ON THE CONVERGENCE RATE OF OPERATOR SPLITTING FOR HAMILTON-JACOBI EQUATIONS WITH SOURCE TERMS

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Abstract. We establish a rate of convergence for a semi-discrete operator splitting method applied to Hamilton-Jacobi equations with source terms. The method is based on sequentially solving a Hamilton-Jacobi equation and an ordinary differential equation. The Hamilton-Jacobi equation is solved exactly while the ordinary differential equation is solved exactly or by an explicit Euler method. We prove that the L^∞ error associated with the operator splitting method is bounded by $\mathcal{O}(\Delta t)$, where Δt is the splitting (or time) step. This error bound is an improvement over the existing $\mathcal{O}(\sqrt{\Delta t})$ bound due to Souganidis [40]. In the one dimensional case, we present a fully discrete splitting method based on an unconditionally stable front tracking method for homogeneous Hamilton-Jacobi equations. It is proved that this fully discrete splitting method possesses a linear convergence rate. Moreover, numerical results are presented to illustrate the theoretical convergence results.

1. Introduction

The purpose of this paper is to study the error associated with an operator splitting procedure for non-homogeneous Hamilton-Jacobi equations of the form

$$
\begin{aligned}
&u_t + H(t,x,u,Du) = G(t,x,u) \quad \text{in} \quad Q_T = \mathbb{R}^N \times (0,T), \\
&u(x,0) = u_0(x) \quad \text{in} \quad \mathbb{R}^N,
\end{aligned}
\tag{1.1}
$$

where $u = u(x,t)$ is the scalar function that is sought, $u_0 = u_0(x)$ is a given initial function, H is a given Hamiltonian, and D denotes the gradient with respect to $x = (x_1, \dots, x_N)$. Hamilton-Jacobi equations arise in a variety of applications, ranging from image processing, via mathematical finance, to the description of evolving interfaces (front propagation problems).

In general problems such as (1.1) do not have classical solutions. In fact, it is well known that solutions of (1.1) generically develop discontinuous derivatives in finite time even with a smooth initial condition. However, under quite general conditions they possess generalized solutions, i.e., solutions that are locally Lipschitz continuous and satisfy the equation almost everywhere. Usually, the generalized solutions are not unique and an additional selection principle, a so-called entropy condition, is needed to single out physically relevant generalized solutions.

To resolve the issue concerning non-uniqueness of generalized solutions, the notion of viscosity solutions was introduced by Crandall and Lions [8], see also [6]. The major advance contained in this notion of weak solution is that indeed uniqueness of the viscosity solution can be proven for a very wide class of equations without requiring a strong convexity assumption as in, e.g., [27]. A viscosity solution is by assumption continuous, but need not be differentiable anywhere. However, a viscosity solution which is locally Lipschitz continuous will satisfy the equation almost everywhere. Generalized solutions obtained by the well-known method of vanishing viscosity belong to the class of viscosity solutions in the sense of [8]. Since the appearance of [8], the theory of viscosity solutions has been intensively studied and extended to a large class of fully nonlinear second order partial differential equations. We refer to Crandall, Ishii, and Lions [7] for an up-to-date overview of the viscosity solution theory for such general partial differential equations.

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It is well known that (homogeneous) Hamilton-Jacobi equations are closely related to (homogeneous) conservation laws. In the one-dimensional case, the notion of viscosity solutions of Hamilton-Jacobi equations is equivalent to the notion of entropy solutions (in the sense of Kružkov [29]) of scalar conservation laws, see [5, 20, 22, 27, 33, 20] for details. In the multi-dimensional case ($d > 1$), this one-to-one correspondence no longer exists. Instead the gradient $p = Du$ satisfies (at least formally) a non-strictly hyperbolic system of conservation laws, see [20, 23, 27, 33] for details. Exploiting this "correspondence" between Hamilton-Jacobi equations and conservation laws, many numerical methods have been developed to accurately capture solutions of Hamilton-Jacobi equations with discontinuous gradients: see [9, 34] for finite difference schemes of upwind type (see also [28]); [1, 26] for finite volume schemes; [36, 37] for ENO schemes; [32, 30] for central schemes; [4, 19] for finite element methods; [20] for relaxation schemes; and [23] for front tracking methods. Using operator splitting, it is also possible to use "homogeneous" Hamilton-Jacobi solvers as building blocks in numerical methods for non-homogeneous problems. In the present context, operator splitting means "splitting off" or isolating the effect of the source term G.

Operator splitting for Hamilton-Jacobi equations, or more generally fully nonlinear second order partial differential equations [7], have been used by Souganidis [40], Barles and Souganidis [3], Sun [42], and Barles [2]. Among these, the paper by Souganidis [40] is the most relevant one for the present work. In that paper, general operator splitting formulas are analyzed and shown to converge to the unique viscosity solution of the governing Hamilton-Jacobi equation as the splitting step tends to zero. The generality in [40] allows for dimensional splitting as well as "splitting of" the source term as we do in the present paper.

In Barles and Souganidis [3], the authors consider fully nonlinear second order elliptic or parabolic partial differential equations and propose an abstract convergence theory for general (monotone, stable, and consistent) approximation schemes. This theory is then applied to splitting methods as well as many other types of numerical methods. In Barles [2], the author studies, among other things, splitting methods for nonlinear degenerate elliptic and parabolic equations arising in option pricing models. In Sun [42], the author studies a dimensional splitting method for a class of second order Hamilton-Jacobi-Bellman equations related to stochastic optimal control problems.

We now summarize the operator splitting procedure analyzed in this paper and state briefly the obtained theoretical result. To ease the presentation, let us for the moment consider the simplified non-homogeneous Hamilton-Jacobi equation

$$u_t + H(Du) = G(u), \qquad u(x,0) = u_0(x), \qquad x \in \mathbb{R}^N,\, t \in (0,T). \tag{1.2}$$

A presentation of the splitting procedure and the corresponding theoretical result in the general case (1.1) can be found in §3. Let $v(x,t) = S(t)v_0(x)$ denote the unique viscosity solution of the homogeneous Hamilton-Jacobi equation

$$v_t + H(Dv) = 0, \qquad v(x,0) = v_0(x), \qquad x \in \mathbb{R}^N,\, t > 0, \tag{1.3}$$

where $S(t)$ is the so-called solution operator associated with (1.3) at time t. Next, let $E(t)$ denote the explicit Euler operator, i.e., $v(x,t) = E(t)v_0(x)$ is defined by

$$v(x,t) = v_0(x) + t\,G(v_0(x)).$$

Our operator splitting method then takes the form

$$u(x, i\Delta t) \approx \big[S(\Delta t)E(\Delta t)\big]^i u_0(x), \tag{1.4}$$

where $\Delta t > 0$ is the splitting (or time) step and $i = 0, \dots, n$ with $n\Delta t = T$.

In this paper, we prove that this splitting approximation converges as $\Delta t \to 0$ to the unique viscosity solution of (1.2). More precisely, we prove that the L^∞ error associated with the time splitting (1.4) is of order Δt:

$$\max_{i=1,\dots,n} \Big\| u(\cdot, i\Delta t) - \big[S(\Delta t)E(\Delta t)\big]^i u_0 \Big\|_{L^\infty} \le K\Delta t, \tag{1.5}$$

for some constant $K > 0$ depending on the data of the problem but not Δt.

In passing, we mention that the proof of (1.5) is inspired by an idea used in Langseth, Tveito, and Winther [31]. In that paper, the authors proved a linear L^1 convergence rate for operator splitting applied to one-dimensional scalar conservation laws with source terms. Having said this, we stress that our method of proof uses "pure" viscosity solution techniques and do not rely on the equivalence between the notions of viscosity [8] and entropy [29] solutions, which exists (only) in the one-dimensional homogeneous case.

As an easy by-product of our analysis, we also obtain an error estimate of the form (1.5) for a variant of (1.4) in which the Euler operator $E(t)$ is replaced by the exact solution operator associated with the ordinary differential equation

$$u_t = G(t,x,u), \qquad v(x,0) = v_0(x), \qquad x \in \mathbb{R}^N,\, t > 0. \tag{1.6}$$

This error estimate is an improvement of an earlier estimate by Souganidis in [40]. In [40], an L^∞ error estimate of order $\sqrt{\Delta t}$ is obtained for a more general operator splitting procedure, which also includes source splitting. This low convergence rate reflects of course the lack of regularity of the viscosity solution and is the "usual" convergence rate obtained for (finite difference and viscous) approximate solutions of Hamilton-Jacobi equations, see [28, 33, 9].

In applications, the exact solution operator $S(t)$ must be replaced by a numerical method. In this paper, we consider the one-dimensional case and replace $S(t)$ by an unconditionally stable front tracking method [15, 22]. Furthermore, we prove that this fully discrete splitting method has a linear convergence rate and present two numerical examples.

We would like to mention that the main results obtained in this paper also hold for weakly coupled systems of Hamilton-Jacobi equations. The details will be presented in a future paper.

Although operator splitting methods have to some extent been studied and used as computationally tools for Hamilton-Jacobi (and related) equations, we feel that these methods have not reached the same degree of popularity as they have for hyperbolic conservation laws. In fact, the first order dimensional splitting method was first introduced by Godunov [14] as a method for solving multi-dimensional conservation laws. Later this method was modified by Strang [41] to achieve formal second order accuracy. Rigorous convergence results (within the Kružkov framework of entropy solutions [29]) for dimensional splitting methods appeared two decades later with the paper by Crandall and Majda [10], see also Holden and Risebro [17]. More recently, L^1 error estimates of order $\sqrt{\Delta t}$ were obtained independently by Teng [44] and Karlsen [21]. Splitting methods for scalar conservation laws with source terms have been analyzed by Tang and Teng [43] and, as already mentioned, Langseth, Tveito, and Winther [31], see also Holden and Risebro [18] for conservation laws with a stochastic source term. Operator splitting methods for conservation laws with parabolic (diffusive) terms have been analyzed by Karlsen and Risebro [24] and Evje and Karlsen [13], see also the lecture notes [12] (and the references therein) for a thorough discussion of viscous splitting methods and their applications. Finally, splitting methods for conservation laws with dispersive terms have been used very recently by Holden, Karlsen, and Risebro [16].

The rest of this paper is organized as follows: In §2, we collect some useful results from the theory of viscosity solutions for Hamilton-Jacobi equations. In §3, we provide a precise description of the operator splitting and state the main convergence results. In §4, we give detailed proofs of the results stated in §3. In §5, we present and analyse a fully discrete operator splitting method for one-dimensional equations. Furthermore, we present numerical examples illustrating the theoretical results. Finally, in §6 we give a proof of a comparison result used in §4.

2. Preliminaries

We start by stating the definition of viscosity solutions as well as some results about existence, uniqueness, and regularity properties of such solutions. These results will be needed in the sections that follows. Proofs of these results (or references to proofs) can be found in [39], see also [40].

Let us introduce some notation. If U is a set, and $f : U \to \mathbb{R}$ is a bounded measurable function on U, then $\|f\| := \text{ess sup}_{x\in U}|f(x)|$. Furthermore let $BUC(U)$, $Lip(U)$, and $Lip_b(U)$ denote the spaces of bounded uniformly continuous functions, Lipschitz functions, and bounded Lipschitz functions on U respectively. Finally, if $f \in Lip(U)$, we denote the Lipschitz constant of f by $\|Df\|$.

For $F \in C([0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$, we consider throughout this section the following general equation

$$u_t + F(t,x,u,Du) = 0 \quad \text{in} \quad Q_T, \tag{2.1}$$

with initial condition

$$u(x,0) = u_0(x) \quad \text{in} \quad \mathbb{R}^N, \tag{2.2}$$

where $u_0 \in BUC(\mathbb{R}^N)$. Note that (1.1) is a special case of (2.1) and (2.2).

