A NOTE ON FRONT TRACKING AND THE EQUIVALENCE BETWEEN VISCOSITY SOLUTIONS OF HAMILTON-JACOBI EQUATIONS AND ENTROPY SOLUTIONS OF SCALAR CONSERVATION LAWS

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ABSTRACT. We give a direct proof of the well known equivalence between the Kružkov-Vol'pert entropy solution for the scalar conservation law $p_t + H(p)_x = 0$ and the Crandall-Lions viscosity solution of the Hamilton-Jacobi equation $u_t + H(u_x) = 0$. In our proof we work directly with the defining entropy and viscosity inequalities and do not, as is usually done, exploit the convergence of the viscosity method. The proof is based on establishing the equivalence directly for a "dense" set of flux functions H and initial data p_0/u_0. In the course of doing so, we translate front tracking for scalar conservation laws to Hamilton-Jacobi equations and derive some of its properties.

1. INTRODUCTION

It is well known that Hamilton-Jacobi equations are closely related to scalar conservation laws. In this paper we give a direct proof of the equivalence between the unique viscosity solution [7, 5] of the Hamilton-Jacobi equation

$$\begin{aligned} &u_t + H(u_x) = 0 \text{ in } \mathbb{R}\times(0,\infty),\\ &u = u_0 \in \mathrm{BUC}(\mathbb{R}) \text{ on } \mathbb{R}\times\{t=0\}, \end{aligned} \tag{1.1}$$

and the unique entropy solution [13, 18] of the corresponding scalar conservation laws

$$\begin{aligned} &p_t + H(p)_x = 0 \text{ in } \mathbb{R}\times(0,\infty),\\ &p = p_0 \in BV(\mathbb{R}) \text{ on } \mathbb{R}\times\{t=0\}, \end{aligned} \tag{1.2}$$

In the course of doing so, we extend, or perhaps more correctly, translate, the front tracking method [8, 9] for conservation laws (1.2) to Hamilton-Jacobi equations (1.1) and derive some of its properties.

We recall that a function $u \in \mathrm{BUC}(\mathbb{R}\times(0,T))$ is a viscosity solution of the initial value problem (1.1) if $u = u_0$ in $\mathbb{R}\times\{t=0\}$ and u is simultaneously a (viscosity) subsolution and a (viscosity) supersolution in $\mathbb{R}\times(0,\infty)$:

(Subsolution): For each $\phi \in C^\infty(\mathbb{R}\times(0,T)$,

$$\begin{cases} \text{if } u-\phi \text{ has a local maximum at a point } (x_0,t_0)\in\mathbb{R}\times(0,\infty),\\ \text{then } \phi_t(x_0,t_0) + H(\phi_x(x_0,t_0)) \le 0. \end{cases}$$

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(Supersolution): For each $\phi \in C^\infty(\mathbb{R} \times (0,T)$,

$$\begin{cases} \text{if } u-\phi \text{ has a local minimum at a point } (x_0,t_0) \in \mathbb{R}\times(0,\infty), \\ \text{then } \phi_t(x_0,t_0) + H(\phi_x(x_0,t_0)) \geq 0. \end{cases}$$

We remark that it is possible to give an equivalent definition of a viscosity solution based the notions of sub- and super-differentials [5]. The existence, uniqueness, and stability properties of viscosity solutions were proved first by Crandall and Lions [7] and Crandall, Evans, and Lions [5]. It is also known [7, 5, 14] that the unique viscosity solution of (1.1) coincides with the vanishing viscosity solution $\lim_{\varepsilon\to 0} u^\varepsilon$ (here the limit is with respect to the uniform topology on compacts), where u^ε is the unique smooth solution of the approximate problem

$$\begin{aligned} u^\varepsilon_t + H(u^\varepsilon_x) &= \varepsilon u^\varepsilon_{xx} \text{ in } \mathbb{R}\times(0,\infty), \\ u^\varepsilon &= u_0 \in \mathrm{BUC}(\mathbb{R}) \text{ on } \mathbb{R}\times\{t=0\}. \end{aligned} \tag{1.3}$$

We recall that $p \in L^\infty(\mathbb{R}\times(0,\infty))$ is an entropy solution of the initial value problem (1.2) if $\|p(\cdot,t)-p_0\|_{L^1_{\mathrm{loc}}(\mathbb{R})} \to 0$ essentially as $t\to 0$ and, for all nonnegative test functions $\phi \in C^\infty(\mathbb{R}\times(0,\infty))$ and constants $k\in\mathbb{R}$,

$$\iint \Big(|p-k|\phi_t + \operatorname{sign}(u-k)\,(H(u)-H(k))\phi_x \Big)\, dxdt \geq 0.$$

The existence, uniqueness, and stability properties of entropy solutions were proved first by Kružkov [13] and by Vol'pert [18] in the smaller BV class. Again, the unique entropy solution of (1.2) coincides with the vanishing viscosity solution $\lim_{\varepsilon\to 0} p^\varepsilon$ (here the limit is with respect to L^1 convergence on compacts), where p^ε is the unique smooth solution of the approximate problem

$$\begin{aligned} p^\varepsilon_t + H(p^\varepsilon)_x &= \varepsilon p^\varepsilon_{xx} \text{ in } \mathbb{R}\times(0,\infty), \\ p^\varepsilon &= p_0 \in BV(\mathbb{R}) \text{ on } \mathbb{R}\times\{t=0\}. \end{aligned} \tag{1.4}$$

As already mentioned, the question of viscosity solutions of (1.1) is equivalent to the question of entropy solutions of (1.2). More precisely, if u is the unique viscosity solution of (1.1), then

$$p = \frac{\partial}{\partial x} u$$

is the unique entropy solution of (1.2). Conversely, if p is the unique entropy solution of (1.2), then u defined via

$$u(x,t) = \int^x p(\xi,t)\, d\xi$$

is the unique viscosity solution (1.1).

