TIME–PERIODIC SOLUTIONS FOR A SCALAR CONSERVATION LAW

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Abstract: We consider the existence of time periodic solutions for a nonlinear hyperbolic scalar conservation law with a time periodic outer force. The uniform asymptotic behavior of the Lax–Friedrichs difference approximation gives fixed points of the Poincare map and the convergence of the approximate periodic solution made from such fixed points is proved by the compensated compactness theory.

Keywords: time periodic solution, conservation law, periodic outer force, compensated compactness, Lax–Friedrichs difference scheme, Brouwer's fixed point theorem.

1 Introduction

In this paper we study a scalar conservation law with an outer force term

$$u_t + f(u)_x = g(t,x), \tag{1}$$

where the function $f(u)$ is smooth and convex. The global existence of weak solutions of the Cauchy problem (1) for any large initial data was proved in [11]. If initial data $u_0(x)$ and an outer force $g(t,x)$ are x-periodic functions then the solution $u(t,x)$ may be periodic and may be regarded as a solution of the initial-boundary value problem (1) and

$$\begin{cases} u(0,x) = u_0(x) \quad (0 < x < 1), \\ u(t,0) = u(t,1) \quad (t > 0). \end{cases} \tag{2}$$

We consider the following problem:

> If $g(t,x)$ is a time-periodic function with period T, does a time-periodic solution exist with the same period, assuming the necessary condition
>
> $$\int_0^T dt \int_0^1 g(t,x)dx = 0 \tag{3}$$
>
> for the outer force term $g(t,x)$?

For the existence of a time periodic solution, we need the decay of the solution to the homogeneous equation

$$u_t + f(u)_x = 0, \tag{4}$$

with data (2). The decay to the mean value

$$\bar{u} = \int_0^1 u(t,x)dx = \int_0^1 u_0(x)dx$$

was obtained under some regularity assumption for the solution at a (slow) rate $1/t$ in [7].

For the viscous equation

$$\begin{cases} u_t + f(u)_x = \varepsilon u_{xx} \qquad (0 < x < 1, \quad t > 0), \\ u(0,x) = u_0(x) \qquad (0 < x < 1), \\ u(t,0) = u(t,1), \quad u_x(t,0) = u_x(t,1) \qquad (t > 0), \end{cases} \tag{5}$$

and the homogeneous equation with positive linear term

$$\begin{cases} u_t + f(u)_x + \varepsilon u = 0 \qquad (0 < x < 1, \quad t > 0), \\ u(0,x) = u_0(x) \qquad (0 < x < 1), \\ u(t,0) = u(t,1) \qquad (t > 0), \end{cases} \tag{6}$$

solutions decay at a faster rate assuming ε is a positive constant. The fast decay for each equation gives a sharp enough estimate to demonstrate the existence of a fixed point of the Poincare map $u(0,x) \mapsto u(T,x)$ for each equation with the time-periodic outer force $g(t,x)$. For systems of conservation laws, Matsumura–Nishida ([9]) proved the existence of periodic solutions for viscous isothermal gas equations for any large periodic outer force, and Matsumura–Yanagi ([10]), Yanagi ([16]) extended the results to the case of viscous isentropic gases. Feireisl ([6]) proved the existence of periodic solutions for systems of hyperbolic conservation laws with positive linear term. For such systems, solutions of the approximation for the homogeneous systems decay fast uniformly, and he made the sequence of the periodic solutions for the viscous approximation and showed the convergence by using the compensated compactness theory.

However, even in the scalar case, we cannot obtain the time-periodic solution of our problem as the limit of the periodic solutions of equation added $g(t,x)$ to (5) and (6) as $\varepsilon \to 0$. This is because the estimates for fast decay depend on the constant ε and are not valid in the limit $\varepsilon \to 0$, and, according to [7], it does not imply that the decay is fast for the limit equation (4).

On the other hand, the standard method for the proof of the existence of the weak solution to the equation (1) is to show the convergence of some approximate solutions which are constructed by a difference scheme method or the artificial viscosity method (5).

Tadmor ([13]) proved the slow decay for the Lax–Friedrichs difference approximations, which does not depend on the mesh size. Note that this is obtained from Oleinik's entropy condition ([11]) and from the periodicity of the boundary condition. We can to solve our problem by such a uniform estimate.

We remark that any such uniform estimates have not been obtained for the approximations of systems of conservation laws for large initial data, and therefore the existence of periodic solutions for systems of equations is still open.

Our result is the following.

THEOREM 1 *Under assumptions (14) and (15) for f and g, the problem (1), (7) has a time periodic weak solution $u(t,x)$ for any average $\bar{u}$, and the solution satisfies the entropy condition (13).*

We remark that the periodic solution of the problem (1), (7) is not unique with respect to g and $\bar{u}$.

The outline of the proof of Theorem 1 is the followings. We construct a Lax–Friedrichs difference approximation for the problem (1), (7) and we will obtain the estimate of uniform bounds for it by the methods similar to those of Tadmor ([13]) in §4. The Poincare map is regarded as on the finite dimensional space for the difference approximation because the approximate solution has values at each finite point for fixed t. We show the map takes a closed convex set into the same set. Hence we can use Brouwer's fixed point theorem for the existence of the fixed point of the map from the continuity of the approximation. Since these fixed points are uniformly bounded with respect to the any mesh lengths, we obtain a subsequence that converges to a weak solution by the compensated compactness theory ([15]). We check the compactness of the entropy for the approximation in §5 for the last convergence, and show the limit is a weak solution of the problem (1), (7) satisfying the entropy condition in §6.

2 Preliminaries

A function $u(t,x)$ is called the *weak solution* of the time-periodic problem (1) and

$$\begin{cases} u(t,0) = u(t,1) & (0 < t < T), \\ u(0,x) = u(T,x) & (0 < x < 1) \end{cases} \tag{7}$$

with period T if the function $u(t,x)$ is bounded measurable in the region $(0,T) \times (0,1)$, and there exists a bounded measurable function $\bar{u}(x)$ such that space-periodic extensions of $u(t,x)$, $g(t,x)$ and $\bar{u}(x)$ satisfy

$$\iint_{0<t<T} (u\phi_t + f(u)\phi_x + g\phi)dxdt + \int_R \bar{u}(x)\{\phi(0,x) - \phi(T,x)\}dx = 0 \tag{8}$$

for any $\phi(t,x) \in C_0^1([0,T] \times R_x)$.

Note that the definition has other equivalent forms. One is the following:

$$\iint_{[0,T]\times[0,1]} (u\phi_t + f(u)\phi_x + g\phi)dxdt + \int_0^1 \bar{u}(x)\{\phi(0,x) - \phi(T,x)\}dx = 0 \tag{9}$$

for any $\phi(t,x) \in C_0^1([0,T] \times [0,1])$, $\phi(t,0) = \phi(t,1) \quad (0 < t < T)$. Another is expressed in terms of the space-periodic extension of $\bar{u}$ and for the space and time periodic extension of $u(t,x)$. That is, it is required to satisfy

$$\iint_{t>0} (u\phi_t + f(u)\phi_x + g\phi)dxdt + \int_R \bar{u}(x)\phi(0,x)dx = 0, \tag{10}$$

for any $\phi(t,x) \in C_0^1([0,\infty) \times R)$. The conclusion is that $u(t,x)$ satisfies (9) if and only if the followings valid

1. u is solution of (1) in weak sense

$$u_t + f(u)_x = g \qquad \text{distribution sense in } (0,T) \times (0,1).$$

2. $u(t,x)$ converges in weak sense to $u(x)$ as t tends to zero and as t tends to T

$$\frac{1}{\varepsilon}\int_0^{\varepsilon} u(t,x)dt, \quad \frac{1}{\varepsilon}\int_{T-\varepsilon}^{T} u(t,x)dt \to u_0(x) \quad \text{as} \quad \varepsilon \downarrow 0$$

in $L^\infty(0,1)$ weak$*$.