Definition 2.1 (Viscosity Solution). *Let $F \in C([0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$.*

1) *A function $u \in C(Q_T)$ is a viscosity subsolution of (2.1) if for every $\phi \in C^1(Q_T)$, if $u - \phi$ attains a local maximum at $(x_0,t_0) \in Q_T$, then $\phi_t(x_0,t_0) + F(t_0,x_0,u(x_0,t_0),D\phi(x_0,t_0)) \leq 0$.*
2) *A function $u \in C(Q_T)$ is a viscosity supersolution of (2.1) if for every $\phi \in C^1(Q_T)$, if $u - \phi$ attains a local minimum at $(x_0,t_0) \in Q_T$, then $\phi_t(x_0,t_0) + F(t_0,x_0,u(x_0,t_0),D\phi(x_0,t_0)) \geq 0$.*
3) *A function $u \in C(Q_T)$ is a viscosity solution of (2.1) if it is both a viscosity sub- and supersolution of (2.1).*
4) *A function $u \in C(\bar{Q}_T)$ is viscosity solution of the initial value problem (2.1) and (2.2) if u is a viscosity solution of (2.1) and $u(x,0) = u_0(x)$ in $\mathbb{R}^N$.*

In order to have existence and uniqueness of (2.2), we need further conditions on F.

(F1) $F \in C([0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ is uniformly continuous on $[0,T] \times \mathbb{R}^N \times [-R,R] \times B_N(0,R)$ for each $R > 0$, where $B_N(0,R) := \{x \in \mathbb{R}^N : |x| \leq R\}$.

(F2) There is a constant $C > 0$ such that $C = \sup_{\bar{Q}_T} |F(t,x,0,0)| < \infty$.

(F3) For each $R > 0$ there is a $\gamma_R \in \mathbb{R}$ such that for $(t,x) \in \bar{Q}_T$, $p \in \mathbb{R}^N$, $-R \leq s \leq r \leq R$, $F(t,x,r,p) - F(t,x,s,p) \geq \gamma_R(r-s)$.

(F4) For each $R > 0$ there is a constant $C_R > 0$ such that for $|r| \leq R$, $x,y,p \in \mathbb{R}^N$, $t \in [0,T]$, $|F(t,x,r,p) - F(t,y,r,p)| \leq C_R(1+|p|)|x-y|$.

We now state a comparison theorem for viscosity solutions.

Theorem 2.1 (Comparison). *Let $F : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy (F1), (F3), and (F4). Let $u,v \in BUC(\bar{Q}_T)$ be viscosity solutions of (2.1) with initial data $u_0, v_0 \in BUC(\mathbb{R}^N)$ respectively. Let $R_0 = \max(\|u\|,\|v\|)$ and $\gamma = \gamma_{R_0}$. Then for every $t \in [0,T]$,*

$$\|u(\cdot,t) - v(\cdot,t)\| \leq e^{-\gamma t}\|u_0 - v_0\|.$$

The next theorem concerns existence of viscosity solutions.

Theorem 2.2 (Existence). *Let $F : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy (F1), (F2), (F3), and (F4). For every $u_0 \in BUC(\mathbb{R}^N)$ there is a $T = T(\|u_0\|) > 0$ and $u \in BUC(\bar{Q}_T)$ such that u is the unique viscosity solution of (2.1) and (2.2). If, moreover, γ_R in (F3) is independent of R, then (2.1) and (2.2) has a unique viscosity solution on $\bar{Q}_T$ for every $T > 0$.*

The following two results are about the behavior of viscosity solutions under additional regularity assumptions on u_0 and u.

Proposition 2.1. *Let $F : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy (F1), (F2), (F3), and (F4). If $u_0 \in Lip_b(\mathbb{R}^N)$, and $u \in BUC(\mathbb{R}^N)$ is the unique viscosity solution of (2.1) and (2.2) in $\bar{Q}_T$, then $u \in Lip_b(\bar{Q}_T)$.*

Proposition 2.2. *Let $F : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy (F1), and (F3) with $\gamma_R \leq 0$ for every $R > 0$. Assume that for $u_0 \in BUC(\mathbb{R}^N)$, $u \in BUC(\bar{Q}_T)$ is a viscosity solution of (2.1) and (2.2). Let $R > \|u\|$ and $\gamma = \gamma_R$. Then the following statements are true for every $t,s \in [0,T]$:*

(a) If H satisfies (F2), then $\|u(\cdot,t)\| \leq e^{-\gamma t}(\|u_0\| + tC)$, where C is given by (F2).

(b) If F satisfies (F4) and $u(\cdot,t) \in Lip_b(\mathbb{R}^N)$ for every $t \in [0,T]$ with $L := \sup_{[0,T]} \|Du(\cdot,t)\|$, then

$$\|Du(\cdot,t)\| \leq e^{-\gamma t}(\|Du_0\| + tC_R(1+L)),$$

where C_R are given by (F4). Moreover

$$L \leq e^{T(2C_R e^{-\gamma T}-\gamma)}(\|Du_0\| + TC_R).$$

(c) If $u_0 \in Lip_b(\mathbb{R}^N)$, $\quad \|u(\cdot,t) - u_0\| \leq te^{-\gamma t} \sup_{\substack{(x,t)\in\bar{Q}_T \\ |r|\leq\|u_0\| \\ |p|\leq\|Du_0\|}} |F(t,x,r,p)|$.

(d) If $u(\cdot,t) \in Lip_b(\mathbb{R}^N)$ for every $t \in [0,T]$ and $L := \sup_{[0,T]} \|Du(\cdot,t)\|$, then $u \in Lip_b(\bar{Q}_T)$ and

$$\|u(\cdot,t) - u(\cdot,s)\| \leq |t-s|e^{-\gamma T} \sup_{\substack{(x,t)\in\bar{Q}_T \\ |r|\leq\|u\| \\ |p|\leq L}} |F(t,x,r,p)|.$$

Finally, we will need the following stability result whose proof is given in the appendix.

Proposition 2.3. *Let $F : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy (F1), and (F3), and let f be a nonnegative, bounded function that belongs to $C(\mathbb{R}^N \times [0,T])$. Assume that $u \in Lip_b(\bar{Q}_T)$ is the viscosity solution of (2.1), and $v \in Lip_b(\bar{Q}_T)$ is a viscosity solution of*

$$|v_t + F(t,x,v,Dv)| \leq f(x,t) \quad \text{in } Q_T. \tag{2.3}$$

Let $R_0 = \max(\|u\|, \|v\|)$ and $\gamma = \gamma_{R_0}$. Then for $0 \leq s \leq t \leq T$,

$$e^{\gamma t}\|u(\cdot,t) - v(\cdot,t)\| \leq e^{\gamma s}\|u(\cdot,s) - v(\cdot,s)\| + \int_s^t e^{\gamma\sigma}\|f(\cdot,\sigma)\|\,d\sigma.$$

Remark 2.3. This is essentially Theorem V.2 (iii) in [8]. The proof we give in the appendix is different from the proof given in [8]. We use techniques from [39], and the proof resembles the proof of Proposition 1.4 in [39].

3. Statement of the results

We will study the convergence of operator splitting applied to the Hamilton-Jacobi equation (1.1), where $u_0 \in Lip_b(\mathbb{R}^N)$ and H and G satisfies the following conditions.

Conditions on H.

- (H1) $H \in C([0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ is uniformly continuous on $[0,T] \times \mathbb{R}^N \times [-R,R] \times B_N(0,R)$ for each $R > 0$.
- (H2) There is a constant $C_H > 0$ such that $C_H = \sup_{\bar{Q}_T} |H(t,x,0,0)| < \infty$.
- (H3) There is a constant $L_H > 0$ such that for $(t,x) \in \bar{Q}_T$, $p \in \mathbb{R}^N$, $r,s \in \mathbb{R}$, $|H(t,x,r,p) - H(t,x,s,p)| \leq L_H|r-s|$.
- (H4) For each $R > 0$ there is a constant $C_{H,R} > 0$ such that for $|r| \leq R$, $x,y,p \in \mathbb{R}^N$, $t \in [0,T]$, $|H(t,x,r,p) - H(t,y,r,p)| \leq C_{H,R}(1+|p|)|x-y|$.
- (H5) For each $R > 0$ there is a constant $N_{H,R} > 0$ such that for $|r| \leq R$, $x,p \in \mathbb{R}^N$, $t,\bar{t} \in [0,T]$, $|H(t,x,r,p) - H(\bar{t},x,r,p)| \leq N_{H,R}(1+|p|)|t-\bar{t}|$.
- (H6) For each $R > 0$ there is a constant $M_R > 0$ such that for $|r|, |p|, |q| \leq R$, $(x,t) \in \bar{Q}_T$, $|H(t,x,r,p) - H(t,x,r,q)| \leq M_R|p-q|$.

Conditions on G.

(G1) $G \in C([0,T] \times \mathbb{R}^N \times \mathbb{R})$ is uniformly continuous on $[0,T] \times \mathbb{R}^N \times [-R,R]$ for each $R > 0$.

(G2) There is a constant $C_G > 0$ such that $C_G = \sup_{\bar{Q}_T} |G(t,x,0)| < \infty$.

(G3) There is a constant $L_G > 0$ such that for $(t,x) \in \bar{Q}_T$, $r,s \in \mathbb{R}$, $|G(t,x,r) - G(t,x,s)| \leq L_G|r-s|$.

(G4) For each $R > 0$ there is a constant $C_{G,R} > 0$ such that for $|r| \leq R$, $x,y \in \mathbb{R}^N$, $t \in [0,T]$, $|G(t,x,r) - G(t,y,r)| \leq C_{G,R}|x-y|$.

(G5) For each $R > 0$ there is a constant $N_{G,R} > 0$ such that for $|r| \leq R$, $x \in \mathbb{R}^N$, $t, \bar{t} \in [0,T]$, $|G(t,x,r) - G(\bar{t},x,r)| \leq N_{G,R}|t - \bar{t}|$.

Conditions (H1), (H2), and (H4) are conditions (F1), (F2), and (F4) from §2 in the case $F(t,x,u,Du) = H(t,x,u,Du)$. The condition corresponding to (F3) is replaced by the stronger condition (H3). The other conditions on H are needed for proving error estimates. The conditions on G are just the corresponding conditions when there is no Du dependence.

By these assumptions the function $F(t,x,r,p) = H(t,x,r,p) - G(t,x,r)$ satisfies conditions (F1)-(F4). Condition (H3) and (G3) implies condition (F3), with $\gamma_R = -L_G - L_H$. Note the minus sign! Also note that this constant is independent of R. So by Theorem 2.2 there exist a unique viscosity solution u of (1.1) on any time interval $[0,T]$, $T > 0$. By Proposition 2.1, $u \in Lip_b(\bar{Q}_T)$.

First we will state an error bound for the splitting procedure when the ordinary differential equation is approximated by the explicit Euler method. To define the operator splitting, let $E(t,s) : Lip_b(\mathbb{R}^N) \to Lip_b(\mathbb{R}^N)$ denote the Euler operator defined by

$$E(t,s)v_0(x) = v_0(x) + (t-s)G(s,x,v_0(x)) \tag{3.1}$$

for $0 \leq s \leq t \leq T$ and $v_0 \in Lip_b(\mathbb{R}^N)$. Furthermore, let $S(t,s) : Lip_b(\mathbb{R}^N) \to Lip_b(\mathbb{R}^N)$ be the solution operator of the Hamilton-Jacobi equation

$$\begin{aligned} &v_t + H(t,x,v,Dv) = 0 \quad \text{in } \mathbb{R}^N \times (s,T), \\ &v(x,s) = v_0(x) \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{3.2}$$

where $v_0 \in Lip_b(\mathbb{R}^N)$. Note that S is well-defined on the time interval $[s,T]$, since (3.2) is basically a special case of (1.1). More precisely, there exists a unique viscosity solution $v \in Lip_b(\mathbb{R}^N \times [s,T'])$, for any $T' > 0$.