We remark that this equivalence has been exploited by many authors trying to translate successful numerical methods for hyperbolic conservation laws to methods for Hamilton-Jacobi equations. In this context, we should mention that in the multi-dimensional case this one-to-one correspondence no longer exists, instead the gradient $p=\nabla u$ satisfies (at least formally) a non-strictly hyperbolic system of conservation laws [12, 14, 11]. For other connections between the theories of viscosity and entropy solutions, we refer the reader to [2].

The usual proof in the one dimensional case (see, e.g., [12, 14, 4]) of the relation $p=u_x$ exploits the known results about existence, uniqueness, and convergence of viscosity method. In particular, one notices that the regularity of p^ε permits the relation $p^\varepsilon = u^\varepsilon_x$ and after sending $\varepsilon \downarrow 0$, one gets the desired result $p=u_x$. On the other hand, there exist very few (quoteable!) references which prove this relation without without using the convergence of the viscosity method but instead use the defining viscosity and entropy inequalities directly. In fact, the only reference that we are aware of which takes the direct approach is Caselles [3]. Caselles treats, however, only the time independent case.

The main purpose of the present paper is to give a direct proof in the time dependent case, at least for sufficiently regular solutions. With a "direct proof", it is understood that we do not use the convergence of the viscosity approximations (1.4) and (1.3). The second purpose is to derive a front tracking algorithm for Hamilton-Jacobi equations in one dimension. Our main result is contained in the following theorem:

Theorem 1.1. *Let u be the unique viscosity solution of the Hamilton-Jacobi equation* (1.1) *and let p be the unique entropy solution to the conservation law*

$$p_t + H(p)_x = 0,$$

with initial data

$$p(x,0) = \frac{d}{dx} u_0(x).$$

If $u_0 \in \mathrm{BUC}(\mathbb{R})$, or $p(x,0) \in BV(\mathbb{R})$, then $u_x(x,t) = p(x,t)$ almost everywhere.

To show Theorem 1.1, we use the front tracking method suggested by Dafermos in [8], and properly defined by Holden, Holden and Høegh-Krohn in [9]. This is a numerical method for scalar conservation laws (1.2), which yields exact entropy solutions if the initial data p_0 is piecewise constant and the flux function H piecewise linear. We then note that this method translates into a method that gives the exact viscosity solutions to the Hamilton-Jacobi equation (1.1) if u_0 and H are piecewise linear and Lipschitz continuous. This gives Theorem 1.1 in the case of piecewise linear/constant initial data, and piecewise linear Hamiltonians/flux functions. This front tracking method uses the solution of the Riemann problem for conservation laws and Hamilton-Jacobi equations, and we state an explicit formula for this solution if the Hamiltonian is piecewise linear. This formula is related to similar formulas found in Bardi and Osher [1], and in Subbotin and Shagalova [17].

To extend the result to more general problems, we take the L^∞/L^1 closure of the set of piecewise linear/constant initial data, and the sup/Lip norm closure of the set of piecewise linear Hamiltonians/flux functions, utilizing stability estimates from [10] and [16] for conservation laws and Hamilton-Jacobi equations respectively. In doing this we also obtain explicit error estimates for the front tracking method both for conservation laws and for Hamilton-Jacobi equations. Furthermore, as an extra bonus, from [9] we have that front tracking will yield the (entropy, viscosity) solution for all $t > 0$ by a finite number of operations.

2. Front tracking

We start this section by describing front tracking for scalar conservation laws. This method was first proposed by Dafermos [8], and later shown to be a viable method for conservation laws by Holden, Holden and Høegh-Krohn [9].

We wish to solve the scalar conservation law

$$p_t + G(p)_x = 0, \quad p(x,0) = p_0(x) \tag{2.1}$$

where G is a piecewise linear continuous function, and p_0 is a piecewise constant function with bounded support taking a finite number of values. By *breakpoints* of G we mean the points where G' is discontinuous. To solve the initial value problem (2.1), we start by solving the Riemann problem, i.e., where p_0 is given by

$$p_0(x) = \begin{cases} p_l & \text{for } x < 0, \\ p_r & \text{for } x \geq 0. \end{cases} \tag{2.2}$$

Let now G_* be the lower convex envelope of G between p_l and p_r,

$$G_*(p; p_l, p_r) = \sup\Big\{ g(p) \;\Big|\; g(p) \text{ is convex and } g(p) \leq G(p) \text{ between } p_l \text{ and } p_r \Big\}. \tag{2.3}$$

