Diffusive BGK Approximations for Nonlinear Multidimensional Parabolic Equations*

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Abstract

We introduce a class of discrete velocity BGK type approximations to multidimensional scalar nonlinearly diffusive conservation laws. We prove the well-posedness of these models, a priori bounds and kinetic entropy inequalities that allow to pass into the limit towards the unique entropy solution recently obtained by Carrillo. Examples of such BGK models are provided.

Key-words: Hyperbolic and parabolic nonlinear conservation laws – singular perturbation problems – BGK models – kinetic entropies

1 Introduction

Consider the Cauchy problem for the possibly degenerate parabolic equation

$$\partial_t u + \sum_{j=1}^{d} \partial_{x_j}[A_j(u)] = \Delta_x[B(u)], \qquad (x,t) \in \mathbb{R}^d \times]0,\infty[, \tag{1.1}$$

with initial data

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^d. \tag{1.2}$$

We assume that the functions A_j and B are Lipschitz continuous and $B' \geq 0$. The aim of this paper is to present a new class of approximations for equation (1.1) which satisfy some important requirements:

*Work partially supported by European TMR project HCL # ERBFMRXCT960033

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i) to have the same framework in both the hyperbolic ($B' \equiv 0$) and the parabolic ($B' > 0$) regime;

ii) to converge to the weak solutions to problem (1.1), and when it is needed also to the admissible solutions in the sense of Carrillo [11]. In this regards we recall that, even for smooth initial data, it is well known that a weak solution can lose its Lipschitz continuity (respectively, continuity) in finite time if $B' = 0$ at a given point (respectively, on some open interval). It is very important to design theoretical and numerical approximations which are able to take into account this kind of singularities;

iii) to keep the monotonicity and comparison properties, which are typical of equation (1.1).

Some other features seem to be advisable for a good approximation:

iv) an easy numerical implementation, even in the multidimensional case and in more complicate situations (source terms, variable coefficients...);

v) possibility to improve numerical schemes by using high order algorithms and different methods (finite differences, finite volumes...);

vi) a natural extension to general (strongly coupled) systems of diffusive conservation laws.

In the present paper we do not consider these three last issues, which will be investigated in [3] for numerical purposes and from an analytical point of view in [24] for the last point.

Let us now introduce our general kinetic BGK framework. Consider the following class of semilinear systems of conservation laws with source,

$$\partial_t f_i^\epsilon + \sum_{j=1}^{d} \gamma_{ij}^\epsilon \partial_{x_j} f_i^\epsilon = \frac{1}{\epsilon}\left(M_i(\epsilon, u^\epsilon) - f_i^\epsilon\right), \quad i = 1, \dots, N, \tag{1.3}$$

with initial condition

$$f_i^\epsilon(x, 0) = f_{0i}^\epsilon(x), \quad i = 1, \dots, N, \tag{1.4}$$

where

$$u^\epsilon(x,t) = \sum_{i=1}^{N} f_i^\epsilon(x,t), \tag{1.5}$$

and

$$\gamma_{ij}^\epsilon := \lambda_{ij} + \frac{\theta_{ij}}{\sqrt{\epsilon}} \tag{1.6}$$

for some fixed constants $\lambda_i = (\lambda_{i1}, \dots, \lambda_{id})$, $\theta_i = (\theta_{i1}, \dots, \theta_{id}) \in \mathbb{R}^d$ for $i = 1, \dots, N$, and for a suitable function $M : [0,1] \times \mathbb{R} \to \mathbb{R}^N$.

In order to prove the convergence we make the following assumptions on the parameters $\lambda_{ij}, \theta_{ij}$ and on the function M on some fixed interval I:

(M_1) $\sum_{i=1}^N M_i(\epsilon, w) = w$ for all $\epsilon \in]0,1]$ and for all $w \in I$;

(M_2) $\sum_{i=1}^N \gamma_{ij}^\epsilon M_i(\epsilon, w) = A_j(w)$ for all $\epsilon \in]0,1]$ and for all $w \in I$;

(M_3) $\sum_{i=1}^N \theta_{ij}\theta_{ik} M_i(0, w) = \delta_{jk} B(w)$ for all $w \in I$;

(M_4) $M_i(\epsilon, w) \to M_i(0, w)$ when $\epsilon \to 0$, uniformly for $w \in I$.

If M satisfies the above conditions we call it a local Maxwellian Function for equation (1.1) on the interval I. Let us remark that in what follows we need to use the decay at zero of the solution as $|x| \to \infty$ and therefore it is more convenient to normalize the maxwellian functions to have $M(\epsilon, 0) = 0$ for all $\epsilon \in [0,1]$.

It is easy to see that if the sequence u^ϵ converges to some limit function u in a suitable (strong) topology, then the limit function is a weak solution to equation (1.1). The rigorous proof of this fact will be given below (see Theorems 3.1 and 4.1). Here we prefer to give a more intuitive, although formal, justification. Setting $v_j^\epsilon := \sum_{i=1}^N \gamma_{ij}^\epsilon f_i^\epsilon$, we have

$$\partial_t u^\epsilon + \sum_{j=1}^d \partial_{x_j} v_j^\epsilon = 0,$$
$$\partial_t v_j^\epsilon + \sum_{k=1}^d \partial_{x_k} \sum_{i=1}^N \gamma_{ik}^\epsilon \gamma_{ij}^\epsilon f_i^\epsilon = \frac{1}{\epsilon}\left(A_j(u^\epsilon) - v_j^\epsilon\right),$$

for $j = 1, \dots, d$. Therefore we deduce that

$$v_j^\epsilon = A_j(u^\epsilon) - \sum_{i=1}^N \sum_{k=1}^d \theta_{ij}\theta_{ik} \partial_{x_k} f_i^\epsilon + O(\sqrt{\epsilon}),$$

and in the limit, using the condition (M_3), we find equation (1.1). Notice that a natural convergence rate in (M_4) would be as $\epsilon^{1/2}$.
We recall that semilinear systems of conservation laws with a finite number of velocities were introduced in [33] to give approximate solutions to scalar conservation laws in several space dimensions. According to [33], we need also a stability condition for (1.3), namely

$$w \mapsto M_i(\epsilon, w) \quad \text{is nondecreasing in } I \text{ for any } i = 1, ..., N,\ \epsilon \in]0,1]. \tag{1.7}$$

Maxwellian functions satisfying this will be called monotone Maxwellian functions (MMF).

We will prove that if M is a MMF on $I = \{u \in \mathbb{R} : |u| \leq \|u_0\|_\infty\}$ and $f_0^\epsilon(x) = M(\epsilon, u_0)$ then $\{u^\epsilon\}$ converges in $C([0,\infty); L^1_{loc}(\mathbb{R}^d))$ to the unique entropy solution to (1.1)-(1.2). By the way we obtain a new proof of existence of entropy solutions to (1.1)-(1.2) in the sense of [11], see also Section 4 below. The same tools as in [33] are involved, but we need here to take care of the singular linear transport term.

Let us recall now some basic facts about nonlinear parabolic and hyperbolic problems. It is well-known that these problems are in general not well–posed since, if the diffusion coefficient is not uniformly parabolic, i.e. $B'(u) = 0$ for some values of u, the solutions can lose their regularity in finite time and general continuations in the weak sense do not need to be unique. Nevertheless if the diffusion coefficient is not too degenerate, that is when $meas\{u|\, B'(u) = 0\} = 0$, the weak solutions are in general continuous and unique, see [40]. Another case where uniqueness of weak solutions can be established without any additional assumption, is when the first order terms A_j are identically equal to zero, as proved in [10].

On the other hand, when there exists some positive measure set where $B'(u) = 0$, the problem becomes strongly degenerate, solutions are in general discontinuous and there is no uniqueness in the class of bounded measurable solutions. In this case, following the ideas by S.N. Kružkov we have to add some supplementary conditions, the so–called *entropy conditions*, in order to select a unique solution, with "good" mathematical properties.. This program has been carried out successfully by Kružkov himself for the pure hyperbolic case $B \equiv 0$ in [21], but for many years the general parabolic case was not completely solved. Actually it is quite easy to prove existence of entropy solutions to problem (1.1)-(1.2 by the artificial viscosity method [38], nonlinear semigroup methods [6] or monotone finite difference schemes [14]. On the contrary, uniqueness has been proved only recently by Wu and Yin [39], for BV solutions in the one dimensional case and finally by Carrillo [11] in the general setting.

In the last years a lot of papers were devoted to the study of the zero relaxation limit of various hyperbolic systems [27, 13, 20, 32, 33], see [34] and references therein. A general framework for discrete velocities approximations was proposed in [20, 2] to build numerical schemes for systems of conservation laws. Entropy conditions have been studied in [8] and a proof of convergence for systems with a convex invariant domain has been done in [37].

As concerning the nonlinear parabolic problems in a kinetic setting, the first idea is just to add a linear diffusion term in the transport equation, to have a sort of parabolic BGK approximation, see [9, 25] for some continuous velocities model and, more recently, [16] for a discrete velocities model.

Here we consider a quite different scaling to approach (quasilinear) parabolic problems by means of (semilinear) hyperbolic ones. On a physical ground this scaling corresponds to a "parabolic" scaling of the (x,t) variables, like $(\frac{x}{\epsilon}, \frac{t}{\sqrt{\epsilon}})$, in the kinetic equations, see for example [12], Ch. 11, and references therein for a discussion of the diffusive limits of the Boltzmann equation. Actually in our approach we are able to obtain in the limit both the hyperbolic and parabolic problems. Here, we just give an incomplete list of papers where diffusive limits have been investigated, see [13, 18, 19, 5, 28, 29].