The operator splitting solution $\{v(x,t_i)\}_{i=1}^n$, where $t_i = i\Delta t$ and $t_n \leq T$, is defined by

$$\begin{aligned} &v(x,t_i) = S(t_i,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1})(x), \\ &v(x,0) = v_0(x). \end{aligned} \tag{3.3}$$

Note that this approximate solution is defined only at discrete t-values. The first result in this paper states that the operator splitting solution, when (3.2) is solved exactly, converges linearly in Δt to the viscosity solution of (1.1).

Theorem 3.1. *Let $u(x,t)$ be the viscosity solution of (1.1) on the time interval $[0,T]$ and $v(x,t_i)$ be the operator splitting solution (3.3). There exists a constant $K > 0$, depending only on T, $\|u_0\|$, $\|Du_0\|$, $\|v_0\|$, $\|Dv_0\|$, H, and G, such that for $i = 1, \dots, n$,*

$$\|u(\cdot,t_i) - v(\cdot,t_i)\| \leq K(\|u_0 - v_0\| + \Delta t).$$

We will prove this theorem in the next section.

Our second result gives a convergence rate for operator splitting when the explicit Euler operator E is replaced by an exact solution operator $\bar{E}$. More precisely, let $\bar{E}(t,s) : Lip_b(\mathbb{R}^N) \to Lip_b(\mathbb{R}^N)$ be the exact solution operator of the ordinary differential equation

$$\begin{aligned} &v_t = G(t,x,v) \quad \text{in } R^N \times (s,T), \\ &v(x,s) = v_0(x) \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{3.4}$$

where $v_0 \in Lip_b(\mathbb{R}^N)$. Note that $\bar{E}$ is well defined on the time interval $[s,T]$. In fact, the assumptions (G1)-(G5) made on G are sufficient for (3.4) to have a unique solution $u \in C^1([s,T']; Lip_b(\mathbb{R}^N))$, for any $T' > 0$.

Let us define the following operator splitting solution $\{\bar{v}(x,t_i)\}_{i=1}^n$, where $t_i = i\Delta t$ and $t_n \leq T$, by

$$\begin{aligned} &\bar{v}(x,t_i) = S(t_i,t_{i-1})\bar{E}(t_i,t_{i-1})\bar{v}(\cdot,t_{i-1})(x), \\ &\bar{v}(x,0) = v_0(x). \end{aligned} \tag{3.5}$$

The following result is consequence of Theorem 3.1.

Corollary 3.1. *Let $u(x,t)$ be the viscosity solution of (1.1) on the time interval $[0,T]$ and $\bar{v}(x,t_i)$ be the operator splitting solution (3.5). There exists a constant $\bar{K} > 0$, depending only on T, $\|u_0\|$, $\|Du_0\|$, $\|v_0\|$, $\|Dv_0\|$, H, and G, such that for $i = 1, \dots, n$,*

$$\|u(\cdot,t_i) - \bar{v}(\cdot,t_i)\| \leq \bar{K}(\|u_0 - v_0\| + \Delta t).$$

We also prove the corollary in the next section.

Remark 3.2. Corollary 3.1 improves Theorem 4.1 (b) in [40] for the splitting defined in (3.5). Note that the generality in [40] allows for a G function also depending on the gradient. The convergence rate $\mathcal{O}(\sqrt{\Delta t})$ is obtained for this more general operator splitting.

4. Proofs of Theorem 3.1 and Corollary 3.1

In this section, we provide detailed proofs of Theorem 3.1 and Corollary 3.1, starting with the proof of Theorem 3.1. An important step in this proof is to introduce a suitable comparison function.

a) Introducing a comparison function.
Before we can introduce the comparison function, we need an auxiliary result. For $0 \leq s \leq t \leq T$, let $w(\cdot,t) = S(t,s)w_0$ denote the viscosity solution of the Hamilton-Jacobi equation (3.2) with initial condition w_0. For a given function $\psi \in C^1(\mathbb{R}^N \times [s,T])$, we introduce the function

$$q(x,t) := w(x,t) + \psi(x,t).$$

Assuming that w is C^1, it follows that q is a C^1 solution of the following initial value problem

$$\begin{aligned} &q_t + H(t,x,q-\psi,Dq - D\psi) = \psi_t \quad \text{in } \mathbb{R}^N \times (s,T), \\ &q(x,s) = w_0(x) + \psi(x,s) \quad \text{in } \mathbb{R}^N. \end{aligned} \tag{4.1}$$

Moreover, this is still true if w and q are only required to be viscosity solutions of equations (3.2) and (4.1) respectively.

Lemma 4.1. *Let w be a viscosity solution of equation (3.2) and $\psi \in C^1(\mathbb{R}^N \times [s,T])$, then $q := w + \psi$ is a viscosity solution of equation (4.1).*

Proof. Assume $\phi \in C^1(\mathbb{R}^N \times (s,T))$ and that $q - \phi$ has a local maximum at $(x_0,t_0) \in \mathbb{R}^N \times (s,T)$. This means that $w - (\phi - \psi)$ has a local maximum at (x_0,t_0). Since $(\phi - \psi)$ is a C^1 test-function and w is by assumption a viscosity solution of (3.2), the definition of a viscosity subsolution yields

$$(\phi_t - \psi_t)(x_0,t_0) + H(t_0,x_0,(q-\psi)(x_0,t_0),(D\phi - D\psi)(x_0,t_0)) \leq 0,$$

where we replaced $w(x_0,t_0)$ by $(q-\psi)(x_0,t_0)$. The inequality holds for any test function ϕ and for any local maximum of $q - \phi$. So q is a viscosity subsolution of (4.1). Similarly you can show that q is a viscosity supersolution of (4.1). ◻

Let j be such that $1 \leq j \leq n$. Recall that to compute the operator splitting solution v at time $t_j = j\Delta t$, we do j steps. In each step we first apply the Euler operator E for a time interval of length Δt. Then we use the resulting function as an initial condition for problem (3.2) which is also solved for a time interval of length Δt. The main step in the proof of Theorem 3.1 is to estimate the error between u and v for one single time interval of length Δt. Hence we are interested in estimating

$$\|u(\cdot,t_i) - S(t_i,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1})\|, \qquad i=1,\dots,n,$$

where $v(x,0)=v_0(x)$.

Now fix $i=1,\dots,n$, and define the function $\zeta:\mathbb{R}^N\times[t_{i-1},t_i]\to\mathbb{R}$ as follows

$$\zeta(x,t) := S(t,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1})(x).$$

Observe that

$$\zeta(x,t_i) = v(x,t_i).$$

To estimate the difference between $u(\cdot,t_i)$ and $v(\cdot,t_i)$, we need to introduce the comparison function $q^\delta:\mathbb{R}^N\times[t_{i-1},t_i]\to\mathbb{R}$ defined by

$$q^\delta(x,t) = \zeta(x,t) + \psi^\delta(x,t), \tag{4.2}$$

where $\psi^\delta:\mathbb{R}^N\times[t_{i-1},t_i]\to\mathbb{R}$ is defined by

$$\psi^\delta(x,t) = -(t_i - t)\int_{\mathbb{R}^N}\eta_\delta(z)G(t_{i-1},x-z,v(x-z,t_{i-1}))\,dz. \tag{4.3}$$

Here $\eta_\delta(x) := \frac{1}{\delta^N}\eta(\frac{x}{\delta})$, where η is the standard mollifier satisfying

$$\eta\in C_0^\infty(\mathbb{R}^N), \quad \eta(x)=0 \text{ when } |x|>1, \quad \int_{\mathbb{R}^N}\eta(x)\,dx = 1.$$

The introduction of the function q^δ is inspired by the comparison function used in [31].

For each $x\in\mathbb{R}^N$ we see that $q^\delta(x,t_i)=v(x,t_i)$ and we will later show that

$$q^\delta(x,t_{i-1}) \to v(x,t_{i-1}) \ \text{ as } \ \delta\to 0.$$

The difference

$$u(\cdot,t_i) - v(\cdot,t_i) = u(\cdot,t_i) - q^\delta(\cdot,t_i)$$

will be estimated by deriving a bound on the difference

$$u(\cdot,t) - q^\delta(\cdot,t), \qquad \forall t\in[t_{i-1},t_i].$$

To this end, observe that q^δ is a viscosity solution to

$$q_t^\delta + H(t,x,q^\delta-\psi^\delta,Dq^\delta - D\psi^\delta) = \psi_t^\delta \qquad \text{in } \ \mathbb{R}^N\times(t_{i-1},t_i), \tag{4.4}$$

$$q^\delta(x,t_{i-1}) = \zeta(x,t_{i-1}) + \psi^\delta(x,t_{i-1}) \qquad \text{in } \ \mathbb{R}^N. \tag{4.5}$$

This is a consequence of Lemma 4.1 since $\psi^\delta\in C^\infty(\mathbb{R}^N\times[t_{i-1},t_i])$. Now we proceed by deriving a priori estimates for u, v, ψ^δ, and q^δ that are independent of Δt.

b) A priori estimates for u, v, ψ^δ, and q^δ.

We start by analyzing S and E. Let $w\in Lip_b(\mathbb{R}^N)$. Assume that

$$R_1 := \max\{\sup_{0\le s\le t\le T}\|E(t,s)w\|, \sup_{0\le s\le t\le T}\|S(t,s)w\|\} < \infty. \tag{4.6}$$

For $0\le s\le t\le T$, let $\bar w(x,t-s) = S(t,s)w(x)$. This function is a viscosity solution of equation (3.2) on $[0,T-s]$ when $H(t,x,r,p)$ is replaced by $H(\tau+s,x,r,p)$. The initial condition is $\bar w(x,0) = w(x)$. Applying Proposition 2.2 (a), (b), and (c) to $\bar w$ and then using $S(t,\tau+s)w(x) = \bar w(x,\tau)$, we get the following estimates

$$\|S(t,s)w\| \le e^{L_H(t-s)}(\|w\| + (t-s)C_H), \tag{4.7}$$

$$\|D\{S(t,s)w\}\| \le e^{(L_H+K_1(R_1))(t-s)}\{\|Dw\| + (t-s)C_{H,R_1}(1+TK_1(R_1))\}, \tag{4.8}$$

$$\|S(t,s)w - w\| \le (t-s)e^{L_H(t-s)}\sup\{|H(t,x,r,p)| : (x,t)\in\bar Q_T, |r|\le\|w\|, |p|\le\|Dw\|\}, \tag{4.9}$$

where

$$K_1(R) = C_{H,R}\exp\left\{2C_{H,R}Te^{L_HT} + TL_H\right\}, \qquad R>0. \tag{4.10}$$

Note that $\gamma = -L_H$, and that in the expression (4.8), the constant L in Proposition 2.2 (b) is replaced by its bound.

Let us turn to E. The following estimates are consequences of the definition (3.1) of E and the properties of G, w:

$$\|E(t,s)w\| \leq (1 + L_G(t-s))\|w\| + (t-s)C_G, \tag{4.11}$$

$$\|D\{E(t,s)w\}\| \leq (1 + L_G(t-s))\|Dw\| + (t-s)C_{G,R_1}, \tag{4.12}$$

$$\|E(t,s)w - w\| \leq (t-s)(C_G + L_G\|w\|). \tag{4.13}$$

Now we see that assumption (4.6) holds. Just replace $t-s$ by T in expressions (4.7) and (4.11).

Let us introduce some notations which will be useful in what follows:

$$\begin{aligned} \bar{L} &:= 2\max(L_H, L_G), \\ C &:= C_H + C_G, \\ C_R &:= C_R^H + C_R^G \quad \text{for} \quad R > 0, \\ N_R &:= N_{H,R} + N_{G,R} \quad \text{for} \quad R > 0. \end{aligned} \tag{4.14}$$

Lemma 4.2. *There exists a constant R_2 independent of Δt such that $\max_{1\leq i\leq n}\|v(\cdot,t_i)\| < R_2$. Moreover, for every $1 \leq i \leq n$,*

(a) $\|v(\cdot,t_i)\| \leq e^{\bar{L}t_i}(\|v_0\| + t_iC)$,
(b) $\|Dv(\cdot,t_i)\| \leq e^{(\bar{L}+K_1(R_2))t_i}\{\|Dv_0\| + t_iC_{R_2}(1 + TK_1(R_2))\}$.

