Hyperbolic Systems with Supercharacteristic Relaxations and Roll Waves

by

SHI JIN[1] AND MARKOS A. KATSOULAKIS[2]

[1] School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332
jin@math.gatech.edu

[2] Department of Mathematics and Statistics
University of Massachusetts
Amherst, MA 01003
markos@math.umass.edu

ABSTRACT. We study a distinguished limit for general 2×2 hyperbolic systems with relaxation, which is valid in both the subcharacteristic and supercharacteristic cases. This is a weakly nonlinear limit, which leads the underlying relaxation systems into Burgers equation with a source term; the sign of the source term depends on the characteristic interleaving condition. In the supercharacteristic case, the problem admits a periodic solution known as the roll-wave, generated by a small perturbation around equilibrium constants. Such a limit is justified in the presence of artificial viscosity, using the energy method.

1. Introduction.

Hyperbolic systems with relaxations occur in the study of a variety of physical phenomena, for example in river equations [Sto] and linear and nonlinear waves [Whi], in relaxing gas flow with thermal and chemical nonequilibrium [VK], and in the kinetic theory of rarefied gas dynamics [Cer]. These problems can be described mathematically by a system of evolution equations

$$\partial_t U + \nabla \cdot F(U) = R(U)\,, \qquad U \in \mathbb{R}^N\,. \tag{1.1}$$

Here, as in [CLL], we use the term relaxation to denote the relaxation term $R(U)$ that has $N - n\,(n < N)$ relaxation times ε_i and determines uniquely the local equilibria $U = e(u)$ for

The research of S. Jin is partially supported by the National Science Foundation Grant No. DMS-9704957. The research of M. A. Katsoulakis is partially supported by the National Science Foundation Grant No. DMS-9801769.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

n independent conserved quantities u. As $\varepsilon_i \to 0$, u formally satisfies an $n \times n$ equilibrium system

$$\partial_t u + \nabla \cdot f(u) = 0\,. \tag{1.2}$$

One illustrative example is the 2×2 p-system with relaxation

$$\begin{aligned} &\partial_t u^\varepsilon + \partial_x v^\varepsilon = 0\,, \\ &\partial_t v^\varepsilon + \partial_x p(u^\varepsilon) = -\frac{1}{\varepsilon}[v^\varepsilon - f(u^\varepsilon)]\,, \end{aligned} \tag{1.3}$$

where $\varepsilon > 0$ is the relaxation parameter. This system is hyperbolic if $p'(u) > 0$ and its characteristic speeds are

$$\Lambda_\pm(u,v) = \pm\sqrt{p'(u)}\,.$$

Here the analogue of (1.2), in the $\varepsilon \to 0$ limit, is the scalar conservation law

$$\partial_t u + \partial_x f(u) = 0\,, \tag{1.4}$$

which describes the dynamics at the local equilibrium $v \approx f(u)$, as $\varepsilon \to 0$. As was shown in [CLL], this approximation yields entropy solutions of (1.4), when the *subcharacteristic* condition holds:

$$-\sqrt{p'(u)} < f'(u) < \sqrt{p'(u)}\,. \tag{1.5}$$

One way to formally justify (1.5), is to consider the Chapman-Enskog expansion $v^\varepsilon = f(u^\varepsilon) + \varepsilon v_1 + O(\varepsilon^2)$ around the line of equilibrium $v = f(u)$. Then, after neglecting the $O(\varepsilon^2)$ terms, the solution u^ε of (1.3a) solves

$$\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = \varepsilon \partial_x \left[(p'(u) - f'(u)^2)\partial_x u\right]\,,$$

which is well-posed, provided (1.5) holds.

The linear stability of relaxation problems was first studied by Whitham in [Whi], and nonlinear analysis begun with Liu [Liu]. In particular, lots of efforts have been focused on studying the limiting behavior of $U(t/\varepsilon, x/\varepsilon)$ of (1.1), as $\varepsilon \to 0$, which is governed by the relaxed equation (1.2). See [CLL, CR, LM, KT, Nat1, Nat3, Ser, Teng, Yon]. Semilinear hyperbolic systems of type (1.1), with $F(U)$ linear in U, have also been introduced as a building block to construct the physically relevant weak solution of (1.2) in the zero relaxation limit and the relaxation schemes for (1.2) (see [JX], [KT], [Nat2]). All these results are based on the subcharacteristic condition [Liu], requiring that the characteristics of the system (1.1) strictly interleave with those of the relaxed equilibrium system (1.2), for instance condition (1.5). Violation of such a condition leads to linear instability of the system (1.1), [Whi]. The only exception occurs in the relaxation approximation to Hamilton-Jacobi equations [JX2], where the relaxed equilibrium equation introduces a curvature term when the subcharacteristic condition is at the critical case [JK, JKX]. In addition, a renormalized Chapman-Enskog expansion was introduced in [Sle] for problems where the subcharacteristic condition cannot be satisfied, as one includes higher order terms in ε, in a Chapman-Enskog expansion.

In physical applications, the subcharacteristic condition (1.5) is not always satisfied and the violation of this condition leads to physical instability. A notable example occurs in a uniform open-channel flow down an incline. Such a physical problem is described by the river equations–the shallow water equations with source terms modeling the balance between the slope and the friction of the river bottom. The two variables are the depth h and the mean velocity v, solving the system

$$\begin{aligned} h_t + (hv)_x &= 0\,, \\ (hv)_t + \big(hv^2 + \frac{1}{2}gh^2\big)_x &= \frac{1}{\varepsilon}\big(ghS - C_f v^2\big)\,, \end{aligned} \tag{1.6}$$

where g is the gravitational constant, S is the slope of the river bottom and C_f is the friction coefficient. The zero-relaxation limit $\varepsilon \to 0$, is associated with a flooding state in which case $v = \sqrt{gSh/C_f}$ and h solves the scalar conservation law $h_t + (hv)_x = 0$. The subcharacteristic condition is formally derived by a Chapman-Enskog type expansion setting $v = \sqrt{gSh/C_f} + \varepsilon v_1 + O(\varepsilon^2)$. By ignoring the higher order terms, (1.6a) yields the diffusive conservation law [Jin],

$$h_t + \Big(\big(\frac{gS}{C_f}h\big)^{1/2} h\Big)_x = \varepsilon\big(Ah(h^{3/2})_x\big)_x\,,$$

where the constant A is positive, i.e. the equation is well-posed, if

$$S < 4C_f\,.$$

This subcharacteristic condition, implying that the Froude number is smaller than 2, is satisfied when the model describes flow in a river bed where the slope S is usually small compared to the friction coefficient C_f; this is not the case for steep channels, where the uniform flow eventually evolves to a series of breaking waves or bores that are separated by sections of gradually varying flow, in a staircase pattern, Fig 1.1. Under suitable conditions these formations constitute *discontinuous* periodic traveling waves, and are known as "roll

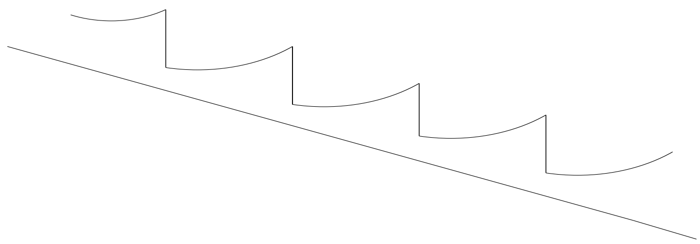

FIG.1.1. Roll waves.

waves" [Sto]. We note that the phenomenon of roll waves has been actually observed (and photographed!) in inclined spillways and riverbeds, see Cornish [Cor].

The mechanism leading to the instability was revealed by Jeffreys [Jef] in an early study of roll waves. Dressler constructed nonlinear periodic solutions that consist of piecewise smooth profiles separated by discontinuities (shocks) using the river equations [Dre]. Novik [Nov] proposed to model the roll waves by a Burgers equation with a source term. Continuous roll waves in the unstable regime were constructed by Needham and Merkin using the river equations with a diffusive term in the momentum equation [NM]. More recently, Kevorkian and Yu [KY, YK] derived an integro-partial differential equation for the evolution of the amplitude for a Froude number bigger than–but close to–the critical value 2. In a fundamental paper, Kranenburg formally derived the viscous Burgers equation with a source term, conjectured earlier by Novik, for small amplitude from the viscous river equations [Kra].

Motivated by the work of Kranenburg, we systematically study such a weakly nonlinear limit (also called generalized geometric optics limit in the literature, see for example [Hun]) for general hyperbolic systems with relaxation (1.1), and in particular when a *supercharacteristic* condition holds, i.e. the subcharacteristic condition fails. While the classical stiff relaxation scaling in (1.3) and (1.6) reflects the large space and time behavior, the weakly nonlinear limit to be studied in this article is the small space and time scaling depending on variables

$$\xi = \frac{x - ct}{\varepsilon}, \qquad \tau = t$$

where c is the characteristic speed. Our study shows that, in the supercharacteristic case, roll wave solutions occur for general 2×2 nonlinear hyperbolic system (with $F(U)$ a nonlinear function) with relaxation. In particular, we derive the same Burgers' equation with source term in the weakly nonlinear limit. In the limiting equation, the sign of the source term depends on the interleaving of the characteristic speeds of the original system and the relaxed equilibrium system. When the subcharacteristic condition is satisfied, the source term is a damping term. The violation of this condition induces exponential growth that balances the N-wave decay of the nonlinear convection, leading to the development of roll waves. This weakly nonlinear limit is rigorously justified by the energy method after adding to the 2×2 system an artificial viscosity.