For discrete velocities model, theoretical investigations start with the papers [22, 30] for the diffusive limit of Carleman equations. The general theory of convergence to equilibrium for two velocities models can be found in [26]. For some recent numerical investigations we refer to [23, 31, 15].

The paper is organized as follows. In Section 2 we obtain existence, $L^\infty-$bounds independent of ϵ and comparison theorems for solutions to problem (1.3)-(1.4). The condition (1.7) will be crucial. In Section 3 we prove the compactness of the sequence $\{f^\epsilon\}$ and then in Section 4 the convergence result with the entropy inequalities. Finally, in Section 5 we present different examples of relaxation approximations to equation (1.1).

2 Global existence and uniform estimates

In this section we shall prove that if M is a Monotone Maxwellian Function on a suitable interval I, depending on the L^∞ norm of the initial data, then the Cauchy problem (1.3)-(1.4) has a globally bounded solution. We observe that to prove the results of this section we do not need assumptions (M_3) and (M_4). Actually these results were already proved in [33] and are reported here to make the paper self-contained.

First let us recall some general results concerning semilinear diagonal hyperbolic systems [17]. Consider the Cauchy problem

$$\partial_t f + \sum_{j=1}^{d} \Lambda_j \partial_{x_j} f = G(f) \tag{2.1}$$

with the initial condition

$$f(x,0) = f_0(x), \tag{2.2}$$

where $f := (f_1, f_2, \dots, f_N) : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^N, \Lambda_j = diag(\lambda_{1j}, \dots, \lambda_{Nj}) \in \mathcal{M}^{N\times N}$, the function $G(f) = (g_1(f), \dots, g_N(f)) : \mathbb{R}^N \to \mathbb{R}^N$ is a given locally Lipschitz continuous function and $f_0 = (f_{01}, \dots, f_{0N}) \in L^\infty(\mathbb{R}^d)^N$.

Definition 2.1 *A function $f \in L^\infty(\mathbb{R}^d \times (0,T))^N (T > 0)$ is a weak solution to the Cauchy problem (2.1)-(2.2) if, for all $\varphi \in C_0^\infty(\mathbb{R}^d \times (0,T))$ and every $i = 1, \dots, N$, it holds*

$$\int_0^T \int_{\mathbb{R}^d} \left[f_i \left(\partial_t \varphi + \sum_{j=1}^{d} \lambda_{ij} \partial_{x_j} \varphi \right) + g_i(f)\varphi \right] dxdt = 0, \tag{2.3}$$

and, for any open set $\Omega \subseteq \mathbb{R}^d$,

$$\lim_{T\to 0^+} \frac{1}{T}\int_0^T \int_\Omega |f_i(x,t) - f_{0i}(x)|\, dxdt = 0.$$

Proposition 2.1 *For any* $f_0 \in L^\infty(\mathbb{R}^d)^N$*, there exists* $T > 0$ *(only depending on* $\|f_0\|_\infty$*) such that there exists a unique weak solution* f *to (2.1)-(2.2) in* $\mathbb{R}^d \times (0,T)$ *and* $f \in C([0,T); L^1_{loc}(\mathbb{R}^d)^N)$*. Moreover, there are only two possibilities: either* $f \in L^\infty(\mathbb{R}^d \times (0,T))^N$ *for any* $T > 0$*, or there exists* $T^* < +\infty$ *such that, for any* $T < T^*$*,* f *is defined on* $\mathbb{R}^d \times (0,T)$ *and*

$$\lim_{T\to T^{*-}} \|f\|_{L^\infty(\mathbb{R}^d\times(0,T))} = +\infty.$$

To have monotonicity properties of the system (2.1) we require the quasimonotonicity of the source term G [17, 36, 35], which is one of the main tools in the present paper.

Proposition 2.2 *Let* f *and* $\hat f$ *be two weak solutions to problem (2.1)-(2.2) in* $\mathbb{R}^d \times (0,T)$ *for the initial data* f_0 *and* $\hat f_0$*, respectively. Let* $Q \subseteq \mathbb{R}^N$ *be an interval (with non empty interior) such that the following quasimonotonicity condition holds:*

each component g_i *of* G *is non decreasing in* f_j*, for* $i \neq j$*, for any* $f \in Q$*.*

If $f, \hat f \in Q$ *a.e. in* $\mathbb{R}^d \times (0,T)$ *and* $f_0 \leq \hat f_0$ *for almost every* $x \in \mathbb{R}^d$*, then* $f \leq \hat f$ *for almost every* $(x,t) \in \mathbb{R}^d \times (0,T)$*.*

Now we can present our first result.

Theorem 2.1 *Let* $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ *be fixed and set* $f_0^\epsilon = M(\epsilon, u_0)$*. Assume that* M *is a MMF on the interval* $I := \{u \in \mathbb{R} : |u| \leq \|u_0\|_\infty\}$*. Then, for all* $\epsilon \in]0,1]$*, there exists a unique global solution* $f^\epsilon \in C([0,\infty); L^1_{loc}(\mathbb{R}^d))^N \cap L^\infty(\mathbb{R}^d \times]0,\infty[)^N$ *to the Cauchy problem (1.3)-(1.4). Moreover the following estimates hold:*

$$M_i(\epsilon, -\|u_0\|_\infty) \leq f_i^\epsilon \leq M_i(\epsilon, \|u_0\|_\infty), \qquad i = 1,\dots,N; \tag{2.4}$$

$$\|u^\epsilon\|_\infty \leq \|u_0\|_\infty. \tag{2.5}$$

Proof. Set

$$M_i^\#(\epsilon,u) = \begin{cases} M_i(\epsilon, -\|u_0\|_\infty), & u < -\|u_0\|_\infty, \\ M_i(\epsilon, u), & |u| \leq \|u_0\|_\infty, \\ M_i(\epsilon, \|u_0\|_\infty)), & u > \|u_0\|_\infty, \end{cases}$$

for $i = 1,\dots,N$. Assume u_0 is a smooth function and let $f^\# \in Lip_{loc}(\mathbb{R}^d \times (0,\infty))^N$ be the corresponding (smooth) solution to problem (1.3)–(1.4) with M replaced by $M^\#$ and $f_0^\epsilon = M(\epsilon, u_0)$. Let

$$T^\delta = sup\{T \geq 0 : |u^{\epsilon\#}(x,t)| \leq \|u_0\|_\infty + \frac{\delta}{2}, \text{ a.e. in } \mathbb{R}^d \times (0,T)\},$$

where $u^{\epsilon\#} = \sum_{i=1}^N f_i^{\epsilon\#}$. Since the functions $M_i^\#$ are monotone nondecreasing on $\mathbb{R}$, the system (1.3), with $M^\#$, verifies the assumptions in Proposition 2.2. The solutions to the associated ordinary differential problem

$$\begin{cases} \dot{p}_{\epsilon i}^{\pm}(t) = \frac{1}{\epsilon}(M_i^\#(\epsilon, v_\epsilon^\pm) - p_{\epsilon i}^\pm), \\ p_{\epsilon i}^\pm(0) = M_i(\epsilon, \pm\|u_0\|_\infty), \end{cases}$$

for $i = 1, \ldots, N$ and $v_\epsilon^\pm = \sum_{i=1}^N p_{\epsilon i}$, verify

$$v_\epsilon^\pm(t) = v_\epsilon^\pm(0) = \sum_{i=1}^N M_i(\epsilon, \pm\|u_0\|_\infty) = \pm\|u_0\|_\infty,$$
$$p_{\epsilon i}^\pm(t) = M_i(\epsilon, \pm\|u_0\|_\infty), \quad i = 1, \ldots, N.$$

Then, by Proposition 2.2 and standard continuation arguments we obtain the claim for $f^\#$ and $u^\#$. Since $M = M^\#$ on the range of $u^\#$, by density arguments we obtain the conclusion. $\square$

Under the hypothesis of the previous theorem we have the following contraction property in L^1.

Theorem 2.2 *Under the assumptions of Theorem 2.1, let $\hat{f}^\epsilon$ be another global solution with the same properties, associated to the initial condition $\hat{f}_0^\epsilon$. Then*

(a) if $f_0^\epsilon(x) \le \hat{f}_0^\epsilon(x)$ for almost every $x \in \mathbb{R}^d$, then $f^\epsilon(x,t) \le \hat{f}^\epsilon(x,t)$ for almost every $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$;

(b) for any $0 \le s \le t$

$$\sum_{i=1}^N \int_{\mathbb{R}^d} |f_i^\epsilon(x,t) - \hat{f}_i^\epsilon(x,t)|\, dx \le \sum_{i=1}^N \int_{\mathbb{R}^d} |f_i^\epsilon(x,s) - \hat{f}_i^\epsilon(x,s)|\, dx. \tag{2.6}$$