Similarly we let $G^*(p; p_l, p_r)$ denote the upper concave envelope of G between p_l and p_r. Now set

$$\tilde{G}(p; p_l, p_r) = \begin{cases} G_*(p; p_l, p_r) & \text{if } p_l < p_r, \\ G^*(p; p_l, p_r) & \text{if } p_l \geq p_r. \end{cases} \tag{2.4}$$

Since G is piecewise linear and continuous, also $\tilde{G}$ will be piecewise linear and continuous. Now set $p_1 = p_l$ and assume that $\tilde{G}$ has $N-1$ breakpoints between p_l and p_r, call these $p_2, \dots, p_{N-1}$ and set $p_N = p_r$, such that $p_i < p_{i+1}$ if $p_l < p_r$ and $p_i > p_{i+1}$ if $p_l > p_r$. Now set $\sigma_0 = -\infty$, $\sigma_N = \infty$ and

$$\sigma_i = \frac{G_{i+1} - G_i}{p_{i+1} - p_i}, \quad \text{for } i = 1, \dots, N-1, \tag{2.5}$$

where $G_i = \tilde{G}(p_i; p_l, p_r) = G(p_i)$. Define Ω_i as

$$\Omega_i = \Big\{ (x,t) \;\Big|\; 0 \leq t \leq T, \text{ and } t\sigma_{i-1} < x \leq t\sigma_i \Big\}. \tag{2.6}$$

Then the following proposition holds:

Proposition 2.1. *Set*

$$p(x,t) = p_i \quad \textit{for } (x,t) \in \Omega_i, \tag{2.7}$$

then p is the entropy solution of the Riemann problem (2.2).

Proof. We show the proposition in the case where $p_l < p_r$, the other case being completely similar. First note that the definition of the lower envelope implies that for $k \in [p_i, p_{i+1}]$,

$$\begin{aligned} G(k) &\geq G_i + (k - p_i)\sigma_i \\ &\geq G_{i+1} + (k - p_{i+1})\sigma_i \\ &\geq \frac{1}{2}(G_{i+1} + G_i) + \left(k - \frac{1}{2}(p_{i+1} + p_i) \right). \end{aligned} \tag{2.8}$$

We wish to show that for each nonnegative test function ϕ,

$$-\iint\limits_{\Omega_T} |p - k|\, \phi_t + q(p,k)\phi_x \, dx dt + \int\limits_{\mathbb{R}} |p(x,T) - k|\, \phi(x,T) - |p_0(x) - k|\, \phi(x,0)\, dx \leq 0, \tag{2.9}$$

where $\Omega_T = \mathbb{R} \times [0,T]$ and $q(p,k) = G(p \vee k) - G(p \wedge k) = \text{sign}\,(p-k)\,(G(p) - G(k))$. Here we use the notation $p \vee k = \max(p,k)$ and $p \wedge k = \min(p,k)$.

The first term in (2.9) is given by

$$-\sum_{i=1}^{N}\iint_{\Omega_i} |p_i-k|\,\phi_t + q(p_i,k)\phi_x\,dxdt = -\int_{\mathbb{R}} |p(x,T)-k|\,\phi(x,T) - |p_0(x)-k|\,\phi(x,0)\,dx$$
$$- \sum_{i=1}^{N-1}\int_0^T \Big\{\sigma_i\left(|p_{i+1}-k| - |p_i-k|\right)$$
$$- \left(q(p_{i+1},k)-q(p_i,k)\right)\Big\}\phi(\sigma_i t,t)\,dt,$$

by Green's formula applied to each Ω_i. Considering the integrand in the last term, we find that

$$\sigma_i\left(|p_{i+1}-k|-|p_i-k|\right) - \left[G(p_{i+1}\vee k) - G(p_{i+1}\wedge k) - G(p_i\vee k) + G(p_i \wedge k)\right] = 0$$

if $k > p_{i+1}$ or $k < p_i$, otherwise we find that

$$\sigma_i\left(|p_{i+1}-k|-|p_i-k|\right) - \left[G(p_{i+1}\vee k) - G(p_{i+1}\wedge k) - G(p_i\vee k) + G(p_i \wedge k)\right] =$$
$$\left(G(k) - \left[\sigma_i\left(k-\frac{1}{2}(p_{i+1}+p_i)\right) + \frac{1}{2}\left(G_{i+1}+G_i\right)\right]\right) \geq 0$$

by (2.8), which implies that p defined by (2.7) is an entropy solution of the Riemann problem. □

Now we construct the solution of the more general initial value problem (2.1). Since the initial value function p_0 is piecewise constant, it defines a series of Riemann problems. We can construct the solutions of these, which amounts to defining the speeds σ_i, $i = 1,\dots,N-1$, for each Riemann problem. Then $p(x,t)$ will be piecewise constant, with discontinuities on straight lines emanating from each discontinuity in p_0. We call these discontinuities *fronts*. Clearly, p can be defined until two fronts collide. At this point we can again solve the Riemann problem defined by the value to the left and right of the collision point, thereby continuing the solution until the next collision and so on. The next proposition sums up the properties of this method, called *front tracking*:

Proposition 2.2. *Let $G(p)$ be a continuous and piecewise linear continuous function with a finite number of breakpoints in the interval $[-M,M]$, where M is some constant. Assume that $p_0(x)$ is a piecewise constant function with a finite number of discontinuities taking values in the interval $[-M,M]$. Then the initial value problem*

$$p_t + G(p)_x = 0, \qquad p(x,0) = p_0(x)$$

has an entropy solution which can be constructed by front tracking. The constructed solution $p(x,t)$ is a piecewise constant function of x for each t, and $p(x,t)$ takes values in the finite set

$$\Big\{u_0(x)\Big\}\bigcup\Big\{\textit{breakpoints of } G\Big\}.$$

Furthermore, there are only a finite number of collisions between fronts in p.