3. $f(u(t,\varepsilon))$ and $f(u(t,1-\varepsilon))$ converge in weak sense to the same value

$$\frac{1}{\varepsilon}\int_0^{\varepsilon} f(u(t,x))dx, \quad \frac{1}{\varepsilon}\int_{1-\varepsilon}^{1} f(u(t,x))dx \to \bar{f}(t) \quad \text{as} \quad \varepsilon \downarrow 0$$

in $L^\infty(0,T)$ weak$*$ for some function $\bar{f}(t)$ of $L^\infty(0,T)$.

It seems that the function u which satisfies above conditions is a solution of the problem (1) satisfying

$$\begin{cases} f(u(t,0)) = f(u(t,1)) & (0 < t < T), \\ u(0,x) = u(T,x) & (0 < x < 1) \end{cases} \tag{11}$$

instead of (7). Certainly, both definitions are equivalent for the weak solution. However, these include different means for the entropy condition. The boundary condition of the original problem (7) seems to say that the space-periodic extension satisfies the equation (1), but the problem (11) does not seem to require it. Hence in the case that boundary values of the weak solution $u(t,0)$ and $u(t,1)$ are different, these should satisfy the entropy condition, that is,

$$f'(u(t,1)) > 0 > f'(u(t,0))$$

for (7), but should not for (11). It remains an open problem whether the boundary condition (11) is well-posed.

A smooth function pair of u

$$(U(u), F(u))$$

is an *entropy pair* for the scalar conservation law

$$u_t + f(u)_x = 0 \tag{12}$$

if

$$U(u(t,x))_t + F(u(x,t))_x = 0$$

for a smooth solution of (12). This is equivalent that U and F satisfy

$$F'(u) = U'(u)f'(u).$$

The function U called *entropy* and F called *entropy flux.* A weak solution u satisfies the *entropy condition* if

$$U(u(t,x))_t + F(u(x,t))_x \le U'(u(t,x))g(t,x) \quad \text{in } (0,T)\times R \tag{13}$$

for any smooth entropy pair with convex entropy U.

We suppose that the function $f(u)$ is smooth,

$$f''(u) \ge \delta > 0 \quad (u \in R), \qquad \int_0^1 dx \int_0^T g(t,x)dt = 0, \tag{14}$$

and $g(t,x)$ is a time-periodic function with period T. The last relation for $g(t,x)$ is need for the existence of a time periodic solution.

We also assume that the space-extension of $g(t,x)$ satisfies

$$g(t,x) - g(t,y) \le G_1(x-y) \qquad (x > y) \tag{15}$$

for any x, y and t, where G_1 is a constant. The condition is necessary by a technical reason.

It is well-known that the solution of the *Riemann problem* for the scalar conservation law

$$\begin{cases} u_t + f(u)_x = 0 \quad (t > 0, x \in R), \\ u(0,x) = \begin{cases} u_l & (x < 0), \\ u_r & (x > 0) \end{cases} \end{cases} \tag{16}$$

is each of the two typical waves. In the case $u_l < u_r$ the solution is the *rarefaction wave*

$$u(t,x) = \begin{cases} u_l & (x/t < f'(u_l)), \\ (f')^{-1}(x/t) & (f'(u_l) < x/t < f'(u_r)), \\ u_r & (x/t > f'(u_r)) \end{cases}$$

and in the case $u_l > u_r$ the solution is the *shock wave*

$$u(t,x) = \begin{cases} u_l & (x/t < s), \\ u_r & (x/t > s), \end{cases}$$

where the *shock speed* s is determined by the *Rankine–Hugoniot relation*

$$f(u_r) - f(u_l) = s(u_r - u_l)$$

and Lax's entropy condition

$$f'(u_l) > s > f'(u_r)$$

(cf. [7], [12]).

We note that the average on the line $t = \Delta t$

$$\frac{1}{2\Delta x}\int_{-\Delta x}^{\Delta x} u(\Delta t, x)dx$$

for the solution of the Riemann problem is equal to the *Lax–Friedrichs difference approximation*

$$\frac{u_r + u_l}{2} - \frac{\Delta t}{2\Delta x}\{f(u_r) - f(u_l)\}$$

provided the *Courant–Friedrichs–Lewy* (CFL) *condition*

$$\frac{\Delta x}{\Delta t} > \max\{|f'(u_l)|, |f'(u_r)|\}.$$

3 Approximate Solution

In this section, we construct an approximate solution of the Lax–Friedrichs difference scheme type for the initial-boundary value problem (1), (2) in a standard way (cf. [3],[4],[14].)

Let the initial data $u(0,x)$ be a bounded measurable function and

$$\|u(0,\cdot)\|_{L^\infty} \leq M.$$

We suppose that the inverse of the x-mesh length Δx and the ratio of the period T and the t-mesh length are even integers $2L$ and $2N$,

$$2L\Delta x = 1, \qquad 2N\Delta t = T.$$

It is necessary that the ratio of Δx and t-mesh length Δt is a sufficiently large constant for the CFL condition. We take the value such that

$$\frac{\Delta x}{\Delta t} \geq \Lambda \equiv \max_{|u|\leq M+TG_0} |f'(u)|, \tag{17}$$

where T is time-period of function g, and G_0 is the maximum value of $|g|$

$$G_0 = \max_{[0,T]\times[0,1]} |g(t,x)|.$$

Let E_j^n be an interval and J_n an index set such that

$$\begin{cases} E_j^n &=& ((j-1)\Delta x, (j+1)\Delta x] \quad (j \in J_n, n = 0,1,2,\ldots), \\ J_n &=& \begin{cases} \{\ldots,-5,-3,-1,1,3,5,\ldots\} & \text{if } n \text{ is even,} \\ \{\ldots,-4,-2,0,2,4,\ldots\} & \text{if } n \text{ is odd,} \end{cases} \end{cases}$$

and we denote $R(t,x;u_l,u_r)$ as the solution of the Riemann problem (16).