Proof. The claim in (a) can be obtained as in the previous proof, by using Proposition 2.2. To prove (b) we use the Duhamel's formula

$$f_i^\epsilon(x,t) = e^{-\frac{t-s}{\epsilon}} f_i^\epsilon(x - \gamma_i^\epsilon(t-s), s) + \frac{1}{\epsilon}\int_s^t e^{-\frac{t-\tau}{\epsilon}} M_i(\epsilon, u^\epsilon(x - \gamma_i^\epsilon(t-\tau), \tau))\, d\tau$$

for $i = 1, \ldots, N$, $x \in \mathbb{R}^d$, $t \ge s \ge 0$, where γ_i^ϵ is the vector in $\mathbb{R}^d$ with components γ_{ij}^ϵ. A similar formula holds for $\hat{f}$. Then multiplying by the function $\mathrm{sgn}(f_i^\epsilon - \hat{f}_i^\epsilon)$, integrating and taking the sum on the index $i = 1, \ldots, N$ we obtain

$$\sum_{i=1}^N \int_{\mathbb{R}^d} |f_i^\epsilon(x,t) - \hat{f}_i^\epsilon(x,t)|\, dx \le e^{-\frac{t-s}{\epsilon}} \sum_{i=1}^N \int_{\mathbb{R}^d} |f_i^\epsilon(x,s) - \hat{f}_i^\epsilon(x,s)|\, dx$$
$$+ \int_s^t \frac{e^{-\frac{t-\tau}{\epsilon}}}{\epsilon} \int_{\mathbb{R}^d} \sum_{i=1}^N |M_i(\epsilon, u^\epsilon(x,\tau)) - M_i(\epsilon, \hat{u}^\epsilon(x,\tau))|\, dx\, d\tau.$$

Since $u^\epsilon, \hat{u^\epsilon} \in I$, the monotonicity property of M yields

$$\sum_{i=1}^{N} |M_i(\epsilon, u^\epsilon) - M_i(\epsilon, \hat{u^\epsilon})| \leq \sum_{i=1}^{N} |f_i^\epsilon - \hat{f}_i^\epsilon|.$$

Now, setting $\mu^\epsilon(t) := \int_{\mathbb{R}^d} \sum_{i=1}^N |f_i^\epsilon(x,t) - \hat{f}_i^\epsilon(x,t)|\, dx$, we have for $0 \leq s \leq t$,

$$\mu^\epsilon(t) \leq e^{-\frac{t-s}{\epsilon}} \mu^\epsilon(s) + \frac{1}{\epsilon} \int_s^t e^{-\frac{t-\tau}{\epsilon}} \mu^\epsilon(\tau)\, d\tau$$

which implies the conclusion. $\square$

3 Convergence

In this section we prove the convergence, as $\epsilon \to 0$, of a subsequence of $u^\epsilon = \sum_{i=1}^N f_i^\epsilon$ to a weak solution u to the Cauchy problem (1.1)-(1.2). Moreover we prove that the function $f = \lim_{\epsilon \to 0} f^\epsilon$ is a Maxwellian distribution, $M(0,u) = f$. The proof of the convergence theorem will follow after some preliminary results.

As a direct consequence of Theorem 2.2 we have the following proposition.

Proposition 3.1 *Let $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ be fixed and set $f_0^\epsilon = M(\epsilon, u_0)$. Assume that M is a MMF on the interval $I := \{u \in \mathbb{R} : |u| \leq \|u_0\|_\infty\}$ and let f^ϵ be the solution to problem (1.3)-(1.4). Then there exists a positive constant $h_0 > 0$ and a continuous nondecreasing function $\omega \in C([0, h_0])$, not depending on ϵ and with $\omega(0) = 0$, such that, for every $t \geq 0$*

$$\int_{\mathbb{R}^d} \sum_{i=1}^{N} |f_i^\epsilon(x+h,t) - f_i^\epsilon(x,t)|\, dx \leq \omega(|h|), \tag{3.1}$$

for any $h \in \mathbb{R}^d$, $|h| \leq h_0$.

As regard to equicontinuity in t, we prove the following two propositions.

Proposition 3.2 *Under the assumptions of Proposition 3.1, let $u^\epsilon = \sum_{i=1}^N f_i^\epsilon$. Then there exists a positive constant τ_0 and a continuous nondecreasing function $\overline{\omega} : [0, \tau_0] \to \mathbb{R}$, not depending on ϵ, with $\overline{\omega}(0) = 0$, such that for $0 \leq t \leq t + \tau, \tau \in (0, \tau_0)$,*

$$\int_{\mathbb{R}^d} |u^\epsilon(x,t+\tau) - u^\epsilon(x,t)| dx \leq \overline{\omega}(\tau). \tag{3.2}$$

Proof. Thanks to Proposition 2.2 and Lemma 5 by Kružkov [21], which in particular gives the condition to transport the control of the space modulus of continuity in L^1 to time equicontinuity with values in L^1, it is enough to prove that

$$\left| \int_{\mathbb{R}^d} \left(u^\epsilon(x,t+\tau) - u^\epsilon(x,t)\right) \varphi(x)dx \right| \leq C\tau \|\varphi\|_{C^2(\mathbb{R}^d)}, \text{ for any } \varphi \in C_0^2(\mathbb{R}^d). \tag{3.3}$$

Let γ_i^ϵ the vector in $\mathbb{R}^d$ with components γ_{ij}^ϵ. We have

$$\begin{aligned} \left| \int_{\mathbb{R}^d} (u^\epsilon(x,t+\tau) - u^\epsilon(x,t))\varphi(x)\, dx \right| &= \left| \int_{\mathbb{R}^d} \int_t^{t+\tau} u_t^\epsilon(x,s)\varphi(x)\, ds\, dx \right| \\ &= \left| \int_{\mathbb{R}^d} \int_t^{t+\tau} \sum_{i=1}^N \sum_{j=1}^d \gamma_{ij}^\epsilon f_i^\epsilon(x,s) \varphi_{x_j}(x)\, dx\, ds \right|. \end{aligned}$$

In order to estimate the last term, let multiply by γ_{ij}^ϵ the equation for f_i^ϵ; then by the Duhamel's formula we obtain

$$\begin{aligned} & \left| \int_t^{t+\tau} \int_{\mathbb{R}^d} \sum_{i=1}^N \gamma_{ij}^\epsilon f_i^\epsilon(x,r) \varphi_{x_j}(x)\, dx\, dr \right| \\ \leq & \left| \int_t^{t+\tau} \int_{\mathbb{R}^d} e^{-\frac{r}{\epsilon}} \sum_{i=1}^N \gamma_{ij}^\epsilon M_i(\epsilon, u_0(x-\gamma_i^\epsilon r)) \varphi_{x_j}(x)\, dx\, dr \right| \\ & + \frac{1}{\epsilon} \left| \int_t^{t+\tau} \int_{\mathbb{R}^d} \int_0^r e^{-\frac{r-s}{\epsilon}} \sum_{i=1}^N \gamma_{ij}^\epsilon M_i(\epsilon, u^\epsilon(x-\gamma_i^\epsilon(r-s), s)) \varphi_{x_j}(x)\, ds\, dx\, dr \right| \\ =: & \ I_1^\epsilon + \frac{1}{\epsilon} I_2^\epsilon. \end{aligned}$$

As for I_1^ϵ we have

$$\begin{aligned} I_1^\epsilon \leq & \left| \int_t^{t+\tau} \int_{\mathbb{R}^d} e^{-\frac{r}{\epsilon}} \sum_{i=1}^N \gamma_{ij}^\epsilon M_i(\epsilon, u_0(x)) \varphi_{x_j}(x)\, dx\, dr \right| \\ & + \left| \int_t^{t+\tau} \int_{\mathbb{R}^d} e^{-\frac{r}{\epsilon}} \sum_{i=1}^N \gamma_{ij}^\epsilon M_i(\epsilon, u_0(x)) \left(\varphi_{x_j}(x+\gamma_i^\epsilon r) - \varphi_{x_j}(x) \right)\, dx\, dr \right| \\ \leq & \ C_1 \|\varphi\|_{C^2(\mathbb{R}^d)} \int_t^{t+\tau} e^{-\frac{r}{\epsilon}} \left(1 + \frac{r}{\epsilon}\right)\, dr \leq C_2 \tau \|\varphi\|_{C^2(\mathbb{R}^d)}, \end{aligned}$$

for suitable constants C_1 and C_2. Concerning I_2^ϵ, in similar way we obtain

$$\begin{aligned} \frac{1}{\epsilon} I_2^\epsilon \leq & \ C_3 \|\varphi\|_{C^2(\mathbb{R}^d)} \int_t^{t+\tau} \int_0^r \frac{e^{-\frac{r-s}{\epsilon}}}{\epsilon} \left(1 + \frac{r-s}{\epsilon}\right)\, ds\, dr \\ \leq & \ C_4 \tau \|\varphi\|_{C^2(\mathbb{R}^d)}, \end{aligned}$$

for suitable constants C_3, C_4 and (3.3) follows. $\square$

Proposition 3.3 *Under the assumptions of Proposition 3.1, let f^ϵ be the solution to problem (1.3)-(1.4). Then for every $\nu > 0$, there exists a positive constant τ_0 and a continuous nondecreasing function $\overline{\omega}_\nu : [0,\tau_0] \to \mathbb{R}$, not depending on ϵ, with $\overline{\omega}_\nu(0) = 0$, such that for $\nu \le t \le t+\tau, \tau \in (0,\tau_0)$,*

$$\int_{\mathbb{R}^d} \sum_{i=1}^{N} |f_i^\epsilon(x,t+\tau) - f_i^\epsilon(x,t)|dx \le \overline{\omega}_\nu(\tau), \tag{3.4}$$

for every $0 < \epsilon < \nu$.

