Convergence of a Relaxation Approximation to a Boundary Value Problem for Conservation Laws*

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Abstract

We propose a semilinear relaxation approximation to the unique entropy solutions of an initial boundary value problem for a scalar conservation law. Without any restriction on the initial–boundary data or on the flux function, we prove uniform a priori estimates and convergence of that approximation as the relaxation parameter tends to zero.

Key words: Hyperbolic conservation laws, initial–boundary value problem, relaxation approximation, entropy solutions.

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1 Introduction

In this paper we investigate a hyperbolic relaxation approximation to the scalar conservation law

$$\partial_t u + \partial_x f(u) = 0, \tag{1.1}$$

for $(x,t) \in \mathbb{R}^+ \times (0,T)$ $(T > 0)$ and for a smooth flux function f, with the initial condition

$$u(x,0) = u_0(x), \tag{1.2}$$

for $x \in \mathbb{R}^+$, and a boundary condition

$$u(0,t) = a_0(t), \tag{1.3}$$

for $t \in (0,T)$. For the initial–boundary value problem (1.1)-(1.3), we shall consider the framework of the well-known entropy solutions [11, 2]. Actually a distinguished feature of hyperbolic mixed value problems is that, since in general boundary data are not assumed in a proper sense, the boundary conditions can be formulated only as compatibility conditions, which involve properties both of the flux function and of the data. Therefore a special care has to be used in the approximation to select the right solutions respect to the given compatibility conditions.

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Our approximation will be given by the following system with a singular perturbation parameter $\epsilon > 0$:

$$\begin{aligned} \partial_t u_1 + \lambda_1 \partial_x u_1 &= \tfrac{1}{\epsilon}(M_1(u) - u_1), \\ \partial_t u_2 + \lambda_2 \partial_x u_2 &= \tfrac{1}{\epsilon}(M_2(u) - u_2), \end{aligned} \tag{1.4}$$

for $(x,t) \in \mathbb{R}^+ \times (0,T)$ $(T > 0)$ and for some fixed constant velocities $\lambda_1 < \lambda_2 \in \mathbb{R}$; here $u = u_1 + u_2$. This system will be supplemented by the initial–boundary data:

$$u_i(x,0) = M_i(u_0(x)), \quad i = 1,2 \ , \tag{1.5}$$

for $x \in \mathbb{R}^+$ and

$$u_i(0,t) = M_i(a_0(t)), \text{ for every } i = 1,2 \text{ such that } \lambda_i > 0 \ , \tag{1.6}$$

for $t \in (0,T)$.

The functions M_i, the so–called Maxwellian functions, will be defined only for u varying in some fixed real compact set I by the following formulas:

$$M_1(u) = \frac{\lambda_2 u - f(u)}{\lambda_2 - \lambda_1}, \qquad M_2(u) = \frac{\lambda_1 u - f(u)}{\lambda_1 - \lambda_2}. \tag{1.7}$$

To have stability with respect to the parameter ϵ, we have to introduce a suitable condition. This condition has been first investigated by Whitham [26] and T.P. Liu [13] and follows by considering a Chapman-Enskog type expansions, see also [14]. In the present case we have that, for u varying in a fixed interval I, the condition reads

$$\lambda_1 \le f'(u) \le \lambda_2. \tag{1.8}$$

The main result of our paper is to show that, under condition (1.8), problem (1.4)-(1.6) relaxes to problem (1.1)-(1.3) when the parameter ϵ goes to zero.

Theorem 1.1 *For every fixed initial–boundary data $u_0 \in BV(\mathbb{R}^+)$ and $a_0 \in BV(0,T)$, $(T > 0)$, let $I = \{u \in \mathbb{R}; min\{ess\ \inf(u_0), ess\ \inf(a_0)\} \le u \le max\{ess\ \sup(u_0), ess\ \sup(a_0)\}\}$. If the Maxwellian functions M_i are given by (1.7) and condition (1.8) is verified for $u \in I$, then for every $\epsilon > 0$ there exists a (unique) globally bounded weak solution $(u_1^\epsilon, u_2^\epsilon) \in BV(\mathbb{R}^+ \times (0,T))^2$ to problem (1.4)-(1.6) with $u^\epsilon := u_1^\epsilon + u_2^\epsilon \in I$ for almost every $(x,t) \in \mathbb{R}^+ \times (0,T)$. Moreover*

$$\lim_{\epsilon \to 0} u^\epsilon = u \qquad \text{in } C([0,T], L^1_{loc}(\mathbb{R}^+)),$$

where u is the unique entropy solution to problem (1.1)-(1.3).

Let us mention here that it is easy to state and prove an analogous result for spatially bounded domain.

Relaxation and discrete kinetic approximations, like the approximation presented here, have been proposed and investigated in recent years, starting from the paper [9] by Shi Jin and Zhouping Xin, mainly to find a class of new efficient approximations to entropy solutions for multidimensional systems of conservation laws, see also [15, 1]. The special features of these approximations are analytical simplicity, meaningful physical interpretation and a great numerical flexibility. On the mathematical side a lot of interesting rigorous results have already been proven [28, 6, 24, 14, 15, 7, 21, 10, 4, 5, 3], both for smooth and weak solutions, which assess the general validity of this approach. For a more complete list of references on general hyperbolic relaxation problems and in particular on the relaxation approximation and its numerical aspects, we refer to the Lecture Notes contained in [16].

