Life-span of classical solutions to quasilinear hyperbolic systems with slow decay initial data

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Abstract

In this paper the author considers the life-span of classical solutions to Cauchy problem for general first order quasilinear strictly hyperbolic systems in two independent variables with "slow" decay initial data. By constructing an example, the author first illustrates that the classical solution to this kind of Cauchy problem may blow up in a finite time, even if the system is weakly linearly degenerate. The author then gives some lower bounds of the life-span of classical solutions in the case that system is weakly linearly degenerate. These estimates imply that, when the system is weakly linearly degenerate, the classical solution exists almost globally in time. Finally, the author proves that Theorems 1.1 and 1.2 in [2] are still valid for this kind of initial data.

Key words and phrases: Quasilinear strictly hyperbolic system, weak linear degeneracy, Cauchy problem, classical solution, life-span.

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1 Introduction

Consider the following quasilinear system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \tag{1.1}$$

where $u = (u_1, \cdots, u_n)^T$ is the unknown vector function of (t, x) and $A(u) = (a_{ij}(u))$ is an $n \times n$ matrix with suitably smooth elements $a_{ij}(u)$ $(i, j = 1, \cdots, n)$.

Suppose that system (1.1) is strictly hyperbolic in a neighbourhood of $u = 0$, namely, for any given u in this domain, $A(u)$ has n distinct real eigenvalues $\lambda_1(u), \lambda_2(u), \cdots, \lambda_n(u)$ such that

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \tag{1.2}$$

For $i = 1, \cdots, n$, let $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ (resp. $r_i(u) = (r_{i1}(u), \cdots, r_{in}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad (\text{resp. } A(u)r_i(u) = \lambda_i(u)r_i(u)). \tag{1.3}$$

We have

$$\det|l_{ij}(u)| \neq 0 \quad (\text{resp. } \det|r_{ij}(u)| \neq 0). \tag{1.4}$$

All $\lambda_i(u)$, $l_{ij}(u)$ and $r_{ij}(u)$ $(i, j = 1, \cdots, n)$ have the same regularity as $a_{ij}(u)$ $(i, j = 1, \cdots, n)$.

Without loss of generality, we may suppose that

$$l_i\,(u)\,r_j\,(u) \equiv \delta_{ij} \quad (i, j = 1, \cdots, n) \tag{1.5}$$

and

$$r_i^T\,(u)\,r_i\,(u) \equiv 1 \quad (i = 1, \cdots, n), \tag{1.6}$$

where δ_{ij} stands for the Kronecker's symbol.

For the following initial data

$$t = 0 : u = \varphi(x), \tag{1.7}$$

where $\varphi(x)$ is a "small" C^1 vector function of x with certain decay properties as $|x| \to +\infty$, we shall investigate the life-span and the breakdown of C^1 solution to Cauchy problem (1.1) and (1.7).

In the case that there exists a constant $\mu > 0$ such that

$$\theta \stackrel{\triangle}{=} \sup_{x \in \boldsymbol{R}} \left\{ (1 + |x|)^{1+\mu} \left(|\varphi(x)| + |\varphi'(x)| \right) \right\} < \infty, \tag{1.8}$$

Li, Zhou and Kong presented a complete result in [1] for the global existence and the blow-up phenomenon of C^1 solution $u = u(t, x)$ to Cauchy problem (1.1) and (1.7). Recently, in [2] Li and Kong gave an asymptotic behaviour of the life-span of C^1 solution as an improvement of Theorem 1.2 in [1] and proved that the singularity is produced by the envelope of characteristics of the same family. The results given in [1-2] were obtained under the assumption that $\mu > 0$. It is natural to propose the following question: what will happen when $\mu = 0$? This paper is devoted to the study of this problem.