Proof. As in the previous proof, according again to Lemma 5 in [21], we have only to prove that there exists a positive constant C such that

$$\left| \int_{\mathbb{R}^d} (f_i^\epsilon(x,t+\tau) - f_i^\epsilon(x,t))\varphi(x)\, dx \right| \le C\tau \|\varphi\|_{C^2(\mathbb{R}^d)}$$

for $i = 1,\dots,N$, $\nu \le t \le t+\tau$, for all $\varphi \in C_0^2(\mathbb{R}^d)$. The Duhamel's formula gives

$$\begin{aligned}\int_{\mathbb{R}^d} f_i^\epsilon(x,t)\varphi(x)\, dx = & \int_{\mathbb{R}^d} e^{-\frac{t}{\epsilon}} f_{0i}^\epsilon(x)\varphi(x+\gamma_i^\epsilon t)\, dx \\ & + \frac{1}{\epsilon}\int_0^t e^{-\frac{t-s}{\epsilon}} \int_{\mathbb{R}^d} M_i(\epsilon, u^\epsilon(x,s))\varphi(x+\gamma_i^\epsilon(t-s))\, dx\, ds.\end{aligned}$$

Then we have

$$\begin{aligned} I^\epsilon(t) &= \left| \int_{\mathbb{R}^d} (f_i^\epsilon(x,t+\tau) - f_i^\epsilon(x,t))\varphi(x)\, dx \right| \\ &\le \left| \int_{\mathbb{R}^d} \left(e^{-\frac{t+\tau}{\epsilon}}\varphi(x+\gamma_i^\epsilon(t+\tau)) - e^{-\frac{t}{\epsilon}}\varphi(x+\gamma_i^\epsilon t)\right) f_{0i}^\epsilon(x)\, dx \right| \\ &\quad + \frac{1}{\epsilon} \left| \int_0^{t+\tau} e^{-\frac{t+\tau-s}{\epsilon}} \int_{\mathbb{R}^d} M_i(\epsilon, u^\epsilon(x,s))\varphi(x+\gamma_i^\epsilon(t+\tau-s))\, dx\, ds \right. \\ &\quad \left. - \int_0^t e^{-\frac{t-s}{\epsilon}} \int_{\mathbb{R}^d} M_i(\epsilon, u^\epsilon(x,s))\varphi(x+\gamma_i^\epsilon(t-s))\, dx\, ds \right| \\ &=: I_1^\epsilon + \frac{1}{\epsilon} I_2^\epsilon, \end{aligned}$$

for $\nu \le t \le t+\tau$. As regard to I_1^ϵ we have, for $\nu \le t \le t+\tau$

$$\begin{aligned} I_1^\epsilon &\le \left| \int_{\mathbb{R}^d} \left(e^{-\frac{t+\tau}{\epsilon}} - e^{-\frac{t}{\epsilon}} \right) \varphi(x+\gamma_i^\epsilon(t+\tau)) f_{0i}^\epsilon(x)\, dx \right| \\ &\quad + \left| \int_{\mathbb{R}^d} e^{-\frac{t}{\epsilon}} (\varphi(x+\gamma_i^\epsilon(t+\tau)) - \varphi(x+\gamma_i^\epsilon t)) f_{0i}^\epsilon(x)\, dx \right| \\ &\le e^{-\frac{\nu}{\epsilon}}\tau \left(\frac{1}{\epsilon} + \frac{c_1}{\sqrt{\epsilon}} \right) \|f_{0i}^\epsilon\|_{L^1(\mathbb{R}^d)} \|\varphi\|_{C^1(\mathbb{R}^d)} \le c_2\tau \|\varphi\|_{C^1(\mathbb{R}^d)} \end{aligned}$$

for suitable constants c_1, c_2. As for I_2^ϵ we have

$$\begin{aligned}
I_2^\epsilon &\leq \left| \int_0^t e^{-\frac{t-s}{\epsilon}} \int_{\mathbb{R}^d} \left[M_i(\epsilon, u^\epsilon(x, s+\tau)) - M_i(\epsilon, u^\epsilon(x,s)) \right] \varphi(x + \gamma_i^\epsilon(t-s))\, dx\, dt \right| \\
&\quad + \left| \int_{-\tau}^0 e^{-\frac{t-s}{\epsilon}} \int_{\mathbb{R}^d} M_i(\epsilon, u^\epsilon(x, s+\tau)) \varphi(x + \gamma_i^\epsilon(t-s))\, dx\, ds \right| \\
&\leq \tau \|\varphi\|_{C^2(\mathbb{R}^d)} \left[\epsilon c_3 \left(1 - e^{-\frac{t}{\epsilon}} \right) + c_4 e^{-\frac{t}{\epsilon}} \|u_0\|_{L^1(\mathbb{R}^d)} \right],
\end{aligned}$$

for suitable constants c_3, c_4; here we used inequality (3.3), the monotonicity of M_i and the fact that $|M_i(\epsilon, w)| \leq |w|$, for every $w \in I$. Then we have

$$\frac{1}{\epsilon} I_2^\epsilon \leq c_5 \tau \|\varphi\|_{C^2(\mathbb{R}^d)}$$

and the claim follows. □

The following proposition gives an estimate of the deviation from the equilibrium in the L^1 norm.

Proposition 3.4 *Under the assumptions of Proposition 3.1, suppose that the initial data u_0 is of bounded variation. Then there exists a positive constant C such that, for every $\epsilon \in]0,1]$ and for every $t > 0$*

$$\sum_{i=1}^N \int_{\mathbb{R}^d} |f_i^\epsilon - M_i(\epsilon, u^\epsilon)| dx \leq C\sqrt{\epsilon} \sum_{i=1}^N \|f_{0i}^\epsilon\|_{BV(\mathbb{R}^d)}. \tag{3.5}$$

Proof. For any $\epsilon \in]0,1]$ take smooth initial data. Then we have

$$\begin{aligned}
&\partial_t \left(f_i^\epsilon - M_i(\epsilon, u^\epsilon) \right) + \frac{1}{\epsilon} \left(f_i^\epsilon - M_i(\epsilon, u^\epsilon) \right) \\
&= - \sum_{j=1}^d \left[\gamma_{ij}^\epsilon \partial_{x_j} f_i^\epsilon + M_i'(\epsilon, u^\epsilon) \sum_{k=1}^N \gamma_{kj} \partial_{x_j} f_k^\epsilon \right].
\end{aligned}$$

Multiplying by $\mathrm{sgn}(f_i^\epsilon - M_i(\epsilon, u^\epsilon))$ and integrating, by standard methods we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\sum_{i=1}^N |f_i^\epsilon - M_i(\epsilon,u^\epsilon)|\,dx\\
&= \int_0^t\int_{\mathbb{R}^d} e^{-\frac{t-s}{\epsilon}}\sum_{i=1}^N \operatorname{sgn}(f_i^\epsilon - M_i(\epsilon,u^\epsilon))\\
&\quad\times\sum_{j=1}^d\left[-\left(\lambda_{ij}+\frac{\theta_{ij}}{\sqrt{\epsilon}}\right)\partial_{x_j}f_i^\epsilon + M_i'(\epsilon,u^\epsilon)\sum_{k=1}^N\left(\lambda_{kj}+\frac{\theta_{kj}}{\sqrt{\epsilon}}\right)\partial_{x_j}f_k^\epsilon\right]dx\,ds\\
&\le \left(C_1+\frac{C_2}{\sqrt{\epsilon}}\right)\left(\int_0^t e^{-\frac{t-s}{\epsilon}}\int_{\mathbb{R}^d}\sum_{i=1}^N\sum_{j=1}^d |\partial_{x_j}f_i^\epsilon|\,dx\,ds\right)\\
&\le C\sqrt{\epsilon}\sum_{i=1}^N \|f_{0i}^\epsilon\|_{BV(\mathbb{R}^d)}.
\end{aligned}
$$

for suitable C_1, C_2 and C. Here we used Proposition 3.1. Then, by density arguments we obtain the claim. □

Now we can prove our convergence theorem.

Theorem 3.1 *Let $u_0 \in L^\infty(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$ be fixed, assume that M is a MMF on the interval $I = \{u \in \mathbb{R} : |u| \le \|u_0\|_\infty\}$ and set $f_{0i}^\epsilon = M_i(\epsilon, u_0)$. Let $f^\epsilon \in C([0,\infty[; L^1_{loc}(\mathbb{R}^d)^N)\cap L^\infty(\mathbb{R}^d\times]0,\infty[)^N$ be the solution to problem (1.3)-(1.4) given by Theorem 2.1. Then there exists a subsequence $\{f^{\epsilon_k}\}$ of $\{f^\epsilon\}$ and a bounded function f such that*

$$f^{\epsilon_k}\to f \quad \text{in } C([0,\infty); L^1_{loc}(\mathbb{R}^d)) \tag{3.6}$$

as $\epsilon_k \to 0$. Moreover the function $u := \sum_{i=1}^N f_i$ is a weak solution to problem (1.1)-(1.2).

Proof. From Propositions 3.1 and 3.3 we obtain the relatively compactness of the sequence $\{f^\epsilon\}$ in $C([\nu,\infty[; L^1_{loc}(\mathbb{R}^d)^N)$, for every $\nu > 0$.