In the case of a semilinear hyperbolic system ($F(U)$ is linear in U), the limiting equation becomes an ordinary differential equation, and the characteristic condition determines whether the solution will decay or grow exponentially. However, for kinetic equations, we always obtain exponential decay independent of the characteristic condition.

This paper is organized as follows. In Section 2 we study the weakly nonlinear asymptotics for the p-system with relaxation (1.3), and derive the limiting Burgers equation with a source term. We then study numerically the p-system with relaxation and the limiting equation under the supercharacteristic condition, and show the instability of roll waves under subharmonic perturbation. In Section 3 this weakly nonlinear limit is justified using the energy method, (1.3) being regularized with artificial viscosities. In Section 4, we discuss weakly nonlinear asymptotics of general 2×2 hyperbolic systems, under the sub- and supercharacteristic condition. In Section 5 we study this limit for several semilinear hyperbolic systems with relaxations, including the discrete velocity kinetic equations. We conclude our study in Section 6.

2. Weakly nonlinear asymptotic limit for a p-system.

Consider the following p-system with relaxation:

$$\begin{aligned} &\partial_t u + \partial_x v = 0\,, \\ &\partial_t v + \partial_x p(u) = f(u) - v\,. \end{aligned} \tag{2.1}$$

Let

$$\xi = \frac{x - ct}{\varepsilon}\,, \qquad \tau = t \tag{2.2}$$

where c is a constant to be determined. Introducing the transformation

$$u(x,t) = u_0 + \varepsilon A(\xi,\tau)\,, \qquad v(x,t) = f(u_0) + \varepsilon B(\xi,\tau)\,, \tag{2.3}$$

where u_0 is any constant, in (2.1), yields the equations for A and B as

$$\begin{aligned} &\partial_\tau A + \tfrac{1}{\varepsilon}\partial_\xi (B - cA) = 0\,, \\ &\partial_\tau B + \tfrac{1}{\varepsilon^2}\partial_\xi p(u_0 + \varepsilon A) - \tfrac{c}{\varepsilon}\partial_\xi B = \tfrac{1}{\varepsilon}[f(u_0 + \varepsilon A) - f(u_0)] - B\,. \end{aligned} \tag{2.4}$$

Let

$$J = \tfrac{1}{\varepsilon}(B - cA)\,,$$

then (2.4) is equivalent to the following system of equations:

$$\begin{aligned} &\partial_\tau A + \partial_\xi J = 0\,, \\ &\partial_\tau J + \tfrac{1}{\varepsilon^2}\partial_\xi\left[\tfrac{1}{\varepsilon}p(u_0+\varepsilon A) - c^2 A\right] - \tfrac{2c}{\varepsilon}\partial_\xi J = \tfrac{1}{\varepsilon^2}[f(u_0+\varepsilon A) - f(u_0)] - J - \tfrac{c}{\varepsilon}A\,. \end{aligned} \tag{2.5}$$

So far everything is exact and we have made no approximations. Now we expand $p(u_0+\varepsilon A)$ and $f(u_0+\varepsilon A)$ in Taylor series around u_0, and (2.5) becomes

$$\begin{aligned} &\partial_\tau A + \partial_\xi J = 0\,, \\ &\partial_\xi[p'(u_0) - c^2]A + \varepsilon p''(u_0)\,\partial_\xi \tfrac{1}{2}A^2 - 2c\,\varepsilon\,\partial_\xi J = \varepsilon[f'(u_0) - c]A + O(\varepsilon^2)\,. \end{aligned} \tag{2.6}$$

Clearly the balance of the $O(1)$ terms in (2.6b) requires

$$c = \pm\sqrt{p'(u_0)}\,. \tag{2.7}$$

With this c, (2.6b) gives

$$\partial_\xi J = \frac{1}{2c}p''(u_0)\partial_\xi\frac{A^2}{2} - \frac{f'(u_0)-c}{2c}A + O(\varepsilon)\,. \tag{2.8}$$

Substituting (2.8) in (2.6a), yields (after ignoring the $O(\varepsilon)$ term) the Burgers equation with a source term,

$$\partial_\tau A + \frac{p''(u_0)}{2c}\partial_\xi\frac{A^2}{2} = \frac{f'(u_0)-c}{2c}A\,. \tag{2.9}$$

What is interesting is that the sign of $(f'(u_0)-c)/2c$ plays a critical role in the solution to (2.9):

(i) If $-c < f'(u_0) < c$, i.e., when the subcharacteristic condition (1.5) is satisfied, the source term is damping, and the solution of (2.9) will decay, at a rate which combines the n-wave decay of the homogeneous part with that of the damping.

(ii) If $f'(u_0) > c$ or $f'(u_0) < -c$, i.e., when a supercharacteristic condition holds, then the source term enforces growth, the balance of which with the nonlinear decay of the convection term yields asymptotically discontinuous periodic waves, similar to the roll-waves described earlier (compare Fig 1.1 and Fig 2.2a). From now on we will also refer to such waves, as roll-waves. Discontinuous periodic solutions of this type for the Burgers equation (2.9) were constructed by Lyberopoulos [Lyb], who also showed that they can be obtained as an asymptotic limit of the entropy solution of (2.9), using the method of generalized characteristics [Daf]. The stability of (2.9) was also investigated in an L^2 framework by Goodman [Gdm].

As (2.9) indicates, the nonlinearity of the flux in (2.1) is critical to the development of roll waves. If $p(u) = au$, as in the Jin-Xin model [JX], then the quadratic term in (2.9) vanishes, and one obtains an ODE with positive eigenvalue in the supercharacteristic condition. Thus the solution will just grow exponentially, and our asymptotics (2.3) eventually will break down. A numerical investigation for this linear case was carried out in [LW].

Note that during the asymptotic expansion we impose no condition on the sign of $(f'(u_0) - c)/2c$, indicating that this expansion is valid (at least formally) for either sign. In Section 3 we will rigorously prove the validity of the expansion in the presence of artificial viscosity, under both the super- and sub-characteristic condition.

We now carry out some numerical experiments to illustrate the development of roll waves and the instability of periodic solutions. We first solve the p-system (2.1) with $p(u) = u + u^2/2$ and $f(u) = 2u$ over the domain $[0, 1]$ with a periodic boundary condition. The spatial derivatives in (2.1) are discretized by the second order relaxed scheme proposed in [JX], and the remaining system of ODEs (the discretized convection with relaxation) was then solved by the second order Runge-Kutta method. In the first example the initial data are $u(x, 0) = 0.01 \sin(8\pi x)$ and $v(x, 0) = 0$. Thus $u_0 = v_0 = 0, c = \sqrt{p'(u_0)} = 1 < f'(u_0) = 2$. We output the numerical solutions from $t = 0$ to $t = 30$ at every integer time. One sees from the top figure in Fig.2.1 that the sinusoidal initial profile grows and eventually forms a periodic roll wave connected by four bores (shocks) in the domain. Eventually these bores merge to form one bigger bore and the roll waves become higher and longer. We then perturb the initial data with a subharmonic disturbance $u(x, 0) = 0.01 \sin(8\pi x) + 0.001 \sin(2\pi x), v(x, 0) = 0$. The results are presented in Fig.2.1b. As one can see, the qualitative behavior of the solutions is the same, however the larger roll waves are formed at a much earlier time. This indicates the instability of the periodic solutions.

We then solve the limiting equation (2.9) with the same data, except now the independent variables have been changed to ξ and τ (so these are not the solutions in the Fig.2.1 at the same location and the same time). The results are depicted in Fig 2.2. where we basically see the same behavior as in the previous example. For more numerical experiments on this equation see [Kra].