Let us notice here that more or less all these works are just concerned with the Cauchy problem. Boundary value problems for systems with relaxation are considered in [27, 18] in the case of regular solutions. For more admissibility results as well as an investigation of the stability of boundary layers for problems with relaxation see [12]. Weak solutions are considered in [8] for a particular relaxation problem arising in chemical engineering with boundary condition, by using the compensated compactness methods. Another related approach to initial–boundary value problems is investigated in [19] by considering a kinetic approximation with a continuous set of velocities, in the spirit of [20], in the case of a flux function f either convex or concave. Finally we recall that problem (1.1)-(1.3) was approximated in [17] by using a sequence of quasilinear Goursat problems for f strictly increasing and the initial–boundary data verifying a special compatibility condition.

In [25], Wei-Cheng Wang and Zhouping Xin gave a first fundamental contribution to the study of relaxation approximations for boundary value problems in the general entropy formulation. In that paper the authors considered the relaxation approximation (1.4), (1.5) with $\lambda_2 = -\lambda_1 = \lambda$ and with the following, slightly different boundary condition for $t \in (0, T)$:

$$u(0,t) = u_1(0,t) + u_2(0,t) = a_0(t). \tag{1.9}$$

The convergence result of that paper is obtained under some quite strong restrictions on the data, namely that, given a constant state $\overline{u}$ such that $f'(\overline{u}) \neq 0$, the initial–boundary data have to be a suitable small perturbations of $\overline{u}$ in the norm $\|\cdot\|_\infty + TV(\cdot)$.

In the present paper we deal with a different approximation for the initial–boundary value problem which converges without any restriction on the flux function or on the initial–boundary data. This approximation extends the ideas and the results obtained for the Cauchy problem in [15].

The main difficulty of this problem is related to the particular way in which boundary conditions are taken into account. For this purpose we have to investigate how to minimize the boundary layer effects in the relaxation approximation to recover the entropy formulation of [2] in the limit.

First of all it is clear that the number of conditions to impose at the boundary is equal to the number of positive velocities among λ_1 and λ_2. To treat the case of transonic boundaries, when $f'(u)$ may be equal to zero for some values of u, and in view of condition (1.8), we deal only with the case $\lambda_1 < 0 < \lambda_2$, and then we can only impose one condition on the boundary. The other cases follow by arguing in the same way, but with simpler arguments, see [23].

Let us consider a general boundary condition of the form

$$\alpha u_1 + u_2 = h(t), \tag{1.10}$$

for some constant α and for a suitable function $h(t)$. Following [27], it is possible to show that a necessary condition for problem (1.4), (1.5), (1.10) to be (locally) uniformly well–posed is that $\alpha \neq \lambda_1/\lambda_2$, as far as $f' \neq 0$. Therefore condition (1.6) ($\alpha = 0$) is in principle admissible, even if the situation with discontinuous solutions and a transonic boundary could be different.

As already noticed in [25], due to to the nonlinearity of the compatibility conditions on the boundary, the main problem that we have to solve is how to obtain compactness estimates to pass into the limit, not only in the functions u_1 and u_2, but also into the traces of that functions. Thanks to condition (1.10), it is enough to give a uniform estimate like the following:

$$\int_0^T |u_1^\epsilon(0,t+h) - u_1^\epsilon(0,t)|\, dt \leq Kh, \tag{1.11}$$

for some fixed $K > 0$ and for every $h > 0$ small enough. This estimate implies a uniform bound of the time total variation of the component u_1^ϵ and takes the greater part of the paper [25], just for the case $\alpha = 1$ and it is the reason to require the mentioned strong restrictions on the initial–boundary data.

Let us note that for the parabolic approximation investigated in [2], this kind of estimates were not needed due to the particular features of that problem.

Here we are going to show that the choice $\alpha = 0$ allows us to prove estimate (1.11) without any restriction on the data or on the flux function and much more. In fact we shall prove that, under the sub–characteristic condition (1.8), our approximation (1.4)-(1.6) has some very nice features, already observed in [14, 15] for the Cauchy problem: comparison property, L^1–contraction and an infinite set of kinetic entropy inequalities, which allow to prove very easily the consistency with entropy inequalities in the limit.

Actually all these nice properties hold true also for $|\alpha| < \frac{|\lambda_1|}{\lambda_2}$, but all the proofs become more complicated and in principle greater oscillations of the solutions are allowed, since the stability estimates are amplified by a factor $\left(1 + \frac{|\alpha|\lambda_2}{|\lambda_1| - |\alpha|\lambda_2}\right)$. In particular our estimates are no longer true in the case considered in [25], $\alpha = \frac{|\lambda_1|}{\lambda_2}$, which is a sort of limit case. These facts indicate that the choice $\alpha = 0$ is a good way to minimize the boundary layer effects of this approximation.

The paper is organized as follows: after a short section of preliminaries, in Section 3 we give our compactness estimates and in Section 4 we show the convergence and the consistency of our approximation.

2 Preliminaries

In this section we shall present some backgrounds which will be of use in the following. For every real number a, b, set $I(a,b) = (\min\{a,b\}, \max\{a,b\})$.