Precisely speaking, in this paper we consider the following initial data

$$t = 0 : u = \varepsilon \psi(x), \tag{1.9}$$

where $\varepsilon > 0$ is a small parameter and $\psi(x)$ is a C^1 vector function satisfying

$$\sup_{x \in \boldsymbol{R}} \left\{ (1 + |x|) \left(|\psi(x)| + |\psi'(x)| \right) \right\} < \infty. \tag{1.10}$$

By constructing an example, we first illustrate that the classical solution to Cauchy problem (1.1) and (1.9) may blow up in a finite time, even if system (1.1) is weakly linearly degenerate and $\varepsilon > 0$ is small enough. That is to say, Theorem 1.1 in [1], the result on the global existence of classical solutions, might be false in the present situation. This shows that the condition $\mu > 0$ is essential and sharp for Theorem 1.1 in [1]. We further give some lower bounds of the life-span of classical solutions in the case that the system is weakly linearly degenerate. On the other hand, we will prove that Theorems 1.1 and 1.2 in [2], the results on the breakdown of classical solutions, are still valid for this kind of initial data, i.e., the initial data (1.9) with (1.10).

For the completeness of statement, we first recall the concepts of the weak linear degeneracy and the normalized coordinates (see [3]).

The i-th characteristic $\lambda_i(u)$ is *weakly linearly degenerate*, if, along the i-th characteristic trajectory $u = u^{(i)}(s)$ passing through $u = 0$, defined by

$$\begin{cases} \frac{du}{ds} = r_i(u), \\ s = 0 : u = 0, \end{cases} \tag{1.11}$$

we have

$$\nabla \lambda_i (u)\, r_i (u) \equiv 0, \quad \forall \ |u| \quad \text{small},$$

namely,

$$\lambda_i \left(u^{(i)} (s) \right) \equiv \lambda_i (0)\,, \quad \forall \ |s| \quad \text{small}. \tag{1.12}$$

If all characteristics are weakly linearly degenerate, then system (1.1) is called to be *weakly linearly degenerate.*

Suppose that $A(u) \in C^k$, where k is an integer ≥ 1. By Lemma 2.5 in [3], there exists an invertible C^{k+1} transformation $u = u(\widetilde{u})$ $(u(0) = 0)$ such that in $\widetilde{u}$-space, for each $i = 1, \cdots, n$, the i-th characteristic trajectory passing through $\widetilde{u} = 0$ coincides with the $\widetilde{u}_i$-axis at least for $|\widetilde{u}_i|$ small, namely,

$$\widetilde{r}_i (\widetilde{u}_i e_i) \equiv e_i, \quad \forall \ |\widetilde{u}_i| \quad \text{small} \quad (i = 1, \cdots, n)\,, \tag{1.13}$$

where

$$e_i = (0, \cdots, 0, \overset{(i)}{1}, 0, \cdots, 0)^T. \tag{1.14}$$

Such a transformation is called the *normalized transformation* and the corresponding unknown variables $\widetilde{u} = (\widetilde{u}_1, \cdots, \widetilde{u}_n)$ are called the *normalized variables* or *normalized coordinates.*

As in [2], we can always find suitable normalized coordinates $\widetilde{u}$ such that

$$\frac{\partial u}{\partial \widetilde{u}}(0) = R(0), \tag{1.15}$$

where $R(u)$ is the matrix composed by the right eigenvectors $r_i(u)$ $(i = 1, \cdots, n)$. Hence, noting (1.5) we have

$$\frac{\partial \widetilde{u}}{\partial u}(0) = L(0), \tag{1.16}$$

where $L(u)$ is the matrix composed by the left eigenvectors $l_i(u)$ $(i = 1, \cdots, n)$.

The main results in this paper are given in Theorems 1.1-1.5.

Theorem 1.1. Suppose that $A(u) \in C^2$ and (1.2) holds in a neighbourhood of $u = 0$. Suppose furthermore that $\psi(x)$ is a C^1 vector function satisfying (1.10). Suppose finally that system (1.1) is weakly linearly degenerate. Then there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, there exists a positive constant κ independent of ε such that the life-span $\widetilde{T}(\varepsilon)$ of the classical solution $u = u(t,x)$ to Cauchy problem (1.1) and (1.9) satisfies

$$\widetilde{T}(\varepsilon) \geq \exp\left(\kappa\varepsilon^{-1}\right). \quad \square \tag{1.17}$$

Definition 1.1. If there exists a positive constant κ independent of ε such that the life-span $\widetilde{T}(\varepsilon)$ satisfies (1.17), then the classical solution $u = u(t,x)$ is called *almost global solution.* $\square$