Proof. Assume there is a constant R_2 independent of Δt such that

$$\max_{1\leq i\leq n}\|v(\cdot,t_i)\| \leq R_2. \tag{4.15}$$

In expressions (4.7) - (4.13) replace R_1 (whenever it appears) by R_2, t by t_i, s by t_{i-1}, and w by $v(\cdot,t_{i-1})$. Successive use of expressions (4.7) and (4.11) yield (a), and similarly (b) follows from (4.8) and (4.12). In (a), replace t_i by T and we see that the assumption (4.15) holds. ☐

From the definition (4.3) of ψ^δ, we see easily that the following lemma is valid:

Lemma 4.3. *For every $1 \leq i \leq n$ and $t \in [t_{i-1}, t_i]$,*

(a) $\|\psi^\delta(\cdot,t)\| \leq (t_i - t)\{C_G + L_G\|v(\cdot,t_{i-1}\|\}$,
(b) $\|D\psi^\delta(\cdot,t)\| \leq (t_i - t)\{C_{G,R_2} + L_G\|Dv(\cdot,t_{i-1})\|\}$.

Now we are in a position to prove a corresponding result for q^δ.

Lemma 4.4. *For every $1 \leq i \leq n$ and $t \in [t_{i-1}, t_i]$,*

(a) $\|q^\delta(\cdot,t)\| \leq e^{2\bar{L}\Delta t}(\|v(\cdot,t_{i-1})\| + 2\Delta tC)$,
(b) $\|Dq^\delta(\cdot,t)\| \leq e^{(2\bar{L}+K_1(R_2))\Delta t}\{\|v(\cdot,t_{i-1})\| + \Delta t\, 2C_{R_2}(1 + TK_1(R_2))\}$,
(c) There exists at constant M independent of t, i, and Δt such that

$$\|q^\delta(\cdot,t) - v(\cdot,t_{i-1})\| \leq M\Delta t.$$

Proof. We only give the proof of (c). The other statements are easy consequences of expressions (4.7), (4.8), (4.11), (4.12), and Lemma 4.3. By estimate (4.9) we get

$$\begin{aligned} &\|S(t,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1}) - E(t_i,t_{i-1})v(\cdot,t_{i-1})\| \leq \Delta t e^{L_H\Delta t} \\ &\quad \times \sup\Big\{|H(t,x,r,p)| \;\Big|\; (t,x) \in \bar{Q}_T,\, |r| \leq \|E(t_i,t_{i-1})v(\cdot,t_{i-1})\|,\, |p| \leq \|D\{E(t_i,t_{i-1})v(\cdot,t_{i-1})\}\|\Big\}. \end{aligned}$$

This supremum is bounded independently of i and Δt. This follows since by Lemma 4.2 and estimates (4.11) and (4.12) there are constants L' and R' independent of i and Δt such that

$$\|E(t_i,t_{i-1})v(\cdot,t_{i-1})\| \leq R' \quad \text{and} \quad \|D\{E(t_i,t_{i-1})v(\cdot,t_{i-1})\}\| \leq L'.$$

Now we can use (H2), (H3), and (H6) to write $|H(t,x,r,p)| \leq C_H + |r|L_H + |p|M_{\max\{L',R'\}}$. So we have showed that

$$\|S(t,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1}) - E(t_i,t_{i-1})v(\cdot,t_{i-1})\| \leq \text{Const}\ \Delta t,$$

where the constant is independent of t, i and Δt. By using expression (4.13) and Lemma 4.2 we can show that

$$\|E(t_i,t_{i-1})v(\cdot,t_{i-1}) - v(\cdot,t_{i-1})\| \leq \text{Const } \Delta t,$$

where the constant is independent of i and Δt. By Lemmas 4.3 and 4.2 we can find a constant independent of t, i and Δt such that

$$\|\psi^\delta\| \leq \text{Const } \Delta t.$$

We finish by noting that by the definition (4.2) of q^δ,

$$\begin{aligned}\|q^\delta(\cdot,t) - v(\cdot,t_{i-1})\| \leq \|&S(t,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1}) - E(t_i,t_{i-1})v(\cdot,t_{i-1})\| \\ &+ \|E(t_i,t_{i-1})v(\cdot,t_{i-1}) - v(\cdot,t_{i-1})\| + \|\psi^\delta\|.\end{aligned}$$

□

Finally we come to (the exact solution) u. Using Proposition 2.2 with $F(t,x,r,p) = H(t,x,r,p) - G(t,x,r)$ (see also the derivation of (4.7) and (4.8)), we get the following result:

Lemma 4.5. *There exist a constant R_3 such that $\max_{[0,T]}\|u(\cdot,t)\| < R_3$. Moreover for $t \in [0,T]$, the following statements hold*

(a) $\|u(\cdot,t)\| \leq e^{\bar{L}t}(tC + \|u_0\|)$,
(b) $\|Du(\cdot,t)\| \leq e^{(\bar{L}+K_2(R_3))t}\{\|Du_0\| + tC_R(1+TK_2(R_3))\}$, *where*

$$K_2(R) = C_R \exp\left\{2C_R T e^{\bar{L}T} + T\bar{L}\right\}.$$

There is a constant R_4 independent of t, i, and Δt such that $\|q^\delta(\cdot,t)\| \leq R_4$. This follows from Lemma 4.4 a) by replacing $\|v(\cdot,t_{i-1})\|$ by R_2 and Δt by T. Similarly there is a constant R_5 independent of t, i, and Δt such that $\|\psi^\delta(\cdot,t)\| \leq R_5$. Define

$$R := \max(R_2, R_3, R_4, R_5). \tag{4.16}$$

By a similar argument there is an L independent of t, i, and Δt such that

$$\max_{1\leq i\leq n}\|Dv(\cdot,t_i)\|,\ \sup_{[t_{i-1},t_i]}\|D\psi^\delta(\cdot,t)\|,\ \sup_{[t_{i-1},t_i]}\|Dq^\delta(\cdot,t)\|,\ \sup_{[0,T]}\|Du(\cdot,t)\| \leq L. \tag{4.17}$$

Furthermore we set

$$\bar{M} := M_{2\max\{L,R\}}. \tag{4.18}$$

We will need the $\bar{M}$ to be this big because of equation (4.1). We are now in a position to prove Theorem 3.1.

c) The proof of Theorem 3.1
We prove Theorem 3.1 by applying Proposition 2.3 to u and q^δ. Let us start by deriving an inequality of the form (2.3) from the equation (4.4) satisfied by the comparison function q^δ.

Let ϕ be a C^1 function, and assume that $q^\delta - \phi$ has a local maximum point in $(t,x) \in [t_{i-1},t_i] \times \mathbb{R}^N$. Then by the definition of viscosity subsolution and equation (4.4) we get

$$\phi_t(x,t) + H(t,x,q^\delta(x,t) - \psi^\delta(x,t), D\phi(x,t) - D\psi^\delta(x,t)) \leq \psi_t^\delta(x,t). \tag{4.19}$$

Now we estimate $\psi_t^\delta(x,t)$ and $H(t,x,q^\delta(x,t)-\psi^\delta(x,t),D\phi(x,t)-D\psi^\delta(x,t))$ as follows:

$$\begin{aligned}
&|\psi_t^\delta(x,t)-G(t_{i-1},x,q^\delta(x,t))|\\
&\quad=\Big|\int_{\mathbb{R}^N}\eta_\delta(z)G(t_{i-1},x-z,v(x-z,t_{i-1}))dz-G(t_{i-1},x,q^\delta(x,t))\Big|\\
&\quad\le\int_{\mathbb{R}^N}\eta_\delta(z)|G(t_{i-1},x-z,v(x-z,t_{i-1}))-G(t_{i-1},x-z,q^\delta(x-z,t))|dz\\
&\qquad+\int_{\mathbb{R}^N}\eta_\delta(z)|G(t_{i-1},x-z,q^\delta(x-z,t))-G(t_{i-1},x,q^\delta(x-z,t))|dz\\
&\qquad+\int_{\mathbb{R}^N}\eta_\delta(z)|G(t_{i-1},x,q^\delta(x-z,t))-G(t_{i-1},x,q^\delta(x,t))|dz\\
&\quad\le\bar{L}M\Delta t+C_R\delta+\bar{L}L\delta,
\end{aligned}$$

where we have used (G3), (G4), and M is given by Lemma 4.4 (c). Using this estimate and (G5), we see that

$$\begin{aligned}
\psi_t^\delta(x,t)&\le G(t,x,q^\delta(x,t))+|G(t_{i-1},x,q^\delta(x,t))-G(t,x,q^\delta(x,t))|\\
&\quad+|{\psi^\delta}_t(x,t)-G(t_{i-1},x,q^\delta(x,t))|\\
&\le G(t,x,q^\delta(x,t))+\Delta t\,\{\bar{L}M+N_R\}+\delta\{C_R+\bar{L}L\}.
\end{aligned}\tag{4.20}$$

We get the following estimate for H:

$$\begin{aligned}
&H(t,x,q^\delta(x,t)-\psi^\delta(x,t),D\phi(x,t)-D\psi^\delta(x,t))\\
&\ge H(t,x,q^\delta(x,t),D\phi(x,t))-\bar{L}|\psi^\delta(x,t)|-\bar{M}|D\psi^\delta(x,t)|\\
&\ge H(t,x,q^\delta(x,t),D\phi(x,t))-\Delta t\{\bar{L}(C+\bar{L}R)+\bar{M}(C_R+\bar{L}L)\},
\end{aligned}\tag{4.21}$$

where we have used (H3), (H6), and Lemma 4.3. Define the constant M_0 by

$$M_0:=\bar{L}\{C+\bar{L}R\}+\bar{M}\{C_R+\bar{L}L\}+\bar{L}M+N_R.\tag{4.22}$$

Substituting (4.20) and (4.21) into (4.19), we get

$$\phi_t(x,t)+H(t,x,q^\delta(x,t),D\phi(x,t))-G(t,x,q^\delta(x,t))\le f(x,t),$$

where

$$f(x,t):=\Delta t\,M_0+\delta\{C_R+\bar{L}L\}.\tag{4.23}$$

In a similar way we can show that if $\bar{\phi}$ is C^1 and $q^\delta-\bar{\phi}$ has a local minimum in $(x,t)\in[t_{i-1},t_i]\times\mathbb{R}^N$, then

$$\bar{\phi}_t(x,t)+H(t,x,q^\delta(x,t),D\bar{\phi}(x,t))-G(t,x,q^\delta(x,t))\ge -f(x,t).$$

This means that q^δ satisfies

$$|q_t^\delta(x,t)+H(t,x,q^\delta(x,t),Dq^\delta(x,t))-G(t,x,q^\delta(x,t))|\le f(x,t)$$

in the viscosity sense, where f is given by (4.23).