The approximation of the initial value $u^\Delta(0,x)$ is defined as a step value function

$$u^\Delta(0,x) = u_j^0 \equiv \frac{1}{m(E_j^0)} \int_{E_j^0} u(0,x)dx \qquad \text{on } E_j^0 \quad (j \in J_0),$$

where $u(0,x)$ is extended to the function on R as the x-periodic function. The value u_j^0 has the periodicity of $u_{j+2L}^0 = u_j^0$. We define $u^\Delta(t,x)$ as the solution of the Riemann problem for the step initial data $u^\Delta(0,x)$

$$u^\Delta(t,x) = R(t, x - j\Delta x, u_{j-1}^0, u_{j+1}^0)$$

in each small regions $(0,\Delta t) \times E_j^1$, $j \in J_1$. The wave must arrive at the top of the region by the CFL condition (17). On the line $t = \Delta t$, we define $u^\Delta(\Delta t, x)$ as the mean value

$$u^\Delta(\Delta t, x) = u_j^1 \equiv \frac{1}{m(E_j^1)} \int_{E_j^1} u^\Delta(\Delta t - 0, x)dx \qquad \text{on } E_j^1 \quad (j \in J_1),$$

and construct $u^\Delta(t,x)$ by solutions of Riemann problems in $(\Delta t, 2\Delta t) \times R$ similarly in $(0,\Delta t) \times R$,

$$u^\Delta(t,x) = R(t - \Delta t, x - j\Delta x, u_{j-1}^1, u_{j+1}^1) \quad \text{in } (\Delta t, 2\Delta t) \times E_j^2 \quad (j \in J_2).$$

On $t = 2\Delta t$, we set $u^\Delta(2\Delta t, x)$ as the sum of the mean value and the term for the outer force

$$\begin{aligned} u^\Delta(2\Delta t, x) &= u_j^2 = u_{0,j}^2 + 2\Delta t g_j^1 \\ \equiv\ & \frac{1}{m(E_j^2)} \int_{E_j^2} u^\Delta(2\Delta t - 0, x)dx + \frac{1}{m(E_j^2)} \int_0^{2\Delta t} dt \int_{E_j^2} g(t,x)dx \\ & \text{on } E_j^2 \quad (j \in J_2). \end{aligned}$$

The last calculation is called the fractional step method.

For $(2n\Delta t, (2n+2)\Delta t] \times R$ $(n = 1, 2, \ldots, N-1)$ we define the approximation $u^\Delta(t,x)$ by the similar way. The following lemma shows that above construction can be continued to $n = N - 1$.

LEMMA 2 *If constants u_l and u_r satisfy*

$$|u_l| \le A, \quad |u_r| \le A,$$

then the solution of the Riemann problem (16) $u(t,x)$ satisfies $|u(t,x)| \le A$.

By Lemma 2, it is easy to show that $|u^\Delta(t,x)| \le M + TG_0$ and waves appeared in the definition of $u^\Delta(t,x)$ cannot access in $(0,T)\times R$ from the CFL condition (17) because speeds of these waves do not over the maximum value of $|f'(u)|$.

4 Decay estimates

In this section, we obtain the estimate for the approximation $u^\Delta(t,x)$ constructed in the last section by the similar way to Tadmor ([13]) for fractional step Lax–Friedrichs difference approximation.

The step values $u_j^n = u(n\Delta t, j\Delta x)$ are able to be described as forms of the Lax–Friedrichs difference scheme

$$\begin{cases} u_j^{2n+1} = \dfrac{u_{j+1}^{2n}+u_{j-1}^{2n}}{2} - \dfrac{\Delta t}{2\Delta x}(f_{j+1}^{2n}-f_{j-1}^{2n}) \qquad (j\in J_{2n+1}), \\ u_j^{2n+2} = \dfrac{u_{j+1}^{2n+1}+u_{j-1}^{2n+1}}{2} - \dfrac{\Delta t}{2\Delta x}(f_{j+1}^{2n+1}-f_{j-1}^{2n+1}) + 2\Delta t g_j^{2n+1} \\ \qquad (j\in J_{2n+2}) \end{cases} \tag{18}$$
$$(n=0,1,2,\ldots),$$

where $f_j^n = f(u_j^n)$ and

$$g_j^n = \frac{1}{4\Delta t\Delta x}\int_{(n-1)\Delta t}^{(n+1)\Delta t} dt \int_{(j-1)\Delta x}^{(j+1)\Delta x} g(t,x)dx.$$

Let v_j^n be the x-backward difference of u_j^n

$$v_j^n = \frac{u_j^n - u_{j-2}^n}{2\Delta x}.$$

Then,

$$\begin{aligned} v_j^{2n+1} &= \frac{u_j^{2n+1}-u_{j-2}^{2n+1}}{2\Delta x} \\ &= \frac{u_{j+1}^{2n}-u_{j-3}^{2n}}{4\Delta x} - \frac{\Delta t}{4(\Delta x)^2}(f_{j+1}^{2n}-2f_{j-1}^{2n}+f_{j-3}^{2n}) \\ &= \frac{v_{j+1}^{2n}+v_{j-1}^{2n}}{2} - \frac{\Delta t}{2\Delta x}f'(u_{j-1}^{2n})(v_{j+1}^{2n}-v_{j-1}^{2n}) \end{aligned}$$

$$
\begin{aligned}
&-\Delta t\left\{(v_{j+1}^{2n})^2\int_0^1(1-\theta)f''(u_{j-1}^{2n}+\theta(u_{j+1}^{2n}-u_{j-1}^{2n}))d\theta\right.\\
&\left.+(v_{j-1}^{2n})^2\int_0^1(1-\theta)f''(u_{j-1}^{2n}+\theta(u_{j-3}^{2n}-u_{j-1}^{2n}))d\theta\right\}\\
\leq\ & \frac{v_{j+1}^{2n}+v_{j-1}^{2n}}{2}-\frac{\Delta t}{2\Delta x}f'(u_{j-1}^{2n})(v_{j+1}^{2n}-v_{j-1}^{2n})-\Delta t\delta\frac{(v_{j+1}^{2n})^2+(v_{j-1}^{2n})^2}{2}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
v_j^{2n+2} \leq\ & \frac{v_{j+1}^{2n+1}+v_{j-1}^{2n+1}}{2}-\frac{\Delta t}{2\Delta x}f'(u_{j-1}^{2n+1})(v_{j+1}^{2n+1}-v_{j-1}^{2n+1})\\
&-\Delta t\left\{\delta\frac{(v_{j+1}^{2n+1})^2+(v_{j-1}^{2n+1})^2}{2}-2G_1\right\},
\end{aligned}
$$

where G_1 is the value in (15). Let $N^n=\max_j v_j^n$. N^n is non-negative because the summation

$$
\sum_{j\in J_n,0\leq j<2L} v_j^n
$$

equals zero from the space periodicity of u_j^n. A function

$$
h(x)=\frac{A}{2}x-\frac{\Delta t\delta}{2}x^2 \qquad (A>0)
$$

increases for $x\leq A/(2\Delta t\delta)$. Hence, if

$$
N^{2n}\leq\frac{1}{2\Delta t\delta}(1-\frac{\Delta t}{\Delta x}\Lambda)
$$

then

$$
N^{2n+1}=\max_j v_j^{2n+1}\leq N^{2n}-\Delta t\delta(N^{2n})^2.
$$

Similarly,

$$
N^{2n+1}\leq\frac{1}{2\Delta t\delta}(1-\frac{\Delta t}{\Delta x}\Lambda)
$$

yields

$$
N^{2n+2}=\max_j v_j^{2n+1}\leq N^{2n}-\Delta t\{\delta(N^{2n})^2-2G_1\}.
$$

The simple estimate for N^n

$$
N^n=\max_j v_j^n=\max_j\frac{u_j^n-u_{j-2}^n}{2\Delta x}\leq\frac{M+TG_0}{\Delta x}
$$

shows that if

$$2\delta(M + TG_0) + \Lambda \leq \frac{\Delta x}{\Delta t}, \tag{19}$$

then

$$\frac{N^{2n+2} - N^{2n}}{2\Delta t} \leq G_1 - \frac{\delta}{2}(N^{2n})^2 \tag{20}$$

for $n \leq N - 1$ since

$$\begin{aligned} N^{2n+1} &\leq N^{2n} - \Delta t\delta(N^{2n})^2 \leq N^{2n}, \\ N^{2n+2} &\leq N^{2n+1} - \Delta t\delta(N^{2n+1})^2 + 2\Delta t G_1 \\ &\leq N^{2n} - \Delta t\delta(N^{2n})^2 + 2\Delta t G_1. \end{aligned}$$