Theorem 1.2. Under the hypotheses of Theorem 1.1, suppose furthermore that $\psi(x)$ satisfies that there exists a constant $\nu \geq 1$ such that

$$\sup_{x \in R}\left\{(1+|x|)^{1+\nu}\,|\psi'(x)|\right\} < \infty. \tag{1.18}$$

Then there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, there exists a positive constant $\tilde{\kappa}$ independent of ε such that the life-span $\widetilde{T}(\varepsilon)$ of the classical solution $u = u(t,x)$ to Cauchy problem (1.1) and (1.9) satisfies

$$\widetilde{T}(\varepsilon) \geq \exp\left(\tilde{\kappa}\varepsilon^{-2}\right). \quad \square \tag{1.19}$$

Remark 1.1. In Section 2 we will show that estimate (1.19) is sharp (see Theorem 2.2). However, it is still open that whether estimate (1.17) is sharp or not. $\square$

When system (1.1) is not weakly linearly degenerate, there exists a nonempty set $J \subseteq \{1, 2, \cdots, n\}$ such that $\lambda_i(u)$ is not weakly linearly degenerate if and only if $i \in J$.

Noting (1.12), for any fixed $i \in J$, either there exists an integer $\alpha_i \geq 0$ such that

$$\left.\frac{d^l \lambda_i\left(u^{(i)}(s)\right)}{ds^l}\right|_{s=0} = 0 \quad (l = 1, \cdots, \alpha_i) \quad \text{but} \quad \left.\frac{d^{\alpha_i+1} \lambda_i\left(u^{(i)}(s)\right)}{ds^{\alpha_i+1}}\right|_{s=0} \neq 0 \tag{1.20}$$

or

$$\left.\frac{d^l \lambda_i\left(u^{(i)}(s)\right)}{ds^l}\right|_{s=0} = 0 \quad (l = 1, 2, \cdots), \tag{1.21}$$

where $u = u^{(i)}(s)$ is defined by (1.11). In the case that (1.21) holds, we define $\alpha_i = +\infty$.

Theorem 1.3. Suppose that (1.2) holds and $A(u)$ is suitably smooth in a neighbourhood of $u = 0$. Suppose furthermore that $\psi(x)$ is a C^1 vector function satisfying (1.10). Suppose finally that system (1.1) is not weakly linearly degenerate and

$$\alpha = \min\{\alpha_i \mid i \in J\} < \infty, \tag{1.22}$$

where α_i is defined by (1.20)-(1.21). Let

$$J_1 = \{i \mid i \in J,\, \alpha_i = \alpha\}. \tag{1.23}$$

If there exists $i_0 \in J_1$ such that

$$l_{i_0}(0)\,\psi(x) \not\equiv 0, \tag{1.24}$$

where $l_{i_0}(u)$ stands for the i_0-th left eigenvector, then there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$ the first order derivatives of the C^1 solution $u = u(t,x)$ to the Cauchy problem (1.1) and (1.9) must blow up in a finite time and the life-span $\widetilde{T}(\varepsilon)$ satisfies

$$\lim_{\varepsilon \to 0}\left(\varepsilon^{\alpha+1}\widetilde{T}(\varepsilon)\right) = M_0, \tag{1.25}$$

where

$$M_0 = \left(\max_{i\in J_1}\sup_{x\in \boldsymbol{R}}\left\{-\frac{1}{\alpha!}\left.\frac{d^{\alpha+1}\lambda_i\left(u^{(i)}(s)\right)}{ds^{\alpha+1}}\right|_{s=0}[l_i(0)\psi(x)]^{\alpha}\, l_i(0)\psi'(x)\right\}\right)^{-1}, \tag{1.26}$$

in which $u = u^{(i)}(\mathrm{s})$ is defined by (1.11). $\square$

Theorem 1.4. Under the assumptions of Theorem 1.3, on the existence domain $0 \le t < \widetilde{T}(\varepsilon)$ of the C^1 solution $u = u(t,x)$ to Cauchy problem (1.1) and (1.9), the solution itself remains bounded, but the first order derivatives of $u = u(t,x)$ tend to the infinity as $t \nearrow \widetilde{T}(\varepsilon)$. Moreover, the singularity occurs at the starting point of the envelope of characteristics of the same family, i.e., the point with minimum t-value on the envelope. $\square$