If $\bar{G}$ is another piecewise linear continuous function with a finite number of breakpoints in $[-M,M]$ and $\bar{p}_0(x)$ another piecewise constant function taking values in $[-M,M]$, set $\bar{p}$ to be the entropy solution to

$$\bar{p}_t + \bar{G}(\bar{p})_x = 0, \qquad \bar{p}(x,0) = \bar{p}_0(x).$$

If p_0 and $\bar{p}_0$ are in $L^1(\mathbb{R}) \cap BV(\mathbb{R})$, then

$$\begin{aligned} \|p(\cdot,T) - \bar{p}(\cdot,T)\|_{L^1(\mathbb{R})} \leq \|p_0 - \bar{p}_0\|_{L^1(\mathbb{R})} \\ + T\left(|p_0|_{BV(\mathbb{R})} \wedge |\bar{p}_0|_{BV(\mathbb{R})}\right) \left\|G - \bar{G}\right\|_{\mathrm{Lip}([-M,M])}. \end{aligned} \tag{2.10}$$

The fact that p is an entropy solution easily follows from Proposition 2.1, for a proof of the remaining part of the proposition, see [10].

Hence, front tracking yields the entropy solution to all initial value problems where

$$G \in \left\{\text{piecewise linear continuous}\right\} =: \mathcal{D}_{\text{flux}},$$

$$p_0 \in \left\{\text{piecewise constant with a finite number of discontinuites}\right\} \bigcap L^1(\mathbb{R}) \bigcap BV(\mathbb{R}) = \mathcal{D}_{\text{initial}}.$$

By taking the Lip-norm closure of $\mathcal{D}_{\text{flux}}$ and the L^1 closure of $\mathcal{D}_{\text{initial}}$, we have existence of an entropy solution for a larger class of problems. In particular we can construct a Cauchy sequence as follows: Let Δx and δ be small numbers. Assume first that $H(p)$ is a C^2 function, set

$$H^\delta(p) = H(i\delta) + (p - i\delta)\frac{H((i+1)\delta) - H(i\delta)}{\delta}, \quad \text{for } i\delta \leq p < (i+1)\delta. \tag{2.11}$$

Then if $\eta > \delta > 0$,

$$\begin{aligned} \left\|H^\eta - H^\delta\right\|_{\mathrm{Lip}([-M,M])} &\leq \sup_{p\in[-M,M]} \left|(H^\eta)'(p) - (H^\delta)'(p)\right| \\ &\leq \sup_{|p-\tilde{p}|\leq\eta} \left|H'(\tilde{p}) - H'(p)\right| \\ &\leq \sup_{|p-\tilde{p}|\leq\eta} \int_p^{\tilde{p}} \left|H''(r)\right| \, dr \\ &\leq \left\|H''\right\|_{L^\infty([-M,M])} \eta \end{aligned}$$

If $p_0(x)$ is in $L^1(\mathbb{R}) \cap BV(\mathbb{R})$, set

$$p^{\Delta x}(x) = \frac{1}{\Delta x}\int_{i\Delta x}^{(i+1)\Delta x} p_0(\xi)\, d\xi, \quad \text{for } i\Delta x \leq x < (i+1)\Delta x. \tag{2.12}$$

Furthermore, we have that

$$\begin{aligned}\left\|p^{\Delta x}-p_0\right\|_{L^1(\mathbb{R})} &= \sum_i \int_{i\Delta x}^{(i+1)\Delta x} \left| p_0(x) - \frac{1}{\Delta x}\int_{i\Delta x}^{(i+1)\Delta x} p_0(z)\,dx \right| dx \\ &\le \sum_i \frac{1}{\Delta x} \int_{i\Delta x}^{(i+1)\Delta x} \int_{i\Delta x}^{(i+1)\Delta x} \left|p_0(x)-p_0(z)\right| dzdx \\ &\le \sum_i \frac{1}{\Delta x} \int_{i\Delta x}^{(i+1)\Delta x} \int_{i\Delta x}^{(i+1)\Delta x} \int_z^x \left|p_0'(y)\right| dy\,dzdx \\ &\le \sum_i \Delta x \int_{i\Delta x}^{(i+1)\Delta x} \left|p_0'(y)\right| dy \\ &\le \Delta x \left|p_0\right|_{BV(\mathbb{R})}\end{aligned}$$