Definition 2.1 *Let $u_0 \in BV(\mathbb{R}^+)$ and $a_0 \in BV(0,T)$ ($T > 0$). We say that a function $u \in BV(\mathbb{R}^+ \times (0,T)$ is an entropy solution to the boundary value problem (1.1)-(1.3) if*

i) for any $\phi \in C_0^1(\mathbb{R}^+ \times [0,T))$, with $\phi \geq 0$, and for any $k \in \mathbb{R}$

$$\int_0^T \int_{\mathbb{R}^+} |u-k|\partial_t\phi + \operatorname{sgn}(u-k)(f(u)-f(k))\partial_x\phi \, dxdt$$
$$+ \int_{\mathbb{R}^+} |u_0 - k|\phi \, dx \geq 0; \tag{2.12}$$

ii) for almost any $t \in (0,T)$

$$\max\{ \operatorname{sgn}(u(0,t) - a_0(t))(f(u(0,t)) - f(k)) \, : \, k \in I(a_0(t), u(0,t))\} = 0. \tag{2.13}$$

Here BV stands for the space of the functions of bounded variation and $u(0,t)$ for the trace of the function $u = u(x,t)$ on the boundary, which is well–defined for $u \in BV$, see [2].

Definition 2.2 *Let $u_0 \in BV(\mathbb{R}^+)$, $a_0 \in BV((0,T))$. The couple of functions $({u_1}^\epsilon, {u_2}^\epsilon) \in (BV(\mathbb{R}^+ \times (0,T)) \cap L^\infty(\mathbb{R}^+ \times (0,T)))^2$ is a weak solution to problem (1.4)-(1.6) if*

i) for every $\phi \in C_0^1(\mathbb{R}^+ \times (0,T))$ we have

$$\int_0^T \int_{\mathbb{R}^+} {u_i}^\epsilon(\partial_t\phi + \lambda_i\partial_x\phi) + \frac{1}{\epsilon}(M_i({u_1}^\epsilon + {u_2}^\epsilon) - {u_i}^\epsilon)\phi \, dxdt = 0 \quad i = 1,2;$$

ii) ${u_i}^\epsilon(x,0) = M_i(u_0(x))$ for almost every $x \in \mathbb{R}^+$, $i = 1,2$;

iii) For every $i = 1,2$ such that $\lambda_i > 0$, ${u_i}^\epsilon(0,t) = M_i(a_0(t))$ for almost every $t \in (0,T)$.

Using Kružkov's techniques [11], it is possible to prove the following theorem (see [22]).

Theorem 2.1 *For every $u_0 \in BV(\mathbb{R}^+)$ and $a_0 \in BV(0,T)$ ($T > 0$), there exists an unique bounded entropy solution $u \in BV(\mathbb{R}^+ \times (0,T))$ to problem (1.1)-(1.3). Moreover, given $\tilde{u}_0 \in BV(\mathbb{R}^+)$, $\tilde{a}_0 \in BV(0,T)$ and the bounded entropy solution $\tilde{u}(x,t)$ to the same problem with initial–boundary data $\tilde{u}_0$ and $\tilde{a}_0$ respectively, the following comparison estimate holds true*

$$\int_0^R [\tilde{u}(x,t) - u(x,t)]_+ \, dx \leq \int_0^{R+Nt} [\tilde{u}_0(x) - u_0(x)]_+ \, dx + \int_0^t [\tilde{a}_0 - a_0]_+ \, d\tau,$$

for any fixed $R > 0$ and for almost any $t \in (0,T)$, where $[\cdot]_+$ is the usual positive part and $N := \sup\limits_{|u| \leq max(||u||_{L^\infty}, ||\tilde{u}||_{L^\infty})} |f'(u)|$.

In this paper we shall prove a similar comparison and stability theorem for the solutions to problem (1.4)-(1.6), which will be essential to prove a priori estimates respect to the parameter ϵ (see Proposition 3.1).

Our main tool will be the next result, which is essentially contained in [11].

Lemma 2.1 *Let u be a bounded weak solution to the linear equation*

$$\partial_t u + \lambda\partial_x u = g(x,t) \ ,$$

for $x \in \mathbb{R}$ and $t \in (0,T)$ $(T > 0)$, and v a solution of the same equation obtained replacing respectively the function g with another function h. Then for every $\phi \in C_0^1(\mathbb{R} \times (0,T))$, with $\phi \geq 0$, there holds

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}} [u-v]_+ \partial_t \phi + \lambda [u-v]_+ \partial_x \phi dxdt \\
&\geq - \int_0^T \int_{\mathbb{R}} H(u(x,t) - v(x,t)) \left(g(x,t) - h(x,t)\right) \phi dxdt,
\end{aligned}
\tag{2.14}
$$

where H is the usual Heaviside function.

3 A priori estimates

In the following we shall consider only the case

$$
\lambda_1 < 0 < \lambda_2,
$$

and we shall prove Theorem 1.1 only in this situation, the other cases being easier, see [23].