Now we are in a position to apply Proposition 2.3 to u and q^δ. Let $\tau\in[t_{i-1},t_i]$ and note that

$$\int_{t_{i-1}}^{\tau}e^{-\bar{L}\sigma}\|f(\cdot,\sigma)\|d\sigma\le\Delta t^2M_0+\Delta t\,\delta\{C_R+\bar{L}L\}.$$

Applying Proposition 2.3 we get

$$e^{-\bar{L}\tau}\|u(\cdot,\tau)-q^\delta(\cdot,\tau)\|\le e^{-\bar{L}t_{i-1}}\|u(\cdot,t_{i-1})-q^\delta(\cdot,t_{i-1})\|+\Delta t^2M_0+\Delta t\,\delta\{C_R+\bar{L}L\}.\tag{4.24}$$

Next, observe that

$$
\begin{aligned}
|v(x,t_{i-1}) - q^\delta(x,t_{i-1})| &= |v(x,t_{i-1}) - E(t_i,t_{i-1})v(\cdot,t_{i-1})(x) - \psi^\delta(x,t_{i-1})| \\
&= |\Delta t\, G(t_{i-1},x,v(x,t_{i-1})) + \psi^\delta(x,t_{i-1})| \\
&\le \Delta t \int_{\mathbb{R}^N} \eta_\delta(z)\Big|G(t_{i-1},x,v(x,t_{i-1})) \\
&\qquad\qquad - G(t_{i-1},x-z,v(x-z,t_{i-1}))\Big|\,dz \\
&\le \Delta t\,\delta\bar{L}\|Dv(\cdot,t_{i-1})\| + \Delta t\,\delta C_R,
\end{aligned}
\tag{4.25}
$$

where the last estimate follows from the triangle inequality, (G3), (G4), and Lipschitz continuity of $v(\cdot,t_{i-1})$. By (4.24) and (4.25), we get

$$
\begin{aligned}
\|u(\cdot,t_i) - v(\cdot,t_i)\| &= \|u(\cdot,t_i) - q^\delta(\cdot,t_i)\| \\
&\le e^{\bar{L}\Delta t}\|u(x,t_{i-1}) - v(x,t_{i-1})\| + \Delta t^2 M_0 e^{\bar{L}t_i} + 2\delta\Delta t\,\{C_R + \bar{L}L\}e^{\bar{L}t_i}.
\end{aligned}
\tag{4.26}
$$

Since $i = 1,\dots,n$ was arbitrary, successive use of (4.26) gives

$$
\begin{aligned}
\|u(\cdot,t_j) - v(\cdot,t_j)\| &\le e^{\bar{L}t_j}\|u_0 - v_0\| + \Delta t^2 M_0 e^{\bar{L}T}\sum_{i=1}^{j} e^{\bar{L}i\Delta t} + 2\delta\Delta t\,\{C_R + \bar{L}L\}e^{\bar{L}T}\sum_{i=1}^{j} e^{\bar{L}i\Delta t} \\
&\le K(\|u_0 - v_0\| + \Delta t) + 2\delta T\{C_R + \bar{L}L\}e^{2\bar{L}T}, \quad \text{for } j = 1,\dots,n,
\end{aligned}
\tag{4.27}
$$

where $K = (1 + M_0 T)e^{2\bar{L}T}$ and M_0 defined in (4.22). So, by the definition of $\bar{L}$ and M_0, Lemmas 4.2 - 4.5, K is a constant depending on H, G, T, $\|u_0\|$, $\|Du_0\|$, $\|v_0\|$, and $\|Dv_0\|$ but not Δt.

Now we are done since sending $\delta \to 0$ in inequality (4.27) produces the desired result.

d) The proof of Corollary 3.1

We end this section by giving the proof of Corollary 3.1. To this end, we need Theorem 3.1 and the following estimate

$$
\|v(x,t_i) - \bar{v}(x,t_i)\| \le \bar{C}\Delta t, \qquad i = 1,\dots,n,
\tag{4.28}
$$

where $\bar{C}$ is a constant depending on G, H, T, $\|u_0\|$, $\|Du_0\|$, $\|v_0\|$, and $\|Dv_0\|$ but not Δt. Equipped with (4.28), we get, for every $i = 1,\dots,n$,

$$
\begin{aligned}
\|u(\cdot,t_i) - \bar{v}(\cdot,t_i)\| &\le \|u(\cdot,t_i) - v(\cdot,t_i)\| + \|v(\cdot,t_i) - \bar{v}(\cdot,t_i)\| \\
&\le K(\|u_0 - v_0\| + \Delta t) + \bar{C}\Delta t.
\end{aligned}
$$

Let $\bar{K} = K + \bar{C}$ and we can immediately conclude that Corollary 3.1 holds.

It remains to show (4.28). Let $w \in Lip_b(\bar{Q}_T)$ be a solution of (3.4) and let $0 \le s \le t \le T$. By the definition of $\bar{E}$, conditions (G2) and (G3), $|\bar{E}(t,s)w(x,t)|_t \le C + \bar{L}|w(x,s)|$ for a.e. x. Grönwalls inequality then yields

$$
\left\|\bar{E}(t,s)w(\cdot,t)\right\| \le e^{\bar{L}(t-s)}\left(\|w(\cdot,s)\| + \Delta tC\right).
$$

If you go through the proof of Lemma 4.2 replacing (4.11) by the above expression, you will get that there is a constant $R' > 0$ independent of Δt, j such that for $t \in [t_j,t_{j+1}]$,

$$
\left\|\bar{E}(t,t_j)\bar{v}(\cdot,t_j)\right\| \le R'.
$$

Now, from the definitions of E and $\bar{E}$ we can write

$$
\begin{aligned}
\left|E(t_i,t_{i-1})v(x,t_{i-1}) - \bar{E}(t_i,t_{i-1})\bar{v}(x,t_{i-1})\right| &\le |v(x,t_{i-1}) - \bar{v}(x,t_{i-1})| \\
&\quad + \int_{t_{i-1}}^{t_i}\left|G(t_{i-1},x,v(x,t_{i-1})) - G(\tau,x,\bar{E}(\tau,t_{i-1})\bar{v}(x,t_{i-1}))\right|d\tau.
\end{aligned}
\tag{4.29}
$$

Using (G3) and (G5) we can estimate the last term the following way,

$$
\begin{aligned}
&\left|G(t_{i-1},x,v(x,t_{i-1})) - G(\tau,x,\bar{E}(\tau,t_{i-1})\bar{v}(x,t_{i-1}))\right| \\
&\quad \le \left|G(t_{i-1},x,v(x,t_{i-1})) - G(t_{i-1},x,\bar{v}(x,t_{i-1}))\right| \\
&\qquad + \left|G(t_{i-1},x,\bar{v}(x,t_{i-1})) - G(t_{i-1},x,\bar{E}(\tau,t_{i-1})\bar{v}(x,t_{i-1}))\right| \\
&\qquad + \left|G(t_{i-1},x,\bar{E}(\tau,t_{i-1})\bar{v}(x,t_{i-1})) - G(\tau,x,\bar{E}(\tau,t_{i-1})\bar{v}(x,t_{i-1}))\right| \\
&\quad \le \bar{L}\left|v(x,t_{i-1}) - \bar{v}(x,t_{i-1})\right| + \bar{L}L_{\bar{v}}(\tau - t_{i-1}) + N_{R'}(\tau - t_{i-1}).
\end{aligned} \tag{4.30}
$$

The constant $L_{\bar{v}}$ denotes the time Lipschitz constant of $\bar{E}(\tau,t_{i-1})\bar{v}(x,t_{i-1})$. Using equation (3.4), conditions (G2) and (G3), and Grönwall's inequality we find the following bound for $L_{\bar{v}}$,

$$
L_{\bar{v}} \le \sup\{|G(t,x,r)| : (t,x) \in \bar{Q}_T, |r| \le R'\} \le e^{\bar{L}T}(\bar{L}R' + TC). \tag{4.31}
$$

By estimates (4.29)–(4.31) we get the existens of a constant C' independent of Δt and i such that

$$
\left|E(t_i,t_{i-1})v(x,t_{i-1}) - \bar{E}(t_i,t_{i-1})\bar{v}(x,t_{i-1})\right| \le e^{\bar{L}\Delta t}\left|v(x,t_{i-1}) - \bar{v}(x,t_{i-1})\right| + C'\Delta t^2,
$$

Now using this and Theorem 2.1 we find that

$$
\begin{aligned}
\|v(\cdot,t_i) - \bar{v}(\cdot,t_i)\| &= \left\|S(t_i,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1}) - S(t_i,t_{i-1})\bar{E}(t_i,t_{i-1})\bar{v}(\cdot,t_{i-1})\right\| \\
&\le e^{\bar{L}\Delta t}\left\|E(t_i,t_{i-1})v(\cdot,t_{i-1}) - \bar{E}(t_i,t_{i-1})\bar{v}(\cdot,t_{i-1})\right\| \\
&\le e^{\bar{L}\Delta t}\Big(\|v(\cdot,t_{i-1}) - \bar{v}(\cdot,t_{i-1})\| + C'\Delta t^2\Big).
\end{aligned} \tag{4.32}
$$

Since that $\bar{v}(x,0) = v_0(x)$, repeated use of inequality (4.32) gives (4.28). This completes the proof.

5. A fully discrete splitting method for one-dimensional equations

In this section we describe a fully discrete operator splitting method that actually possesses a linear convergence rate. There are not many numerical methods that are likely to produce linear convergence, since numerical methods for Hamilton-Jacobi equations are usually based on numerical methods for conservation laws. Most methods for conservation laws (even "higher order" methods) have an L^1 convergence rate of $1/2$ (or less). Roughly speaking, this translates to a L^∞ convergence rate for the Hamilton-Jacobi equations of $1/2$. Therefore the linear error contribution $\mathcal{O}(\Delta t)$ (see Theorem 3.1) coming from the temporal splitting is swamped up by the method-dependent error, unless one uses a method that possesses a convergence rate of at least 1 for the Hamilton-Jacobi equation (3.2). The only methods likely to achieve this are translations of front tracking methods for conservation laws. Since these methods are first order (or higher [35]) only in the one-dimensional case, this section is entirely devoted to one-dimensional equations.

The front tracking method we shall use here was first proposed by Dafermos [11] and later shown to be a viable method for conservation laws by Holden, Holden and Høegh-Krohn [15]. An extension of this method to Hamilton-Jacobi equations was studied in [22].

Without modification it applies to the initial value problem for the scalar conservation law

$$
p_t + H(p)_x = 0,
$$

which is equivalent (see the discussion in §1) to the Hamilton-Jacobi equation

$$
u_t + H(u_x) = 0, \qquad u(x,0) = u_0(x). \tag{5.1}
$$

The Riemann problem for this is the case where

$$
u_0(x) = u_0(0) + \begin{cases} p_l x & \text{for } x < 0, \\ p_r x & \text{for } x \ge 0, \end{cases} \tag{5.2}
$$

where p_l and p_r are constants. We now briefly describe the solution of (5.2). Let $H_\smile(p;p_l,p_r)$ denote the lower convex envelope of H between p_l and p_r, i.e.,

$$
H_\smile(p;p_l,p_r) = \sup\Big\{G(p) \;\Big|\; G'' \ge 0 \text{ and } G(p) \le H(p) \text{ for } p \text{ between } p_l \text{ and } p_r\Big\}. \tag{5.3}
$$

Similarly, let $H_{\frown}(p;p_l,p_r)$ denote the upper concave envelope of H between p_l and p_r. Let also

$$\tilde{H}(p;p_l,p_r)=\begin{cases}H_{\smile}(p;p_l,p_r) & \text{if } p_l\le p_r,\\ H_{\frown}(p;p_l,p_r) & \text{if } p_l>p_r.\end{cases}$$

Note that $\tilde{H}'(p)$ is monotone between p_l and p_r, hence we can define its inverse and set

$$p(x,t)=\begin{cases}p_l & \text{for } x<t\min\left\{\tilde{H}'(p_l),\tilde{H}'(p_r)\right\},\\ \left(\tilde{H}'\right)^{-1}\left(\frac{x}{t}\right) & \text{for } t\min\left\{\tilde{H}'(p_l),\tilde{H}'(p_r)\right\}\le x<t\max\left\{\tilde{H}'(p_l),\tilde{H}'(p_r)\right\},\\ p_r & \text{for } x\ge t\max\left\{\tilde{H}'(p_l),\tilde{H}'(p_r)\right\}.\end{cases}\tag{5.4}$$

Then the viscosity solution of the Riemann problem (5.2) is given by (see [22])

$$u(x,t)=u_0(0)+xp(x,t)-tH(p(x,t)).\tag{5.5}$$

Note that in the case where H is convex, this formula can be derived from the Hopf-Lax formula for the solution to (5.1).

