s)^2}{g(u)} \\ &\ge \varepsilon \left(\frac{f''(u_{i+1})g''(u_{i+1}) - f'''(u_{i+1})g'(u_{i+1})}{2f''(u_{i+1})^2} + \frac{c_3^2\delta k_1}{2c_2} \right) + \mathcal{O}(\varepsilon^{3/2}), \end{aligned}$$

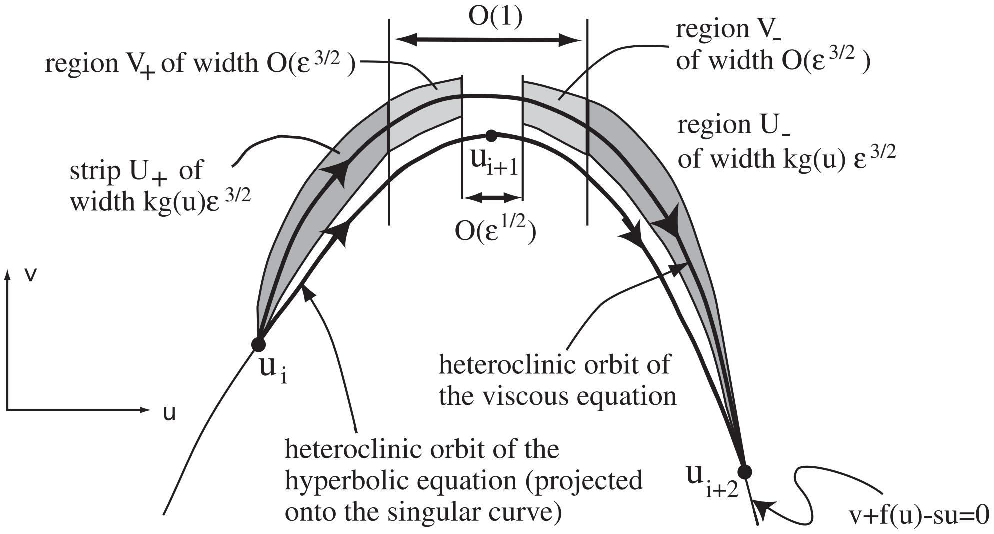

Figure 2: The heteroclinic orbit at $s = s(\varepsilon)$

since

$$\lim_{u \to u_{i+1}} \frac{(f'(u)-s)g'(u) - f''(u)g(u)}{(f'(u)-s)^2} = \frac{f''(u_{i+1})g''(u_{i+1}) - f'''(u_{i+1})g'(u_{i+1})}{2f''(u_{i+1})^2}$$

and

$$-\frac{(f'(u)-s)^2}{g(u)} \geq \frac{c_3^2(u_{i+1}-u)}{c_2} \geq \frac{c_3^2 \delta \sqrt{\varepsilon}}{c_2}.$$

By choosing k_1 large and ε small, if necessary, we can achieve that the vector field along the upper boundary is directed into the interior of V_+. The same argument can be applied to the lower boundary. On the left and right boundary of V_+ the situation is clear and so the claim about V_+ is proven.

$\square$

Up to now, we know that the heteroclinic orbit u_ε found in lemma 4.4 passes through the four regions U_+, V_+, V_- and U_-, where u'_ε is $\mathcal{O}(\varepsilon^{1/2})$-close to the vector field of the hyperbolic traveling wave equation (2).
We parametrize the family of heteroclinic orbits u_ε such that

$$u_\varepsilon(0) = u_{i+1} \text{ for all } \varepsilon.$$

With this parametrization we can find $\eta_- = \eta_-(\varepsilon)$ such that

$$u_\varepsilon(\eta_-(\varepsilon)) = u_{i+1} - \delta\sqrt{\varepsilon}$$

and $\eta_+ = \eta_+(\varepsilon)$ such that

$$u_\varepsilon(\eta_+(\varepsilon)) = u_{i+1} + \delta\sqrt{\varepsilon}.$$

Lemma 4.9 *$u_\varepsilon'(\xi) \geq c$ for $\xi \in [\eta_-, \eta_+]$ with a constant $c > 0$ independent of ε small. In particular, this implies that $\eta_+ - \eta_- = \mathcal{O}(\sqrt{\varepsilon})$.*