First, let us state a comparison and stability result for the relaxation problem (1.4)-(1.6), which will be useful in the following

Proposition 3.1 *Fix $\epsilon > 0$ and assume that the Maxwellian functions are given by (1.7) and condition (1.8) is verified on a given interval I. Let $(u_1^\epsilon, u_2^\epsilon)$ and $(\overline{u}_1^\epsilon, \overline{u}_2^\epsilon)$ be two weak solutions to problem (1.4)-(1.6), with initial data (u_0, a_0) and $(\overline{u}_0, \overline{a}_0)$ respectively, and $u^\epsilon := u_1^\epsilon + u_2^\epsilon$, $\overline{u}^\epsilon := \overline{u}_1^\epsilon + \overline{u}_2^\epsilon \in I$ for almost every $(x,t) \in \mathbb{R}^+ \times (0,T)$. Then for every $K \in \mathbb{R}^+$, and almost every $t \in (0,T)$, the following inequalities hold true:*

$$
\int_0^K \sum_{i=1}^2 [u_i^\epsilon(x,t) - \overline{u_i^\epsilon}(x,t)]_+ dx \leq \int_0^{K+|\lambda_1|t} [u_0(x) - \overline{u}_0(x)]_+ dx + \lambda_2 \int_0^t [a_0(\tau) - \overline{a}_0(\tau)]_+ d\tau; \tag{3.15}
$$

$$
|\lambda_1| \int_0^t [u_1^\epsilon(0,\tau) - \overline{u}_1^\epsilon(0,\tau)]_+ \, d\tau \leq \int_0^{|\lambda_1|t} [u_0(x) - \overline{u}_0(x)]_+ \, dx + \lambda_2 \int_0^t [a_0(\tau) - \overline{a}_0(\tau)]_+ \, d\tau. \tag{3.16}
$$

Proof. To simplify the notations we shall omit the dependence on ϵ of the solutions. Let us consider inequality (2.14) for the couple of functions u_1^ϵ ,$\overline{u}_1^\epsilon$ and u_2^ϵ, $\overline{u}_2^\epsilon$. Setting $w_i(x,t) = u_i(x,t) - \overline{u}_i(x,t)$, $r(x,t) = u_1(x,t) + u_2(x,t)$ and $\overline{r}(x,t) = \overline{u}_1(x,t) + \overline{u}_2(x,t)$, we obtain that, for every smooth nonnegative function ϕ, there holds

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^+} \sum_{i=1}^2 [w_i]_+ (\phi_t + \lambda_i \partial_x \phi) dxdt \\
&\geq -\tfrac{1}{\epsilon} \int_0^T \int_{\mathbb{R}^+} \sum_{i=1}^2 (H(w_i)(M_i(r) - M_i(\overline{r}) - w_i)) \phi dxdt.
\end{aligned}
\tag{3.17}
$$

To prove that the right hand side in inequality (3.17) is nonnegative, we first show that

$$\sum_{i=1}^{2}(H(w_i))(M_i(r) - M_i(\overline{r}) - w_i)) \leq 0.$$

First, let us consider the case $H(w_1) = H(w_2) = 1$. We have

$$\sum_{i=1}^{2}(H(w_i))(M_i(r) - M_i(\overline{r}) - w_i)$$

$$= M_1(r) + M_2(r) - M_1(\overline{r}) - M_2(\overline{r}) - u_1(x,t) + \overline{u}_1(x,t) - u_2(x,t) + \overline{u}_2(x,t)$$

$$= u_1(x,t) + u_2(x,t) - \overline{u}_1(x,t) - \overline{u}_2(x,t) - u_1(x,t) + \overline{u}_1(x,t) - u_2(x,t) + \overline{u}_2(x,t) = 0.$$

Since in the case $\operatorname{sgn} w_1 = \operatorname{sgn} w_2 = -1$ we have nothing to prove, we have only to check the case $w_1 > 0$ e $w_2 \leq 0$. This yields:

$$M_1(u_1 + u_2) - M_1(\overline{u}_1 + \overline{u}_2) - |u_1 - \overline{u}_1|$$

$$= M_1'(\theta)(u_1 - \overline{u}_1) + M_1'(\theta)(u_2 - \overline{u}_2) - |u_1 - \overline{u}_1|,$$

where θ is a suitable intermediate value. Since $u_2 \leq \overline{u}_2$ and, from (1.7) and (1.8), $0 \leq M_i'(\theta) \leq 1$ for $i = 1, 2$, it follows that the above expression is non positive. Therefore

$$\int_0^T \int_{\mathbb{R}^+} \sum_{i=1}^{2}[u_i - \overline{u}_i]_+(\phi_t + \lambda_i \partial_x \phi)\, dxdt \geq 0 \tag{3.18}$$

Choosing a sequence of test functions that approximate the characteristic function of the set $\{(x,\tau) \in \mathbb{R}^+ \times (0,t) \ : \ 0 \leq x \leq K + |\lambda_1|(t-\tau)\}$, yields

$$\int_0^K \sum_{i=1}^{2}[u_i^\epsilon(x,t) - \overline{u_i^\epsilon}(x,t)]_+ dx \leq \int_0^{K+|\lambda_1|t} \sum_{i=1}^{2}[M_i(u_0(x)) - M_i(\overline{u}_0(x))]_+ dx$$

$$+ \int_0^t \sum_{i=1}^{2} \lambda_i [u_i(0,\tau)) - \overline{u}_i(0,\tau))]_+ d\tau$$

and the first claim follows.