Next we compare the solution of the A-J equation (2.5) with that of (2.9), for $\varepsilon = 0.1$. In order to have enough resolution we have to take $\Delta x << \varepsilon$. We have used $\Delta x = \varepsilon$ and the results were not close to that of (2.9). Instead we take $\Delta x = 0.001 = \varepsilon^3$. Since the A-J equation (2.5) has characteristic speed $\sim \frac{1}{\varepsilon}$, one has to use $\Delta t \sim \varepsilon \Delta x$, which is rather

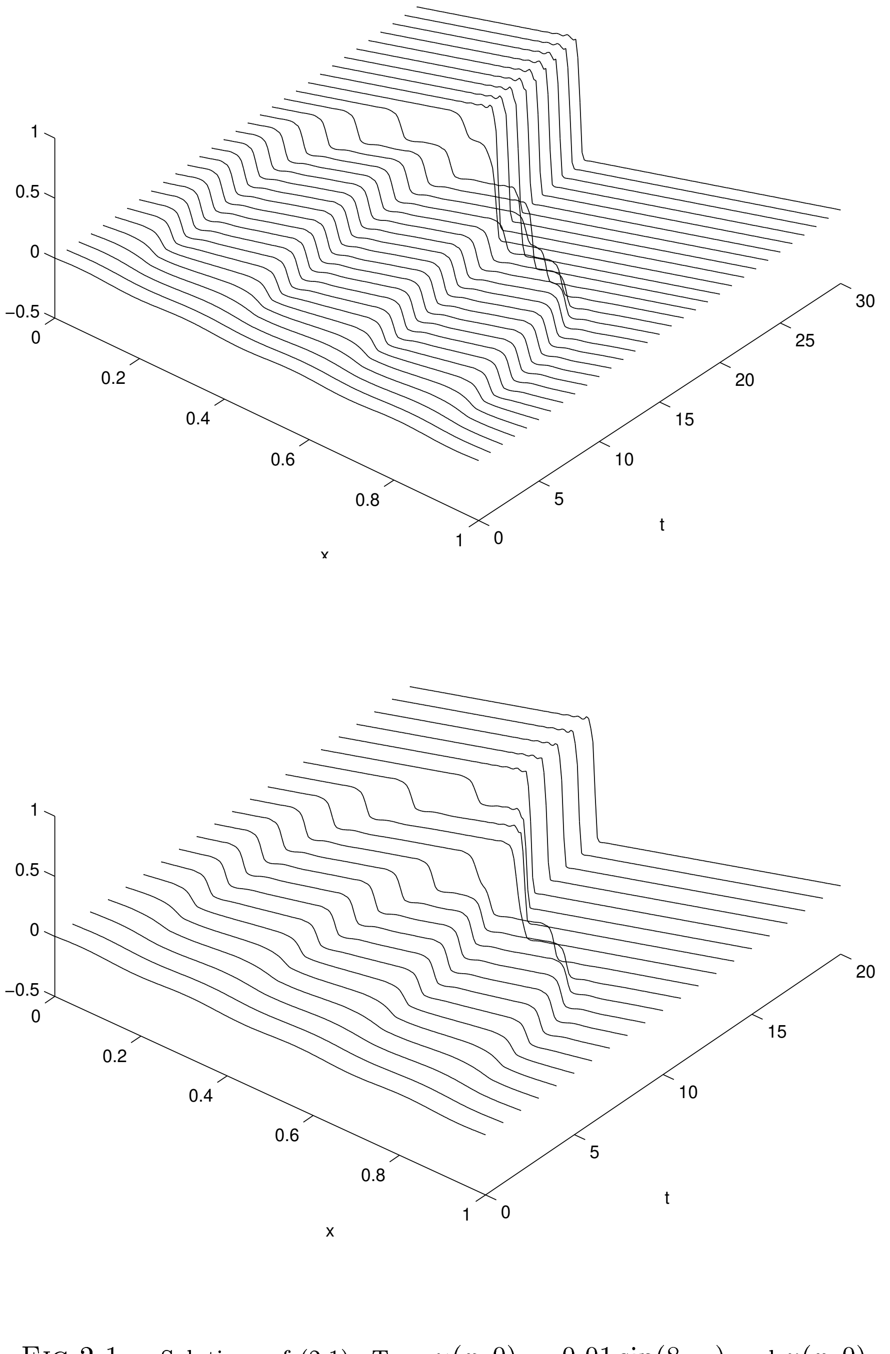

FIG.2.1. Solutions of (2.1). Top: $u(x,0) = 0.01\sin(8\pi x)$ and $v(x,0) = 0$. Bottom: $u(x,0) = 0.01\sin(8\pi x) + 0.001\sin(2\pi x), v(x,0) = 0$.

restrictive. To solve (2.9), however, $\Delta t \sim \Delta x$ is appropriate. The results from $t = 0$ to $t = 10$, output at every integer time, are depicted in Fig.2.3, showing that the solution of (2.9) stays close to the solution of (2.5a), for small ε.

3. Proof of the weakly nonlinear limit.

In this section we justify rigorously the weakly nonlinear limit formally discussed in Section 2, for the Cauchy problem of the relaxation system with artificial viscosity,

$$\begin{cases} \partial_t u^\varepsilon + \partial_x v^\varepsilon = \delta \partial_{xx} u^\varepsilon \\ \partial_t v^\varepsilon + \partial_x p(u^\varepsilon) = f(u^\varepsilon) - v^\varepsilon + \delta \partial_{xx} v^\varepsilon \\ u^\varepsilon(x,0) = u_0(x)\,, \quad v^\varepsilon(x,0) = v_0(x)\,, \end{cases} \tag{3.1}$$

for both the subcharacteristic and supercharacteristic conditions. Set $\delta = \varepsilon^2$, and rescale (3.1) as in (2.3), namely, let $u(x,t) = u_0 + \varepsilon A^\varepsilon\left(\frac{x-ct}{\varepsilon}, t\right)$, $v^\varepsilon(x,t) = f(u_0) + \varepsilon B^\varepsilon\left(\frac{x-ct}{\varepsilon}, t\right)$, with $c = \pm\sqrt{p'(u_0)}$. Then, $A^\varepsilon, B^\varepsilon$ satisfy the system (here we replace the (ξ,τ)-variables with (x,t)):

$$\begin{cases} A^\varepsilon_t + \frac{1}{\varepsilon}(B^\varepsilon - cA^\varepsilon)_x = A^\varepsilon_{xx}\,, \\ B^\varepsilon_t + \frac{1}{\varepsilon^2} p(u_0 + \varepsilon A^\varepsilon)_x - \frac{c}{\varepsilon} B^\varepsilon_x = \frac{1}{\varepsilon}[f(u_0 + \varepsilon A^\varepsilon) - f(u_0) - \varepsilon B^\varepsilon] + B^\varepsilon_{xx}\,, \\ A^\varepsilon(x,0) = A^\varepsilon_0(x), \quad B^\varepsilon(x,0) = B^\varepsilon_0(x)\,. \end{cases} \tag{3.2}$$

Setting

$$J^\varepsilon = \frac{B^\varepsilon - cA^\varepsilon}{\varepsilon}, \qquad K^\varepsilon = B^\varepsilon + cA^\varepsilon\,,$$

we obtain the system

$$\begin{cases} {K^\varepsilon}_t + \frac{1}{\varepsilon} M(K^\varepsilon, J^\varepsilon)_x + \frac{1}{\varepsilon} N(K^\varepsilon, J^\varepsilon) = {K^\varepsilon}_{xx}\,, \\ \varepsilon^2 {J^\varepsilon}_t + M(K^\varepsilon, J^\varepsilon)_x - 2\varepsilon c J^\varepsilon_x + N(K^\varepsilon, J^\varepsilon) = \varepsilon^2 {J^\varepsilon}_{xx}, \end{cases} \tag{3.3}$$

where

$$\begin{aligned} M(K^\varepsilon, J^\varepsilon) &= \frac{1}{\varepsilon}\left[p(u_0 + \varepsilon A^\varepsilon) - p(u_0)\right] - p'(u_0) A^\varepsilon\,, \\ N(K^\varepsilon, J^\varepsilon) &= \frac{\varepsilon}{2}(K^\varepsilon + \varepsilon J^\varepsilon) - \left[f(u_0 + \varepsilon A^\varepsilon) - f(u_0)\right]. \end{aligned}$$

We prove the following Theorem:

Theorem 3.1. *Suppose that $(A^\varepsilon, B^\varepsilon)$ solves (3.2) with initial data satisfying the conditions of Lemma 3.2 below. Then for any given $T > 0$,*