Theorem 1.5. Under the assumptions of Theorem 1.4, for each $i \notin J_1$, the family of i-th characteristics never forms any envelope on the domain $0 \le t \le \widetilde{T}(\varepsilon)$. In particular, each family of weakly linearly degenerate characteristics and then each family of linearly degenerate characteristics never form any envelope on $0 \le t \le \widetilde{T}(\varepsilon)$.

Remark 1.2. Theorems 1.3-1.5 generalize Theorems 1.1-1.3 in [2] respectively to the case $\mu = 0$. $\square$

Similar to Remark 1.6 and Remark 1.7 in [2], we have

Remark 1.3. In the case that

$$J_1 = J = \{1, 2, \cdots, n\} \quad \text{and} \quad \alpha < +\infty, \tag{1.27}$$

for any nontrivial initial data (1.9) (in which $\psi(x) \not\equiv 0$), the conclusion of Theorems 1.3, 1.4 and 1.5 holds. More particularly, when system (1.1) is genuinely nonlinear (i.e., $\alpha = 0$ in (1.27)), we have the same conclusion. □

Remark 1.4. Theorems 1.1-1.5 still hold if

$$\varphi(x) = \varepsilon\psi(x) + \psi_1(x, \varepsilon), \tag{1.28}$$

in which

$$\psi_1(x, \varepsilon), \quad \frac{\partial \psi_1(x, \varepsilon)}{\partial x} = O(\varepsilon^2). \quad \square$$

The arrangement of this paper is as follows: In Section 2 we will construct an example to illustrate the necessity and sharpness of the condition $\mu > 0$ for Theorem 1.1 in [1]. Theorems 1.1 and 1.2 will be proved in Section 3 and Section 4 respectively; Theorem 1.3 will be shown in Section 5 and Theorems 1.4-1.5 in Section 6.

2 An example

Consider the following Cauchy problem

$$\begin{cases} r_t + (1 + rs)r_x = 0, \\ s_t = 0, \end{cases} \tag{2.1}$$

$$t = 0: \quad r = \varepsilon r_0(x), \; s = \varepsilon s_0(x), \tag{2.2}$$

where $r_0(x)$ and $s_0(x)$ are C^1 functions with bounded C^1 norm, $\varepsilon > 0$ is a small parameter.

Obviously, in a neighbourhood of $(r, s) = (0, 0)$, (2.1) is a strictly hyperbolic system with two distinct real eigenvalues

$$\lambda_1(r, s) \triangleq 1 + rs > \lambda_2(r, s) \triangleq 0. \tag{2.3}$$

On the other hand, it is easy to check that system (2.1) is weakly linearly degenerate. Therefore, by Theorem 1.1 in [1] we have

Theorem 2.1. Under the hypotheses mentioned above, if there exists a constant $\mu > 0$ such that

$$\sup_{x \in \mathbb{R}} \left\{ (1+|x|)^{1+\mu} \left(|r_0(x)| + |s_0(x)| + |r_0'(x)| + |s_0'(x)| \right) \right\} < \infty, \tag{2.4}$$

then there exists $\varepsilon_0 > 0$ so small that for any given $\varepsilon \in [0, \varepsilon_0]$, Cauchy problem (2.1)-(2.2) admits a unique global C^1 solution $u = u(t,x)$ for all $t \in \mathbb{R}$. $\square$

Now we consider the following initial data

$$t = 0: \quad r = \varepsilon \overline{r}_0(x) \triangleq \varepsilon (1+x^2)^{-1}, \quad s = \varepsilon \overline{s}_0(x), \tag{2.5}$$

where $\overline{s}_0(x)$ is a C^1 function satisfying

(i) $\overline{s}_0(x) \geq 0, \quad \forall\, x \in \mathbb{R};$

(ii) $\overline{s}_0'(x) \leq 0, \quad \forall\, x \geq 0;$

(iii) $\overline{s}_0'(x) \geq 0, \quad \forall\, x \leq 0;$

(iv) $||\overline{s}_0(x)||_{C^1(\mathbb{R})} \leq \overline{M}$ (where $\overline{M}$ is a positive constant);