The solution $y(t)$ of ordinary differential equation

$$\begin{cases} y' = G_1 - \dfrac{\delta}{2}y^2 \\ \\ y(0) = N^0 \end{cases}$$

tends to the value

$$\alpha = \sqrt{2G_1/\delta} \tag{21}$$

as t tends to infinity. If $N^0 > \alpha$ then $y(t)$ is monotone decreasing convex function, and if $N^0 < \alpha$ then $y(t)$ is monotone increasing concave one. In the latter case, the tangent line of $y(t)$ started $t = t_0$ across the line $y = \alpha$ at

$$t = t_0 + \frac{2}{\delta(y(t_0) + \alpha)}$$

and the time is not smaller than $t_0 + 1/(\delta\alpha)$. Hence, if $N^{2n} \leq \alpha$ then $N^{2m} \leq \alpha$ for $n \leq m \leq N$ provided that

$$2\Delta t \leq \frac{1}{\delta\alpha} \tag{22}$$

from the inequality (20). In the case $N^0 > \alpha$, it is easy to show that

$$N^{2m} \leq y(2m\Delta t) \leq \alpha + \frac{2\alpha}{e^{2m\alpha\delta\Delta t} - 1} \qquad (m \leq N). \tag{23}$$

We consider the estimate for u_j^{2N} using above estimates for v_j^n. For the summation of u_j^{2N}

$$
\begin{aligned}
&2\Delta x \sum_{j=1}^{L} u_{2j-1}^{2N} \\
&= 2\Delta x \sum_{j=1}^{L} \left\{ \frac{u_{2j}^{2N-1} + u_{2j-2}^{2N-1}}{2} - \frac{\Delta t}{2\Delta x}(f_{2j}^{2N-1} - f_{2j-2}^{2N-1}) + 2\Delta t g_{2j-1}^{2N-1} \right\} \\
&= 2\Delta x \sum_{j=1}^{L} u_{2j}^{2N-1} + 4\Delta x \Delta t \sum_{j=1}^{L} g_{2j-1}^{2N-1} \\
&= 2\Delta x \sum_{j=1}^{L} u_{2j-1}^{2N-2} + 4\Delta x \Delta t \sum_{j=1}^{L} g_{2j-1}^{2N-1} \\
&= 2\Delta x \sum_{j=1}^{L} u_{2j-1}^{0} + 4\Delta x \Delta t \sum_{n=1}^{N} \sum_{j=1}^{L} g_{2j-1}^{2n-1} \\
&= \int_0^1 u_0(x)dx + \int_0^T dt \int_0^1 g(t,x)dx \\
&= \bar{u}
\end{aligned}
$$

from (3).

Next simple lemma is used to the estimate for u_j^{2N}.

LEMMA 3 *Let p_j be real values which satisfy*

$$p_1 + p_2 + \cdots + p_K = 0,$$

and let

$$q_j = \begin{cases} p_j - p_{j-1} & (j = 2, 3, \ldots, K), \\ p_1 - p_K & (j = 1). \end{cases}$$

Then

$$\max_j |p_j| \le \sum_{j=1}^{K} |q_j| \le 2K \max_j q_j.$$

The proof is omitted.

Let $K = L$, $p_j = u_{2j-1}^{2N} - \bar{u}$ in Lemma 3, then $q_j = 2v_{2j-1}^{2N}\Delta x$ and

$$|u_j^{2N} - \bar{u}| \le 2L \max_j 2\Delta x v_j^{2N} = 4L\Delta x N^{2N} \le 2\alpha + \frac{4\alpha}{e^{\alpha\delta T} - 1}$$

from (23).

PROPOSITION 4 *Under assumptions (19), (22)*

$$|u_j^{2N} - \bar{u}| \leq 2\alpha + \frac{4\alpha}{e^{\alpha\delta T} - 1}. \tag{24}$$

Proposition 4 give the time-periodic solution of the difference approximation.

Let A be sufficiently large number such that

$$A > 2\alpha + \frac{4\alpha}{e^{\alpha\delta T} - 1},$$

and let $M = |\bar{u}| + A$. We take mesh lengths Δx and Δt to satisfy (19), (22) and

$$\frac{\Delta x}{\Delta t} \leq \Lambda_2 \equiv 2\{2\delta(M + TG_0) + \Lambda\} \tag{25}$$

for the estimation of the entropy (§5). Let D_L be a set

$$\begin{aligned} D_L &= D_L(\bar{u}, A) \\ &= \{(u_1, u_2, \ldots, u_L) \in R^L; \quad \sum_{j=1}^{L} u_j = \bar{u}, \quad \max |u_j - \bar{u}| \leq A\}. \end{aligned}$$

If we take $(u_1^0, u_3^0, \ldots, u_{2L-1}^0)$ from D_L, set u_j^0 periodically for any $j \in J_0$, then we can construct the approximation $u^\Delta(t, x)$ by the way in §3. Obviously D_L is a closed convex set and the range of the mapping

$$D_L \ni (u_1^0, u_3^0, \ldots u_{2L-1}^0) \mapsto (u_1^{2N}, u_3^{2N}, \ldots u_{2L-1}^{2N})$$

is contained in D_L from Proposition 4. Hence, we obtain the fixed point

$$(\bar{u}_1^0, \bar{u}_3^0, \ldots, \bar{u}_{2L-1}^0) \tag{26}$$

of the mapping from Brouwer's fixed point theorem in D_L. Of course, the fixed point depends on Δx.

5 Compactness of entropy

In this section, we show the compactness of

$$\{U(u^{\Delta}(t,x))_t + F(u^{\Delta}(t,x))_x; \quad (u_1^0, u_3^0, \dots, u_{2L-1}^0) \in D_L, L \geq L_0\}$$

in $H^{-1}_{\mathrm{loc}}((0,T)\times(0,1))$ for any smooth entropy pair $(U(u), F(u))$, where L_0 is sufficiently large integer for (19) and (22).

For simplify, we set $u^{2n}_{0,+}(x)$ as the function consists of the step values $u^{2n}_{0,j}$

$$u^{2n}_{0,+}(x) = u^{2n}_{0,j} \quad (x \in E^{2n}_j)$$

and use notations

$$u^n_+(x) = u(n\Delta t + 0, x), \qquad u^n_-(x) = u(n\Delta t - 0, x), \qquad f^n_j = f(u^n_j),$$

and so on.