Proof: Let v_ε denote the v-component of the heteroclinic orbit of (4) which exists at $s = s(\varepsilon)$. According to lemma 4.8, this heteroclinic orbit leaves the strip V_+ at a height

$$\begin{aligned}
v_\varepsilon(\eta_-) &\geq -f(u_\varepsilon) + s(\varepsilon)u_\varepsilon + \frac{\varepsilon g(u_\varepsilon)}{(f'(u_\varepsilon) - s(\varepsilon))} - k\varepsilon^{3/2} \\
&= -\frac{\delta^2}{2} f''(u_{i+1})\varepsilon + \frac{\varepsilon g'(u_{i+1})\sqrt{\varepsilon}\delta + \mathcal{O}(\varepsilon^2)}{(f''(u_{i+1})\delta\sqrt{\varepsilon} + \mathcal{O}(\varepsilon))} + \mathcal{O}(\varepsilon^{3/2}) \\
&= -\frac{\delta^2}{2} f''(u_{i+1})\varepsilon + \frac{\varepsilon g'(u_{i+1})}{f''(u_{i+1})} + \mathcal{O}(\varepsilon^{3/2}) \\
&= \frac{\varepsilon g'(u_{i+1})}{2f''(u_{i+1})} + \mathcal{O}(\varepsilon^{3/2})
\end{aligned}$$

where lemma 4.6 and the definition of δ was used. We can now find $c > 0$ such that

$$v_\varepsilon(\eta_-) \geq c\varepsilon.$$

Similarly, $v_\varepsilon(\eta_+) \geq c\varepsilon$. Since v_ε is monotone increasing on $[\eta_-, 0]$ and monotone decreasing on $[0, \eta_+]$, we know that

$$v_\varepsilon(\xi) + f(u_\varepsilon(\xi)) - s(\varepsilon)u_\varepsilon(\xi) \geq min\{v_\varepsilon(\eta_-), v_\varepsilon(\eta_+)\}$$

for $\xi \in [\eta_-, \eta_+]$. So, on this small part of the heteroclinic orbit we have $u_\varepsilon' = \frac{1}{\varepsilon}(v_\varepsilon + f(u_\varepsilon) - s(\varepsilon)u_\varepsilon) \geq c$.

□

Lemma 4.10 *The traveling waves $u_{0,\lambda}$ of the form (B1) with $\lambda = 0$ admit a viscous profile, more precisely, the heteroclinic connections orbits u_ε with $s = s(\varepsilon)$ satisfy with a suitable parametrization*

$$\|u_\varepsilon - u_{0,0}\|_{L^1(\mathbb{R})} \to 0 \ \ for \ \ \varepsilon \searrow 0.$$

All the other entropy traveling waves $u_{0,\lambda}$ with $\lambda \neq 0$ are not admissible.

Proof: The first part of the lemma is proved if we can show that for any given $n \in \mathbb{N}$

$$\|u_\varepsilon - u_{0,0}\| \leq \frac{1}{n}$$

for all ε sufficiently small. To this end, we split the trajectories in five different parts: For $\xi > \xi_+$ and $\xi < \xi_-$ with $\xi_\pm$ to be determined later, the exponential convergence to u_{i+2} and u_i will give us good estimates. For $\xi \in [\xi_-, \eta_-]$ and $\xi \in [\eta_+, \xi_+]$ the heteroclinic orbit $(u_\varepsilon, v_\varepsilon)$ lies in the regions $U_\pm$ and $V_\pm$ where we have some control over u'_ε. The remaining interval $[\eta_-, \eta_+]$ is so small that it does not affect the L^1-estimate:

$$\int_{\eta_-}^{\eta_+} |u_\varepsilon(\xi) - u_{0,0}(\xi)| \, d\xi = \mathcal{O}(\sqrt{\varepsilon}) \le \frac{1}{5n} \tag{15}$$

for ε small enough.
To make use of the exponential decay near u_i, we determine $\theta > 0$ small such that

$$\frac{g(u)}{f'(u) - s} + k\varepsilon^{1/2} g(u) \ge \frac{1}{2} \frac{g'(u_i)(u - u_i)}{f'(u_i) - s_0} \tag{16}$$

for $u_i \le u \le u_i + \theta$, $|s - s_0| \le \sigma$ and $0 \le \varepsilon \le \varepsilon_1$. Moreover, θ should be so small that

$$\theta \int_{-\infty}^{0} e^{\frac{1}{2}\frac{g'(u_i)}{f'(u_i) - s_0}\xi} \le \frac{1}{10n}. \tag{17}$$

With this θ we can find ξ_- such that $u_{0,0}(\xi_-) < u_i + \theta$ and $u_\varepsilon(\xi_-) < u_i + \theta$ for all ε small enough. This is possible since $u'_\varepsilon = \frac{g(u_\varepsilon)}{f'(u_\varepsilon) - s(\varepsilon)} + \mathcal{O}(\sqrt{\varepsilon})$ is bounded away from zero uniformly in ε for $u_\varepsilon \in [u_i + \theta, u_{i+1}]$.
Since

$$u'_\varepsilon \ge \frac{g(u_\varepsilon)}{f'(u_\varepsilon) - s(\varepsilon)} + k\varepsilon^{1/2} g(u_\varepsilon) \tag{18}$$