Therefore if $\Delta y \ge \Delta x > 0$,

$$\begin{aligned}\left\|p^{\Delta y}-p^{\Delta x}\right\|_{L^1(\mathbb{R})} &\le \left\|p^{\Delta y}-p_0\right\|_{L^1(\mathbb{R})} + \left\|p^{\Delta x}-p_0\right\|_{L^1(\mathbb{R})} \\ &\le (\Delta y + \Delta x)\left|p_0\right|_{BV(\mathbb{R})} \le 2\Delta y \left|p_0\right|_{BV(\mathbb{R})}.\end{aligned}$$

Now let $p^{\delta,\Delta x}$ be the entropy solution to

$$p_t^{\delta,\Delta x} + H^\delta\left(p^{\delta,\Delta x}\right)_x = 0, \quad p^{\delta,\Delta x}(x,0) = p^{\Delta x}(x). \tag{2.13}$$

Then (2.10) implies that $\left\{p^{\delta,\Delta x}\right\}$ is a Cauchy sequence in $L^1(\mathbb{R})$ since

$$\left\|p^{\eta,\Delta y}(\cdot,T) - p^{\delta,\Delta x}(\cdot,T)\right\|_{L^1(\mathbb{R})} \le \left(2\Delta y + T\left\|H''\right\|_{L^\infty([-M,M])}\eta\right)\left|p_0\right|_{BV(\mathbb{R})}. \tag{2.14}$$

Hence, we can define the L^1 limit

$$p = \lim_{(\delta,\Delta x)\to 0} p^{\delta,\Delta x}.$$

Set $q^\delta(p,k) = H^\delta(p \vee k) - H^\delta(p \wedge k)$, then $\lim_{\delta\to 0} q^\delta = q$. Now since $p \mapsto |p-k|$ and $p \mapsto q(p,k)$ are continuous functions, it easily follows that the limit function p is an entropy solution to

$$p_t + H(p)_x = 0, \quad p(x,0) = p_0(x).$$

We can now use the stability estimate (2.10) to relax the C^2 condition on H by an approximation argument, yielding existence and stability if H is Lipschitz continuous.

Next we describe how the front tracking construction translates to the Hamilton-Jacobi equation

$$u_t + G\left(u_x\right) = 0, \quad u(x,0) = u_0(x), \tag{2.15}$$

where G is piecewise linear and continuous, and u_0 is also piecewise linear and continuous (u_0' is bounded and piecewise constant). First we study the Riemann problem for (2.15), which is the initial value problem

$$u_0(x) = u_0(0) + \begin{cases} p_l x & \text{for } x < 0, \\ p_r x & \text{for } x \ge 0, \end{cases} \tag{2.16}$$

where p_l and p_r are constants, c.f. (2.2). Let now $p(x,t)$ denote the solution of the corresponding Riemann problem for the conservation law (2.7). We now claim that the viscosity solution to the Riemann problem (2.16), (2.15), is given by

$$u(x,t) = u_0(0) + xp(x,t) - tG(p(x,t)). \tag{2.17}$$

Although p is discontinuous, a closer look at this formula reveals that u is uniformly continuous. Indeed, for fixed t, $u(x,t)$ is piecewise linear in x, with breakpoints located at the fronts in p. Hence, when computing u, one only needs to keep record of how u changes at the fronts. Along a front with speed σ_i, u is given by

$$u(\sigma_i t,t) = u_0(0) + t\left(\sigma_i p_i - G(p_i)\right) = u_0(0) + t\left(\sigma_i p_{i+1} - G(p_{i+1})\right) \tag{2.18}$$

by the definition of σ_i, (2.5). We postpone showing that u is a viscosity solution, and instead compute an explicit example.

Example 2.1. First we note that the formula (2.17), holds also when G is Lipschitz continuous, if p is interpreted as the solution of the corresponding Riemann problem. Consider Burgers' equation

$$u_t + \frac{1}{2}\left(u_x\right)^2 = 0,$$

and assume that the Riemann initial data are given by $u_0(0) = 0$, $p_l = 0$ and $p_r = 1$. In this case $p_l < p_r$ and we use the convex envelope, which means that $\tilde{G}(p) = 1/2\,p^2$ and $(\tilde{G}')^{-1}(z) = z$. So $p(x,t)$ is given by

$$p(x,t) = \begin{cases} 0 & x < 0, \\ \frac{x}{t} & 0 \le x < t, \\ 1 & x \ge t. \end{cases}$$

Hence (2.17) reads

$$u(x,t) = \begin{cases} 0 & x < 0, \\ \frac{x^2}{2t} & 0 \le x < t, \\ x - \frac{t}{2} & x \ge t. \end{cases}$$

In this case u is differentiable, and corresponds to a rarefaction wave for the conservation law. If we interchange p_l and p_r we have the shock case, now $\tilde{G}(p) = 1/2\,p$, thus

$$p(x,t) = \begin{cases} 1 & x < t/2, \\ 0 & x \ge t/2, \end{cases}$$

and

$$u(x,t) = \begin{cases} x - \frac{t}{2} & x < t/2, \\ 0 & x \ge t/2. \end{cases}$$