We consider for a particular entropy pairs (U^*, F^*) defined by

$$U^* = \frac{u^2}{2}, \qquad F^* = \int_0^u uf'(u) = uf - \int_0^u f.$$

Since the approximation $u^{\Delta}(t,x)$ satisfies the equation (12) almost everywhere,

$$\begin{aligned}
0 &= \int_0^T dt \int_0^1 \{U^*(u^{\Delta}(t,x))_t + F^*(u^{\Delta}(t,x))_x\} dx \\
&= \sum_{n=1}^{N} \left(\int_{(2n-2)\Delta t}^{(2n-1)\Delta t} dt + \int_{(2n-1)\Delta t}^{2n\Delta t} dt \right) \int_0^1 (U^*_t + F^*_x) dx \\
&= \sum_{n=1}^{N} \int_0^1 \left([U^*]^{(2n-1)\Delta t - 0}_{(2n-2)\Delta t + 0} + [U^*]^{2n\Delta t - 0}_{(2n-1)\Delta t + 0} \right) dx \\
&\quad + \int_0^T \sum_{\text{shock}} (\sigma[U^*] - [F^*]) dt \\
&= \sum_{n=1}^{N} \int_0^1 [U^*]^{(2n-1)\Delta t - 0}_{(2n-1)\Delta t + 0} dx + \sum_{n=1}^{N} \int_0^1 [U^*]^{2n\Delta t - 0}_{2n\Delta t + 0} dx + \Sigma^* \\
&\quad + \int_0^1 U^*(T+0,x) dx - \int_0^1 U^*(+0,x) dx \\
&= \sum_{n=1}^{N} \sum_{j=0}^{L-1} \int_{E^{2n-1}_{2j}} [U^*]^{(2n-1)\Delta t - 0}_{(2n-1)\Delta t + 0} dx + \sum_{n=1}^{N} \sum_{j=1}^{L} \int_{E^{2n}_{2j-1}} \{(U^*)^{2n}_- - (U^*)^{2n}_{0,+}\} dx \\
&\quad + \Sigma^* + I^* + J^*,
\end{aligned}$$

where

$$\begin{aligned}
J^* &= \sum_{n=1}^{N}\sum_{j=1}^{L}\int_{E_{2j-1}^{2n}}\{(U^*)_{0,+}^{2n}-(U^*)_{+}^{2n}\}dx,\\
I^* &= \int_0^1 U^*(T+0,x)dx-\int_0^1 U^*(+0,x)dx,\\
\Sigma^* &= \int_0^T\sum_{\text{shock}}(\sigma[U^*]-[F^*])dt,
\end{aligned}$$

$\sum_{\text{shock}}$ is summation for all the shock waves arise in $[0,T]\times[0,1]$, $[U^*]$ and $[F^*]$ is the difference across the wave, and σ is the speed of it. Because of

$$\begin{aligned}
&(U^*)''=1,\quad u_+^{2n-1}=\frac{1}{m(E_{2j}^{2n-1})}\int_{E_{2j}^{2n-1}}u_-^{2n-1}dx \qquad (x\in E_{2j}^{2n-1}),\\
&\text{and}\quad u_{0,+}^{2n}=\frac{1}{m(E_{2j-1}^{2n})}\int_{E_{2j-1}^{2n}}u_-^{2n}dx \qquad (x\in E_{2j-1}^{2n}),
\end{aligned}$$

we have

$$\frac{1}{2}\sum_{n=1}^{N}\int_0^1|u_-^{2n-1}-u_+^{2n-1}|^2dx+\frac{1}{2}\sum_{n=1}^{N}\int_0^1|u_-^{2n}-u_{0,+}^{2n}|^2dx+\Sigma^*=-I^*-J^*.$$

The boundedness of $u^\Delta(t,x)$

$$|u^\Delta(t,x)|\le M+TG_0$$

shows that

$$\begin{aligned}
|J^*| &\le \sum_{n=1}^{N}\sum_{j=1}^{L}\int_{E_{2j-1}^{2n}}|(U^*)_{0,+}^{2n}-(U^*)_{+}^{2n}|dx\\
&= \sum_{n=1}^{N}\sum_{j=1}^{L}\int_{E_{2j-1}^{2n}}\left|(u_{0,+}^{2n}-u_+^{2n})\frac{u_{0,+}^{2n}+u_+^{2n}}{2}\right|dx\\
&= \sum_{n=1}^{N}\sum_{j=1}^{L}\int_{E_{2j-1}^{2n}}2\Delta t\left|g_{2j-1}^{2n-1}\frac{u_{0,+}^{2n}+u_+^{2n}}{2}\right|dx\\
&\le TG_0(M+TG_0),\\
|I^*| &\le \int_0^1|U^*(T+0,x)|dx+\int_0^1|U^*(+0,x)|dx\\
&\le (M+TG_0)^2.
\end{aligned}$$

Hence we obtain the inequality

$$\begin{aligned}
&\frac{1}{2}\sum_{n=1}^{N}\int_0^1 |u_-^{2n-1}-u_+^{2n-1}|^2dx+\frac{1}{2}\sum_{n=1}^{N}\int_0^1 |u_-^{2n}-u_{0,+}^{2n}|^2dx\\
&+\int_0^T\sum_{\text{shock}}(\sigma[U^*]-[F^*])dt \quad\le\quad C(M,T,G_0),
\end{aligned}\tag{27}$$

where $C(M,T,G_0)$ is a constant depends on M, T and G_0.

Let $\phi(t,x)$ be a function in $C_0^\infty((0,T)\times(0,1))$. The integral

$$\iint_{[0,T]\times[0,1]}\{U(u^\Delta)\phi_t+F(u^\Delta)\phi_x\}dtdx \tag{28}$$

is divided into four parts

$$\begin{aligned}
&\iint_{[0,T]\times[0,1]}\{U(u^\Delta)\phi_t+F(u^\Delta)\phi_x\}dtdx\\
&=\iint_{[0,T]\times[0,1]}\{(U\phi)_t+(F\phi)_x\}dtdx\\
&=\Sigma(\phi)+L_1(\phi)+L_2(\phi)+L_3(\phi),
\end{aligned}$$

where

$$\begin{aligned}
\Sigma(\phi) &= \int_0^T\sum_{\text{shock}}(\sigma[U]-[F])dt,\\
L_1(\phi) &= \sum_{n=1}^{N}\sum_{j=1}^{L-1}\phi_{2j}^{2n-1}\int_{E_{2j}^{2n-1}}(U_-^{2n-1}-U_+^{2n-1})dx\\
&+\sum_{n=1}^{N-1}\sum_{j=1}^{L}\phi_{2j-1}^{2n}\int_{E_{2j-1}^{2n}}(U_-^{2n}-U_{0,+}^{2n})dx,\\
L_2(\phi) &= \sum_{n=1}^{N}\int_0^1\phi(U_{0,+}^{2n}-U_+^{2n})dx,\\
L_3(\phi) &= \sum_{n=1}^{N}\left\{\sum_{j=1}^{L-1}\int_{E_{2j}^{2n-1}}(\phi-\phi_{2j}^{2n-1})(U_-^{2n-1}-U_+^{2n-1})dx\right.\\
&+\left.\int_0^{\Delta x}\phi(U_-^{2n-1}-U_+^{2n-1})dx+\int_{1-\Delta x}^1\phi(U_-^{2n-1}-U_+^{2n-1})dx\right\}\\
&+\sum_{n=1}^{N-1}\sum_{j=1}^{L}\int_{E_{2j-1}^{2n}}(\phi-\phi_{2j-1}^{2n})(U_-^{2n}-U_{0,+}^{2n})dx,
\end{aligned}$$