To prove inequality (3.16) we consider in (3.18) a sequence of test functions that approximate the characteristic function of the set $\{(x,\tau) \in \mathbb{R}^+ \times (0,t) \ : \ 0 < x < |\lambda_1|(t-\tau)\}$. There follows

$$|\lambda_1| \int_0^t [u_1^\epsilon(0,\tau) - \overline{u}_1^\epsilon(0,\tau)]_+ \, d\tau \leq \int_0^{|\lambda_1|t} \sum_{i=1}^{2} [M_i(u_0(x)) - M_i(\overline{u}_0(x))]_+ \, dx$$

$$+ \int_0^t \lambda_2 [M_2(a_0(\tau)) - M_2(\overline{a}_0(\tau))]_+ \, d\tau,$$

which proves (3.16). □

Let us present some consequences of this result.

Corollary 3.1 *Assume that the Maxwellian functions are given by (1.7) and condition (1.8) is verified on a given interval I. Let $(u_1^\epsilon, u_2^\epsilon)$ and $(\overline{u}_1^\epsilon, \overline{u}_2^\epsilon)$ be two weak solutions to problem (1.4)-(1.6), with initial data (u_0, a_0) and $(\overline{u}_0, \overline{a}_0)$ respectively, and $u^\epsilon := u_1^\epsilon + u_2^\epsilon, \overline{u}^\epsilon := \overline{u}_1^\epsilon + \overline{u}_2^\epsilon \in I$ for almost every $(x,t) \in \mathbb{R}^+ \times (0,T)$. Assume that*

$$u_0 \leq \overline{u}_0 \qquad \textit{almost everywhere in } \mathbb{R}^+;$$

$$a_0 \leq \overline{a}_0 \qquad \textit{almost everywhere in } (0,T).$$

Then

$$u_i^\epsilon \leq \overline{u}_i^\epsilon \qquad \textit{almost everywhere in } \mathbb{R}^+ \times (0,T),\ i = 1,2.$$

Corollary 3.2 *In the same assumptions of Proposition 3.1, let $(u_1^\epsilon, u_2^\epsilon)$ be a weak solution to problem (1.4)-(1.6) with initial data (u_0, a_0), then, for almost every $(x,t) \in \mathbb{R}^+ \times (0,T)$,*

$$M_i(min\{ess\ \inf(u_0), ess\ \inf(a_0)\}) \leq u_i^\epsilon \leq M_i(max\{ess\ \sup(u_0), ess\ \sup(a_0)\}),$$

for $i = 1,2$.

Proof. It is enough to observe that for every $p \in \mathbb{R}$, the couple $(M_1(p), M_2(p))$ is a weak solution to problem (1.4)-(1.6), having initial and boundary data identically equal to p. The claim follows by Corollary 3.1. □

By using the standard regularity properties and from the continuation principle for the solutions to problem (1.4)-(1.6), we obtain the following result. The proof is omitted.

Theorem 3.1 *For $u_0 \in BV(\mathbb{R}^+)$ and $a_0 \in BV((0,T))$, let $I = \{u \in \mathbb{R}; min\{ess\ \inf(u_0), ess\ \inf(a_0)\} \leq u \leq max\{ess\ \sup(u_0), ess\ \sup(a_0)\}\}$. Assume the Maxwellian functions M_i are given by (1.7) and condition (1.8) is verified for $u \in I$. Then for every $\epsilon > 0$ there exists a unique global weak solution to problem (1.4)-(1.6), with $u^\epsilon := u_1^\epsilon + u_2^\epsilon \in I$. Moreover, when the data u_0, a_0 are in the class C^k with $u_0(0) = a_0(0)$, $f \in C^{k+1}(\mathbb{R})$, then $u_i{}^\epsilon \in Lip(\mathbb{R}^+ \times [0,T)) \cap C^k(\mathbb{R}^+ \times [0,T) \setminus \gamma_i)$, $i = 1,2$, where $\gamma_i := \{(x,t) \in \mathbb{R}^+ \times [0,T) \ :\ x = \lambda_i t\}$.*

In the following we shall deal only with the case of smooth data, since, thanks to Proposition 3.1, we can extend a priori estimates obtained for regular solution to the general BV case. Also we tacitly assume everywhere that the hypotheses of Theorem 3.1 are always satisfied.

Proposition 3.2 *Let $u_0 \in C^2(\overline{\mathbb{R}}^+) \cap BV(\mathbb{R}^+)$ and $a_0 \in C^2([0,T)) \cap BV((0,T))$, with $u_0(0) = a_0(0)$. For every $t \in [0,T)$, the solution $(u_1^\epsilon, u_2^\epsilon)$ to problem (1.4)-(1.6) verifies the following estimate*

$$\int_{\mathbb{R}^+} |\partial_t u_i^\epsilon(x,t)| dx \leq \max\{|\lambda_1|, \lambda_2\}(TV(u_0, \mathbb{R}^+) + \lambda_2 TV(a_0, (0,T))), \tag{3.19}$$

for $i = 1,2$.