Note that the above construction (5.4) and (5.5) only requires that H is Lipschitz continuous, not differentiable. Exploiting this, let δ be a small positive number and set

$$H^{\delta}(p)=H(i\delta)+(p-i\delta)\,\frac{H((i+1)\delta)-H(i\delta)}{\delta}\quad \text{for } i\delta\le p<(i+1)\delta.\tag{5.6}$$

If H is Lipschitz continuous, then H^{δ} is piecewise linear and Lipschitz continuous. Furthermore, also $\tilde{H}^{\delta}$ will be piecewise linear and $\left(\left(\tilde{H}^{\delta}\right)'\right)^{-1}$ will be piecewise constant. Now set u^{δ} to be the viscosity solution of the Riemann problem for the equation

$$u_t^{\delta}+H^{\delta}\left(u_x^{\delta}\right)=0.$$

¿From (5.5) we then see that u^{δ} will be piecewise linear. The discontinuities in u_x^{δ} will move with constant speed in the (x,t) plane.

This construction can be extended to more general initial values. Assume that $u_0^{\delta}(x)$ is a continuous piecewise linear function such that

$$\lim_{\delta\to 0}\left\|u_0^{\delta}-u_0\right\|=0.\tag{5.7}$$

Then one can solve the initial Riemann problems located at the discontinuities of u_{0x}^{δ} according to (5.5). At some $t_1>0$, two of these discontinuities will interact, thereby defining a new Riemann problem at the interaction point. This can now be solved and the process repeated. Note that this amounts to solving the initial value problem for the conservation law

$$p_t^{\delta}+H^{\delta}\left(p^{\delta}\right)_x=0\qquad p^{\delta}(x,0)=u_{0x}^{\delta}(x).$$

In [15] it was shown that this yields a piecewise constant function $p^{\delta}(x,t)$, which is constant on a finite number of polygons in the (x,t) plane. Let $u^{\delta}(x,t)$ denote the result of applying (5.5) at each interaction of discontinuities. From [22], we have the following lemma:

Lemma 5.1. *The piecewise linear function $u^{\delta}(x,t)$ is the viscosity solution of*

$$u_t^{\delta}+H^{\delta}\left(u_x^{\delta}\right)=0,\quad u^{\delta}(x,0)=u_0^{\delta}(x).\tag{5.8}$$

Now we can state our main result:

Theorem 5.1. *Let $u(x,t)$ be the viscosity solution of*

$$u_t+H\left(u_x\right)=G(x,t,u),\quad u(x,0)=u_0(x).\tag{5.9}$$

Let S^{δ} be the solution operator for (5.8), *and let*

$$v^{\delta}(x,t)=S^{\delta}\left(t_i,t_{i-1}\right)E\left(t_i,t_{i-1}\right)v^{\delta}(\cdot,t_{i-1}),\quad \textit{for } t\in(t_{i-1},t_i],\tag{5.10}$$

with

$$v^{\delta}(x,0)=u_0(j\Delta x)+(x-j\Delta x)\frac{u_0((j+1)\Delta x)-u_0(j\Delta x)}{\Delta x},\quad \textit{for } x\in[j\Delta x,(j+1)\Delta x].$$

Then there is a constant K, depending only on $\|u_0\|$, $\|u_{0,x}\|$, H, G and T_m, such that

$$\left\|u(\cdot,t) - v^\delta(\cdot,t)\right\| \leq K\left(\delta + \Delta t + \Delta x\right), \qquad \forall t \in (0, T_m). \tag{5.11}$$

Proof. Let w^δ denote the viscosity solution of

$$w_t^\delta + H^\delta\left(w_x^\delta\right) = G\left(t, x, w^\delta\right), \quad w^\delta(x,0) = u_0(x). \tag{5.12}$$

Then Theorem 3.1 and the fact that w^δ is Lipschitz in time ensures the existence of a suitable constant K such that

$$\left\|w^\delta(\cdot,t) - v^\delta(\cdot,t)\right\| \leq K\left(\left\|v^\delta(\cdot,0) - u_0\right\| + \Delta t\right). \tag{5.13}$$

By the definition of $v^\delta(x,0)$ and since $u_0 \in Lip_b(\mathbb{R})$,

$$\left\|v^\delta(\cdot,0) - u_0\right\| \leq K\Delta x. \tag{5.14}$$

Also, from Proposition 1.4 in [39], we find that

$$\left\|u(\cdot,t) - w^\delta(\cdot,t)\right\| \leq K \sup_{|p|\leq L}\left|H(p) - H^\delta(p)\right| \leq K\delta, \tag{5.15}$$

since we assume that H is locally Lipschitz. The result now follows from (5.13) and (5.15). ☐

Remark 5.2. If H and u_0 are twice continuously differentiable, then the estimates (5.14) and (5.15) can be replaced by

$$\left\|v^\delta(\cdot,0) - u_0\right\| \leq K\Delta x^2 \quad \text{and} \quad \left\|u(\cdot,t) - w^\delta(\cdot,t)\right\| \leq K\delta^2$$

respectively. Thus the final error estimate (5.11) is found to be

$$\left\|u(\cdot,t) - v^\delta(\cdot,t)\right\| \leq K\left(\delta^2 + \Delta x^2 + \Delta t\right). \tag{5.16}$$

Therefore, if H and u_0 are C^2, then δ and Δx can be chosen much larger than Δt without loss of accuracy.

Example 5.1. We now illustrate the above result with a concrete example, and test the operator splitting method (5.10) on the initial value problem

$$u_t + \frac{1}{3}\left(u_x\right)^3 = u, \quad u(x,0) = \begin{cases} \sin(\pi x) & \text{for } |x| \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.17}$$

The approximate solution operators are front tracking for the Hamilton-Jacobi equation

$$u_t + \frac{1}{3}\left(u_x\right)^3 = 0,$$

and Euler's method for the ordinary differential equation $u_t = u$. Figure 1 shows the approximate solution found using $\Delta x = 0.02$ and $\delta = 2\Delta x$, as well as the upwind approximation (5.18) with the same Δx. To the left we see the approximation $u(x, 1/2)$ obtained by two splitting steps, i.e, $\Delta t = 0.25$, and to the right we have used $\Delta t = 0.025$. To check the convergence rate (5.11), we compared the splitting approximations to a difference approximation on a fine grid. We used the upwind stencil

$$u_j^{i+1} = (1 + \Delta t)u_j^i - \frac{\Delta t}{3}\left(\frac{u_j^i - u_{j-1}^i}{\Delta x}\right)^3, \tag{5.18}$$

with (hopefully) self-explanatory notation. For the reference solution we used $\Delta x = 1/250$. In Table 1, we list the percentage relative L^∞ error for three difference sequences of approximations: $\Delta x = 0.04$, $\Delta x = 0.02$, and $\Delta x = 0.01$. In all cases $\delta = 2\Delta x$. We compared the approximations at $t = 1/2$. In the left column are the number of splitting steps ($\Delta t = 1/2\#\text{steps}$) and in the other columns we show the errors. From this table we see that the numerical convergence rate is linear in all three cases, confirming (5.11).

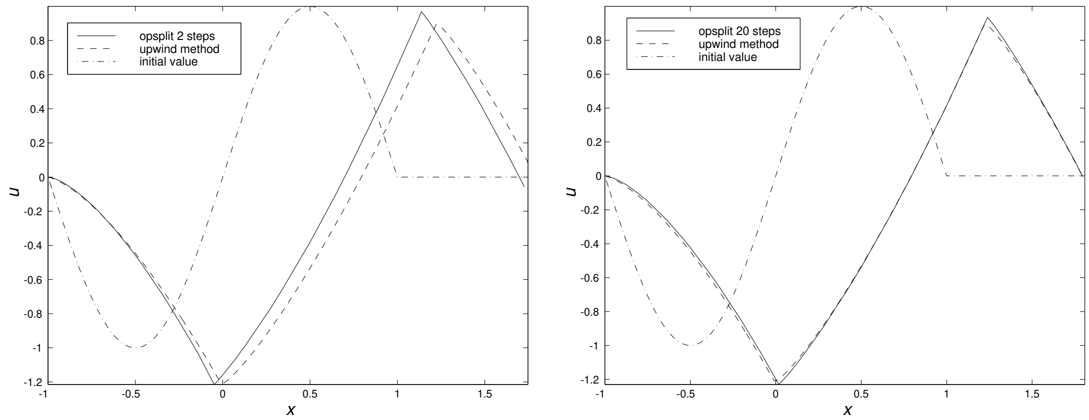

FIGURE 1. Left: $u(x, 1/2)$ with $\Delta t = 0.25$, right: $u(x, 1/2)$ with $\Delta t = 0.025$.

	$100 \times$ relative L^∞-error		
#steps	$\Delta x = 0.04$	$\Delta x = 0.02$	$\Delta x = 0.01$
1	41.2	38.4	39.9
2	22.8	23.2	23.2
4	11.3	14.5	11.8
8	6.2	7.4	5.9
16	3.3	3.0	2.9
32	1.6	1.8	1.4

TABLE 1. Convergence of operator splitting applied to (5.17).

Example 5.2. As another example where we test the convergence rate (5.16), we compute approximate solutions of the initial value problem

$$u_t + \frac{1}{2}(u_x)^2 = u, \quad u(x, 0) = \sin(\pi x). \tag{5.19}$$

As a reference solution, we have used the Engquist-Osher (or generalized upwind) scheme

$$u_j^i = u_j^i(1 + \Delta t) - \frac{1}{2}\left(\min\left(\frac{u_{j+1}^i - u_j^i}{\Delta x}, 0\right)^2 + \max\left(\frac{u_j^i - u_{j-1}^i}{\Delta x}, 0\right)^2\right),$$

with $\Delta x = 1/2000$ (special millennium value). We compared the approximations at $t = 1$. In Figure 2 we show the approximate solutions with 1, 2, 4 and 8 steps as well as the reference solution at $t = 1$. Also, instead of the splitting described above, one can use the Strang splitting

$$u(\cdot, i\Delta t) \approx [E(\Delta t/2)S(\Delta t)E(\Delta t/2)]^i \, u_0.$$

This gives formal second order convergence, and one would expect it to be better than the Godunov splitting in practice. To take advantage of (5.16), we set

$$\Delta t = 1/\#\text{steps}, \quad \Delta x = \sqrt{\Delta t/25}, \quad \text{and} \quad \delta = \sqrt{\Delta t/10}$$

as parameters for the front tracking scheme. In Table 2 we list the results. From this we see that in both cases the convergence rate is linear, but Strang splitting gives a much smaller error.

6. APPENDIX: PROOF OF PROPOSITION 2.3

In this section we present the proof of Proposition 2.3. The proof follows rather closely the proof of Proposition 1.4 in [39].

In what follows, we shall need the following Grönwall type result for viscosity solutions.

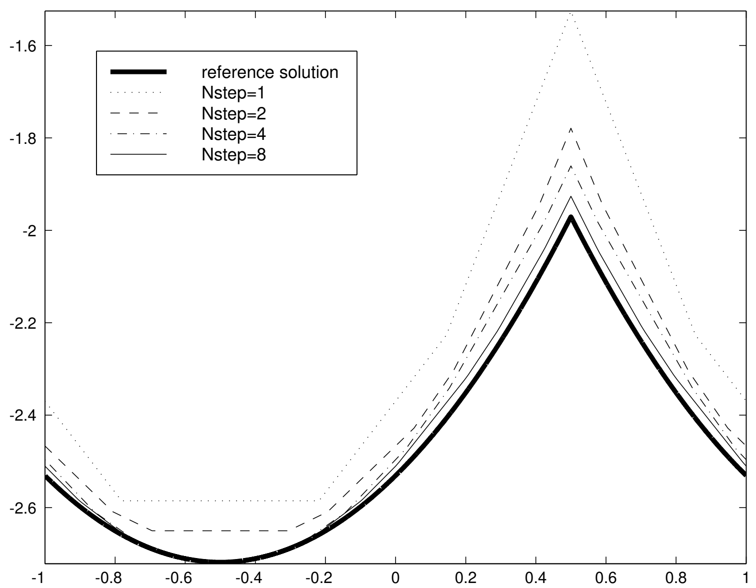

FIGURE 2. Approximate solutions of (5.19) at $t = 1$, with $\Delta t = 1/\text{Nstep}$ and Nstep $= 1, 2, 4, 8$.