and $\phi_j^n = \phi(n\Delta t, j\Delta x)$. We estimate $L_j(\phi)$

$$
\begin{aligned}
|L_1(\phi)| &\le \|\phi\|_{C^0} \max_{|u|\le M+TG_0} |U''(u)| \\
&\quad \times \left(\sum_{n=1}^{N} \int_0^1 |u_-^{2n-1} - u_+^{2n-1}|^2 dx + \sum_{n=1}^{N-1} \int_0^1 |u_-^{2n} - u_{0,+}^{2n}|^2 dx \right) \\
&\le C(U'', M, T, G_0) \|\phi\|_{C^0}, \\
|L_2(\phi)| &\le \|\phi\|_{C^0} \max_{|u|\le M+TG_0} |U'(u)| \sum_{n=1}^{N} \sum_{j=1}^{L} \int_{E_{2j-1}^{2n}} |u_{0,+}^{2n} - u_+^{2n}| dx \\
&\le \|\phi\|_{C^0} \max_{|u|\le M+TG_0} |U'(u)| \sum_{n=1}^{N} \sum_{j=1}^{L} 4\Delta t \Delta x |g_{2j-1}^{2n-1}| \\
&\le C(U', M, T, G_0) \|\phi\|_{C^0}, \\
|L_3(\phi)| &\le \|\phi\|_{C^\beta} (\Delta x)^\beta \\
&\quad \times \left(\sum_{n=1}^{N} \int_0^1 |U_-^{2n-1} - U_+^{2n-1}| dx + \sum_{n=1}^{N-1} \int_0^1 |U_-^{2n} - U_{0,+}^{2n}| dx \right) \\
&\le \|\phi\|_{C^\beta} (\Delta x)^\beta \max_{|u|\le M+TG_0} |U'(u)| \sqrt{2N-1} \\
&\quad \times \left(\sum_{n=1}^{N} \int_0^1 |u_-^{2n-1} - u_+^{2n-1}|^2 dx + \sum_{n=1}^{N-1} \int_0^1 |u_-^{2n} - u_{0,+}^{2n}|^2 dx \right)^{1/2} \\
&\le C(U', M, T, G_0) \sqrt{T\Delta x/\Delta t} \|\phi\|_{C^\beta} (\Delta x)^{\beta - 1/2} \\
&\le C(U', M, T, G_0, \Lambda_2) \|\phi\|_{C^\beta} (\Delta x)^{\beta - 1/2}
\end{aligned}
$$

and $\Sigma(\phi)$

$$
|\Sigma(\phi)| \le \|\phi\|_{C^0} \max_{|u|\le M+TG_0} |U''(u)| \Sigma^* \le C(U'', M, T, G_0) \|\phi\|_{C^0}
$$

by the similar way in [3] (I). Hence the integral (28) is the sum of two linear operators $T_1(\phi)$ and $T_2(\phi)$ which satisfy

$$
|T_1(\phi)| \le C \|\phi\|_{C^0}, \qquad |T_2(\phi)| \le C \|\phi\|_{C^\beta} (\Delta x)^{\beta - 1/2},
$$

where the constant C depends on M, T, G_0, maximum value of U' and U'', and Λ_2. This shows the following proposition by the argument in [3] (I).

PROPOSITION 5 *For any smooth entropy pair (U, F), the set*

$$
\{U(u^\Delta(t,x))_t + F(u^\Delta(t,x))_x; \quad (u_1^0, u_3^0, \ldots, u_{2L-1}^0) \in D_L, L \ge L_0\}
$$

is relatively compact in $H^{-1}_{\mathrm{loc}}((0,T)\times(0,1))$.

Proposition 5 and Tartar's theorem ([15]) yields the existence of subsequence which converges almost everywhere. It also valid for the initial data $(\bar{u}^0_1, \bar{u}^0_3, \ldots, \bar{u}^0_{2L-1})$ (§4).

6 Convergence to an entropy solution

In the last section, we saw the existence of a convergent sequence of approximate. In this section, we show that the limit is a weak solution and satisfies the entropy condition in this section.

To prove the limit is a weak solution, it is sufficient that the integral

$$\begin{aligned} I &= \iint_{[0,T]\times[0,1]} (u^\Delta \phi_t + f^\Delta \phi_x + g\phi)dtdx + \int_0^1 u^\Delta(+0,x)\phi(0,x)dx \\ &\quad - \int_0^1 u^\Delta(T+0,x)\phi(T,x)dx \end{aligned}$$

tends to zero as Δx tends to zero for any $\phi \in C^1_0([0,T]\times[0,1])$ such that $\phi(t,0) = \phi(t,1) \quad (0 < t < T)$, because the data $(\bar{u}^0_1, \bar{u}^0_3, \ldots, \bar{u}^0_{2L-1})$ is uniformly bounded and have a subsequence which converges to a function $\bar{u}(x)$ of $L^\infty(0,1)$.

By the similar calculation in §5, it follows

$$\begin{aligned} I &= \iint_{[0,T]\times[0,1]} g\phi dtdx + \sum_{n=1}^{N}[\phi u^\Delta]^{(2n-1)\Delta t-0}_{(2n-1)\Delta t+0}dx + \sum_{n=1}^{N}[\phi u^\Delta]^{2n\Delta t-0}_{2n\Delta t+0}dx \\ &= \sum_{n=1}^{N}\left(\sum_{j=1}^{L-1}\phi^{2n-1}_{2j}\int_{E^{2n-1}_{2j}}[u^\Delta]dx + \phi^{2n-1}_0\int_0^{\Delta x}[u^\Delta]dx\right. \\ &\quad \left. +\phi^{2n-1}_L\int_{1-\Delta x}^1[u^\Delta]dx\right) + \sum_{n=1}^{N}\sum_{j=1}^{L}\phi^{2n}_{2j-1}\int_{E^{2n}_{2j-1}}(u^{2n}_- - u^{2n}_{0,+})dx \\ &\quad + \sum_{n=1}^{N}\left\{\sum_{j=1}^{L-1}\int_{E^{2n-1}_{2j}}(\phi-\phi^{2n-1}_{2j})[u^\Delta]dx + \int_0^{\Delta x}(\phi-\phi^{2n-1}_0)[u^\Delta]dx\right. \\ &\quad \left. + \int_{1-\Delta x}^1(\phi-\phi^{2n-1}_L)[u^\Delta]dx\right\} \end{aligned}$$

$$
\begin{aligned}
& + \sum_{n=1}^{N}\sum_{j=1}^{L}\int_{E_{2j-1}^{2n}}(\phi-\phi_{2j-1}^{2n})(u_-^{2n}-u_{0,+}^{2n})dx \\
& + \iint_{[0,T]\times[0,1]} g\phi dt dx + \sum_{n=1}^{N}\int_0^1 \phi(u_{0,+}^{2n}-u_+^{2n})dx \\
= \; & I_1 + I_2
\end{aligned}
$$