Proof. Fix $L,\, t \in \mathbb{R}^+$, and differentiate both the equations in (1.4) in the t–variable. By standard methods we can multiply the first equation by $\operatorname{sgn}(\partial_t u_1^\epsilon)$ and the second one by $\operatorname{sgn}(\partial_t u_2^\epsilon)$. Set

$$\partial_t u_1^\epsilon = p, \qquad \partial_t u_2^\epsilon = q,$$

where for simplicity we omit the dependence on ϵ in the unknowns. We can take the sum of the equations in (1.4) and integrate in the domain $D_L^t := \{(x,s) \in \mathbb{R}^+ \times (0,t) \ : \ 0 < x < L + |\lambda_1|(t-s)\}$. Therefore, by applying the divergence theorem to our domain, and using again that the contribution on the boundary $(L + |\lambda_1|(t-s), s)$ is non negative, we obtain the inequality

$$\begin{aligned}\int_0^L (|p|+|q|)(x,t)dx &\le \int_0^{L+|\lambda_1|t} (|p|+|q|)(x,0)dx + \int_0^t (\lambda_1|p| + \lambda_2|q|)(0,s)\, ds \\ &+\tfrac{1}{\epsilon}\int_0^t \int_0^{L+|\lambda_1|(t-s)} \big(M_1'(u)(p+q)\operatorname{sgn}(p) - |p| + M_2'(u)(p+q)\operatorname{sgn}(q) - |q|\big)\, dxds,\end{aligned} \tag{3.20}$$

where $u := u_1^\epsilon + u_2^\epsilon$. Let us observe that from (1.7) and condition (1.8), we have

$$\begin{aligned}&|p|(M_1'(u) + M_2'(u)\operatorname{sgn}(p)\operatorname{sgn}(q) - 1)\\ &+|q|(M_2'(u) + M_1'(u)\operatorname{sgn}(p)\operatorname{sgn}(q) - 1) \le 0.\end{aligned}$$

Moreover it is easily seen that

$$|(u_2^\epsilon)_t(0,t)| = |\frac{d}{dt}M_2(a_0(t))|;$$

and

$$|(u_i^\epsilon)_t(x,0)| = |\lambda_i||\frac{d}{dx}M_i(u_0(x))|,$$

for $i = 1, 2$. Therefore from (3.20) we obtain (3.19). $\square$

Let us present now our crucial estimate on the total variation of the traces of the approximating functions u_1^ϵ.

Proposition 3.3 *Let $u_0 \in C^2(\overline{\mathbb{R}}^+) \cap BV(\mathbb{R}^+)$ and $a_0 \in C^2([0,T)) \cap BV((0,T))$, with $u_0(0) = a_0(0)$ and let $(u_1^\epsilon, u_2^\epsilon)$ be the solution to problem (1.4)-(1.6). Then there exists a constant K independent on ϵ such that*

$$\int_0^T |u_1^\epsilon(0,t+h) - u_1^\epsilon(0,t)|\, dt \le Kh, \tag{3.21}$$

for every $h > 0$ small enough.

Proof. To simplify the notations we omit the dependence on ϵ and h. Set

$$w_i(x,t) = u_i(x,t+h) - u_i(x,t) \qquad i = 1, 2,$$

and

$$C(x,t) := \int_0^1 M_1'(s(w_1(x,t)+w_2(x,t)) + u_1(x,t)+u_2(x,t))\,ds.$$

From (1.7) and (1.8) we have that $0 \le C(x,t) \le 1$ for every $(x,t) \in \mathbb{R}^+ \times (0,T)$. Also

$$M_1(u_1(x,t+h)+u_2(x,t+h)) - M_1(u_1(x,t)+u_2(x,t))$$

$$= C(x,t)(w_1(x,t)+w_2(x,t)).$$

Observe that the functions w_i verify the following differential equation

$$\partial_t w_i + \lambda_i \partial_x w_i = \frac{(-1)^i}{\epsilon}((1-C(x,t))w_1 - C(x,t)w_2) \qquad i=1,2. \tag{3.22}$$

Then multiplying equation (3.22) by $\operatorname{sgn}(w_i)$ and summing the equations, we easily obtain

$$\partial_t(|w_1|+|w_2|) + \partial_x(\lambda_1|w_1| + \lambda_2|w_2|) \le 0. \tag{3.23}$$

Integrating (3.23) in the domain $\{(x,s) \in \mathbb{R}^+ \times (0,t) \ : \ 0 < x < |\lambda_1|(t-s)\}$ yields

$$|\lambda_1| \int_0^t |w_1(0,s)|\,ds \le \int_0^{|\lambda_1|t} \sum_{i=1}^2 |w_i(x,0)|\,dx$$

$$+ \int_0^t \lambda_2 |M_2(a_0(s+h)) - M_2(a_0(s))|\,ds + \int_0^t \sum_{i=1}^2 (\lambda_1 - \lambda_i)|w_i(|\lambda_1|(t-s),s)|\,ds.$$

The conclusion follows from Proposition 3.2. $\square$

Proposition 3.4 *In the same assumptions of Proposition 3.3, the following inequality holds true*

$$\int_{\mathbb{R}^+} |\partial_x u_i^\epsilon(x,t)| dx \le \overline{K}, \tag{3.24}$$

for $i = 1,2$, *where* $\overline{K}$ *is a positive constant depending only on* λ_2, $TV(a_0,\mathbb{R}^+)$ *and* $TV(u_0,\mathbb{R}^+)$