$$A^\varepsilon \to A \ \textit{in}\ L^2(\mathbb{R} \times (0,T)),$$

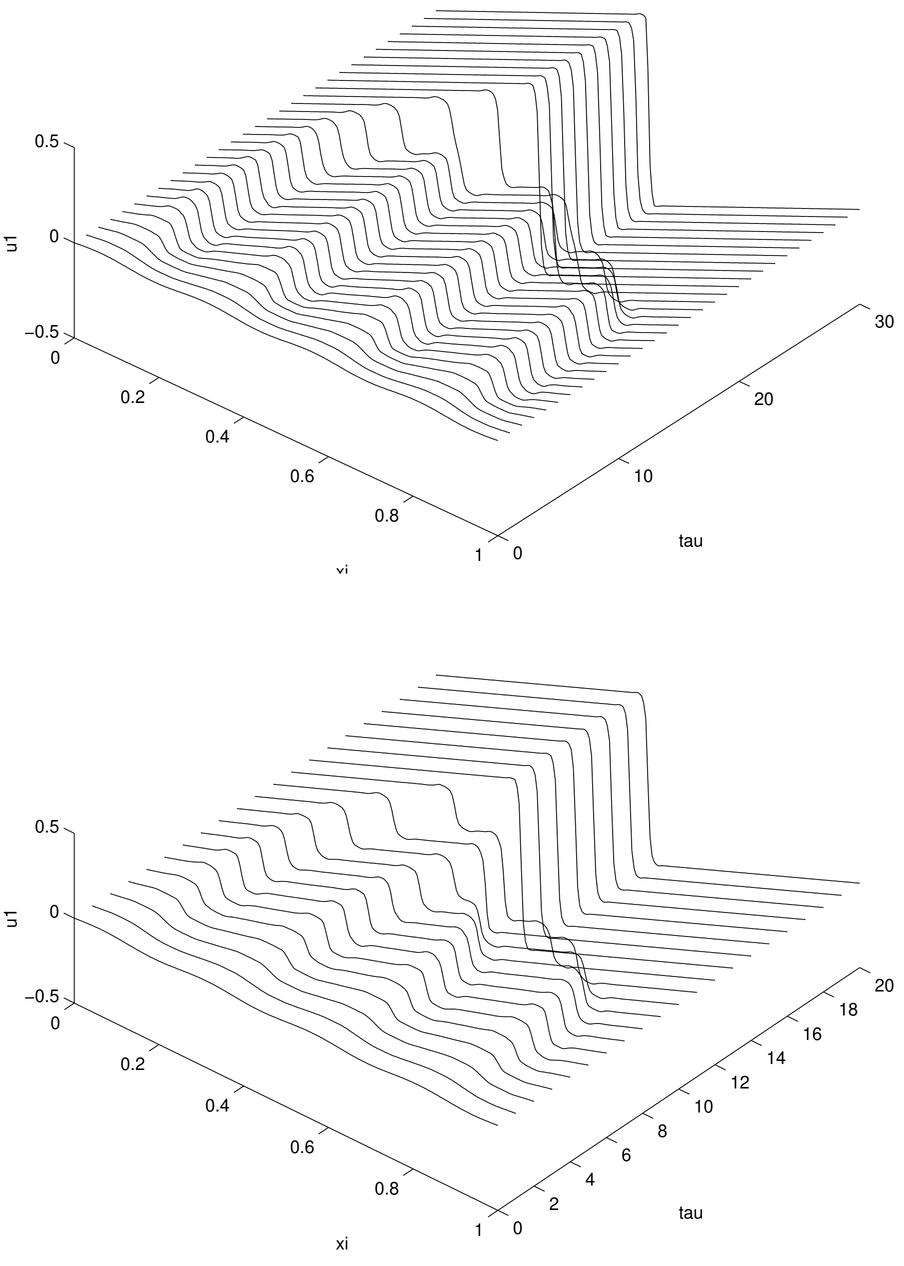

FIG.2.2. Solutions of (2.9). Top: $u(x,0) = 0.01\sin(8\pi x)$ and $v(x,0) = 0$. Bottom: $u(x,0) = 0.01\sin(8\pi x) + 0.001\sin(2\pi x), v(x,0) = 0$.

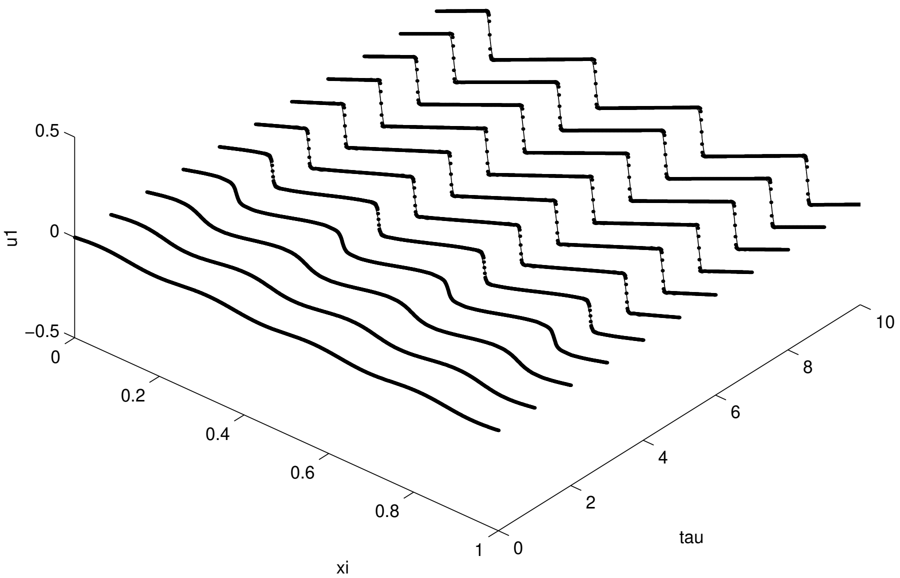

FIG.2.3. Solutions of (2.5) with $\varepsilon = 0.1$ (solid lines) compared with solutions of (2.9) (dotted line).

where A solves the viscous Burgers equation,

$$\partial_t A + \frac{p''(u_0)}{2c}\partial_x \frac{A^2}{2} = \frac{f'(u_0) - c}{2c} A + \partial_{xx} A\,. \tag{3.4}$$

Furthermore, $\sqrt{\varepsilon} J^\varepsilon = \frac{1}{\sqrt{\varepsilon}}(B^\varepsilon - cA^\varepsilon)$ is bounded uniformly in ε, in $L^\infty(\mathbb{R} \times (0,T))$.

The main point in the proof of this Theorem is that due to the weakly nonlinear expansion (2.3), the source term in the system (3.2) is expected to behave linearly to leading order, even if the subcharacteristic condition fails. This observation is indeed verified by the estimates in the following Lemma:

Lemma 3.2. *There exists a positive constant C_0 such that, if*

$$\|A_0^\varepsilon\|_{H^2(\mathbb{R}^2\times(0,T))}, \|\varepsilon J_0\|_{H^2(\mathbb{R}^2\times(0,T))} \le C_0, \tag{3.5}$$

then

$$\int_{-\infty}^{\infty}\big(A^{\varepsilon 2}+A_x^{\varepsilon 2}+{A_{xx}^{\varepsilon}}^2+A_t^{\varepsilon 2}+\varepsilon^2(J^{\varepsilon 2}+{J^{\varepsilon}}_x^{\,2}+{J^{\varepsilon}}_{xx}^{\,2}+{J^{\varepsilon}}_t^{\,2})\big)(x,t)\,dx$$
$$+\int_0^t\int_{-\infty}^{\infty}\big({A_{xxx}^{\varepsilon}}^2+{A_{xt}^{\varepsilon}}^2+\varepsilon({J^{\varepsilon}}_{xxx}^{\,2}+{J^{\varepsilon}}_{xt}^{\,2})\big)(x,s)\,dxds$$
$$\leq C\int_{-\infty}^{\infty}\big(A_0^{\varepsilon 2}+{A_{0x}^{\varepsilon}}^2+{A_{0xx}^{\varepsilon}}^2+{A_{0t}^{\varepsilon}}^2+\varepsilon^2(J_0^{\varepsilon 2}+{J_{0x}^{\varepsilon}}^2+{J_{0xx}^{\varepsilon}}^2+{J_{0t}^{\varepsilon}}^2)\big)(x,0)dx\,,$$

for all $0\leq t\leq T$, where C depends only on C_0. □

Proof of Lemma 3.2. We first obtain a priori estimates for A^ε and J^ε. For simplicity in the notation, we drop the ε-superscripts.

1. The relation (3.3b) implies that,

$$\partial_t\frac{\varepsilon^2}{2}\int_{-\infty}^{\infty}J^2(x,t)dx+\varepsilon^2\int_{-\infty}^{\infty}{J_x}^2(x,t)dx$$
$$-\int_{-\infty}^{\infty}M(K,J)J_x(x,t)dx+\int_{-\infty}^{\infty}N(K,J)J(x,t)dx=0,$$

which in turn yields

$$\partial_t\frac{\varepsilon^2}{2}\int_{-\infty}^{\infty}J^2(x,t)dx+\varepsilon^2\int_{-\infty}^{\infty}({J_x}^2+J^2)(x,t)dx\leq\tfrac{\varepsilon}{2}\|p''\|_{L^\infty}\|A\|_{L^\infty}\int_{-\infty}^{\infty}|A||J_x|(x,t)dx$$
$$+\varepsilon|c-f'(u_0)|\int_{-\infty}^{\infty}|A||J|(x,t)dx+\tfrac{\varepsilon^2}{2}\|f''\|_{L^\infty}\|A\|_{L^\infty}\int_{-\infty}^{\infty}|A||J|(x,t)dx.$$

Using the inequality $2\alpha\beta\leq\frac{1}{a}\alpha^2+a\beta^2$, and after suitably choosing $a>0$ (this technique will be used frequently throughout the proof without repeatedly being mentioned) , one gets the a priori estimate

$$\varepsilon^2\int_{-\infty}^{\infty}J^2(x,t)dx+\varepsilon^2\int_0^t\int_{-\infty}^{\infty}({J_x}^2+J^2)(x,s)dx\,ds\leq\varepsilon^2\int_{-\infty}^{\infty}J^2(x,0)dx$$
$$+\widehat{C}\int_0^t\int_{-\infty}^{\infty}A^2(x,s)dx\,ds\,. \tag{3.8}$$

A bound similar to (3.8) holds for J_x:

$$\varepsilon^2\int_{-\infty}^{\infty}{J_x}^2(x,t)dx+\varepsilon^2\int_0^t\int_{-\infty}^{\infty}({J_{xx}}^2+{J_x}^2)(x,s)\,dx\,ds$$
$$\leq\varepsilon^2\int_{-\infty}^{\infty}{J_x}^2(x,0)\,dx+\widehat{C}\int_0^t\int_{-\infty}^{\infty}A_x^2\,dx\,ds. \tag{3.9}$$