Now we can use the front tracking construction for conservation laws to define a solution to the general initial value problem (2.15). We track the fronts as for the conservation law, but updates u along each front by (2.18). Note that if for some (x,t), $u(x,t)$ is determined by the solution of the Riemann problem at (x_j,t_j), then

$$u(x,t) = u\left(x_j,t_j\right) + \left(x - x_j\right)p(x,t) - \left(t - t_j\right)H(p(x,t)), \tag{2.19}$$

where p is the solution of the initial value problem for (2.1) with initial values given by

$$p(x,0) = \frac{d}{dx}u_0(x).$$

Analogously to Proposition 2.2 we have:

Proposition 2.3. *The piecewise linear function $u(x,t)$ is the viscosity solution of* (2.15)*. Furthermore $u(x,t)$ piecewise linear on a finite number of polygons in $\mathbb{R}\times\mathbb{R}_0^+$. If u_0 is bounded and uniformly continuous (*BUC*), then u is in* $\mathrm{BUC}(\mathbb{R}\times[0,T])$ *for any $T<\infty$. If $\bar{G}$ is another Lipschitz continuous piecewise linear function with a finite number of breakpoints, and $\bar{u}$ is the viscosity solution of*

$$\bar{u}_t+\bar{G}(\bar{u}_x)=0,\qquad \bar{u}(x,0)=\bar{u}_0(x),$$

and u_0 and $\bar{u}_0$ are bounded and uniformly continuous (BUC), then

$$\|u(\cdot,T)-\bar{u}(\cdot,T)\|_{L^\infty(\mathbb{R})}\le\|u_0-\bar{u}_0\|_{L^\infty(\mathbb{R})}+T\sup_{|p|\le M}\left|G(p)-\bar{G}(p)\right|, \tag{2.20}$$

where $M=\min\left(\|u_{0\,x}\|,\|\bar{u}_{0\,x}\|\right)$.

Proof. We first show that u is a viscosity solution. We have that u is determined by solution of a finite number of Riemann problems at the points $(\bar{x}_j,\bar{t}_j)$. Given a point (x,t) in where $t>0$, we can find a j such that $u(x,t)$ is determined by the Riemann problem solved at $(\bar{x}_j,\bar{t}_j)$.

Set $p=u_x$. Assume that $u-\phi$ has a maximum at (x_0,t_0), and assume that $u\left(x_0,t_0\right)$ is determined by the Riemann solution at $(\bar{x}_j,\bar{t}_j)$. Since u is piecewise linear, we can define the following limits

$$\lim_{x\to x_0-}u_x^\delta(x,t_0)-\phi_x(x_0,t_0)\ge 0,\quad \lim_{x\to x_0+}u_x^\delta(x,t_0)-\phi_x(x_0,t_0)\le 0.$$

or

$$p_l\le\phi_x(x_0,t_0)\le p_r, \tag{2.21}$$

where $p_{l,r}=\lim_{x\to x_0\mp}p(x,t_0-)$. Set

$$\sigma=\frac{G(p_l)-G(p_r)}{p_l-p_r}\quad\text{if } p_l\ne p_r,$$

otherwise set $\sigma=\tilde{G}'(p_l)$. Since $\phi_x(x_0,t_0)$ is between p_l and p_r, the construction of $\tilde{G}$ implies that

$$G\left(\phi_x\left(x_0,t_0\right)\right)\ge G\left(p_l\right)+\sigma\left(\phi_x\left(x_0,t_0\right)-p_l\right). \tag{2.22}$$

Now choose (x,t) sufficiently close to (x_0,t_0) such that

$$\sigma=\frac{x_0-x}{t_0-t}$$

and $u(x,t)$ is also determined by the solution of the Riemann problem at $(\bar{x}_j,\bar{t}_j)$, and $t<t_0$. If $t_0>0$ this is always possible, we return to the case where $t_0=0$ below. Then

$$\frac{1}{t_0-t}\left(u\left(x_0,t_0\right)-u(x,t)\right)\ge\frac{1}{t_0-t}\left(\phi\left(x_0,t_0\right)-\phi(x,t)\right). \tag{2.23}$$

Using (2.17) we have that

$$u\left(x_0,t_0\right)=u(x,t)+\left(x_0-x\right)p_l-\left(t_0-t\right)G\left(p_l\right).$$

Hence, by letting $t\to t_0-$, we find that

$$\begin{aligned}\sigma p_l-G(p_l)&\ge\phi_t\left(x_0,t_0\right)+\sigma\phi_x\left(x_0,t_0\right)\\&\ge\phi_t\left(x_0,t_0\right)+G\left(\phi_x\left(x_0,t_0\right)\right)+\sigma p_l-G\left(p_l\right),\end{aligned} \tag{2.24}$$

which implies that u is a subsolution. A similar argument is applied to show that u is supersolution.

Assume now that $t_0 = 0$ and $u - \phi$ has a maximum at $(x_0, 0)$. Set $p_{l,r} = \lim_{x \to x_0 \mp} p(x, 0+)$. Then

$$u(x,t) = u\left(x_0, 0\right) + \left(x - x_0\right) p_l - tG\left(p_l\right)$$

where $\sigma = (x - x_0)/t$ and (x,t) is sufficiently close to $(x_0, 0)$. Now, using (2.23) as before gives the conclusion. Note that this also shows that the solution of the Riemann problem (2.17) is a viscosity solution.