(v) $\overline{s}_0(x) = \begin{cases} \frac{1}{1+x}, & \text{as} \quad x \geq 1, \\ 0, & \text{as} \quad x \leq -1. \end{cases}$

By (2.4), we see that the initial data (2.5) corresponds to the case that $\mu = 0$ in (2.4). However the conclusion of Theorem 2.1 is false because we have

Theorem 2.2. There exists $\varepsilon_0 > 0$ so small that for any given $\varepsilon \in (0, \varepsilon_0]$, the first order derivatives of the C^1 solution to the Cauchy problem (2.1) and (2.5) must blow up in a finite time and there exist two positive constants a and b independent of ε such that the life-span $\widetilde{T}(\varepsilon)$ satisfies

$$\exp\left(a\varepsilon^{-2}\right) \leq \widetilde{T}(\varepsilon) \leq \exp\left(b\varepsilon^{-2}\right). \quad \square \tag{2.6}$$

Remark 2.1. (2.6) shows that estimate (1.19) is sharp. $\square$

Proof of Theorem 2.2. Noting the second equation system in (2.1), we have

$$s(t,x) = \varepsilon \overline{s}_0(x), \quad \forall\, (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{2.7}$$

Substituting it into the first equation in (2.1), we observe that the Cauchy problem (2.1) and (2.5) simply reduces to the following Cauchy problem for a scalar equation

$$r_t + (1 + \varepsilon \overline{s}_0(x) r)\, r_x = 0, \tag{2.8}$$

$$t = 0: \quad r = \varepsilon \overline{r}_0(x) = \varepsilon(1+x^2)^{-1}, \tag{2.9}$$

where $\overline{s}_0(x)$ satisfies the properties (i)-(v). Therefore, in what follows it suffices to consider the Cauchy problem (2.8)-(2.9).

On the existence domain of the C^1 solution to the Cauchy problem (2.8)-(2.9), let $x = x(t,\beta)$ be the characteristic passing through a point $(0,\beta)$ on the x-axis and set

$$\lambda(t,x) = 1 + \varepsilon \overline{s}_0(x) r(t,x). \tag{2.10}$$

By the definition of characteristic curve, $x = x(t,\beta)$ satisfies

$$\begin{cases} \frac{dx}{dt} = \lambda(t,x), \\ t = 0: \quad x = \beta, \end{cases} \tag{2.11}$$

on which

$$r = \varepsilon \overline{r}_0(\beta) = \varepsilon(1+\beta^2)^{-1}. \tag{2.12}$$

Hence, noting (2.10) and using (2.12), we may rewrite (2.11) as

$$\begin{cases} \frac{dx}{dt} = 1 + \varepsilon^2 \overline{s}_0(x)(1+\beta^2)^{-1}, \\ t = 0: \quad x = \beta. \end{cases} \tag{2.13}$$

It follows from (2.12) that along the characteristic $x = x(t,\beta)$

$$r_x(t, x(t,\beta)) = -2\varepsilon\beta \left(1+\beta^2\right)^{-2} / x_\beta(t,\beta). \tag{2.14}$$

On the other hand, we obtain from (2.13) that

$$x_\beta(t,\beta) = A(t,\beta) \exp \triangle(t,\beta), \tag{2.15}$$

where

$$\triangle(t,\beta) = \frac{\varepsilon^2}{1+\beta^2} \int_0^t \overline{s}_0'(x(\tau,\beta)) d\tau \tag{2.16}$$

and

$$A(t,\beta) = 1 - \frac{2\varepsilon^2\beta}{(1+\beta^2)^2} \int_0^t \overline{s}_0(x(\tau,\beta)) \exp\left(- \triangle(\tau,\beta)\right) d\tau. \tag{2.17}$$

Now we estimate $\triangle(t,\beta)$.