$100 \times$ relative L^∞-error		
#steps	Godunov	Strang
1	18.80	3.32
2	7.46	1.73
4	4.04	0.93
8	1.67	0.48
16	0.80	0.21
32	0.48	0.10
64	0.19	0.05

TABLE 2. Convergence of Godunov and Strang splitting.

Lemma 6.1. *Let $T > 0$, $\gamma \in \mathbb{R}$, and v, $h \in C([0,T])$. Suppose that v satisfies*

$$v'(t) + \gamma v(t) \leq h(t) \tag{6.1}$$

in the viscosity sense. Then, for $0 \leq s \leq t \leq T$,

$$e^{\gamma t} v(t) \leq e^{\gamma s} v(s) + \int_s^t e^{\gamma \tau} h(\tau)\, d\tau. \tag{6.2}$$

The proof of this result can be found in §I.11 in [8].

Remark 6.1. Condition (6.1) means that for every $\phi \in C^1((0,T))$, if $v - \phi$ attains a local maximum at $t_0 \in (0,T)$, then $\phi'(t_0) + \gamma v(t_0) \leq h(t_0)$.

In order to prove Proposition 2.3 we will proceed as follows. Assume that we have a certain comparison principle involving $u(x,t) - v(y,t)$ where $|x - y| \leq \varepsilon$, for $\varepsilon > 0$. We start by showing that Proposition 2.3 follows from this comparison principle when we let $\varepsilon \to 0$. Then we prove the comparison principle. It is this proof that is similar to the proof of Proposition 1.4 in [39].

a) A comparison principle to close the proof of Proposition 2.3.
In order to state the comparison principle we need to define some quantities. Let $\varepsilon > 0$, R_0 be as

defined in Proposition 2.3, and $\beta_\varepsilon(x) := \beta(x/\varepsilon)$, where $\beta \in C_0^\infty(\mathbb{R}^N)$ is such that

$$0 \le \beta \le 1, \quad \beta(0) = 1, \quad |D\beta| \le 2, \quad \beta(x) = 0 \text{ when } |x| > 1. \tag{6.3}$$

Let $0 \le s \le \tau \le T$, and define D_ε as follows

$$D_\varepsilon = \{(x,y) : x, y \in \mathbb{R}^N, |x-y| \le \varepsilon\}. \tag{6.4}$$

We will prove the following comparison principle

$$\begin{aligned} &e^{\gamma\tau} \sup_{(x,y)\in D_\varepsilon} \{|u(x,\tau) - v(y,\tau)| + 3R_0 e^{-\gamma(\tau-s)}\beta_\varepsilon(x-y)\} \\ &\le e^{\gamma s} \sup_{(x,y)\in D_\varepsilon} \{|u(x,s) - v(y,s)| + 3R_0\beta_\varepsilon(x-y)\} + \int_s^\tau e^{\gamma\sigma}\|f(\cdot,\sigma)\|\, d\sigma + \tilde{K}\omega(\varepsilon), \end{aligned} \tag{6.5}$$

where γ is defined in Proposition 2.3, $\tilde{K}$ is some constant, and ω is some modulus. We recall that a modulus ω is a positive, nondecreasing, continuous function satisfying $\lim_{r\to 0}\omega(r) = 0$.

Now note that

$$\|u(\cdot,\tau) - v(\cdot,\tau)\| + 3R_0 e^{-\gamma(\tau-s)} \le \sup_{(x,y)\in D_\varepsilon} \{|u(x,\tau) - v(y,\tau)| + 3R_0 e^{-\gamma(\tau-s)}\beta_\varepsilon(x-y)\}.$$

Using this fact, the comparison principle (6.5), and letting $\varepsilon \to 0$, we get

$$e^{\gamma\tau}\{\|u(\cdot,\tau) - v(\cdot,\tau)\| + 3R_0 e^{-\gamma(\tau-s)}\} \le e^{\gamma s}\{\|u(\cdot,s) - v(\cdot,s)\| + 3R_0\} + \int_s^\tau \|f(\cdot,\sigma)\|\, d\sigma,$$

which is Proposition 2.3. We will now prove the comparison principle (6.5).

b) An alternative statement of the comparison principle.
We start by defining $m^\pm$,

$$m^\pm(\tau) = \sup_{(x,y)\in D_\varepsilon} \{(u(x,\tau) - v(y,\tau))^\pm + 3R_0 e^{-\gamma(\tau-s)}\beta_\varepsilon(x-y)\}, \tag{6.6}$$

where $(\cdot)^- = \min(\cdot, 0)$ and $(\cdot)^+ = \max(\cdot, 0)$.

The comparison principle (6.5) follows if we can show

$$e^{\gamma\tau} m^\pm(\tau) \le e^{\gamma s} m^\pm(s) + \int_s^\tau e^{\gamma\sigma}\|f(\cdot,\sigma)\|\, d\sigma + \tilde{K}\omega(\varepsilon).$$

Thanks to Lemma 6.1, since $m^\pm \in C([0,T])$ it is sufficient to show that $m^\pm$ is a viscosity solution in $(0,T)$ of

$$(m^\pm)'(\tau) + \gamma m^\pm(\tau) \le \|f(\cdot,\tau)\| + \omega(\varepsilon).$$

We only prove this for m^+, since the proof for m^- is similar.

So let $n \in C^1((0,T))$ and let $\hat{\tau} \in (0,T)$ be a strict local maximum of $m^+ - n$ in $I := [\hat{\tau}-\alpha, \hat{\tau}+\alpha]$ for some $\alpha > 0$. We want to show that

$$n'(\hat{\tau}) + \gamma m^+(\hat{\tau}) \le \|f(\cdot,\tau)\| + \omega(\varepsilon). \tag{6.7}$$

If $m^+(\hat{\tau}) = 3R_0 e^{-\gamma(\hat{\tau}-s)}$, then $\hat{\tau}$ is the maximum of $3R_0 e^{-\gamma(\tau-s)} - n(\tau)$ in I, and (6.7) is obviously satisfied. So assume

$$m^+(\hat{\tau}) > 3R_0 e^{-\gamma(\hat{\tau}-s)}. \tag{6.8}$$

c) "Doubling of variables".
For $\delta' > 0$, let $\Phi : \mathbb{R}^N \times \mathbb{R}^N \times I \times I \to \mathbb{R}$ be defined by

$$\begin{aligned} \Phi(x,y,\tau,r) = (u(x,\tau) - v(y,r))^+ + 3R_0 e^{-\gamma(\frac{\tau+r}{2}-s)}\beta_\varepsilon(x-y) \\ + (3R_0 + 2R_n)\gamma_{\delta'}(\tau - r) - n\Big(\frac{\tau+r}{2}\Big), \end{aligned} \tag{6.9}$$

where $R_n := \sup_{t\in I}|n(t)|$, $\gamma \in C_0^\infty(\mathbb{R})$ is such that $0 \le \gamma \le 1$, $\gamma(r) = 0$ when $|r| > 1$, and $\gamma_{\delta'}(r) := \gamma(r/\delta')$. Since Φ is bounded on $\mathbb{R}^N \times \mathbb{R}^N \times I \times I$ for every $\delta' > 0$, there is a point $(x_1, y_1, \tau_1, r_1) \in \mathbb{R}^N \times \mathbb{R}^N \times I \times I$ such that

$$\Phi(x_1, y_1, \tau_1, r_1) > \sup_{\mathbb{R}^N\times\mathbb{R}^N\times I\times I} \Phi - \delta'.$$

Next select $\zeta \in C_0^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ satisfying $0 \le \zeta \le 1$, $\zeta(x_1, y_1) = 1$, $|D\zeta| \le 1$, and define $\Psi : \mathbb{R}^N \times \mathbb{R}^N \times I \times I \to \mathbb{R}$ by

$$\Psi(x, y, \tau, r) = \Phi(x, y, \tau, r) + 2\delta'\zeta(x, y). \tag{6.10}$$

Since $\Psi = \Phi$ off the support of ζ and

$$\Psi(x_1, y_1, \tau_1, r_1) = \Phi(x_1, y_1, \tau_1, r_1) + 2\delta' > \sup_{\mathbb{R}^N \times \mathbb{R}^N \times I \times I} \Phi + \delta',$$

there exists a $(x_0, y_0, \tau_0, r_0) \in \mathbb{R}^N \times \mathbb{R}^N \times I \times I$ such that

$$\Psi(x_0, y_0, \tau_0, r_0) \ge \Psi(x, y, \tau, r) \text{ for every } (x, y, \tau, r) \in \mathbb{R}^N \times \mathbb{R}^N \times I \times I. \tag{6.11}$$

d) Some properties of the maximum point (x_0, y_0, τ_0, r_0).
We claim that the following properties hold:

Lemma 6.2. *(i) If* $\delta' < \frac{R_0}{2}$*, then* $|\tau_0 - r_0| \le \delta'$.
(ii) $|x_0 - y_0| \le \varepsilon$ *when*

$$2\delta' + \sup_{I \times I}\{|n(r) - n(t)| : |r - t| < \delta'/2\} < R_0. \tag{6.12}$$

(iii) $\tau_0, r_0 \to \hat{\tau}$ *as* $\delta' \to 0$.
(iv) As $\delta' \to 0$,

$$\begin{aligned}
&(u(x_0, \tau_0) - v(y_0, r_0))^+ + 3R_0 e^{-\gamma(\frac{\tau_0 + r_0}{2} - s)}\beta_\varepsilon(x_0 - y_0) \\
&= u(x_0, \tau_0) - v(y_0, r_0) + 3R_0 e^{-\gamma(\frac{\tau_0 + r_0}{2} - s)}\beta_\varepsilon(x_0 - y_0) \to m^+(\hat{\tau}).
\end{aligned}$$

Proof. *(i)* Assume to the contrary that $\delta' < \frac{R_0}{2}$ and $|\tau_0 - r_0| > \delta'$. So $\gamma_{\delta'}(\tau_0 - r_0) = 0$, and by (6.11) we get

$$\begin{aligned}
2R_0 + 3R_0 e^{-\gamma(\hat{\tau} + \alpha - s)} - n\Big(\frac{\tau_0 + r_0}{2}\Big) + 2\delta' &\ge \Psi(x_0, y_0, \tau_0, r_0) \ge \Psi(x, x, \hat{\tau} + \alpha, \hat{\tau} + \alpha) \\
&\ge 3R_0 e^{-\gamma(\hat{\tau} + \alpha - s)} + 3R_0 + 2R_n - n(\hat{\tau} + \alpha),
\end{aligned}$$

i.e.,

$$2\delta' \ge R_0 + 2R_n - n(\hat{\tau} + \alpha) + n\Big(\frac{\tau_0 + r_0}{2}\Big) \ge R_0 \quad \text{so that} \quad \delta' \ge \frac{R_0}{2},$$

which is a contradiction.
(ii) Let δ' be so small that (6.12) hold. If $|x_0 - y_0| > \varepsilon$, then (6.3), (6.11), and (i) implies

$$\begin{aligned}
2R_0 + 3R_0 + 2R_n - n\Big(\frac{\tau_0 + r_0}{2}\Big) + 2\delta' &\ge \Psi(x_0, y_0, \tau_0, r_0) \ge \Psi(x, x, \tau_0, \tau_0) \\
&\ge 3R_0 e^{-\gamma(\tau_0 - s)} + 3R_0 + 2R_n - n(\tau_0),
\end{aligned}$$

i.e.,

$$2\delta' + n(\tau_0) - n\Big(\frac{\tau_0 + r_0}{2}\Big) \ge R_0,$$

which is a contradiction.
(iii) Since I is compact, there is a $\bar{\tau} \in I$ such that $\tau_0, r_0 \to \bar{\tau}$ along a subsequence as $\delta' \to 0$ (we denote the subsequence in the same way as the sequence). If we assume (6.12), then it follows from (6.6), (6.11), and (ii), that for every $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ and $\tau \in I$,

$$\begin{aligned}
&e^{-\gamma(\frac{\tau_0 + r_0}{2} - s)}(v(y_0, \tau_0) - v(y_0, r_0))^+ + m^+(\tau_0) - n\Big(\frac{\tau_0 + r_0}{2}\Big) + 2\delta' + 3R_0 + 2R_n \\
&\quad \ge \Psi(x_0, y_0, \tau_0, r_0) \ge \Psi(x, y, \tau, \tau) \\
&\quad \ge 3R_0 + 2R_n + (u(x, \tau) - v(y, \tau))^+ + 3R_0\beta_\varepsilon(x - y) - n(\tau),
\end{aligned}$$

i.e., since x and y are arbitrary,

$$e^{-\gamma(\frac{\tau_0 + r_0}{2} - s)}(v(y_0, \tau_0) - v(y_0, r_0))^+ + m^+(\tau_0) - n\Big(\frac{\tau_0 + r_0}{2}\Big) + 2\delta' \ge m^+(\tau) - n(\tau).$$

