NOTE

An Explanation for the Sonic Point Glitch

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1 Introduction

Although much attention has been paid to capture the discontinuous shock and contact waves, the proper resolution of rarefaction wave has also proved difficult. For example, even in the smooth flow region, such as a high speed expansion wave passing through a corner, the sonic glitch or the so-called "dogleg" phenomena have occasionally been observed [7], see Fig.(1). Toro observed that the sonic glitch is associated with almost all upwinding shock capturing schemes applied to the nonlinear hyperbolic equations [6], i.e., the Godunov, Osher, Roe, Steger-Warming, van Leer, and Liou-Stefen methods. The well-accepted name for the sonic glitch is the so-called "entropy glitch", which seems implicitly to point out the relation between the glitch and the entropy condition. Recently, Moschetta and Gressier showed that the glitch is not due to the non-smoothness of the flux function in the approximate Riemann solutions at the sonic point [4]. Based on the different numerical performance of the gas-kinetic schemes and the upwinding schemes around the sonic point, they tried to fix the glitch by borrowing a kinetic pressure correction (KPC) term from the gas-kinetic scheme and applied it to the upwinding schemes. No explicit reason for the glitch formation was given there.

In this note, based on the Godunov method for the inviscid Burgers' and the Euler equations, for the first time we are going to point out the physical reason underlying the sonic glitch. The disparity in the numerical and physical wave speeds around the sonic point triggers the sonic wiggle in the numerical solution.

2 Formation of the Sonic Glitch

2.1 Rarefaction Wave in the Burgers' equation

For the Burgers' equation $u_t + uu_x = 0$, the sonic point in a rarefaction wave corresponds to a point with $u = 0$, and the location of this sonic point is fixed in space due to its diminishing wave speed.

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Suppose at time $t = t_0$, the exact rarefaction wave of the Burgers' equation is given by

$$u(x,t) = \frac{x}{t_0}, \quad x \in (-1,1),$$

where $x = 0$ is the sonic point. Numerically, in order to describe the above flow distribution, a uniform mesh with constant cell size Δx is constructed first. For example, cell j represents a small volume $x \in [x_{j-1/2}, x_{j+1/2}]$ and the cell center is located at x_j. After mesh construction, the exact solution at time $t = t_0$ is represented numerically as a constant state inside each cell,

$$u_j = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} (x/t_0)dx = \frac{x_j}{t_0},$$

where the states on the right hand side of the sonic point have $x_j > 0$ and $x_j < 0$ on the left. At a later time $t = t_0 + \Delta t$, the exact solution becomes

$$u(x, t_0 + \Delta t) = \frac{x}{t_0 + \Delta t},$$

and the corresponding exact cell-averaged value is

$$\tilde{u}_j^e = \frac{x_j}{t_0 + \Delta t} = \frac{x_j}{t_0}(\frac{1}{1 + \Delta t/t_0}), \tag{2.1}$$

where e means the exact solution.

Numerically, the Godunov method updates the flow variable inside each cell by solving the exact Riemann solver at the cell boundary. With two constant states across a cell interface,

$$u_0(x,t_0) = \begin{cases} u_j, & x < x_{j+1/2}, \\ u_{j+1}, & x > x_{j+1/2}, \end{cases}$$

the upwinding flux is

$$f_{j+1/2} = \begin{cases} \frac{1}{2}u_j^2 & \text{in the region } x_{j+1/2} > 0, \\ \frac{1}{2}u_{j+1}^2, & \text{in the region } x_{j+1/2} < 0. \end{cases}$$