Here $\widehat{C}$ denotes (possibly different) constants depending on $\|A\|_{L^\infty}$. In addition,

$$\begin{aligned}
&\varepsilon^2 \int_{-\infty}^{\infty} {J_{xx}}^2(x,t)dx + \varepsilon^2 \int_0^t \int_{-\infty}^{\infty} ({J_{xxx}}^2 + {J_{xx}}^2)(x,s)\,dx\,ds \\
&\qquad \le \varepsilon^2 \int_{-\infty}^{\infty} {J_{xx}}^2(x,0)\,dx + \widehat{C}_1 \int_0^t \int_{-\infty}^{\infty} A_{xx}^2\,dx\,ds,
\end{aligned} \tag{3.10}$$

where $\widehat{C}_1$ denotes a constant depending on $\|A\|_{L^\infty}$ and $\|A_x\|_{L^\infty}$. These bounds are guaranteed provided suitable Sobolev bounds hold, since,

$$A^2(x,t) = \int_{-\infty}^{x} 2AA_x dx \le 2\left(\int_{-\infty}^{\infty} A^2 dx \int_{-\infty}^{\infty} {A_x}^2 dx\right)^{1/2},$$

and

$${A_x}^2(x,t) = \int_{-\infty}^{x} 2A_xA_{xx} dx \le 2\left(\int_{-\infty}^{\infty} {A_x}^2 dx \int_{-\infty}^{\infty} {A_{xx}}^2 dx\right)^{1/2}.$$

2. We now estimate the L^2 norm of εJ_t: equation (3.3b) yields

$$\varepsilon^2 J_{tt} + M(A,J)_{tx} - 2c\varepsilon J_{xt} + N(A,J)_t = \varepsilon^2 J_{xxt}. \tag{3.11}$$

Proceeding as in the previous estimates, we have

$$\begin{aligned}
\partial_t \tfrac{\varepsilon^2}{2} \int_{-\infty}^{\infty} {J_t}^2(x,t)dx + \varepsilon^2 \int_{-\infty}^{\infty} ({J_{xt}}^2 + {J_t}^2)(x,t)dx \le \varepsilon^2 \|p''\|_{L^\infty} \|A\|_{L^\infty} \int_{-\infty}^{\infty} |A_t||J_{xt}|dx \\
+ \varepsilon|c - f'(u_0)| \int_{-\infty}^{\infty} |A_t||J_t|(x,t)dx + \varepsilon^2 \|f''\|_{L^\infty} \|A\|_{L^\infty} \int_{-\infty}^{\infty} |A_t||J_t|(x,t)dx\,.
\end{aligned}$$

By Gronwall's inequality,

$$\begin{aligned}
&\varepsilon^2 \int_{-\infty}^{\infty} {J_t}^2(x,t)dx + \varepsilon^2 \int_0^t \int_{-\infty}^{\infty} ({J_{xt}}^2 + {J_t}^2)(x,s)dx\,ds \\
&\qquad \le \varepsilon^2 \int_{-\infty}^{\infty} {J_t}^2(x,0)dx + \check{C} \int_0^t \int_{-\infty}^{\infty} {A_t}^2(x,s)dx\,ds,
\end{aligned} \tag{3.12}$$

for $0 \le t \le T$; here $\check{C}$ depends on $\|A\|_{L^\infty}, \|p''\|_{L^\infty}, T$.

3. We now turn towards the estimates for K (and consequently A and B). Equation (3.3a) is rewritten as,

$$\begin{aligned}
K_t + \frac{p''(u_0)}{2}\left(\frac{1}{4c^2}K^2\right)_x + \frac{c - f'(u_0)}{2c}K - K_{xx} = -\varepsilon\left(1 - \frac{c - f'(u_0)}{2c}\right)J \\
+ \frac{p''(u_0)}{2c}\left(\tfrac{\varepsilon}{2c}JK - \frac{1}{4c}\varepsilon^2 J^2\right)_x + Q_0(A)A_x + Q_1(A)\,,
\end{aligned} \tag{3.13}$$

where $A = \frac{1}{2c}(K - \varepsilon J)$ and

$$Q_0(A) = \frac{1}{\varepsilon}\left[\varepsilon p''(u_0)A - p'(u_0 + \varepsilon A) + p'(u_0)\right] = O(\varepsilon \|p'''\|_{L^\infty}(|K| + |\varepsilon J|)^2)$$
$$Q_1(A) = \frac{1}{\varepsilon}\left[f(u_0 + \varepsilon A) - f(u_0) - \varepsilon f'(u_0)A\right] = O(\varepsilon \|f''\|_{L^\infty}(|K| + |\varepsilon J|)^2).$$

4. We first estimate the L^2 norm of K: (3.13) implies

$$\begin{aligned}\int_{-\infty}^{\infty} K^2(x,t)dx + \int_0^t \int_{-\infty}^{\infty} {K_x}^2(x,s)dx\,ds \le \int_{-\infty}^{\infty} K^2(x,0)dx \\ + C\int_0^t \int_{-\infty}^{\infty} \left(K^2 + \varepsilon^2 J^2\right)(x,s)dx\,ds,\end{aligned} \tag{3.14}$$

where $C = C(D)$, assuming that for some constant D

$$\|K, \varepsilon J\|_{L^\infty} \le D\,. \tag{3.15}$$

Relations (3.8), (3.15) along with Gronwall's inequality imply

$$\begin{aligned}\int_{-\infty}^{\infty} \left(K^2 + \varepsilon^2 J^2\right)(x,t)dx + \int_0^t \int_{-\infty}^{\infty} \left(K_x^2 + \varepsilon^2 J_x^2 + \varepsilon^2 J^2\right)(x,s)dx\,ds \\ \le C\int_{-\infty}^{\infty} \left(K^2 + \varepsilon^2 J^2\right)(x,0)dx\,.\end{aligned} \tag{3.16}$$

5. We now estimate $\int_{-\infty}^{\infty} {K_x}^2(x,t)dx$: as in Step 4 we have

$$\begin{aligned}\int_{-\infty}^{\infty} \left(K_x^2 + \varepsilon^2 J_x^2\right)(x,t)dx + \int_0^t \int_{-\infty}^{\infty} \left(K_{xx}^2 + \varepsilon^2 J_{xx}^2 + \varepsilon^2 J_x^2\right)(x,s)dx\,ds \\ \le C\int_{-\infty}^{\infty} \left(K_x^2 + \varepsilon^2 J_x^2\right)(x,0)dx\,,\end{aligned} \tag{3.17}$$

for all $0 \le t \le T$, where $C = C(D)$ depends on T and the constant D in (3.15). Note that according to the discussion preceeding Step 2, and the a priori estimates (3.16-17), we can guarantee the bound (3.15) provided the norm $B_0 = \|K(\cdot,0)\|_{L^2} + \|K_x(\cdot,0)\|_{L^2} + \|\varepsilon J(\cdot,0)\|_{L^2} + \|\varepsilon J_x(\cdot,0)\|_{L^2}$ is sufficiently small. Hence the estimates (3.16) and (3.17) indeed hold true.

Similarly we estimate $\int_{-\infty}^{\infty} {K_t}^2(x,t)dx$:

$$\begin{aligned}\int_{-\infty}^{\infty} \left(K_t^2 + \varepsilon^2 J_t^2\right)(x,t)dx + \int_0^t \int_{-\infty}^{\infty} \left(K_{xt}^2 + \varepsilon^2 J_{xt}^2 + \varepsilon^2 J_t^2\right)(x,s)dx\,ds \\ \le C\int_{-\infty}^{\infty} \left(K_t^2 + \varepsilon^2 J_t^2\right)(x,0)dx\,.\end{aligned} \tag{3.18}$$

Higher derivatives can also be estimated following Steps 4 and 5:

$$\begin{aligned}\int_{-\infty}^{\infty}\left(K_{xx}^2+\varepsilon^2 J_{xx}^2\right)(x,t)dx+&\int_0^t\int_{-\infty}^{\infty}\left(K_{xxx}^2+\varepsilon^2 J_{xxx}^2+\varepsilon^2 J_{xx}^2\right)(x,s)dx\,ds\\ &\leq C(D)\int_{-\infty}^{\infty}\left(K_{xx}^2+\varepsilon^2 J_{xx}^2\right)(x,0)dx\,. \end{aligned}\tag{3.19}$$

The estimates (3.16-19) yield the result.

Proof of Theorem 3.1.