Remember that $v \in Lip_b(\bar{Q}_T)$ and let $\delta' \to 0$, we then get

$$m^+(\bar{\tau}) - n(\bar{\tau}) \geq m^+(\tau) - n(\tau) \text{ for envery } \tau \in I.$$

But then $\bar{\tau} = \hat{\tau}$, since $\hat{\tau}$ is a strict maximum of $m^+ - n$ on I.
(iv) As before, we use (6.11) to obtain the following:

$$\begin{aligned}
&(u(x_0,\tau_0) - v(y_0, r_0))^+ + 3R_0 e^{-\gamma(\frac{\tau_0+r_0}{2}-s)}\beta_\varepsilon(x_0 - y_0) + 3R_0 + 2R_n - n\Big(\frac{\tau_0 + r_0}{2}\Big) + 2\delta' \\
&\quad \geq \Psi(x_0, y_0, \tau_0, r_0) \geq \Psi(x, y, \hat{\tau}, \hat{\tau}) \\
&\quad \geq (u(x,\hat{\tau}) - v(y,\hat{\tau}))^+ + 3R_0 e^{-\gamma(\hat{\tau}-s)}\beta_\varepsilon(x - y) + 3R_0 + 2R_n - n(\hat{\tau}).
\end{aligned}$$

Here $x, y \in \mathbb{R}^N$ are arbitrary, so

$$(u(x_0,\tau_0) - v(y_0, r_0))^+ + 3R_0 e^{-\gamma(\frac{\tau_0+r_0}{2}-s)}\beta_\varepsilon(x_0 - y_0) \geq m^+(\hat{\tau}) + n\Big(\frac{\tau_0 + r_0}{2}\Big) - n(\hat{\tau}) - 2\delta',$$

and this implies that

$$\liminf_{\delta' \to 0}\{(u(x_0,\tau_0) - v(y_0, r_0))^+ + 3R_0 e^{-\gamma(\frac{\tau_0+r_0}{2}-s)}\beta_\varepsilon(x_0 - y_0)\} \geq m^+(\hat{\tau}).$$

Now by the above limit inferior and since $\hat{\tau}$ is the global maximum in I of $m^+ - n$, we get

$$\begin{aligned}
m^+(\hat{\tau}) - n(\hat{\tau}) &\geq \limsup_{\delta' \to 0}\{(u(x_0,\tau_0) - v(y_0, r_0))^+ + 3R_0 e^{-\gamma(\frac{\tau_0+r_0}{2}-s)}\beta_\varepsilon(x_0 - y_0)\} - n(\hat{\tau}) \\
&\geq \liminf_{\delta' \to 0}\{(u(x_0,\tau_0) - v(y_0, r_0))^+ + 3R_0 e^{-\gamma(\frac{\tau_0+r_0}{2}-s)}\beta_\varepsilon(x_0 - y_0)\} - n(\hat{\tau}) \\
&\geq m^+(\hat{\tau}) - n(\hat{\tau}).
\end{aligned}$$

Finally, if along some subsequence $\lim_{\delta' \to 0}(u(x_0, \tau_0) - v(y_0, r_0))^+ = 0$, then $m^+(\hat{\tau}) \leq 3R_0 e^{-\gamma(\hat{\tau}-s)}$, which contradicts (6.8). So now we have proved the claim. □

e) Using the equations to close the proof of the comparison principle.
By Lemma 6.2 d) let δ' be so small that $(u(x_0, \tau_0) - v(y_0, r_0))^+ = u(x_0, \tau_0) - v(y_0, r_0)$. Now observe that by (6.11), $(x_0, \tau_0) \in Q_T$ is a local maximum for $u - \phi$, and $(y_0, r_0) \in Q_T$ is a local minimum of $v - \bar{\phi}$, where we define

(6.13)

$$\phi(x,\tau) := -3R_0 e^{-\gamma(\frac{\tau+r_0}{2}-s)}\beta_\varepsilon(x - y_0) - (3R_0 + 2R_n)\gamma_{\delta'}(\tau - r_0) - 2\delta'\zeta(x, y_0) + n\Big(\frac{\tau + r_0}{2}\Big),$$

(6.14)

$$\bar{\phi}(y, r) := 3R_0 e^{-\gamma(\frac{\tau_0+r}{2}-s)}\beta_\varepsilon(x_0 - y) + (3R_0 + 2R_n)\gamma_{\delta'}(\tau_0 - r) + 2\delta'\zeta(x_0, y) - n\Big(\frac{\tau_0 + r}{2}\Big).$$

Recall that u and v are viscosity solutions of equation (1.1) and inequality (2.3), respectively. By the definition of viscosity sub- and supersolutions, we get

$$\begin{aligned}
&\phi_t(x_0, \tau_0) + F(\tau_0, x_0, u(x_0, \tau_0), D\phi(x_0, \tau_0)) \leq 0, \\
&\bar{\phi}_t(y_0, r_0) + F(r_0, y_0, v(y_0, r_0), D\bar{\phi}(y_0, r_0)) \geq -f(y_0, r_0).
\end{aligned}$$

Now we compute $\phi_t(x_0, \tau_0)$ and $\bar{\phi}_t(y_0, r_0)$ and subtract the two inequalities, yielding

$$\begin{aligned}
&\gamma 3R_0 e^{-\gamma(\frac{\tau_0+r_0}{2}-s)}\beta_\varepsilon(x_0 - y_0) + n'\Big(\frac{\tau_0 + r_0}{2}\Big) \\
&\quad \leq F(r_0, y_0, v(y_0, r_0), D\bar{\phi}(y_0, r_0)) \\
&\qquad - F(\tau_0, x_0, u(x_0, \tau_0), D\phi(x_0, \tau_0)) + f(y_0, r_0).
\end{aligned} \tag{6.15}$$

We will estimate the various terms on the right hand side of this inequality in order to obtain inequality (6.7). We assume that δ' is so small that (6.12) is satisfied.

Define $L := \max\{\sup_{[0,T]} \|Du(\cdot,t)\|, \sup_{[0,T]} \|Dv(\cdot,t)\|\}$. Since $u, v \in Lip_b(\bar{Q}_T)$, $L < \infty$. Since $(u-\phi)(x_0,\tau_0) \geq (u-\phi)(x_0+th,\tau_0)$ for all $t \in \mathbb{R}$, $h \in \mathbb{R}^N$, we have by (6.13)

$$\begin{aligned}\phi(x_0,\tau_0) - \phi(x_0+th,\tau_0) = & -3R_0 e^{-\gamma(\frac{\tau_0+r_0}{2}-s)}(\beta_\varepsilon(x_0-y_0) - \beta_\varepsilon(x_0+th-y_0)) \\ & - 2\delta'(\zeta(x_0,y_0) - \zeta(x_0+th,y_0)) \\ \leq & u(x_0,\tau_0) - u(x_0+th,\tau_0) \leq L|t||h|.\end{aligned}$$

By letting $t \to 0^+$ and $t \to 0^-$ we see that

$$|3R_0 e^{-\frac{\gamma}{2}(\tau_0+s_0)} D\beta_\varepsilon(x_0,y_0) + 2\delta' D_x\zeta(x_0,y_0)| \leq L.$$

This means that $\|D\phi\| \leq L$ and in a similar way we can show that $\|D\bar{\phi}\| \leq L$.

Let ω_F be the modulus given by (F1) when $R = \max(R_0, L)$. Furthermore, let ω_u denote the modulus of continuity of u. To derive the desired estimates, we will also use condition (F3). To use this condition, we have to distinguish between two cases: (i) $u(x_0,\tau_0) - v(y_0,r_0)$ is nonnegative and (ii) $u(x_0,\tau_0) - v(y_0,r_0)$ is nonpositive. Since the result is the same and the calculations are similar in both cases, we only treat case (i).

We compute $D\phi(x_0,\tau_0)$ and $D\bar{\phi}(y_0,s_0)$ and use (F1), (F3), and the fact that $u, v \in Lip_b$. The result is

$$\begin{aligned}& F(\tau_0, x_0, u(x_0,\tau_0), D\phi(x_0,\tau_0)) \\ & \quad = F(r_0,y_0,v(y_0,r_0),D\bar{\phi}(y_0,s_0)) \\ & \qquad + F(\tau_0,y_0,v(y_0,r_0),D\bar{\phi}(y_0,s_0)) - F(r_0,y_0,v(y_0,r_0),D\bar{\phi}(y_0,s_0)) \\ & \qquad + F(\tau_0,x_0,v(y_0,r_0),D\bar{\phi}(y_0,s_0)) - F(\tau_0,y_0,v(y_0,r_0),D\bar{\phi}(y_0,s_0)) \\ & \qquad + F(\tau_0,x_0,u(x_0,\tau_0),D\bar{\phi}(y_0,s_0)) - F(\tau_0,x_0,v(y_0,r_0),D\bar{\phi}(y_0,s_0)) \\ & \qquad + F(\tau_0,x_0,u(x_0,\tau_0),D\phi(x_0,\tau_0)) - F(\tau_0,x_0,u(x_0,\tau_0),D\bar{\phi}(y_0,s_0)) \\ & \quad \geq F(r_0,y_0,v(y_0,r_0),D\bar{\phi}(y_0,s_0)) - \omega_F(|\tau_0-r_0|) - \omega_F(|x_0-y_0|) \\ & \qquad + \gamma(u(x_0,\tau_0) - v(y_0,r_0)) - \omega_F(2\delta'(|D_x\zeta(x_0,y_0)| + |D_y\zeta(x_0,y_0)|)) \\ & \quad \geq F(r_0,y_0,v(y_0,r_0),D\bar{\phi}(y_0,s_0)) - \omega_F(\delta') - \omega_F(\varepsilon) \\ & \qquad + \gamma|u(x_0,r_0) - v(y_0,r_0)| - |\gamma|\omega_u(\delta') - \omega_F(4\delta'),\end{aligned} \tag{6.16}$$

where we also have used $|D\zeta| \leq 1$.

By (6.15) and (6.16), we get

$$\begin{aligned}n'\Big(\frac{\tau_0+r_0}{2}\Big) + \gamma\{|u(x_0,r_0) - v(y_0,r_0)| + 3R_0 e^{-\gamma(\frac{\tau_0+r_0}{2}-s)}\beta_\varepsilon(x_0-y_0)\} \\ \leq f(y_0,r_0) + \omega_F(\delta') + \omega_F(\varepsilon) + |\gamma|\omega_u(\delta') + \omega_F(4\delta').\end{aligned}$$

Now, by letting $\delta' \to 0$, we get inequality (6.7). This follows from Lemma 6.2 and the fact that $(u(x_0,s_0) - v(y_0,s_0))^+ \leq |u(x_0,s_0) - v(y_0,s_0)|$. This ends the proof of the comparison principle.

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