Therefore, on the right hand side of the sonic point $x > 0$, the flow variable is updated through

$$\begin{aligned} u_j^{n+1} &= u_j^n + \frac{\Delta t}{\Delta x}(f_{j-1/2} - f_{j+1/2}) \\ &= u_j^n + \frac{\Delta t}{\Delta x}\left(\frac{1}{2}(\frac{x_{j-1}}{t_0})^2 - \frac{1}{2}(\frac{x_j}{t_0})^2\right) \\ &= \frac{x_j}{t_0}(1 - \frac{\Delta t}{t_0}) + \frac{\Delta t \Delta x}{2t_0^2}, \end{aligned}$$

and on the left side $x < 0$,

$$\begin{aligned} u_j^{n+1} &= u_j^n + \frac{\Delta t}{\Delta x}(f_{j-1/2} - f_{j+1/2}) \\ &= u_j^n + \frac{\Delta t}{\Delta x}\left(\frac{1}{2}(\frac{x_j}{t_0})^2 - \frac{1}{2}(\frac{x_{j+1}}{t_0})^2\right) \\ &= \frac{x_j}{t_0}(1 - \frac{\Delta t}{t_0}) - \frac{\Delta t \Delta x}{2t_0^2}. \end{aligned}$$

The difference between the above numerical solution and the exact one (Eq.(2.1)) is

$$(\Delta u)_j = u_j^{n+1} - \tilde{u}_j^e \simeq \begin{cases} \frac{\Delta t \Delta x}{2t_0^2}, & \text{in region} \quad x > 0, \\ -\frac{\Delta t \Delta x}{2t_0^2} & \text{in region} \quad x < 0, \end{cases} \tag{2.2}$$

which describes a uniform shift and this shift is independent of the relative location between the point x_j and the sonic point $x = 0$ [3]. Unfortunately, the shift has opposite signs in the $x > 0$ and $x < 0$ regions. As a result, in the $x > 0$ region, the numerical solution moves upward in comparison with the exact one, and moves downward in $x < 0$ region. Therefore, a jump with the magnitude of $\Delta t \Delta x / t_0^2$ will appear at the sonic point after one evolution step. Fig.(2) clearly shows the shift between the numerical and exact solutions, which is qualitatively consistent with the above analysis.

An alternative simple way to understand the glitch formation can be the following. Due to the upwinding flux at a cell interface, in both regions away from the sonic point the magnitude of the numerical wave propagation speed at the cell interface is smaller than the exact physical wave speed there, and the speed difference is independent of the distance to the sonic point. So, the propagation of the numerical solution undergoes a delay around the sonic point and generates a wiggle there. For a rarefaction wave without a sonic point in it, such as a wave with u going from 1 to 2, the uniform upward shift $(\Delta u)_j$ in this case will not generate any glitch in this rarefaction wave.

From the above analysis, we can understand that the glitch is not left from any initial discontinuous data of a shock tube problem. Even started from the continuous exact rarefaction wave, a glitch can still be formed. The glitch is closely associated with the upwinding flux. If a viscous governing equation is solved, there will not have a clear upwinding wave propagation direction at a cell interface. Unfortunately, for most upwinding schemes, the numerical dissipation goes to a minimum value around the sonic point [2]. For the Lax-Friedrichs scheme, the update of the variable u_j^{n+1} is

$$u_j^{n+1} = \frac{x_j}{t_0}(1 - \frac{\Delta t}{t_0}),$$

in both $x > 0$ and $x < 0$ regions. There is neither additional term $\Delta x \Delta t / 2t_0^2$, nor the sonic glitch, see Fig.(5.15) in [6].

2.2 Rarefaction Wave in the Euler Equations

For a standard shock tube problem with the initial data ($\rho_l = 1.0, u_l = 0.75, p_l = 1.0$) and ($\rho_r = 0.125, u_r = 0.0, p_r = 0.1$), the Godunov method clearly generates a glitch at the left running rarefaction wave around the sonic point $\lambda_1 = u - a \simeq 0$, where u is the fluid velocity and a is the sound speed, see Fig.(3). The reason for the glitch formation in the Euler solution is basically the same as that in the Burgers' equation. For the above shock tube problem, the left running rarefaction wave in Fig.(3) is constructed by the characteristic wave with speed $\lambda_1 = u - a$. For this wave, the exact solution of the velocity field u is a similarity solution [1],

$$u = \frac{2}{\gamma + 1}(a_l + \frac{x}{t}),$$

[3] the ignored higher order terms in (2.2) are always proportional to x_j/t_0, which describes a solid rotation relative to the point $x = 0$. No glitch will be formed from higher order terms.