According to Lemma 3.2, A^ε is bounded in $H^1(0,T;H^2(\mathbb{R}))$. By the Sobolev Imbedding Theorem, there is a subsequence $A^{\varepsilon'}$ converging strongly in L^2 to some $A\in H^1(0,T;H^2(\mathbb{R}))$. In addition, since $(A^\varepsilon, J^\varepsilon)$ are smooth,

$$\varepsilon J^{\varepsilon 2}(x,t)=\int_{-\infty}^{x}\varepsilon(J^{\varepsilon 2})_x dx=2\int_{-\infty}^{x}\varepsilon J^\varepsilon J_x^\varepsilon dx\leq\int_{-\infty}^{\infty}\varepsilon^2 J^{\varepsilon 2}dx+\int_{-\infty}^{\infty}J_x^{\varepsilon 2}dx\,.$$

Moreover from (3.3a), ${J^\varepsilon}_x={A^\varepsilon}_{xx}-{A^\varepsilon}_t$, hence

$$\varepsilon J^{\varepsilon 2}(x,t)\leq\int_{-\infty}^{\infty}\varepsilon^2 J^{\varepsilon 2}(x,t)dx+\int_{-\infty}^{\infty}({A_{xx}^\varepsilon}^2+A_t^{\varepsilon 2})dx.$$

From the bounds (3.16-19) we obtain that

$$|\sqrt{\varepsilon}J^\varepsilon(x,t)|\leq C,$$

thus $\varepsilon J^\varepsilon\to 0$ a.e. in $\mathbb{R}\times[0,T]$. Consequently $(A^\varepsilon,\varepsilon J^\varepsilon)\to(A,0)$ a.e. in $\mathbb{R}\times[0,T]$ and A solves (3.4) (c.f. (3.13)).

4. Weakly nonlinear asymptotics for general 2×2 hyperbolic systems with relaxation.

In this Section we study the weakly nonlinear asymptotic limits introduced in Section 2, under sub- and supercharacteristic conditions for general 2×2 hyperbolic relaxation systems:

$$\begin{aligned}&\partial_t u+\partial_x f_1(u,v)=0\,,\\ &\partial_t v+\partial_x f_2(u,v)+r(u,v)=0\,.\end{aligned}\tag{4.1}$$

We assume that (4.1) is strictly hyperbolic with characteristic speeds

$$\Lambda_\pm(u,v)=\frac{1}{2}\left[f_{1u}+f_{2v}\pm\sqrt{(f_{1u}-f_{2v})^2+4f_{1v}f_{2u}}\right].$$

For the source term $r = r(u,v)$, we assume there is a unique stable equilibrium $v = e(u)$, such that

$$r(u,e(u)) = 0\,, \qquad r_v(u,e(u)) > 0\,. \tag{4.2}$$

In addition we assume that (4.1) is nontrivially coupled, i.e.

$$f_{1v}(u,e(u)) \neq 0\,,$$

for all $u \in \mathbb{R}$. The solution $u(x/\varepsilon, t/\varepsilon), v(x/\varepsilon, t/\varepsilon)$ of (4.1), as $\varepsilon \to 0$, relaxes to the entropy solution of the scalar conservation law

$$\partial_t u + \partial_x f(u) = 0\,,$$

where $f(u) := f_1(u,e(u))$ and the corresponding characteristic speed is

$$\lambda(u) = f'(u) = f_{1u}(u,e(u)) + e'(u) f_{1v}(u,e(u))\,. \tag{4.3}$$

The relevant subcharacteristic condition for (4.1), is (see [CLL]),

$$\Lambda_- < \lambda < \Lambda_+\,. \tag{4.4}$$

Clearly, Eqs (4.1) are general enough to cover all 2 hyperbolic systems with relaxation characterized at the beginning of the introduction section, in particular, the p-system discussed in section 2 and the river equations (1.6).

Following Sections 2 and 3, we study here the asymptotics of (4.1), around the equilibrium constant states $(u_0, e(u_0))$. We first define the new variables $\xi = \frac{x-ct}{\varepsilon}$ $, \tau = t$ where c is a constant to be determined. Applying

$$u(x,t) = u_0 + \varepsilon A(\xi,\tau)\,, \qquad v(x,t) = f(u_0) + \varepsilon B(\xi,\tau)\,.$$

in (4.1) yields the equations for A and B:

$$\begin{aligned} &\partial_\tau A + \tfrac{1}{\varepsilon^2}\partial_\xi[f_1(u_0+\varepsilon A, e(u_0)+\varepsilon B) - \varepsilon c A] = 0\,,\\ &\partial_\tau B + \tfrac{1}{\varepsilon^2}\partial_\xi[f_2(u_0+\varepsilon A, e(u_0)+\varepsilon B) - \varepsilon c B] + \tfrac{1}{\varepsilon} r(u_0+\varepsilon A, e(u_0)+\varepsilon B) = 0\,. \end{aligned} \tag{4.5}$$

We now carry out the formal analysis as in Section 2. In (4.5), we expand f_1, f_2 and r in Taylor series and match the $O(\frac{1}{\varepsilon})$ terms. After sending $\varepsilon \to 0$, this gives

$$\begin{aligned} &(f_{1u} - c)\partial_\xi A + f_{1v}\partial_\xi B = 0\,,\\ &f_{2u}\partial_\xi A + (f_{2v} - c)\partial_\xi A = 0\,, \end{aligned} \tag{4.6}$$

where the partial derivatives of f_1 and f_2 are evaluated at $(u_0, e(u_0))$. Since we are seeking nontrivial solutions of (4.6), we need to have $(f_{1u} - c)(f_{2v} - c) - f_{1v} f_{2u} = 0$. Thus,

$$c = \Lambda_{\pm}(u_0, e(u_0)) . \tag{4.7}$$

With this choice of c, (4.5a) implies that

$$B(\xi, \tau) = \frac{c - f_{1u}}{f_{1v}} A(\xi, \tau) + \varepsilon J(\xi, \tau) , \tag{4.8}$$

for some J. After the Taylor expansions around $(u_0, e(u_0))$ are carried out, (4.5) yields

$$\begin{aligned} &\partial_\tau A + f_{1v} \partial_\xi J + \partial_\xi [\frac{1}{2} e^T \nabla^2 f_1 e] + O(\varepsilon) = 0 , \\ &\varepsilon \partial_\tau J + (f_{1u} + f_{2v} - 2c) \partial_\xi J + \partial_\xi \left[\frac{1}{2} e^T \nabla^2 \left(f_2 - \frac{c - f_{1u}}{f_{1v}} f_1 \right) e \right] + \left(r_u + r_v \frac{c - f_{1u}}{f_{1v}} \right) + \\ &\qquad + \varepsilon r_v J + O(\varepsilon) = 0 , \end{aligned}$$

where

$$e^T = \left(1, \frac{c - f_{1u}}{f_{1v}}\right) ,$$

and all derivatives are evaluated at the point $(u_0, e(u_0))$. By dropping the $O(\varepsilon)$ terms, one obtains

$$\begin{aligned} &\partial_\tau A + f_{1v} \partial_\xi J + \partial_\xi [\frac{1}{2} e^T \nabla^2 f_1 e] = 0 , \\ &(f_{1u} + f_{2v} - 2c) \partial_\xi J + \partial_\xi \left[\frac{1}{2} e^T \nabla^2 \left(f_2 - \frac{c - f_{1u}}{f_{1v}} f_1\right) e\right] + \left(r_u + r_v \frac{c - f_{1u}}{f_{1v}}\right) = 0 . \end{aligned} \tag{4.9}$$

Solving (4.9b) for $\partial_\xi J$, then applying it in (4.9a), one obtains a Burgers equation with a source term,

$$\partial_\tau A + \mu \partial_\xi \frac{A^2}{2} = \nu A , \tag{4.10}$$

where

$$\begin{aligned} \mu &= \frac{1}{\pm\sqrt{(f_{1u} - f_{2v})^2 + 4 f_{1v} f_{2u}}} e^T \left(f_{1v} \nabla^2 f_2 + (\Lambda_{\pm} - f_{1v}) \nabla^2 f_1 \right) e , \\ \nu &= \frac{r_v}{\pm\sqrt{(f_{1u} - f_{2v})^2 + 4 f_{1v} f_{2u}}} (\lambda - \Lambda_{\pm}) . \end{aligned} \tag{4.11}$$

As in the case of the p-system, from (4.11) one can see that:

(i) If the subcharacteristic condition (4.4) is satisfied, then $\nu < 0$ and the source term in (4.10) has a damping effect.

(ii) If a supercharacteristic condition holds, i.e. either $\Lambda_- > \lambda$ or $\Lambda_+ < \lambda$, then $\nu > 0$ and the source term in (4.10) induces growth, the balance of which with the nonlinear decay of the convection term yields roll-wave type of solutions, as was the case in (2.9).

It is also worth noting that (4.11) (or (2.9)) admits roll-wave solution regardless of the sign of μ (or $p''(u_0)$ in (2.9)), since one can always use the change of variable $A \mapsto -A$ which changes only the sign of the convection term but not the sign of other terms–including the viscosity term, if there is such a term.

Furthermore, we readily see that the analysis in Lemma 3.2 easily extends to (4.5) and consequently the results of Section 3 are valid for general 2×2 hyperbolic relaxation systems.