since $\sigma[u^\Delta]-[f^\Delta]=0$ for shocks from the Rankine–Hugoniot relation and $\phi_0^{2n-1}=\phi_L^{2n-1}$, where

$$
\begin{aligned}
I_1 &= \sum_{n=1}^{N}\left\{\sum_{j=1}^{L-1}\int_{E_{2j}^{2n-1}}(\phi-\phi_{2j}^{2n-1})[u^\Delta]dx + \int_0^{\Delta x}(\phi-\phi_0^{2n-1})[u^\Delta]dx\right. \\
&\quad \left. + \int_{1-\Delta x}^{1}(\phi-\phi_L^{2n-1})[u^\Delta]dx\right\} \\
&\quad + \sum_{n=1}^{N}\sum_{j=1}^{L}\int_{E_{2j-1}^{2n}}(\phi-\phi_{2j-1}^{2n})(u_-^{2n}-u_{0,+}^{2n})dx, \\
I_2 &= \iint_{[0,T]\times[0,1]} g\phi dt dx + \sum_{n=1}^{N}\int_0^1 \phi(u_{0,+}^{2n}-u_+^{2n})dx.
\end{aligned}
$$

I_1 tends to zero because

$$
\begin{aligned}
|I_1| &\le \|\phi\|_{C^1}\Delta x\left(\sum_{n=1}^{N}\int_0^1 |u_-^{2n-1}-u_+^{2n-1}|dx + \sum_{n=1}^{N}\int_0^1 |u_-^{2n}-u_{0,+}^{2n}|dx\right) \\
&\le \|\phi\|_{C^1}\Delta x\sqrt{2N} \\
&\quad \times\left(\sum_{n=1}^{N}\int_0^1 |u_-^{2n-1}-u_+^{2n-1}|^2dx + \sum_{n=1}^{N}\int_0^1 |u_-^{2n}-u_{0,+}^{2n}|^2dx\right)^{1/2} \\
&\le C(\|\phi\|_{C^1},M,T,G_0,\Lambda_2)\sqrt{\Delta x}.
\end{aligned}
$$

For I_2, we obtain

$$
\begin{aligned}
I_2 &= \iint_{[0,T]\times[0,1]} g\phi dt dx - \sum_{n=1}^{N}\sum_{j=1}^{L}\int_{E_{2j-1}^{2n}} 2\Delta t g_{2j-1}^{2n-1}\phi dx \\
&= \sum_{n=1}^{N}\sum_{j=1}^{L}\int_{(2n-2)\Delta t}^{2n\Delta t} dt\int_{E_{2j-1}^{2n}} g(t,x)\{\phi(t,x)-\tilde{\phi}_{2j-1}^{2n}\}dx,
\end{aligned}
$$

where

$$
\tilde{\phi}_j^n = \frac{1}{m(E_j^n)}\int_{E_j^n}\phi(n\Delta t,x)dx.
$$

The function $\phi - \tilde{\phi}^{2n}_{2j-1}$ tends to zero because

$$\begin{aligned} |\phi - \tilde{\phi}^{2n}_{2j-1}| &\le \frac{1}{2\Delta x}\int_{E^{2n}_{2j-1}} |\phi(t,x) - \phi(2n\Delta t, y)|dy \ \le \ \|\phi\|_{C^1}(2\Delta x + \Delta t) \\ &\le \|\phi\|_{C^1}\Delta x(2 + 1/\Lambda). \end{aligned}$$

Therefore we obtain

$$|I| \le C(\|\phi\|_{C^1}, M, T, G_0, \Lambda_2)\sqrt{\Delta t}.$$

Lastly, we show that the limit satisfies the entropy condition (13). Let (U, F) be a smooth entropy pair and U be a convex. The weak form of the entropy condition is the inequality

$$\iint_{t>0} (U\phi_t + F\phi_x + U'g\phi)dtdx \ge 0$$

for any $\phi \in C^1((0,\infty)\times R)$. It is sufficient for the proof to show that

$$\bar{I} = \int_0^T dt \int_0^1 (U(u^\Delta)\psi_t + F(u^\Delta)\psi_x + U'(u^\Delta)g\psi_x)dx \ge o(1)$$

for $\psi \in C^1([0,T]\times[0,1])$ such that

$$\psi \ge 0, \quad \psi(0,x) = \psi(T,x) \quad (x \in [0,1]), \quad \psi(t,0) = \psi(t,1) \quad (t \in [0,T]).$$

For convex entropy U we can see

$$\sigma[U] - [F] \ge 0$$

(cf. [3]). Hence,

$$\begin{aligned} \bar{I} &= \sum_{n=1}^{N}\int_0^1 [U\psi]^{(2n-1)\Delta t - 0}_{(2n-1)\Delta t + 0}dx + \sum_{n=1}^{N}\int_0^1 [U\psi]^{2n\Delta t - 0}_{2n\Delta t + 0}dx + \iint U'g\psi dtdx \\ &\quad + \int_0^1 (U\psi)(T+0,x)dx - \int_0^1 (U\psi)(+0,x)dx \\ &\quad + \int_0^T \sum_{\text{shock}} \phi(\sigma[U] - [F])dt + \iint U'g\psi dtdx \\ &\ge \bar{L}_1 + \bar{L}_2 + \bar{L}_3 + \iint U'g\psi dtdx \end{aligned}$$

since $u^{\Delta}(T+0,x)=u^{\Delta}(+0,x)$, where

$$\begin{aligned}
\bar{L}_1 &= \sum_{n=1}^{N}\left\{\sum_{n=1}^{L}\psi_{2j}^{2n-1}\int_{E_{2j}^{2n-1}}[U]dx+\psi_0^{2n-1}\int_0^{\Delta x}[U]dx\right.\\
&\quad\left.+\psi_{2L}^{2n-1}\int_{1-\Delta x}^{1}[U]dx\right\}+\sum_{n=1}^{N}\sum_{j=1}^{L}\psi_{2j-1}^{2n}\int_{E_{2j-1}^{2n}}(U_{-}^{2n}-U_{0,+}^{2n})dx,\\
\bar{L}_2 &= \sum_{n=1}^{N}\int_0^1\psi(U_{0,+}^{2n}-U_{+}^{2n})dx,\\
\bar{L}_3 &= \sum_{n=1}^{N}\left\{\sum_{n=1}^{L}\int_{E_{2j}^{2n-1}}(\psi-\psi_{2j}^{2n-1})[U]dx+\int_0^{\Delta x}(\psi-\psi_0^{2n-1})[U]dx\right.\\
&\quad\left.+\int_{1-\Delta x}^{1}(\psi-\psi_{2L}^{2n-1})[U]dx\right\}\\
&\quad+\sum_{n=1}^{N}\sum_{j=1}^{L}\int_{E_{2j-1}^{2n}}(\psi-\psi_{2j-1}^{2n})(U_{-}^{2n}-U_{0,+}^{2n})dx.
\end{aligned}$$