where a_l is the sound speed of the initial left state. The exact solution of the sound speed is

$$\begin{aligned} a &= a_l \left(1 - \frac{\gamma-1}{2}(\frac{u}{a_l})\right) \\ &= \frac{2}{\gamma+1}a_l + \frac{2}{\gamma+1}(\frac{x}{t}). \end{aligned}$$

Therefore, similar to the Burgers' equation, the characteristic wave speed in the rarefaction wave is proportional to (x/t) with the relation,

$$\lambda_1 = u - a = \frac{3-\gamma}{\gamma+1}(\frac{x}{t}) = \alpha(\frac{x}{t}), \tag{2.3}$$

where $\alpha = (3-\gamma)/(\gamma+1)$. For this left running rarefaction wave, there is a Riemann invariant,

$$R = u + \frac{2a}{\gamma-1} = u_l + \frac{2a_l}{\gamma-1}.$$

The governing equation for R is

$$R_t + (u-a)R_x = 0.$$

Since the characteristic wave speed λ_1 is linearly distributed (Eq.(2.3)) in the rarefaction wave, after numerical discretization at a cell interface the wave propagation speed of the Riemann solver becomes

$$(u-a)_{j+1/2} = \begin{cases} (u-a)_j = \alpha(\frac{x_j}{t}), & \text{in the supersonic region} \quad (u-a) > 0, \\ (u-a)_{j+1} = \alpha(\frac{x_{j+1}}{t}), & \text{in the subsonic region} \quad (u-a) < 0. \end{cases}$$

So, similar to the Burgers' equation, the magnitude of the numerical wave speed at a cell interface $x_{j+1/2}$ is smaller than the exact one (Eq.(2.3)) in both the supersonic and subsonic regions. The velocity difference $(u-a)_{j+1/2} - (u-a)_{exact}$ will be proportional to Δx in both the supersonic and subsonic regions. As a result, the glitch will be formed around the sonic point $\lambda_1 = 0$. Fig.(3) clearly demonstrates the disparity in the characteristic wave speed, where the numerical distribution $(u-a)$ is above the exact solution in the supersonic side and below the exact one in the subsonic side. In the 2D simulation, as shown in Fig.(1), due to the implementation of 1D Riemann solver in both directions, across the sonic line the disparity in the wave speed generates the "dogleg" phenomena. The non-smoothness of the flow distribution at the sonic point is basically from the upwinding flux evaluation method in the shock capturing schemes. At the sonic point, there is no any intrinsic difference between the Godunov method and many other Riemann solvers, such as Steger-Warming, van Leer, Roe, and Osher schemes, because the numerical dissipation is reduced to a minima in all these methods [2]. In order to avoid the sonic glitch, there are a few ways to work with.

(1). Change the numerical wave speed. For example, the Beam scheme constructs a numerical sound speed $\tilde{a} = \sqrt{3p/\rho}$ and the left moving particle has a velocity $u - \tilde{a}$, which does not go to zero in the rarefaction wave of Fig.(2) [5].

(2). Do not use the discrete wave speeds $u-a, u, u+a$ in the flux construction at all. For example, the gas-kinetic scheme uses a continuous gas distribution function in the flux evaluation [8].

(3). Solve a viscous governing equation, such as the Lax scheme. There is no explicit upwinding property in the flux evaluation.

(4). Use Eulerian-Lagrangian method, where the characteristics wave speed is changed in a local reference of frame following the fluid motion [3].

In conclusion, we point out that the sonic glitch has no any direct connection with the violation of the entropy condition of a numerical scheme. Even for an entropy-satisfying scheme, such as the Godunov method, there is still existing a sonic glitch. As analyzed in this note, the glitch is coming from the disparity wave speeds across the sonic point.