Let ε_0 be so small that

$$\varepsilon_0^2 \overline{M} \leq \frac{1}{2}, \tag{2.18}$$

where $\overline{M}$ is given in property (iv). Hence, noting (2.10), (2.12) and property (i), on the existence domain of the C^1 solution to the Cauchy problem (2.8)-(2.9) we have

$$1 \leq \lambda(t,x) \leq \frac{3}{2}. \tag{2.19}$$

Then noting (2.19) and the first equation in (2.13), we obtain from (2.16) that

$$\begin{aligned}\triangle(t,\beta) &= \tfrac{\varepsilon^2}{1+\beta^2}\int_0^t \overline{s}_0'(x(\tau,\beta))\tfrac{1}{\lambda(\tau,x(\tau,\beta))}\lambda(\tau,x(\tau,\beta))d\tau \\ &= \tfrac{\varepsilon^2}{1+\beta^2}\int_\beta^{x(t,\beta)} \overline{s}_0'(x)\tfrac{1}{1+\varepsilon^2\overline{s}_0(x)(1+\beta^2)^{-1}}dx.\end{aligned} \tag{2.20}$$

Noting (2.19) again and using properties (iv)-(v), we get

$$\begin{aligned}|\triangle(t,\beta)| &\leq \varepsilon^2\int_{-\infty}^{\infty}|\overline{s}_0'(x)|\,dx \leq \varepsilon^2\left\{\int_{-1}^{1}|\overline{s}_0'(x)|\,dx + \int_1^\infty \tfrac{1}{(1+x)^2}dx\right\} \\ &\leq \varepsilon^2\left\{2\overline{M}+\tfrac{1}{2}\right\}, \quad \forall\, t\geq 0, \quad \forall\, \beta\in I\!R.\end{aligned} \tag{2.21}$$

Thus we obtain

$$|\triangle(t,\beta)| \leq C_1\varepsilon^2, \quad \forall\, t\geq 0, \quad \forall\, \beta\in I\!R, \tag{2.22}$$

where C_1 is a positive constant independent of ε.

Moreover, by (2.11) and (2.19) we have

$$\beta + t \leq x(t,\beta) \leq \beta + \frac{3}{2}t, \quad \forall\, t\geq 0, \quad \forall\, \beta\in I\!R. \tag{2.23}$$

We next estimate $A(t,\beta)$.

For any $\beta\in I\!R$, noting (2.22), property (i), (2.19), the first equation in (2.13) and (2.23), we obtain from (2.17) that

$$\begin{aligned}A(t,\beta) &\geq 1 - \tfrac{2\varepsilon^2|\beta|}{(1+\beta^2)^2}\exp\left(C_1\varepsilon^2\right)\int_0^t \overline{s}_0(x(\tau,\beta))d\tau \\ &= 1 - \tfrac{2\varepsilon^2|\beta|}{(1+\beta^2)^2}\exp\left(C_1\varepsilon^2\right)\int_0^t \overline{s}_0(x(\tau,\beta))\tfrac{\lambda(\tau,x(\tau,\beta))}{\lambda(\tau,x(\tau,\beta))}d\tau \\ &= 1 - \tfrac{2\varepsilon^2|\beta|}{(1+\beta^2)^2}\exp\left(C_1\varepsilon^2\right)\int_\beta^{x(t,\beta)} \overline{s}_0(x)\tfrac{1}{1+\varepsilon^2\overline{s}_0(x)(1+\beta^2)^{-1}}dx \\ &\geq 1 - \tfrac{2\varepsilon^2|\beta|}{(1+\beta^2)^2}\exp\left(C_1\varepsilon^2\right)\int_{-1}^{|\beta|+\frac{3}{2}t} \overline{s}_0(x)dx.\end{aligned} \tag{2.24}$$

Case I: $|\beta| + \frac{3}{2}t \leq 1$.