In order to prove that the convergence holds also around $t = 0$, we consider a sequence $\{u_{0n}\}\subset C_0^\infty(\mathbb{R}^d)$ converging to u_0 in $L^1(\mathbb{R}^d)$ and the corresponding solutions $u^{\epsilon n}$ and $f^{\epsilon n}$. We have

$$\sum_{i=1}^N \|f_{0i}^{\epsilon n}\|_{BV(\mathbb{R}^d)} \le c_1 n\sum_{i=1}^N \|f_{0i}\|_{L^1(\mathbb{R}^d)}, \tag{3.7}$$

where c_1 is a suitable constant. For every $\epsilon < 1$, we choose $n = \epsilon^{-\frac{1}{4}}$ in the following inequality

$$\sum_{i=1}^N\int_{\mathbb{R}^d}|M_i(\epsilon,u^\epsilon)-f_i^\epsilon|\,dx \;\le\; \sum_{i=1}^N\int_{\mathbb{R}^d}|f_i^{\epsilon n}-M_i(\epsilon,u^{\epsilon n})|\,dx$$

$$+\sum_{i=1}^{N}\int_{\mathbb{R}^d}|f_i^\epsilon - f_i^{\epsilon n}|\,dx \quad + \quad \sum_{i=1}^{N}\int_{\mathbb{R}^d}|M_i(\epsilon,u^{\epsilon n}) - M_i(\epsilon,u^\epsilon)|\,dx.$$

Now, using Proposition 3.4 and (3.7), we get that the first term on the right-hand side goes to zero for $\epsilon \to 0$, for $t \in [0,\infty)$, while, thanks to Theorem 2.2 and to the Lipschitz continuity of M with respect to u we derive the same result for the last two terms. Therefore, we conclude that for $i = 1,\dots,N$

$$\int_{\mathbb{R}^d}|f_i^\epsilon(x,t) - M_i(\epsilon,u^\epsilon(x,t))|\,dx \to 0 \text{ when } \epsilon \to 0, \qquad \text{for } t \in [0,\infty[. \tag{3.8}$$

But according to Propositions 3.1 and 3.2, u^ϵ is compact in $C([0,\infty[,L^1_{loc}(\mathbb{R}^d))$, thus $M_i(\epsilon,u^\epsilon)$ is compact in $C([0,\infty[,L^1_{loc}(\mathbb{R}^d))$ (use (M_4)). Therefore, f_i^ϵ is also compact in $C([0,\infty[,L^1_{loc}(\mathbb{R}^d))$ by (3.8).

The last result of our statement, namely the weak consistency of our approximation, will follow from Theorem 4.1 in the next section. □

4 Consistency with entropy inequalities

This section is devoted to the proof that the limit u of the relaxation system obtained as $\epsilon \to 0$ is indeed the right entropy solution to (1.1). We have the following result.

Theorem 4.1 *The function $u \in C([0,\infty[,L^1_{loc}(\mathbb{R}^d)) \cap L^\infty$ obtained in Theorem 3.1 satisfies for any convex Lipschitz function $\eta : \mathbb{R} \to \mathbb{R}$*

$$\partial_t[\eta(u)] + \operatorname{div}[G_\eta(u)] - \Delta[B_\eta(u)] \le 0 \quad \textit{in } \mathbb{R}^d \times]0,\infty[, \tag{4.1}$$

with

$$G'_\eta = \eta' A', \qquad B'_\eta = \eta' B', \tag{4.2}$$

and

$$\nabla_x[B(u)] \in L^2(\mathbb{R}^d \times]0,\infty[). \tag{4.3}$$

According to [11], conditions (4.1) and (4.3) ensure the well-posedness of the Cauchy problem (1.1)-(1.2). Therefore, by uniqueness, it is not necessary to extract any subsequence, the whole family u^ϵ converges to u.

Proof of Theorem 4.1. In what follows, the notation

$$g^\epsilon(x,t) = O(\epsilon^\alpha) \tag{4.4}$$

for some $\alpha \in \mathbb{R}$ has to be understood in the sense of distributions in (x,t). More explicitly, it means that for any $\varphi \in C_c^\infty(\mathbb{R}^d \times]0,\infty[)$

$$|\langle g^\epsilon,\varphi\rangle| \le C_\varphi \epsilon^\alpha, \quad \epsilon \le 1, \tag{4.5}$$

for some constant C_φ independent of ϵ. Similarly, the notation $g^\epsilon = o(\epsilon^\alpha)$ means that $g^\epsilon/\epsilon^\alpha \to 0$ in the sense of distributions as $\epsilon \to 0$.

Let us first prove weak consistency with (1.1). We have obtained in the proof of Theorem 3.1 that (up to a subsequence)

$$f_i^\epsilon - M_i(\epsilon, u^\epsilon) = o(1). \tag{4.6}$$

Putting this in (1.3) gives

$$\begin{aligned} M_i(\epsilon, u^\epsilon) - f_i^\epsilon &= \epsilon \left(\partial_t M_i(\epsilon, u^\epsilon) + \operatorname{div} \gamma_i^\epsilon M_i(\epsilon, u^\epsilon) + o(\epsilon^{-1/2})\right) \\ &= \epsilon \operatorname{div} \gamma_i^\epsilon M_i(\epsilon, u^\epsilon) + o(\epsilon^{1/2}). \end{aligned} \tag{4.7}$$

Therefore,

$$\begin{aligned} \sum_i \gamma_{ij}^\epsilon f_i^\epsilon &= \sum_i \gamma_{ij}^\epsilon M_i(\epsilon, u^\epsilon) - \sum_i \gamma_{ij}^\epsilon \epsilon \operatorname{div} \gamma_i^\epsilon M_i(\epsilon, u^\epsilon) + o(1) \\ &= A_j(u^\epsilon) - \partial_{x_j} B(u^\epsilon) + o(1), \end{aligned} \tag{4.8}$$

where we have used (M_2) and

$$\epsilon \sum_i \gamma_{ij}^\epsilon \gamma_{ik}^\epsilon M_i(\epsilon, w) = B(w)\delta_{jk} + o(1), \tag{4.9}$$

which comes from (M_3) and (M_4). Now, by summing up (1.3) with respect to i and by using (4.8) we get

$$\partial_t u^\epsilon + \sum_j \partial_j A_j(u^\epsilon) - \Delta B(u^\epsilon) = o(1), \tag{4.10}$$

and when $\epsilon \to 0$ we recover (1.1).

Now, in order to obtain entropy inequalities, we are going to use the convex kinetic entropies $H_{\eta i}^\epsilon$ associated to each convex entropy η. According to [8], they can be obtained as follows. For any convex and globally Lipschitz continuous function $\eta : \mathbb{R} \to \mathbb{R}$, we have the identity

$$\eta(w) = \int_{\mathbb{R}} \frac{1}{2} \left(|w-k| - |k|\right) \eta''(k)\, dk + \frac{1}{2} \left(\eta'(-\infty) + \eta'(\infty)\right) w + \eta(0). \tag{4.11}$$

Since for a Kružkov entropy

$$\eta_k(w) = |w-k| - |k| \tag{4.12}$$

the kinetic entropy is given by $H_{ki}^\epsilon(f) = |f - M_i(\epsilon,k)| - |M_i(\epsilon,k)|$, for a general entropy function η we can take

$$\begin{aligned} H_{\eta i}^\epsilon(f) = &\int_{\mathbb{R}} \frac{1}{2} \left(|f - M_i(\epsilon,k)| - |M_i(\epsilon,k)|\right) \eta''(k)\, dk \\ &+ \frac{1}{2} f \;\left(\eta'(-\infty) + \eta'(\infty)\right), \end{aligned} \tag{4.13}$$

which is convex in f and has a Lipschitz constant independent of ϵ. In order to simplify technical discussions, we are going to make first a simplifying assumption. The modifications for the general case will then be indicated.

Non degenerate case. Let us assume that

$$M_i(\epsilon,\cdot) \text{ is strictly increasing in } I=\{u\in\mathbb{R}:|u|\leq\|u_0\|_\infty\}, \tag{4.14}$$

and let us consider only entropies $\eta\in C^2$. In any case, we can let η tend to a Kružkov entropy at the end and recover "irregular" entropies. Now for such C^2 entropies, (4.14) ensures that $H^\epsilon_{\eta i}\in C^1([M_i(\epsilon,-\|u_0\|_\infty),M_i(\epsilon,\|u_0\|_\infty)])$. From (4.13), we have our basic identity

$${H^\epsilon_{\eta i}}'\left(M_i(\epsilon,w)\right)=\eta'(w),\quad w\in I. \tag{4.15}$$

From this and from (M_1), (M_2) we easily deduce that

$$\sum_i H^\epsilon_{\eta i}\left(M_i(\epsilon,w)\right)=\eta(w)-\eta(0), \tag{4.16}$$

$$\sum_i \gamma_i^\epsilon H^\epsilon_{\eta i}\left(M_i(\epsilon,w)\right)=G_\eta(w), \tag{4.17}$$

the entropy flux defined by

$$G_\eta'=\eta'A',\qquad G_\eta(0)=0. \tag{4.18}$$

Similarly, from (M_3) we get

$$\sum_i \theta_{ij}\theta_{ik}H^0_{\eta i}\left(M_i(0,w)\right)=B_\eta(w)\delta_{jk}, \tag{4.19}$$

with

$$B_\eta'=\eta'B',\qquad B_\eta(0)=0. \tag{4.20}$$

Together with (M_4), it yields

$$\sum_i \epsilon\gamma^\epsilon_{ij}\gamma^\epsilon_{ik}H^\epsilon_{\eta i}\left(M_i(\epsilon,w)\right)=B_\eta(w)\delta_{jk}+o(1). \tag{4.21}$$

Let us now multiply (1.3) by ${H^\epsilon_{\eta i}}'(f_i^\epsilon)$, which is possible since $H^\epsilon_{\eta i}\in C^1$. We get

$$\partial_t H^\epsilon_{\eta i}(f_i^\epsilon)+\operatorname{div}\gamma_i^\epsilon H^\epsilon_{\eta i}(f_i^\epsilon)={H^\epsilon_{\eta i}}'(f_i^\epsilon)\frac{M_i(\epsilon,u^\epsilon)-f_i^\epsilon}{\epsilon}. \tag{4.22}$$