Proof. As in the proof of Proposition 3.2, we take the x–derivative of both the equations in (1.4) and then multiply respectively the first one by $\operatorname{sgn}(\partial_x u_1^\epsilon)$ and the second one by $\operatorname{sgn}(\partial_x u_2^\epsilon)$. Set

$$\partial_x u_1^\epsilon = m, \qquad \partial_x u_2^\epsilon = n.$$

Summing up and integrating, for every fixed L, $t \in \mathbb{R}^+$, on the domain $D_L^t = \{(x,\tau) \in \mathbb{R}^+ \times (0,t) \ : \ 0 < x < L + |\lambda_1|(t-\tau)\}$, yields

$$\int_0^L (|m|+|n|)(x,t)dx \le \int_0^{L+|\lambda_1|t} (|m|+|n|)(x,0)dx + \int_0^t (\lambda_1|m| + \lambda_2|n|)(0,\tau)d\tau \ .$$

Observe that

$$|\partial_x u_i^\epsilon(x,0)| = |\frac{d}{dx} M_i(u_0(x))|,$$

for $i = 1, 2$, and there holds the equality

$$\partial_t(u^1_\epsilon + u^2_\epsilon) + \partial_x(\lambda_1 u^\epsilon_1 + \lambda_2 u^\epsilon_2) = 0.$$

Therefore we have

$$\int_0^t (\lambda_1|m| + \lambda_2|n|)(0,\tau)\,d\tau = \int_0^t (-|\lambda_1 m| + |\lambda_2 n|)(0,\tau)\,d\tau$$

$$\leq \int_0^t (|\lambda_1 m + \lambda_2 n|)(0,\tau)\,d\tau = \int_0^t (|\partial_t(u^1_\epsilon + u^2_\epsilon)|)(0,\tau)\,d\tau \leq TV(a_0, \mathbb{R}^+) + K,$$

where K is given in Proposition 3.3. This concludes the proof. $\square$

The following result is proved in [15] for the Cauchy problem (1.4)–(1.5) and can be transposed with no changes to the boundary value problem (1.4)-(1.6).

Proposition 3.5 *Under the same assumptions of Proposition 3.3, there exists a constant $C > 0$, such that for every $L \in \mathbb{R}^+$, $t \in (0,T)$ and $\epsilon > 0$ there holds*

$$\int_0^L \sum_{i=1}^2 |(M_i(u^\epsilon_1 + u^\epsilon_2) - u^\epsilon_i)(x,t)|dx \leq C\epsilon\ .$$

As a consequence of the above results, we obtain the following uniform bounds.

Theorem 3.2 *Let $u_0 \in BV(\mathbb{R}^+)$, $a_0 \in BV(0,T)$. Then the solution $(u^\epsilon_1, u^\epsilon_2)$ to problem (1.4)-(1.6) satisfies the following uniform estimates for every $t \in (0,T)$:*

$$TV(u^\epsilon_i(\cdot,t)) \leq C;$$

$$\int_{\mathbb{R}} |u^\epsilon_i(x,t+h) - u^\epsilon_i(x,t)|dx \leq C|h|;$$

$$\int_0^L \sum_{i=1}^2 |(M_i(u^\epsilon_1 + u^\epsilon_2) - u^\epsilon_i)(x,t)|dx \leq C\epsilon;$$

$$TV(u^\epsilon_i(0,\cdot);(0,T)) \leq C,$$

for every $h \in \mathbb{R}$ and $i = 1, 2$, where C is a positive constant only depending on the total variation of the data and on λ_2.

4 Convergence results

In this Section we prove Theorem 1.1. First we need to show that our solutions satisfy an infinite set of "kinetic" entropy inequalities, see [20, 15].

Lemma 4.1 *Let $u_0 \in BV(\mathbb{R}^+)$, $a_0 \in BV(0,T)$ and let $(u_1^\epsilon, u_2^\epsilon)$ be the weak solution to problem (1.4)-(1.6). Then for every $\phi \in C_0^1(\mathbb{R}^+ \times [0,T))$, with $\phi \geq 0$, and for every $k \in \mathbb{R}$, there holds*

$$\int_0^T \int_{\mathbb{R}^+} \sum_{i=1}^2 |u_i^\epsilon - M_i(k)|(\partial_t \phi + \lambda_i \partial_x \phi)\, dxdt$$

$$+ \int_{\mathbb{R}^+} \sum_{i=1}^2 |M_i(u_0) - M_i(k)| \phi(x,0)\, dx + \int_0^T \sum_{i=1}^2 \lambda_i |u_i^\epsilon - M_i(k)| \phi(0,t)\, dt \tag{4.25}$$

$$\geq \tfrac{1}{\epsilon} \int_0^T \int_{\mathbb{R}^+} (\sum_{i=1}^2 |u_i^\epsilon - M_i(k)| - |u_1^\epsilon + u_2^\epsilon - k|)\phi\, dxdt \geq 0.$$

Proof. Multiplying (1.4) by $\operatorname{sgn}(u_i^\epsilon - M_i(k))\phi(x,t)$, summing and integrating in the domain $\mathbb{R}^+ \times (0,T)$, one obtains

$$\int_0^T \int_{\mathbb{R}^+} \sum_{i=1}^2 |u_i^\epsilon - M_i(k)|(\partial_t \phi + \lambda_i \partial_x \phi)\, dxdt + \int_{\mathbb{R}^+} \sum_{i=1}^2 |M_i(u_0) - M_i(k)| \phi(x,0)\, dx$$

$$+ \int_0^T \sum_{i=1}^2 \lambda_i |u_i^\epsilon(0,t) - M_i(k)| \phi(0,t)\, dt$$

$$= \tfrac{1}{\epsilon} \int_0^T \int_{\mathbb{R}^+} (\sum_{i=1}^2 \operatorname{sgn}(u_i^\epsilon - M_i(k))(u_i^\epsilon - M_i(u_1^\epsilon + u_2^\epsilon))\phi\, dxdt$$

$$= \tfrac{1}{\epsilon} \int_0^T \int_{\mathbb{R}^+} (\sum_{i=1}^2 ((|u_i^\epsilon - M_i(k)| - \operatorname{sgn}(u_i^\epsilon - M_i(k))(M_i(u_1^\epsilon + u_2^\epsilon) - M_i(k)))\phi\, dxdt.$$