By properties (iv)-(v), it follows from (2.24) that

$$\begin{aligned} A(t,\beta) &\geq 1-\tfrac{2\varepsilon^2|\beta|}{(1+\beta^2)^2}\exp\left(C_1\varepsilon^2\right)\int_{-1}^{1}\overline{s}_0(x)dx \\ &\geq 1-4\overline{M}\varepsilon^2\exp\left(C_1\varepsilon^2\right) \\ &\geq 1-4\overline{M}\varepsilon_0^2\exp\left(C_1\varepsilon_0^2\right) \\ &\geq \tfrac{1}{2}, \quad \forall\, t\in \mathbb{R}^+, \quad \forall\, \beta\in\left\{\beta\in\mathbb{R}\,\middle|\, |\beta|+\tfrac{3}{2}t\leq 1\right\}, \end{aligned} \tag{2.25}$$

provided that $\varepsilon_0>0$ is suitably small.

Case II: $|\beta|+\frac{3}{2}t>1$.

Noting properties (iv)-(v), we obtain from (2.24) that

$$\begin{aligned} A(t,\beta) &\geq 1-\tfrac{2\varepsilon^2|\beta|}{(1+\beta^2)^2}\exp\left(C_1\varepsilon^2\right)\left\{\int_{-1}^{1}\overline{s}_0(x)dx+\int_{1}^{|\beta|+\frac{3}{2}t}\tfrac{1}{1+x}dx\right\} \\ &\geq \tfrac{3}{4}-C_2\tfrac{|\beta|}{(1+\beta^2)^2}\varepsilon^2\left\{\ln\left(1+|\beta|+\tfrac{3}{2}t\right)-\ln 2\right\} \\ &\geq \tfrac{3}{4}-C_2\tfrac{|\beta|}{(1+\beta^2)^2}\varepsilon^2\left\{\ln\left(2\left(|\beta|+\tfrac{3}{2}t\right)\right)-\ln 2\right\} \\ &= \tfrac{3}{4}-C_2\tfrac{|\beta|}{(1+\beta^2)^2}\varepsilon^2\ln\left(|\beta|+\tfrac{3}{2}t\right) \\ &\geq \tfrac{3}{4}-C_2\tfrac{\varepsilon^2}{1+\beta^2}\ln\left(|\beta|+\tfrac{3}{2}t\right), \end{aligned} \tag{2.26}$$

provided that $\varepsilon_0>0$ is suitably small, here and hereafter C_j $(j=2,3,\cdots)$ stand for positive constants independent of t,β and ε. It is easy to see that when $\varepsilon_0>0$ is suitably small, for any fixed $\varepsilon\in(0,\varepsilon_0]$ we have

$$\frac{1}{3}\exp\left(\frac{1}{4C_2\varepsilon^2}\right)\leq\frac{2}{3}\left\{\exp\left[\frac{1+\beta^2}{4C_2\varepsilon^2}\right]-|\beta|\right\}, \quad \forall\,\beta\in\mathbb{R}.$$

Choosing C_3 to satisfy

$$\exp\left(C_3\varepsilon^{-2}\right)\leq\frac{1}{3}\exp\left(\frac{1}{4C_2\varepsilon^2}\right),$$

we have

$$\frac{C_2\varepsilon^2}{1+\beta^2}\ln\left(|\beta|+\frac{3}{2}t\right)\leq\frac{1}{4}, \quad \forall\, t\in\left[0,\exp\left(C_3\varepsilon^{-2}\right)\right].$$

Then it follows from (2.26) that

$$A(t,\beta)\geq\frac{1}{2}, \quad \forall\, t\in\left[0,\exp\left(C_3\varepsilon^{-2}\right)\right], \quad \forall\,\beta\in\left\{\beta\in\mathbb{R}\,\middle|\,|\beta|+\frac{3}{2}t>1\right\}. \tag{2.27}$$

Thus, combining (2.25) and (2.27) gives

$$A(t,\beta)\geq\frac{1}{2}, \quad \forall\, t\in\left[0,\exp\left(C_3\varepsilon^{-2}\right)\right], \quad \forall\,\beta\in\mathbb{R}. \tag{2.28}$$

Therefore, noting (2.15), (2.22) and (2.28), we obtain from (2.14) that

$$|r_x(t,\beta)| \le C_4\varepsilon, \quad \forall\, t \in \left[0, \exp\left(C_3\varepsilon^{-2}\right)\right], \quad \forall\, \beta \in \mathbb{R}. \tag{2.29}$$

(2.29) implies that

$$\widetilde{T}(\varepsilon) \ge \exp\left(a\varepsilon^{-2}\right), \tag{2.30}$$

where $a = C_3$ is a positive constant independent of ε.