If we choose η such that $\eta(0)=0$, $\eta'(0)=0$, then $H^\epsilon_{\eta i}(M_i(\epsilon,0))=0$, ${H^\epsilon_{\eta i}}'(M_i(\epsilon,0))=0$ and thus $H^\epsilon_{\eta i}\geq 0$. Integrating (4.22) over a cone and letting it tend to infinity gives after summing up with respect to i

$$\begin{aligned}&\sum_i\int_{\mathbb{R}^d}H^\epsilon_{\eta i}(f_i^\epsilon(x,T))dx+\sum_i\int_0^T\!\!\int_{\mathbb{R}^d}\left[{H^\epsilon_{\eta i}}'(f_i^\epsilon)-{H^\epsilon_{\eta i}}'(M_i(\epsilon,u^\epsilon))\right]\frac{f_i^\epsilon-M_i(\epsilon,u^\epsilon)}{\epsilon}dtdx\\&=\sum_i\int_{\mathbb{R}^d}H^\epsilon_{\eta i}(f_i^\epsilon(x,0))dx,\end{aligned} \tag{4.23}$$

where we have used (4.15). Then since $u_0 \in L^1$,

$$\sum_i \int H^\epsilon_{\eta i}\left(f^\epsilon_i(x,0)\right)\, dx = \int \sum_i H^\epsilon_{\eta i}\left(M_i(\epsilon, u_0(x))\right)\, dx = \int \eta(u_0(x))\, dx < \infty, \quad (4.24)$$

and therefore the left-hand side of (4.23) is finite and bounded independently of T, ϵ. Next, by choosing $\eta(w) = w^2/2$, $w \in I$, and by denoting by $w^\epsilon_i(x,t)$ a function such that

$$f^\epsilon_i(x,t) = M_i(\epsilon, w^\epsilon_i(x,t)), \quad (4.25)$$

which is possible by (2.4), we have thanks to (4.15)

$$\begin{aligned} & {H^\epsilon_{\eta i}}'(f^\epsilon_i) - {H^\epsilon_{\eta i}}'(M_i(\epsilon, u^\epsilon)) \\ &= {H^\epsilon_{\eta i}}'(M_i(\epsilon, w^\epsilon_i(x,t)) - {H^\epsilon_{\eta i}}'(M_i(\epsilon, u^\epsilon(x,t))) \\ &= \eta'(w^\epsilon_i(x,t)) - \eta'(u^\epsilon(x,t)) \\ &= w^\epsilon_i(x,t) - u^\epsilon(x,t). \end{aligned} \quad (4.26)$$

Therefore, since $M_i(\epsilon,\cdot)$ has Lipschitz constant 1,

$$\begin{aligned} & ({H^\epsilon_{\eta i}}'(f^\epsilon_i) - {H^\epsilon_{\eta i}}'(M_i(\epsilon, u^\epsilon)))(f^\epsilon_i - M_i(\epsilon, u^\epsilon)) \\ &= (w^\epsilon_i(x,t) - u^\epsilon(x,t))(M_i(\epsilon, w^\epsilon_i(x,t)) - M_i(\epsilon, u^\epsilon(x,t))) \\ &\geq \left[M_i(\epsilon, w^\epsilon_i(x,t)) - M_i(\epsilon, u^\epsilon(x,t))\right]^2 \\ &= \left[f^\epsilon_i - M_i(\epsilon, u^\epsilon)\right]^2. \end{aligned} \quad (4.27)$$

Thus we obtain the estimate

$$\sum_i \int_0^\infty\!\!\int_{\mathbb{R}^d} \left[f^\epsilon_i - M_i(\epsilon, u^\epsilon)\right]^2\, dtdx \leq \epsilon \int \frac{u_0(x)^2}{2} dx. \quad (4.28)$$

This estimate has to be compared to (3.5). Let us now consider again a convex Lipschitz entropy $\eta \in C^2$ satisfying $\eta(0) = 0$, $\eta'(0) = 0$, and let us prove the entropy inequality (4.1). The constraint $\eta'(0) = 0$ can be removed afterwards since we have proved that u is a weak solution to (1.1). We are going to perform the same analysis as in (4.6)-(4.10), but starting from (4.22) instead of (1.3). We have from (4.22), (4.28) and (4.21)

$$\begin{aligned} & \sum_i \gamma^\epsilon_{ij} {H^\epsilon_{\eta i}}'(f^\epsilon_i)(M_i(\epsilon, u^\epsilon) - f^\epsilon_i) \\ &= \partial_t \sum_i \epsilon \gamma^\epsilon_{ij} H^\epsilon_{\eta i}(f^\epsilon_i) + \operatorname{div} \sum_i \epsilon \gamma^\epsilon_{ij} \gamma^\epsilon_i H^\epsilon_{\eta i}(f^\epsilon_i) \\ &= \operatorname{div} \sum_i \epsilon \gamma^\epsilon_{ij} \gamma^\epsilon_i H^\epsilon_{\eta i}(M_i(\epsilon, u^\epsilon)) + O(\epsilon^{1/2}) \\ &= \partial_{x_j} B_\eta(u^\epsilon) + o(1). \end{aligned} \quad (4.29)$$

But we notice that by convexity of $H^\epsilon_{\eta i}$,

$$\begin{aligned} 0 &\leq H^\epsilon_{\eta i}(M_i(\epsilon, u^\epsilon)) - H^\epsilon_{\eta i}(f^\epsilon_i) - {H^\epsilon_{\eta i}}'(f^\epsilon_i)(M_i(\epsilon, u^\epsilon) - f^\epsilon_i) \\ &\leq \left({H^\epsilon_{\eta i}}'(M_i(\epsilon, u^\epsilon)) - {H^\epsilon_{\eta i}}'(f^\epsilon_i)\right)\left(M_i(\epsilon, u^\epsilon) - f^\epsilon_i\right), \end{aligned} \quad (4.30)$$

and according to (4.23) this is bounded in L^1_{tx} by $C\epsilon$. Therefore, with (4.17) and (4.29)

$$\begin{aligned}\sum_i \gamma_{ij}^\epsilon H_{\eta i}^\epsilon(f_i^\epsilon) &= \sum_i \gamma_{ij}^\epsilon H_{\eta i}^\epsilon(M_i(\epsilon,u^\epsilon)) - \sum_i \gamma_{ij}^\epsilon {H_{\eta i}^\epsilon}'(f_i^\epsilon)\,(M_i(\epsilon,u^\epsilon)-f_i^\epsilon) + O(\epsilon^{1/2})\\ &= G_{\eta j}(u^\epsilon) - \partial_{x_j}B_\eta(u^\epsilon) + o(1).\end{aligned} \tag{4.31}$$

We have also according to (4.28) and (4.16)

$$\sum_i H_{\eta i}^\epsilon(f_i^\epsilon) = \sum_i H_{\eta i}^\epsilon(M_i(\epsilon,u^\epsilon)) + O(\epsilon^{1/2}) = \eta(u^\epsilon) + O(\epsilon^{1/2}). \tag{4.32}$$

Finally, by summing up (4.22) with respect to i and by (4.31), (4.32), we obtain

$$\begin{aligned}&\partial_t\eta(u^\epsilon) + \operatorname{div} G_\eta(u^\epsilon) - \Delta B_\eta(u^\epsilon)\\ &= \sum_i \left[{H_{\eta i}^\epsilon}'(f_i^\epsilon) - {H_{\eta i}^\epsilon}'(M_i(\epsilon,u^\epsilon))\right]\frac{M_i(\epsilon,u^\epsilon)-f_i^\epsilon}{\epsilon} + o(1),\end{aligned} \tag{4.33}$$

and since the term on the right-hand side is non-positive, it provides at the limit

$$\partial_t\eta(u) + \operatorname{div} G_\eta(u) - \Delta B_\eta(u) \le 0. \tag{4.34}$$

In order to prove the regularity of $B(u)$, we notice that according to (4.28), $[M_i(\epsilon,u^\epsilon) - f_i^\epsilon]/\epsilon^{1/2}$ is bounded in L^2_{tx}. But (4.7) tells that it tends to $\operatorname{div}\theta_i M_i(0,u)$ in the sense of distributions. Therefore, $\operatorname{div}\theta_i M_i(0,u) \in L^2(\mathbb{R}^d \times]0,\infty[)$, and

$$\frac{M_i(\epsilon,u^\epsilon)-f_i^\epsilon}{\epsilon^{1/2}} \rightharpoonup \operatorname{div}\theta_i M_i(0,u) \qquad \text{in } L^2(\mathbb{R}^d\times]0,\infty[) - weak, \tag{4.35}$$

$$\sum_i \int_0^\infty\!\!\int_{\mathbb{R}^d} |\operatorname{div}\theta_i M_i(0,u)|^2\,dtdx \le \int_{\mathbb{R}^d} \frac{u_0(x)^2}{2}\,dx. \tag{4.36}$$

Then, by (M_3),

$$\partial_{x_k}B(u) = \sum_i \theta_{ik}\operatorname{div}\theta_i M_i(0,u) \quad \in L^2(\mathbb{R}^d\times]0,\infty[). \tag{4.37}$$

General case. Let us here indicate what are the modifications to the proof above when we do not make the non-degeneracy assumption (4.14), and when dealing with non smooth entropies η. The summation formulas (4.16), (4.17), (4.19) remain valid, this can be seen by the integral formulas (4.11) and (4.13) which allow to only check that they hold for linear and for Kružkov entropies. In that case it reduces to a simple computation, for example for (4.16)

$$\begin{aligned}&\sum_i |M_i(\epsilon,w) - M_i(\epsilon,k)|\\ &= \sum_i \operatorname{sgn}(w-k)[M_i(\epsilon,w)-M_i(\epsilon,k)]\\ &= \operatorname{sgn}(w-k)(w-k)\\ &= |w-k|.\end{aligned} \tag{4.38}$$

Therefore, (4.21) is also valid.