5. Semilinear hyperbolic systems with relaxation.

5.1 Discrete-velocity kinetic models.

We now derive the weakly nonlinear limit to a couple of discrete-velocity kinetic models. First consider the Broadwell model [Bro]:

$$\begin{aligned} \partial_t f^+ + \partial_x f^+ &= \tfrac{1}{\varepsilon}(f^0 f^0 - f^+ f^-)\,, \\ \partial_t f^0 &= \tfrac{1}{\varepsilon}(f^+ f^- - f^0 f^0)\,, \\ \partial_t f^- - \partial_x f^- &= \tfrac{1}{\varepsilon}(f^0 f^0 - f^+ f^-)\,, \end{aligned} \tag{5.1}$$

where f^+, f^0, and f^- denote the mass densities of gas particle with speed $1, 0$ and -1 respectively, and $\varepsilon > 0$ is the mean free path. After introducing the fluid moment variables

$$\rho = f^+ + 4f^0 + f^-\,, \quad m = f^+ - f^-\,, \quad u = \frac{m}{\rho}\,, \quad z = f^+ + f^-\,, \tag{5.2}$$

system (5.1) can be rewritten as

$$\begin{aligned} &\partial_t \rho + \partial_x m = 0\,, \\ &\partial_t m + \partial_x z = 0\,, \\ &\partial_t z + \partial_x m = \tfrac{1}{\varepsilon}\left[\tfrac{1}{8}(\rho - z)^2 + \tfrac{1}{2}(m^2 - z^2)\right]\,. \end{aligned} \tag{5.3}$$

As $\varepsilon \to 0$, the moment variables ρ and m solve a model Euler equations, under the subcharacteristic condition $|u| < 1$ [CP, Xin]. Now, as in previous sections, we introduce

$$\xi = \frac{x - ct}{\varepsilon^2}\,, \qquad \tau = \frac{t}{\varepsilon}\,. \tag{5.4}$$

We can proceed as before. However, here we instead use a Hilbert expansion, which nevertheless gives the same result. Introducing the ansatz

$$\begin{aligned} \rho &= \rho_0 + \varepsilon\rho_1(\xi,\tau) + \varepsilon^2\rho_2(\xi,\tau) + \cdots\,, \\ m &= m_0 + \varepsilon m_1(\xi,\tau) + \varepsilon^2 m_2(\xi,\tau) + \cdots\,, \\ z &= z_0 + \varepsilon z_1(\xi,\tau) + \varepsilon^2 z_2(\xi,\tau) + \cdots\,, \end{aligned} \tag{5.5}$$

where

$$z_0 = \rho_0 \left[\frac{2}{3}\sqrt{1+3u_0^2} - \frac{1}{3} \right] \tag{5.6}$$

is the local Maxwellian, which zeros the collision term on the right hand side of (5.3). Substituting the ansatz (5.5) into (5.3),

$$\begin{aligned}
-c\partial_\xi \rho_1 + \partial_\xi m_1 + \varepsilon[\partial_\tau \rho_1 - c\partial_\xi \rho_2 + \partial_\xi m_2] &= O(\varepsilon^2) \\
-c\partial_\xi m_1 + \partial_\xi z_1 + \varepsilon[\partial_\tau m_1 - c\partial_\xi m_2 + \partial_\xi z_2] &= O(\varepsilon^2) \\
-c\partial_\xi z_1 + \partial_\xi m_1 + \varepsilon[\partial_\tau z_1 - c\partial_\xi z_2 + \partial_\xi m_2] & \\
= \tfrac{1}{8}[\rho_0 + \varepsilon\rho_1 + \varepsilon^2\rho_2 + \cdots - z_0 - \varepsilon z_1 - \varepsilon^2 z_2 - \cdots]^2 & \\
+ \tfrac{1}{2}[(m_0 + \varepsilon m_1 + \varepsilon^2 m_2 + \cdots)^2 - (z_0 + \varepsilon z_1)^2 + \varepsilon^2 z_2 + \cdots] + O(\varepsilon^2) & \\
= \tfrac{\varepsilon}{4}[\rho_0\rho_1 - \rho_0 z_1 - \rho_1 z_0 + z_0 z_1] + \varepsilon[m_0 m_1 - z_0 z_1] + O(\varepsilon^2) &
\end{aligned} \tag{5.7}$$

To balance the leading term one needs

$$c = \pm 1 \,. \tag{5.8}$$

With this choice of c, (5.7) clearly shows

$$m_1 = c\rho_1 + O(\varepsilon)\,, \qquad z_1 = cm_1 + O(\varepsilon) = c^2\rho_1 + O(\varepsilon)\,. \tag{5.9}$$

Multiplying the second equation of (5.7) by c, then adding to the third equation, gives (using $c^2 = 1$)

$$c\partial_\tau m_1 + \partial_\tau z_1 = \tfrac{1}{4}[\rho_0\rho_1 - \rho_0 z_1 - z_0\rho_1 + z_0 z_1] + m_0 m_1 - z_0 z_1\,. \tag{5.10}$$

Using (5.9) in (5.10), and ignoring the $O(\varepsilon)$ terms, give the asymptotic equation for ρ_1:

$$\partial_\tau \rho_1 = \tfrac{1}{2}(cm_0 - z_0)\rho_1\,. \tag{5.11}$$

We now verify that

$$\tfrac{1}{2}(cm_0 - z_0) \le 0\,. \tag{5.12}$$

By (5.6),

$$cm_0 - z_0 = \rho_0 \left[cu_0 - \frac{2}{3}\sqrt{1+3u_0^2} + \frac{1}{3} \right]$$

where $u_0 = m_0/\rho_0$. To show

$$3cu_0 + 1 \le 2\sqrt{1+3u_0}\,,$$

it suffices to show

$$(3cu_0+1)^2 \le 4(1+3u_0)\,,$$

or

$$g(u_0) = 36u_0^4 + 15u_0^2 - 6cu_0 + 3 \ge 0\,. \tag{5.13}$$

Note

$$g(u_0) = (6u_0^2+1)^2 + 3(u_0-c)^2 - 1 \ge 1 + 3(u_0-c)^2 - 1 = 3(u_0-c)^2 \ge 0\,.$$

Thus (5.12) is true and (5.11) has exponentially decaying solutions, independent of the sign of $|u_0|-1$. This is rather different from the relaxation models with nonlinear fluxes, studied in the previous sections.

Next, we consider the Carleman model, which is a two velocity kinetic model [Car]:

$$\begin{aligned} \partial_t f^+ + \partial_t f^+ &= \tfrac{1}{\varepsilon}(f^- - f^+)\,, \\ \partial_t f^- - \partial_t f^- &= \tfrac{1}{\varepsilon}(f^+ - f^-)\,, \end{aligned} \tag{5.14}$$

where f^+,and f^- denote the mass densities of gas particle with speed 1 and -1 respectively, $\varepsilon > 0$ is the mean free path. Note that the Carleman model always satisfies the subcharacteristic condition, for positive solutions $f^\pm$, [KT]. Introducing the moment variables

$$\rho = f^+ + f^-\,, \qquad j = f^+ - f^- \tag{5.15}$$

then (5.14) can be written as

$$\begin{aligned} \partial_t\rho + \partial_x j &= 0\,, \\ \partial_t j + \partial_x \rho &= -2\rho j\,. \end{aligned} \tag{5.16}$$

Employing the ansatz

$$\rho = \rho_0 + \varepsilon\rho_1(\xi,\tau) + \varepsilon^2\rho_2(\xi,\tau) + \cdots\,, \qquad j = j_0 + \varepsilon j_1(\xi,\tau) + \varepsilon^2 j_2(\xi,\tau) + \cdots\,, \tag{5.17}$$

one can show that $c^2 = 1$ and asymptotically ρ_1 solves

$$\partial_t\rho = -\rho_0\rho\,. \tag{5.18}$$

The solutions of this ODE are exponentially decaying for $\rho_0 > 0$. We note here that the Euler limit of (5.14) is the trivial ODE $\partial_t\rho = 0$, while the weakly nonlinear limit introduced here gives a non-trivial ODE (5.18).

5.2 Semilinear hyperbolic systems with relaxations.

Another class of semilinear hyperbolic systems was introduced to regularize or relax a homogeneous systems of conservation laws. See for example [JX, Sul]. Here we study the weakly nonlinear limit for two such systems, namely, the Jin-Xin model [JX] and the Suliciu model [Sul]. First consider the 2×2 relaxation system in [JX]:

$$\begin{aligned} \partial_t u + \partial_x v &= 0\,, \\ \partial_t v + c^2 \partial_x u &= f(u) - v\,. \end{aligned} \tag{5.19}$$

The solution $u(x/\varepsilon, t/\varepsilon), v(x/\varepsilon, t/\varepsilon)$ of this system, as $\varepsilon \to 0$, relaxes to the entropy solution of

$$\partial_t u + \partial_x f(u) = 0\,,$$

[JX, Nat1], under the subcharacteristic condition

$$-c < f'(u) < c\,,$$

As we pointed out in Section 2, due to the lack of nonlinearity in the flux of (5.19), the weakly nonlinear scaling (2.2), (2.3), gives rise to the ODE

$$\partial_\tau A = \frac{f'(u_0) - c}{2c} A\,.$$

Clearly, the solutions decay exponentially when the subcharacteristic condition is satisfied, while grow exponentially otherwise, thus the weakly nonlinear asymptotic expansion (2.3) breaks down in the supercharacteristic case.