Next we show the stability estimate (2.20). This is a consequence of Proposition 1.4 in [16], which in our context says that

$$\begin{aligned}\sup_{(x,y)\in D_\varepsilon} \left\{\left|u(x,t) - \bar{u}(y,t)\right| + 3R\beta_\varepsilon(x-y)\right\} \le \sup_{(x,y)\in D_\varepsilon} &\left\{\left|u_0(x) - \bar{u}_0(y)\right| + 3R\beta_\varepsilon(x-y)\right\} \\ &+ t \sup_{|p|\le M} \left|G(p) - \bar{G}(p)\right|,\end{aligned} \tag{2.25}$$

where D_ε is the set

$$D_\varepsilon = \left\{ (x,y) \;\middle|\; |x-y| \le \varepsilon \right\},$$

and $\beta_\varepsilon(x) = \beta(x/\varepsilon)$ for some C_c^∞ function $\beta(x)$ with $\beta(0) = 1$ and $\beta(x) = 0$ for $|x| > 1$. Furthermore, $R = \max\left(\|u\|, \|\bar{u}\|\right)$. Consequently,

$$\begin{aligned}\left\|u(\cdot,t) - \bar{u}(\cdot,t)\right\|_{L^\infty(\mathbb{R})} + \sup_{(x,y)\in D_\varepsilon} &\left\{3R\beta_\varepsilon(x-y) - \left|\bar{u}(x,t) - \bar{u}(y,t)\right|\right\} \\ &\le \left\|u_0 - u_0^\delta\right\|_{L^\infty(\mathbb{R})} + 3R + t \sup_{|p|\le M} \left|G(p) - \bar{G}(p)\right|.\end{aligned}$$

The inequality of the lemma now follows by noting that $\bar{u}$ is in $\mathrm{BUC}(\mathbb{R} \times [0,T])$, and taking the limit as $\varepsilon \to 0$ on the left side. □

Example 2.2. Now we present an example where front tracking is used to compute an approximation to the viscosity solution of the initial value problem

$$u_t + \frac{1}{3}\left(u_x\right)^3 = 0, \quad u(x,0) = \begin{cases} \sin(\pi x) & \text{if } |x| \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.26}$$

Here we approximate the Hamiltonian $H(p) = 1/3p^3$ and $u(x,0)$ by piecewise linear functions. Figure 1 shows the computed viscosity solution at $t = 1/2$, obtained using 50 linear segments to approximate $u(x,0)$ in the interval $x \in [-1,1]$, and using linear segments of length $1/25$ to approximate $H(p)$. To investigate the convergence properties of front tracking as the approximation of the initial function and of H becomes better, we compared the results with a reference solution using 300 linear segments when approximating $u(x,0)$ and linear segments of length $1/150$ in the approximation of H. Table 1 gives the percentage relative error in the L^∞ norm (e) in the right column, and the number of linear segments in u_0 (N) in the left. We used linear segments of length $2/N$ to approximate the Hamiltonian. In this example, the convergence rate is superlinear, and seems to be closer to 2 than to 1, see Remark 2.2 at the end of this paper.

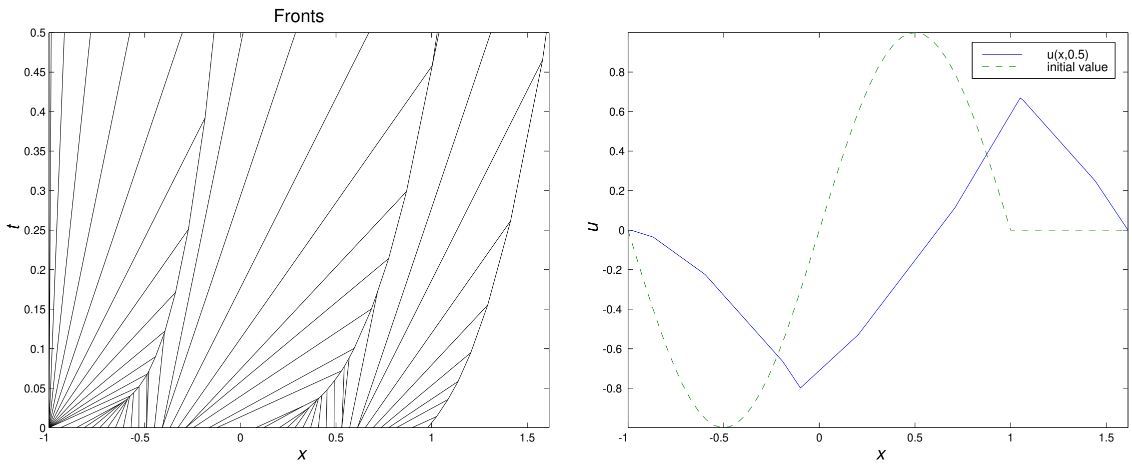

FIGURE 1. Left: Fronts in the (x,t) plane. Right: $u(x,0)$ (dashed) and $u(x,1/2)$ (solid line).

N	e
10	88.9
20	17.6
40	4.8
80	1.2
160	0.2

TABLE 1. $100 \times$ relative L^∞ error.