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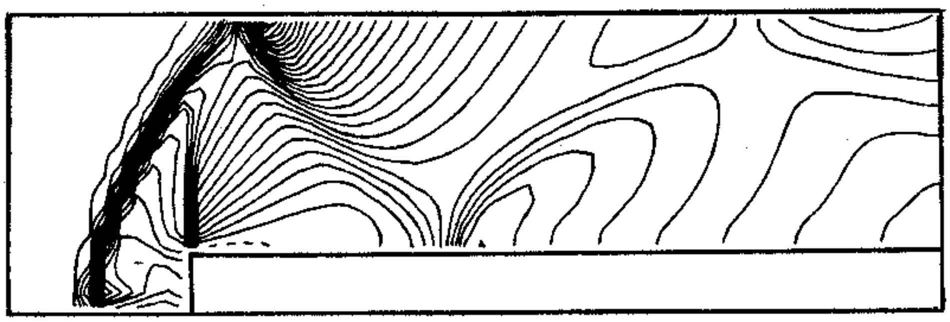

Figure 1: Unsteady flow in a step channel. Density distribution calculated by the 1st-order Godunov method [7]. The "dogleg" phenomena around the sonic line is clearly shown.

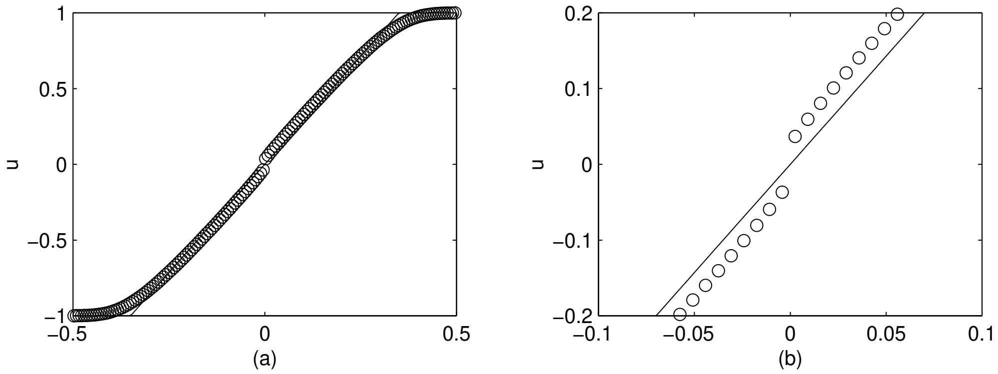

Figure 2: Numerical rarefaction wave of the Burgers' equation. The 1st-order Godunov method is used here. (a). solid line: exact solution, o: numerical solution. (b). u distribution around the sonic point

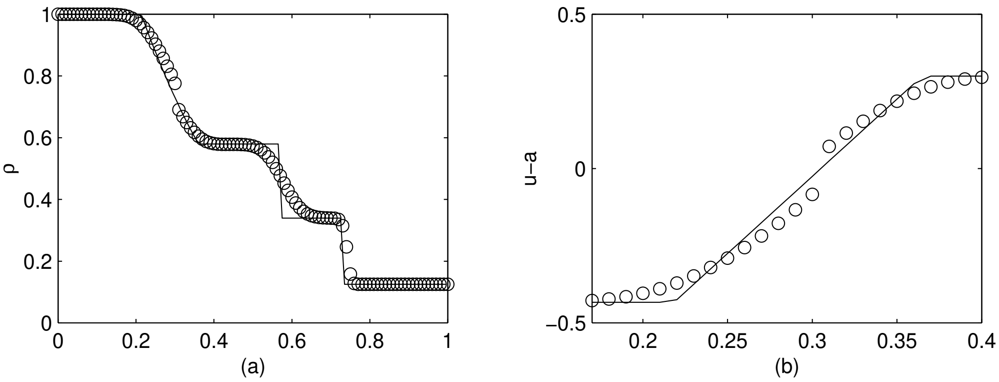

Figure 3: Numerical rarefaction wave of the Euler equations. The 1st-order Godunov method is used here. (a). density distribution. solid line: exact solution, o: numerical solution. (b). characteristic wave speed $u - a$ in the rarefaction wave. The smoothed corners in this case severely effect the rarefaction wave, however $(u - a)$ distribution around the sonic point is still consistent with the analysis in Section (2.2).