By $U''\geq 0$ and $\psi_0^{2n-1}=\psi_{2L}^{2n-1}$, $\bar{L}_1\geq 0$. It follows by the similar way in §5 that

$$\begin{aligned}
|\bar{L}_3| &\leq \|\psi\|_{C^1}\Delta x\max|U'|\\
&\quad\times\left(\sum_{n=1}^{N}\int_0^1|u_{-}^{2n-1}-u_{+}^{2n-1}|dx+\sum_{n=1}^{N}\int_0^1|u_{-}^{2n}-u_{0,+}^{2n}|dx\right)\\
&\leq C(\|\psi\|_{C^1},U',M,T,G_0,\Lambda_2)\sqrt{\Delta x}.
\end{aligned}$$

Hence we have

$$\bar{I}\geq\iint_{[0,T]\times[0,1]}U'g\psi dtdx+\bar{L}_2+O(\sqrt{\Delta x}). \tag{29}$$

From Taylor's expansion theorem

$$\begin{aligned}
\bar{L}_2 &= \sum_{n=1}^{N}\int_0^1\psi(2n\Delta t,x)\{U'(u_{+}^{2n})(u_{0,+}^{2n}-u_{+}^{2n})\}dx\\
&\quad+\sum_{n=1}^{N}\int_0^1\psi(2n\Delta t,x)(u_{0,+}^{2n}-u_{+}^{2n})^2dx\\
&\quad\times\int_0^1(1-\theta)U''(u_{+}^{2n}+\theta(u_{0,+}^{2n}-u_{+}^{2n}))d\theta\\
&\geq -\sum_{n=1}^{N}\sum_{j=1}^{L}2\Delta t g_{2j-1}^{2n-1}U'(u_{2j-1}^{2n})\int_{E_{2j-1}^{2n}}\psi(2n\Delta t,x)dx.
\end{aligned}$$

First two terms of the right hand side of (29) become

$$
\begin{aligned}
&\iint_{[0,T]\times[0,1]} U' g\psi dt dx + \bar{L}_2 \\
= \ & \sum_{n=1}^{N}\sum_{j=1}^{L}\int_{(2n-2)\Delta t}^{2n\Delta t} dt \int_{E_{2j-1}^{2n}} (U'\psi)_{2j-1}^{2n} g dx \\
& + \sum_{n=1}^{N}\sum_{j=1}^{L}\int_{(2n-2)\Delta t}^{2n\Delta t} dt \int_{E_{2j-1}^{2n}} U'(\psi - \psi_{2j-1}^{2n}) g dx \\
& + \sum_{n=1}^{N}\sum_{j=1}^{L}\int_{(2n-2)\Delta t}^{2n\Delta t} dt \int_{E_{2j-1}^{2n}} +\{U' - U'(u_{2j-1}^{2n})\}\psi_{2j-1}^{2n} g dx + \bar{L}_2 \\
\geq \ & \sum_{n=1}^{N}\sum_{j=1}^{L} 4\Delta t \Delta x (U'\psi)_{2j-1}^{2n} g_{2j-1}^{2n-1} \\
& + \sum_{n=1}^{N}\sum_{j=1}^{L} \psi_{2j-1}^{2n} \int_{(2n-2)\Delta t}^{2n\Delta t} dt \int_{E_{2j-1}^{2n}} \{U' - U'(u_{2j-1}^{2n})\} g dx \\
& - \sum_{n=1}^{N}\sum_{j=1}^{L} 2\Delta t g_{2j-1}^{2n-1} U'(u_{2j-1}^{2n}) \int_{E_{2j-1}^{2n}} \psi(2n\Delta t, x) dx + O(\Delta x) \\
\geq \ & \sum_{n=1}^{N}\sum_{j=1}^{L} \psi_{2j-1}^{2n} \int_{(2n-2)\Delta t}^{2n\Delta t} dt \int_{E_{2j-1}^{2n}} \{U' - U'(u_{2j-1}^{2n})\} g dx + O(\Delta x) \\
= \ & \sum_{n=1}^{N} \int_{(2n-2)\Delta t}^{2n\Delta t} dt \int_0^1 \{U' - U'(u_+^{2n})\}\psi_+^{2n} g dx + O(\Delta x) \\
= \ & \sum_{n=1}^{N} \int_{(2n-2)\Delta t}^{2n\Delta t} dt \int_0^1 \{U' - U'(u_{0,+}^{2n})\}\psi_+^{2n} g dx + O(\Delta x),
\end{aligned}
$$

where ψ_+^{2n} is a step value function defined by

$$\psi_+^{2n}(x) = \psi_{2j-1}^{2n} \qquad (x \in E_{2j-1}^{2n}).$$

Remember the following lemma proved in [8].

LEMMA 6 *Let $u(t,x)$ be the solution of the Riemann problem (16), $\bar{u}$ be the average of $u(\Delta t, x)$*

$$u(t,x) = R(t, x; u_l, u_r), \qquad \bar{u} = \frac{1}{2\Delta x}\int_{-\Delta x}^{\Delta x} u(\Delta t, x) dx$$

and the maximum wave speed $\Lambda = \max\{|f'(u_l)|, |f'(u_r)|\}$ satisfy $\Lambda < \Delta x/\Delta t$. Then

$$\int_{-\Delta x}^{\Delta x} |\bar{u} - u(t,x)| dx \leq C \int_{-\Delta x}^{\Delta x} |\bar{u} - u(\Delta t, x)| dx \qquad (0 \leq t \leq \Delta t),$$

where C is a positive constant

$$C = (1 - \Lambda\Delta t/\Delta x)^{-1}.$$

For the approximation $u^{\Delta}(t,x)$ the constant C is

$$\left(1 - \frac{\Lambda}{2\delta(M+TG_0)+\Lambda}\right)^{-1} = 1 + \frac{\Lambda}{2\delta(M+TG_0)}$$

from (19). By Lemma 6 we estimate the rest term

$$\begin{aligned}
&\left|\sum_{n=1}^{N}\int_{(2n-2)\Delta t}^{2n\Delta t} dt \int_0^1 \{U' - U'(u_{0,+}^{2n})\}\psi_+^{2n} g dx\right| \\
\leq\ & \sum_{n=1}^{N}\int_{(2n-1)\Delta t}^{2n\Delta t} dt \int_0^1 |U' - U'(u_{0,+}^{2n})||\psi_+^{2n} g| dx \\
&+\sum_{n=1}^{N}\int_{(2n-2)\Delta t}^{(2n-1)\Delta t} dt \int_0^1 |U' - U'(u_+^{2n-1})||\psi_+^{2n} g| dx \\
&+\sum_{n=1}^{N}\int_{(2n-2)\Delta t}^{(2n-1)\Delta t} dt \int_0^1 |U'(u_+^{2n-1}) - U'(u_{0,+}^{2n})||\psi_+^{2n} g| dx \\
\leq\ & C(M,\delta,M,T,G_0,\|\psi\|_{C^0},\max|U''|)\Delta t \\
&\times\sum_{n=1}^{N}\int_0^1 (|u_-^{2n} - u_{0,+}^{2n}| + |u_-^{2n-1} - u_+^{2n-1}| + |u_-^{2n} - u_{0,+}^{2n}|) dx \\
\leq\ & C\sqrt{\Delta x}.
\end{aligned}$$

Therefore, we obtain $\bar{I} \geq -C\sqrt{\Delta x}$.

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