Now we are able to explicitly construct viscosity solution to all problems of the type (2.15) where G and u_0 are piecewise linear and Lipschitz continuous with a finite number of breakpoints. By using the above stability estimate and taking appropriate closures, we obtain existence of viscosity solutions for a larger class of problems. Let now H^δ be defined by (2.11) and let

$$u_0^{\Delta x} = u_0(i\Delta x) + (x - i\Delta x)\frac{u_0((i+1)\Delta x) - u_0(i\Delta x)}{\Delta x}, \quad \text{for } i\Delta x \le x < (i+1)\Delta x, \tag{2.27}$$

where we assume that u_0 bounded and uniformly continuous. Set $u^{\delta,\Delta x}$ to be the viscosity solution of

$$u_t^{\delta,\Delta x} + H^\delta\left(u_x^{\delta,\Delta x}\right) = 0, \quad u^{\delta,\Delta x}(x,0) = u^{\Delta x}(x).$$

Then for $\eta > \delta > 0$ and $\Delta y > \Delta x > 0$,

$$\begin{aligned}\left\|u^{\delta,\Delta x}(\cdot,T) - u^{\eta,\Delta y}(\cdot,T)\right\|_{L^\infty(\mathbb{R})} &\le \left\|u^{\Delta x} - u^{\Delta y}\right\|_{L^\infty(\mathbb{R})} + T\sup_{|p|\le M}\left|H^\delta(p) - H^\eta(p)\right| \\ &\le \Delta y \left\|u_0\right\|_{\mathrm{Lip}} + \eta\left\|H\right\|_{\mathrm{Lip}}\end{aligned}$$

Thus, the sequence $\left\{u^{\delta,\Delta x}\right\}$ is a Cauchy sequence in L^∞. Since H^δ converges uniformly to H on $[-M,M]$, we can now use Theorem I.2 in [7] to conclude that

$$u(x,t) = \lim_{(\delta,\Delta x)\to 0} u^{\delta,\Delta x}(x,t)$$

is a viscosity solution of

$$u_t + H(u_x) = 0, \quad u(x,0) = u_0(x). \tag{2.28}$$

Also in this case we can use our stability estimate (2.20) and an approximation arguement to extend the class of permissible H, thereby obtaining the convergence in the case where H is Lipschitz continuous.

Now we can state our main theorem

Theorem 2.1. *Let u be the unique viscosity solution of the Hamilton-Jacobi equation* (2.28)*, where u_0 is in* BUC($\mathbb{R}$)*, and let p be the unique entropy solution to the conservation law*

$$p_t + H(p)_x = 0, \tag{2.29}$$

with initial data

$$p(x,0) = \frac{d}{dx} u_0(x). \tag{2.30}$$

Then for $t > 0$ $u_x(x,t) = p(x,t)$ almost everywhere.

Proof. Fix z, by construction we have that

$$\begin{array}{ccccccc} u^{\delta,\Delta x}(x,t) & = & u^{\delta,\Delta x}(z,t) & + & \displaystyle\int_z^x p^{\delta,\Delta x}(y,t)\,dy & & \\ \downarrow & & \downarrow & & \downarrow & & \text{as } (\delta,\Delta x) \to 0, \\ u(x,t) & = & u(z,t) & + & \displaystyle\int_z^x p(y,t)\,dy, & & \end{array}$$

by the Lebesgue convergence theorem. Hence the theorem holds. □

Remark 2.2. As a byproduct of the two stability estimates (2.20) and (2.10) we obtain convergence rate for front tracking for scalar conservation laws and for Hamilton-Jacobi equations. If H is Lipschitz continuous and p_0 is of bounded variation and in $L^1(\mathbb{R})$, then

$$\left\| p^{\delta,\Delta x}(\cdot,T) - p(\cdot,T) \right\|_{L^1(\mathbb{R})} \le \mathrm{Const}_T \cdot (\Delta x + \delta), \tag{2.31}$$

a well known result see, e.g., [15]. If u_0 is in BUC($\mathbb{R}$), then

$$\left\| u^{\delta,\Delta x}(\cdot,T) - u(\cdot,T) \right\|_{L^\infty(\mathbb{R})} \le \mathrm{Const}_T \cdot (\Delta x + \delta). \tag{2.32}$$

If u_0 and H is in C^2, we are able to do a little better, for if $p \in [i\delta, (i+1)\delta]$ then

$$\begin{aligned} \left| H(p) - H^\delta(p) \right| &\le \int_{i\delta}^p \left| H'(\tilde{p}) - H'(z) \right| \,dz \\ &\le \int_{i\delta}^p \int_{z\wedge\tilde{p}}^{z\vee\tilde{p}} \left| H''(y) \right| \,dy\,dz \\ &\le \delta^2 \sup_{|p|\le M} \left| H''(p) \right|. \end{aligned}$$

Similarly

$$\left| u^{\Delta x}(x) - u_0(x) \right| \le \Delta x^2 \sup_x \left| u_0''(x) \right|.$$

Consequently,

$$\left\| u^{\delta,\Delta x}(\cdot,T) - u(\cdot,T) \right\|_{L^\infty(\mathbb{R})} \le \mathrm{Const}_T \cdot (\Delta x^2 + \delta^2), \tag{2.33}$$

thereby explaining the observed convergence rate in Example 2.2.

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