In the following, we use the functions η' and ${H^\epsilon_{\eta i}}'$. Since they are discontinuous, we have to precise their value at discontinuities, and we just prescribe them to take an arbitrary value between left and right limits. Indeed, according to [7], since $\partial_t f_i^\epsilon + \sum_{j=1}^d \gamma_{ij}^\epsilon \partial_{x_j} f_i^\epsilon = (M_i(\epsilon, u^\epsilon) - f_i^\epsilon)/\epsilon \in L^1_{loc}$, this quantity vanishes almost everywhere where f_i^ϵ remains in a set of measure zero. Therefore, ${H^\epsilon_{\eta i}}'(f_i^\epsilon)(M_i(\epsilon, u^\epsilon) - f_i^\epsilon)$ is well defined and does not depend on the representative of ${H^\epsilon_{\eta i}}'$ that is chosen. Then, (4.22) holds and we notice that

$$\left({H^\epsilon_{\eta i}}'(f_i^\epsilon) - \eta'(u^\epsilon)\right)(f_i^\epsilon - M_i(\epsilon, u^\epsilon)) \geq 0. \tag{4.39}$$

Indeed by the representation formulas (4.11), (4.13), it is enough to prove that it holds for a Kružkov entropy (4.12). In that case, writing

$$f_i^\epsilon - M_i(\epsilon, u^\epsilon) = f_i^\epsilon - M_i(\epsilon, k) + M_i(\epsilon, k) - M_i(\epsilon, u^\epsilon), \tag{4.40}$$

we obtain

$$\begin{aligned} &[\operatorname{sgn}(f_i^\epsilon - M_i(\epsilon, k)) - \operatorname{sgn}(u^\epsilon - k)]\,[f_i^\epsilon - M_i(\epsilon, u^\epsilon)] \\ = {}& |f_i^\epsilon - M_i(\epsilon, k)| + \operatorname{sgn}(f_i^\epsilon - M_i(\epsilon, k))(M_i(\epsilon, k) - M_i(\epsilon, u^\epsilon)) \\ &+ |M_i(\epsilon, k) - M_i(\epsilon, u^\epsilon)| - \operatorname{sgn}(u^\epsilon - k)(f_i^\epsilon - M_i(\epsilon, k)) \\ \geq {}& 0, \end{aligned} \tag{4.41}$$

which holds whatever value in $[-1, 1]$ takes the sign at 0. Similarly, we have

$$H^\epsilon_{\eta i}(f_i^\epsilon) \geq H^\epsilon_{\eta i}(M_i(\epsilon, u^\epsilon)) + \eta'(u^\epsilon)(f_i^\epsilon - M_i(\epsilon, u^\epsilon)), \tag{4.42}$$

which can be checked again for Kružkov entropies by using (4.40). According to [8], this inequality generalizes (4.15), it can be written in terms of subdifferentials $\partial H^\epsilon_{\eta i}(M_i(\epsilon, w)) \ni \eta'(w)$, $w \in I$. Then we take η satisfying $\eta(0) = 0$, $\eta \geq 0$, which implies that $H^\epsilon_{\eta i}(M_i(\epsilon, 0)) = 0$, $H^\epsilon_{\eta i} \geq 0$, and we obtain (4.23) with ${H^\epsilon_{\eta i}}'(M_i(\epsilon, u^\epsilon))$ replaced by $\eta'(u^\epsilon)$.

In order to generalize (4.27), we take η even such that $\eta(w) = w^2/2$, $w \in I$ and $\eta''(w) = 0$, $w \notin I$. By (4.11), (4.13) and since $M_i' \leq 1$,

$$\begin{aligned} &\left({H^\epsilon_{\eta i}}'(f_i^\epsilon) - \eta'(u^\epsilon)\right)(f_i^\epsilon - M_i(\epsilon, u^\epsilon)) \\ = {}& \int_{\mathbb{R}} \frac{1}{2}[\operatorname{sgn}(f_i^\epsilon - M_i(\epsilon, k)) - \operatorname{sgn}(u^\epsilon - k)]\,[f_i^\epsilon - M_i(\epsilon, u^\epsilon)]\eta''(k)\,dk \\ \geq {}& \int_I \frac{1}{2}[\operatorname{sgn}(f_i^\epsilon - M_i(\epsilon, k)) - \operatorname{sgn}(u^\epsilon - k)]\,[f_i^\epsilon - M_i(\epsilon, u^\epsilon)]M_i'(\epsilon, k)\,dk \\ = {}& \frac{1}{2}\Big[\Big(-|f_i^\epsilon - M_i(\epsilon, k)| + \operatorname{sgn}(u^\epsilon - k)(M_i(\epsilon, u^\epsilon) - M_i(\epsilon, k))\Big)(f_i^\epsilon - M_i(\epsilon, u^\epsilon))\Big]_{-\|u_0\|_\infty}^{\|u_0\|_\infty} \\ = {}& (f_i^\epsilon - M_i(\epsilon, u^\epsilon))^2. \end{aligned} \tag{4.43}$$

Thus we obtain the estimate (4.28). The remainder of the proof is identical, except that we replace ${H^\epsilon_{\eta i}}'(M_i(\epsilon, u^\epsilon))$ by $\eta'(u^\epsilon)$ everywhere. $\square$

Remark. The right-hand side of (4.33) formally tends to

$$-\sum_i {H^0_{\eta i}}''(M_i(0,u))\,(\operatorname{div}\theta_i M_i(0,u))^2 = -\eta''(u)B'(u)|\nabla u|^2, \tag{4.44}$$

which can be seen by computing ${H^0_{\eta i}}''(M_i(0,u))$ by differentiating (4.15). This term is the one that should stand in the right-hand side of (4.1) for smooth solutions to (1.1).

5 Examples

This section is devoted to present some examples of relaxation systems of kind (1.3)-(1.4) to approximate parabolic equations in the class (1.1).

Example 1 We first deal with the one-dimensional case

$$\partial_t u + \partial_x A(u) = \partial_{xx} B(u), \qquad (x,t) \in \mathbb{R}\times]0,\infty[, \tag{5.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{5.2}$$

where $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, $A, B \in Lip_{loc}(\mathbb{R})$ and $B'(u) \geq 0$. We want to present an approximation system with M not depending on ϵ. The compatibility conditions read now

$$\begin{array}{rl}
i) & \sum_{i=1}^N \quad M_i(u) = u;\\
ii) & \sum_{i=1}^N \quad \lambda_i M_i(u) = A(u);\\
iii) & \sum_{i=1}^N \quad \theta_i^2 M_i(u) = B(u);\\
iv) & \sum_{i=1}^N \quad \theta_i M_i(u) = 0.
\end{array}$$

Taking $N = 4$ and fixing $\lambda > 0$ we have as a possible choice of the coefficients

$$\lambda_1 = -\lambda, \quad \lambda_2 = \lambda, \quad \lambda_3 = \lambda_4 = 0;$$

$$\theta_1 = \theta_2 = 0, \quad \theta_3 = -\sqrt{\lambda}, \quad \theta_4 = \sqrt{\lambda},$$

$$\begin{aligned}
M_1(u) &= \frac{1}{2}\left(u - \frac{A(u)+B(u)}{\lambda}\right),\\
M_2(u) &= \frac{1}{2}\left(u + \frac{A(u)-B(u)}{\lambda}\right),\\
M_3(u) &= M_4(u) = \frac{B(u)}{2\lambda}.
\end{aligned}$$

We have convergence provided that

$$\lambda \geq \max_I |A'| + \max_I B',$$

where $I = \{u \in \mathbb{R} : |u| \leq \|u_0\|_{\mathrm{L}^\infty(\mathbb{R})}\}$.

Example 2 Consider now the d-dimensional case

$$\partial_t u + \sum_{j=1}^{d} \partial_{x_j} A_j(u) = \Delta B(u), \qquad (x,t) \in \mathbb{R}^d \times]0,\infty[, \tag{5.3}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^d, \tag{5.4}$$

where $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, $A_j, B \in Lip_{loc}(\mathbb{R})$ and $B'(u) \geq 0$.