The conclusion follows, since

$$\textstyle\sum_{i=1}^2 \operatorname{sgn}(u_i^\epsilon - M_i(k))(M_i(u_1^\epsilon + u_2^\epsilon) - M_i(k)) \leq \sum_{i=1}^2 |M_i(u_1^\epsilon + u_2^\epsilon) - M_i(k)|$$

$$\textstyle= |u_1^\epsilon + u_2^\epsilon - k| \leq \sum_{i=1}^2 |u_i^\epsilon - M_i(k)|. \ \square$$

Proof of Theorem 1.1 The first part of the statement follows by Theorem 3.1. Furthermore, thanks to Theorem 3.2 we can extract a subsequence $(u_1^{\epsilon_n}, u_2^{\epsilon_n})$ converging in the norm $C([0,T)\,,\, L^1_{loc}(\mathbb{R}^+))^2$ to a limit (u_1, u_2). Moreover, using Proposition 3.5, we have that $u_i = M_i(u)$, for $i = 1, 2$. Now, from Proposition 3.3, we can extract from the sequence $u_1^{\epsilon_n}(0,\cdot)$ a further subsequence, denoted again by $u_1^{\epsilon_n}$, which converges in $L^1(0,T)$ to a limit function w_1. Letting ϵ to zero in (4.25), we obtain for every $\phi \in C_0^1(\mathbb{R}^+ \times [0,T))$, with $\phi \geq 0$, and every $k \in \mathbb{R}$,

$$\int_0^T \int_{\mathbb{R}^+} \sum_{i=1}^{2} |M_i(u) - M_i(k)|(\partial_t \phi + \lambda_i \partial_x \phi)\, dxdt + \int_{\mathbb{R}^+} \sum_{i=1}^{2} |M_i(u_0) - M_i(k)|\phi(x,0)\, dx$$

$$+ \int_0^T \lambda_1 |w_1 - M_1(k)|\phi(0,t)\, dt + \int_0^T \lambda_2 |M_2(a_0) - M_2(k)|\phi(0,t)\, dt \geq 0.$$

From (1.7) we obtain, for every $k \in \mathbb{R}$ and every $\phi \in C_0^1(\mathbb{R}^+ \times [0,T))$, with $\phi \geq 0$,

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^+} |u-k|\partial_t \phi + \operatorname{sgn}(u-k)(f(u) - f(k))\partial_x \phi\, dxdt + \int_{\mathbb{R}^+} |u_0 - k|\phi(x,0)\, dx \geq \\ &- \int_0^T \lambda_1 |w_1 - M_1(k)|\phi(0,t)\, dt - \int_0^T \lambda_2 |M_2(a_0) - M_2(k)|\phi(0,t)\, dt. \end{aligned} \tag{4.26}$$

Now, choosing $\phi \in C_0^1((0,\infty) \times [0,\infty))$, it follows that u is an entropy solution of the conservation law (1.1), which assumes the initial data, i.e.: verifies i) of Definition 2.1. To check the boundary condition (2.13), take

$$\phi(x,t) = \xi(t) \max\{0, 1 - \frac{x}{\eta}\}, \qquad \eta \in \mathbb{R}^+ ,$$

and let η tend to zero. There holds

$$\operatorname{sgn}(u(0,t) - k)(f(u(0,t)) - f(k)) \leq \lambda_1 |w_1 - M_1(k)| + \lambda_2 |M_2(a_0) - M_2(k)| \tag{4.27}$$

for every $k \in \mathbb{R}$ and almost every $t \in \mathbb{R}^+$. A first elementary consequence of (4.27) is the equality

$$f(u(0,t)) = \lambda_2 M_2(a_0(t)) + \lambda_1 w_1(t). \tag{4.28}$$

Now let us prove that (4.27) and (4.28) imply (2.13). Suppose that $u(0,t) \geq a_0(t)$. In such situation we have to prove that

$$f(u(0,t)) \leq f(k), \qquad \text{for every } k \in [a_0(t), u(0,t)]. \tag{4.29}$$

There are now two possibilities. Let $w_1(t) \leq M_1(k)$. From (4.27) we obtain

$$\begin{aligned} f(u(0,t)) - f(k) &\leq \lambda_1(M_1(k) - w_1(t)) + \lambda_2(M_2(k) - M_2(a_0(t)) \\ &= f(k) - f(u(0,t)) \end{aligned}$$

which gives (4.29).

Otherwise, let $w_1(t) \geq M_1(k)$. Then $\lambda_1 w_1(t) \leq \lambda_1 M_1(k)$. Using (4.28), this implies again inequality (4.29). The case $a_0(t) > u(0,t)$ follows from similar arguments. The conclusion follows since, by the uniqueness of the limit solution, all the sequence converges. □

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