for $u_\varepsilon \in [u_i, u_{i+1} - d]$ we have, using (16),

$$u_\varepsilon(\xi) - u_i \le e^{\frac{1}{2}\frac{g'(u_i)}{f'(u_i) - s_0}(\xi_- - \xi)} (u_\varepsilon(\xi_-) - u_i) \le e^{\frac{1}{2}\frac{g'(u_i)}{f'(u_i) - s_0}(\xi_- - \xi)} \theta$$

since $u_\varepsilon(\xi_-) \in [u_i, u_{i+1} - d]$.
Again by (16),

$$u_{0,0}(\xi) - u_i \le e^{\frac{1}{2}\frac{g'(u_i)}{f'(u_i) - s_0}(\xi_- - \xi)} (u_{0,0}(\xi_-) - u_i) \le e^{\frac{1}{2}\frac{g'(u_i)}{f'(u_i) - s_0}(\xi_- - \xi)} \theta$$

By (17) this implies that

$$\int_{-\infty}^{\xi_-} |u_\varepsilon(\xi) - u_{0,0}(\xi)| \, d\xi \int_{-\infty}^{\xi_-} |u_\varepsilon(\xi) - u_i| \, d\xi + \int_{-\infty}^{\xi_-} |u_i - u_{0,0}(\xi)| \, d\xi \le \frac{1}{5n}. \tag{19}$$

In exactly the same way we can show that

$$\int_{\xi_+}^{+\infty} |u_\varepsilon(\xi) - u_{0,0}(\xi)| \, d\xi \le \frac{1}{5n} \tag{20}$$

where ξ_+ is determined similarly as ξ_-. For the estimate on the intermediate interval $[\xi_-, \eta_-]$, we note first that $|u_\varepsilon(\eta_-) - u_{0,0}(\eta_-)| = \mathcal{O}(\sqrt{\varepsilon})$ by lemma 4.9. Since the heteroclinic trajectory $(u_\varepsilon, v_\varepsilon)$ passes through U_+ and V_+ we have for $\xi \in [\xi_-, \eta_-]$

$$\begin{aligned}
& |u_\varepsilon(\xi) - u_{0,0}(\xi)| - |u_\varepsilon(\eta_-) - u_{0,0}(\eta_-)| \\
\le & \int_{\eta_-}^{\xi} |u'_\varepsilon(\zeta) - u'_{0,0}(\zeta)| \, d\zeta \\
\le & \int_{\eta_-}^{\xi} \left| \frac{g(u_{0,0})}{f'(u_{0,0}) - s_0} - \frac{g(u_\varepsilon)}{f'(u_\varepsilon) - s(\varepsilon)} \right| + \mathcal{O}(\sqrt{\varepsilon}) \, d\zeta \\
\le & \int_{\eta_-}^{\xi} \left| \frac{g(u_{0,0})}{f'(u_{0,0}) - s_0} - \frac{g(u_\varepsilon)}{f'(u_\varepsilon) - s_0 + \mathcal{O}(\varepsilon)} \right| + \mathcal{O}(\sqrt{\varepsilon}) \, d\zeta \\
\le & \int_{\eta_-}^{\xi} L |u_\varepsilon(\zeta) - u_{0,0}(\zeta)| \, d\zeta + \mathcal{O}(\sqrt{\varepsilon})
\end{aligned}$$

where L is a Lipschitz constant for the function $\frac{g}{f'-s_0}$ on $[\xi_-, \eta_-]$. By the Gronwall inequality, this implies $|u_\varepsilon(\xi) - u_{0,0}(\xi)| = \mathcal{O}(\sqrt{\varepsilon})$ again for $\xi \in [\xi_-, \eta_-]$. After integration, this yields

$$\int_{\xi_-}^{\eta_-} |u_\varepsilon(\xi) - u_{0,0}(\xi)| \, d\xi = \mathcal{O}(\sqrt{\varepsilon}) \le \frac{1}{5n} \tag{21}$$

for ε small. Exactly by the same reasoning, we get

$$\int_{\eta_+}^{\xi_+} |u_\varepsilon(\xi) - u_{0,0}(\xi)| \, d\xi \le \frac{1}{5n} \tag{22}$$

for all sufficiently small ε. Adding up (19)-22), we arrive at

$$\int_{-\infty}^{+\infty} |u_\varepsilon(\xi) - u_{0,0}(\xi)| \, d\xi \le \frac{1}{n}$$

which proves the first part of the lemma.

The other entropy traveling waves $u_{0,\lambda}$ with $\lambda \neq 0$ cannot be admissible since u_ε is the only heteroclinic wave that connects u_i and u_{i+2} with a wave speed near $f'(u_{i+1})$. However, since $u_\varepsilon \to u_{0,0}$ in L^1 we cannot have $u_{\varepsilon_n} \to u_{0,\lambda}$ for some sequence $\varepsilon_n \searrow 0$ and $\lambda \neq 0$.

□

5 Acknowledgements

The author thanks P. Szmolyan for bringing his recent work on singular perturbation to his attention and C. Mascia for some comments.

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