As in the first example we obtain an approximation system with the functions M_i not depending on ϵ with $N = 3d+1$. In fact, fix $\lambda > 0$ and set $\lambda_{ij} = -\delta_{ij}\lambda$ for $i = 1,\dots,d$, $\lambda_{ij} = 0$ for $i = d+1,\dots,N-1$, $\lambda_{Nj} = \lambda$ for $j = 1,\dots,d$; $\theta_{ij} = 0$ for $i \neq j+d, j+2d$, $\theta_{j+d,j} = -\sqrt{\lambda}$, $\theta_{j+2d,j} = \sqrt{\lambda}$, for $j = 1,\dots,d$. The corresponding Maxwellian function is now given, for $i = 1,\dots,d$, by

$$M_i(u) = \frac{1}{d+1}\left[u + \sum_{j\neq i} \frac{A_j(u)}{\lambda} - d\left(\frac{B(u) + A_i(u)}{\lambda}\right)\right],$$

$$M_{i+d} = M_{i+2d} = \frac{B(u)}{2\lambda},$$

and finally

$$M_N = \frac{1}{d+1}\left(u + \sum_{j=1}^{d} \frac{A_j(u)}{\lambda} - d\frac{B(u)}{\lambda}\right).$$

The quasimonotonicity of the system is ensured by the condition

$$\lambda \geq d\left(\max_I \{\sum_{j=1}^{d} |A_j'| + B'\}\right).$$

Example 3

Here we present a general procedure to construct a relaxation system (1.3) for equation (5.3). Fix $N, \overline{N} \in \mathbb{N}$ such that $d < \overline{N} < N - d$; for any vector $V \in \mathbb{R}^N$ we denote by $V^I \in \mathbb{R}^{\overline{N}}$ and $V^{II} \in \mathbb{R}^{N-\overline{N}}$ respectively the vector obtained by the first $\overline{N}$ components of V and the vector obtained by the last $N - \overline{N}$ ones. Let ν, μ vectors in $\mathbb{R}^N$ such that

$$\nu_i = 1 \quad \text{for } i = 1,\dots,\overline{N}, \quad \nu_i = 0 \quad \text{for } i = \overline{N}+1,\dots,N,$$

$$\mu_i = -\alpha \quad \text{for } i = 1, \ldots, \overline{N}, \quad \mu_i = 1 \quad \text{for } i = \overline{N}+1, \ldots, N,$$

with $\alpha > 0$ and $\overline{N}\alpha = N - \overline{N}$.

Now we fix $\{\lambda_j^I\}_{j=1,\ldots,d}$ an orthogonal system in the orthogonal space to ν^I and λ_j^{II}, for $j = 1, \ldots, d$, vectors in the orthogonal space to μ^{II}; moreover we fix $\beta > 0$, $\{\sigma_j\}_{j=1,\ldots,d}$ an orthonormal system of vectors in $\mathbb{R}^{N-\overline{N}}$ in the orthogonal space to μ^{II} and we set θ_j^I equal to the null vector for all $j = 1, \ldots, d$ and $\theta_j^{II} = \beta\sigma_j$ for $j = 1, \ldots, d$.

As for the Maxwellian functions we set, for $i = 1, \ldots, N$,

$$M_i(u) = \left(\frac{u}{\overline{N}} + \sum_{k=1}^{d} \frac{A_k(u)\lambda_{ik}}{|\lambda_k^I|^2} \right) \nu_i + \frac{B(u)}{\beta^2}\mu_i.$$

The monotonicity conditions require that

$$M_i'(u) = \frac{1}{\overline{N}} + \sum_{k=1}^{d} \frac{A_k'(u)}{|\lambda_k^I|^2}\lambda_{ik} - \alpha\frac{B'(u)}{\beta^2} \geq 0, \quad \text{for } i = 1, \ldots, \overline{N},$$

then we have conditions on λ_{ij} for $i = 1, \ldots, \overline{N}, j = 1, \ldots, d$ and on $|\theta_j|$.

In the one-dimensional case it is possible to use a similar procedure, with $1 \leq \overline{N} < N-1$ to construct a relaxation approximation to (5.3)-(5.4) with Maxwellian functions depending on ϵ. We fix λ^I, λ^{II} vectors in the orthogonal spaces respectively to ν^I and μ^{II}; moreover we set θ^I the null vector in $\mathbb{R}^{\overline{N}}$ and θ^{II} an orthogonal vector to μ^{II}. Finally the Maxwellian functions are defined by

$$M(u) = \frac{u}{\overline{N}}\nu + \frac{\gamma A(u)}{|\gamma|^2} + \frac{B(u)}{|\theta|^2}\mu.$$

For instance take $N = 3$. Let $\overline{N} = 1$, $\lambda_1 = \theta_1 = 0$, $\lambda_2 = -\lambda_3 = \lambda > 0$, $\theta_2 = -\theta_3 = \theta > 0$; then we have

$$M_1(\epsilon, u) = u - \frac{B(u)}{\theta^2},$$

$$M_2(\epsilon, u) = \frac{\left(\lambda + \frac{\theta}{\sqrt{\epsilon}}\right)}{2\left|\lambda + \frac{\theta}{\sqrt{\epsilon}}\right|^2} A(u) + \frac{B(u)}{2\theta^2},$$

and

$$M_3(\epsilon, u) = -\frac{\left(\lambda + \frac{\theta}{\sqrt{\epsilon}}\right)}{2\left|\lambda + \frac{\theta}{\sqrt{\epsilon}}\right|^2} A(u) + \frac{B(u)}{2\theta^2}.$$

If the equation (5.3) does not degenerates or some particular conditions linking A' and B' hold, like

$$B'(u) \geq |A'(u)|, \tag{5.5}$$

the monotonicity conditions are verified for ϵ small enough if $\theta^2 > \max_I B'$.

Example 4

In [19] the authors prove that the problem

$$(E_\epsilon) \qquad \begin{cases} u_t^\epsilon + v_x^\epsilon = 0, \\ \epsilon^2 v_t^\epsilon + a u_x^\epsilon = A(u^\epsilon) - v^\epsilon, \quad (x,t) \in \mathbb{R}\times]0,\infty[, \quad a > 0 \end{cases}$$

as $\epsilon \to 0$ reduces to the problem

$$(E_0) \qquad \begin{cases} u_t + A(u)_x = a u_{xx}, \\ v = A(u) - a u_x, \qquad (x,t) \in \mathbb{R}\times]0,\infty[, \quad a > 0; \end{cases}$$

more precisely they investigate the diffusion limit of system (E_ϵ), with initial data prescribed around a traveling wave of (E_ϵ). The same system has been considered for numerical purposes in [4, 5]. Here we want to show that problem (E_ϵ) can be obtained by our relaxation scheme for the parabolic equation in (E_0), by a suitable choice of parameters λ_i, θ_i and functions M_i, now depending on ϵ, satisfying conditions $(M_1) - (M_4)$ and then we have convergence of $\{u^\epsilon\}$ to u for initial data in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Set $N = 2$,

$$\lambda_1 = \lambda_2 = 0, \qquad \theta_1 = \sqrt{a}, \qquad \theta_2 = -\sqrt{a},$$

and

$$M_1(\epsilon, u) = \frac{u}{2} + \frac{\sqrt{\epsilon}A(u)}{2\sqrt{a}}, \quad M_2(\epsilon, u) = \frac{u}{2} - \frac{\sqrt{\epsilon}A(u)}{2\sqrt{a}}.$$

It is readily seen that conditions $(M_1) - (M_4)$ are immediately verified for ϵ small enough and we obtain the following relaxation system

$$\begin{cases} f_{1_t}^\epsilon + \sqrt{\frac{a}{\epsilon}} f_{1_x}^\epsilon &= \frac{1}{\epsilon}\left(\frac{u^\epsilon}{2} + \frac{\sqrt{\epsilon}A(u^\epsilon)}{2\sqrt{a}} - f_1^\epsilon\right), \\ \\ f_{2_t}^\epsilon - \sqrt{\frac{a}{\epsilon}} f_{2_x}^\epsilon &= \frac{1}{\epsilon}\left(\frac{u^\epsilon}{2} - \frac{\sqrt{\epsilon}A(u^\epsilon)}{2\sqrt{a}} - f_2^\epsilon\right). \end{cases}$$

Setting

$$u^\epsilon = f_1^\epsilon + f_2^\epsilon, \qquad v^\epsilon = \sqrt{\frac{a}{\epsilon}}(f_1^\epsilon - f_2^\epsilon),$$

we obtain system (E_ϵ). The stability condition reads now

$$|A'(u)| \leq \sqrt{\frac{a}{\epsilon}}\ ,$$

which is always verified on the interval I for ϵ small enough.

Example 5

In this last example we want to present an approximation to problem (5.3)-(5.4) with $N = 5$ for $d = 1$. A similar model can be easily extended for $d > 1$. Notice that the use of more velocities could be useful in the numerical approximations, see [2, 33]. Set $\theta > 0$,

$$\lambda_4 = \lambda_5 = 0,$$

$$\theta_1 = \theta_2 = \theta_3 = 0, \qquad \theta_4 = -\theta_5 = -\sqrt{\theta},$$

and

$$M_1(u) = \frac{\lambda_3\left(u - \frac{B(u)}{\theta}\right) - A(u) + (\lambda_2 - \lambda_3)M_2(u)}{\lambda_3 - \lambda_1},$$

$$M_3(u) = \frac{-\lambda_1\left(u - \frac{B(u)}{\theta}\right) + A(u) + (\lambda_1 - \lambda_2)M_2(u)}{\lambda_3 - \lambda_1},$$

$$M_4(u) = M_5(u) = \frac{B(u)}{2\theta};$$

then we have a convergent approximation, for $\lambda_1 < \lambda_2 < \lambda_3$, if we choose as $M_2(u)$ any Lipschitz continuous function such that

$$0 \le M_2'(u) \le \inf\left\{\frac{-A'(u) + \lambda_3\left(1 - \frac{B'}{\theta}\right)}{\lambda_3 - \lambda_2}, \frac{A'(u) - \lambda_1\left(1 - \frac{B'}{\theta}\right)}{\lambda_2 - \lambda_1}\right\},$$

for $u \in I$, for some $I \subseteq \mathbb{R}$.

For $\lambda_3 = -\lambda_1 = \lambda > 0$, $\lambda_2 = 0$ it is possible to choose

$$M_1' = \frac{(A')_-}{\lambda}, \qquad M_2' = 1 - \frac{|A'|}{\lambda} - \frac{B'}{\theta}, \qquad M_3' = \frac{(A')_+}{\lambda}.$$

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