Similar behavior occurs in the Suliciu model, introduced for modeling van der Waals phase transitions [Sul]:

$$\begin{aligned} \partial_t u + \partial_t v &= 0\,, \\ \partial_t v + \partial_t p &= 0\,, \\ \partial_t p + E \partial_t v &= q(u) - p\,, \end{aligned} \tag{5.20}$$

when u is the specific volume, v the velocity, p the pressure, $q(u)$ the equation of state, and $E > 0$ being the Young's modulus. The subcharacteristic condition for this model is $E > q'(u)$ for all u. By introducing the ansatz

$$\begin{aligned} u &= u_0 + \varepsilon u_1(\xi, \tau) + \varepsilon^2 u_2(\xi, \tau) + \cdots\,, \\ v &= v_0 + \varepsilon v_1(\xi, \tau) + \varepsilon^2 v_2(\xi, \tau) + \cdots\,, \\ p &= q(u_0) + \varepsilon p_1(\xi, \tau) + \varepsilon^2 p_2(\xi, \tau) + \cdots\,, \end{aligned} \tag{5.21}$$

one can show that $c = \pm\sqrt{E}$, and u_1 solves

$$\partial_t u = \frac{q'(u_0) - E}{2E} u\,.$$

The solutions to this ODE decay exponentially, provided the subcharacteristic condition $q'(u_0) < E$ is satisfied, while they grow exponentially when it fails, in a similar manner to the semilinear Jin-Xin model.

The difference between the discrete-velocity kinetic models and the semilinear hyperbolic systems with relaxations under the weakly nonlinear limit is rather interesting. Although the Chapman-Enskog expansion for both classes gives similar limiting behavior (to the model Euler and Navier-Stokes equations) under the subcharacteristic condition, the limiting behavior under the weakly nonlinear limit is drastically different. For kinetic models the weakly nonlinear limit always yields stable ODEs, while for semilinear hyperbolic systems the stability depends on the characteristic condition.

6. Conclusion.

In this paper we study a weakly nonlinear limit for general hyperbolic systems with relaxations. Under such a limit, we obtain the asymptotic equations which turn out to be Burgers equation with a source term. Of particular interest is the sign of the source, which depends on the characteristic relation known as the subcharacteristic condition. If this condition is satisfied, the source term is damping. If this condition is violated (the supercharacteristic case), the source term introduces an exponential growth which balances the nonlinear decay of the convection term, leading to physically interesting "roll-wave" solutions. This limit is derived for a general class of 2×2 nonlinear hyperbolic systems with relaxation, as well as for semilinear hyperbolic systems with relaxation and discrete velocity kinetic models. Using the energy method, we also rigorously justified this asymptotic limit for a 2×2 p-system with relaxation, after viscous regularization.

In the future we will seek analytic tools to study this limit without artificial viscosity, and to understand the roll-wave solutions for more general physical systems and in higher space dimensions.

References

[Bro] J.E. Broadwell, *Shock structure in a simple discrete velocity gas*, Phys. Fluids 7, 1013-1037, 1964.

[CP] R.E. Caflisch and G.C. Papanicolaou, *The fluid-dynamical limit of nonlinear model of Boltzmann equations*, Comm. Pure Appl. Math. 32, 589-616, 1979.

[Car] T. Carleman, *Problèmes Mathématiques dans la Thèorie Cinètique des Gaz*, Almqvist-Wiksells, Uppsala, 1957.

[Cer] C. Cercignani, The Boltzmann Equation and its Applications, Springer-Verlag, New York, 1988.

[CLL] G.Q. Chen, C.D. Levermore and T.P. Liu, *Hyperbolic Conservation Laws with Stiff Relaxation Terms and Entropy*, Comm. Pure Appl. Math., 47, 789-830, 1993.

[CR] J.F. Collet and M. Rascle, *Convergence of the Relaxation to a Scalar Nonlinear Hyperbolic Equation in on Space Dimension*, Z. Angew. Math. Phys. 47, 400-409, 1996.

[Cor] V. Cornish, Ocean waves and kindred geophysical phenomena, Cambridge University Press, 1934.

[Daf] C. M. Dafermos, *Generalized characteristics and structures of solutions of hyperbolic conservation laws*, Indiana Univ. Math. J. 26, 1097-1119, 1977.

[Dre] R.F. Dressler, *Mathematical solution of the problem of roll-waves in inclined open channels*, Comm. Pure Appl. Math. 2, 149-194, 1949.

[Gdm] J. Goodman, *Stability of the Kuramoto-Sivashinsky and related systems*, Comm. Pure Appl. Math. 47, 293-306, 1994.

[Hun] J.K. Hunter, *Asymptotic equations for nonlinear hyperbolic waves*, in Surveys in Appl. Math., vol. 2, eds. J.B. Keller, G. Papanicolaou and D.W. McLaughlin, 1993.

[Jef] H. Jeffreys, *The flow of water in an inclined channel of rectangular section,* Phil. Mag. 49, 793-807, 1925.

[Jin] S. Jin, Numerical Transfer in Diffusive Regimes, Ph.D. thesis, University of Arizona, 1991.

[JK] S. Jin and M.A. Katsoulakis, *Relaxation Approximations to Front Propagation*, J. Diff. Eqn. 138, 380-387 (1997),

[JKX] S. Jin, M.A. Katsoulakis and Z.P. Xin, *Relaxation Schemes for Curvature-Dependent Front Propagation*, Comm. Pure Appl. Math., to appear.

[JX] S. Jin and Z.P. Xin, *The Relaxation Schemes for Systems of Conservation Laws in Arbitrary Space Dimensions*, Comm. Pure Appl. Math. 48, 235-276 (1995),

[JX2] S. Jin and Z.P. Xin, *Numerical Passage from Systems of Conservation Laws to Hamilton-Jacobi Equations, and a Relaxation Scheme*, SIAM J. Num. Anal. 35, 2385-2404, 1998.

[KT] M. A. Katsoulakis and A.E. Tzavaras, *Contractive relaxation systems and the scalar multidimensional conservation law*, Comm. PDE 22, 195-233, 1997.

[KY] J. Kevorkian and J. Yu, *Weakly nonlinear waves for a class of linearly unstable hyperbolic conservation laws with source terms,* SIAM J. Appl. Math. 55, 446-484, 1995.

[Kra] C. Kranenburg, *On the evolution of roll waves*, J. Fluid Mech. 245, 249-261, 1992.

[LM] C. Lattanzio and P. Marcati, *The zero relaxation limit for the hydrodynamic Whitham traffic flow model*, J. Diff. Eqn, 141, 150-178, 1997.

[Lax] P.D. Lax, Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves, CBMS-NSF Regional Conference Series in Applied Mathematics, Philadelphia, 1973.

[LW] R.J. LeVeque and J.H. Wang, *A linear hyperbolic system with stiff source terms*, Proc. Fourth Int'l Conf. Hyperbolic Problems, Taomina, Italy, April 1992.

[Liu] T.P. Liu, *Hyperbolic conservation laws with relaxation*, Comm. Math. Phys. 108, 153–175, 1987.

[Lyb] A.N. Lyberopoulos, *Asymptotic oscillations of solutions of scalar conservation laws with convexity under the action of a linear excitation*, Quar. Appl. Math. XLVIII, 755-765, 1990.

[Nat1] R. Natalini, *Convergence to Equilibrium for the Relaxation Approximations of Conservation Laws*, Comm. Pure Appl. Math, 49, 795-823, 1996.

[Nat2] R. Natalini, *A discrete kinetic approximation of entropy solutions to multidimensional scalar conservation laws*, J. Diff. Eqn. 148, 292-317, 1998.

[Nat3] R. Natalini, *Recent mathematical results on hyperbolic relaxation problems*, preprint, 1998.

[NM] D.J. Needham and J.H. Merkin, *On roll waves down an open inclined channel*, Proc. Roy. Soc. Lond. A 394, 259-278, 1984.

[Nov] O.B. Novik, *Model description of roll-waves*, J. Appl. Math. Mech. 35, 938-951, 1971.

[Ser] D. Serre, *Relaxation semi-linèaire et cinètique des systèmes de lois de conservation*, preprint, 1998.

[Sle] M. Slemrod, *A renormalization method for the Chapman-Enskog expansion*, Physica D, 109, 257-273, 1997.

[Smo] J. Smoller, Shock Waves and Reaction-Diffusion Equations, Springer-Verlag, New York, 1982.

[Sto] J.J. Stoker, Water Waves, John Wiley & Sons, 1958.

[Sul] I. Suliciu, *On modeling phase transitions by means of rate-type constitutive equations. Shock wave structure.* Int'l I. Eng. Sci. 28, 829-841, 1990.

[Ten] Z.H. Teng, *First-order L^1-convergence for relaxation approximations to conservation laws*, Comm. Pure Appl. Math. 51, 857-895, 1998.

[VK] W. Vicenti and C. Kruger, Introduction to Physical Gas Dynamics, Robert E. Kreger, 1982

[Whi] J. Whitham, Linear and Nonlinear Waves, Wiley, New York, 1974

[Xin] Z. P. Xin, *The fluid dynamical limit of the Broadwell model of the nonlinear Boltzmann equation in the presence of shock*, Comm. Pure Appl. Math. 44, 679-714, 1974

[KY] J. Yu and J. Kevorkian, *Nonlinear evolution of small disturbances into roll waves in an inclined open channel*, J. Fluid Mech. 243, 575-594, 1992.

[Yon] Wen-an Yong, Singular perturbations of first order hyperbolic systems with stiff source terms, preprint 1998.