Similarly, for any given $\beta \ge 1$ we have

$$\begin{aligned} A(t,\beta) &\le 1 - \tfrac{2\varepsilon^2\beta}{(1+\beta^2)^2}\exp\left(-C_1\varepsilon^2\right)\int_0^t \overline{s}_0(x(\tau,\beta))d\tau \\ &= 1 - \tfrac{2\varepsilon^2\beta}{(1+\beta^2)^2}\exp\left(-C_1\varepsilon^2\right)\int_\beta^{x(t,\beta)} \tfrac{1}{1+x}\tfrac{1}{1+\varepsilon^2\overline{s}_0(x)(1+\beta^2)^{-1}}dx \\ &\le 1 - \tfrac{4\varepsilon^2\beta}{3(1+\beta^2)^2}\exp\left(-C_1\varepsilon^2\right)\int_\beta^{x(t,\beta)} \tfrac{1}{1+x}dx \\ &= 1 - \tfrac{4\varepsilon^2\beta}{3(1+\beta^2)^2}\exp\left(-C_1\varepsilon^2\right)\left[\ln(1+x(t,\beta)) - \ln(1+\beta)\right], \quad \forall\, t \ge 0. \end{aligned} \tag{2.31}$$

Particularly, in what follows we consider the case that $\beta = 1$.

Noting (2.23), from (2.31) we get

$$A(t,1) \le 1 - \frac{1}{3}\varepsilon^2\exp\left(-C_1\varepsilon^2\right)\left[\ln(2+t) - \ln 2\right], \quad \forall\, t \ge 0. \tag{2.32}$$

Then it follows from (2.32) that

$$A(t_0,1) \le 0, \tag{2.33}$$

where

$$t_0 = 2\exp\left\{\frac{3}{\varepsilon^2}\exp\left(C_1\varepsilon^2\right)\right\} - 2. \tag{2.34}$$

Noting (2.15), (2.22) and (2.33), from (2.14) we see that the C^1 solution to the Cauchy problem (2.8)-(2.9) must blow up at t_0 at the latest. This implies that

$$\widetilde{T}(\varepsilon) \le t_0 \le \exp\left(b\varepsilon^{-2}\right), \tag{2.35}$$

where b is a positive constant independent of ε.

The combination of (2.30) and (2.35) gives (2.6). The proof is completed. Q.E.D.

Remark 2.2. Theorem 2.2 makes it clear that the condition that $\mu > 0$ is essential and sharp for Theorem 1.1 in [1]. □

3 Almost global existence of C^1 solution (I) — Proof of Theorem 1.1

Theorem 1.1 will be proved in a way similar to the proof of Theorem 1.1 in [1]. In what follows we only point out the essentially different part in the proof and $\varepsilon_0 > 0$ is always supposed to be suitably small.

As in [1], we may suppose that

$$0 < \lambda_1(0) < \lambda_2(0) < \cdots < \lambda_n(0). \tag{3.1}$$

By the existence and uniqueness of local C^1 solution to Cauchy problem for quasilinear hyperbolic systems (cf. Chapter 1 in Li and Yu [4]), in order to prove Theorem 1.1 it suffices to establish a uniform *a priori* estimate on the C^0 norm of the C^1 solution $u = u(t,x)$ to the Cauchy problem (1.1) and (1.9) on any fixed domain $0 \le t \le T$ with

$$0 < T \le \exp\left(\kappa\varepsilon^{-1}\right), \tag{3.2}$$

where κ is a positive constant independent of ε and will be determined later.

By (3.1), there exist positive constants δ and δ_0 so small that

$$\lambda_{i+1}(u) - \lambda_i(v) \ge 4\delta_0, \quad \forall\, |u|,\, |v| \le \delta \quad (i = 1, \cdots, n-1) \tag{3.3}$$

and

$$|\lambda_i(u) - \lambda_i(v)| \le \frac{\delta_0}{2}, \quad \forall\, |u|,\, |v| \le \delta \quad (i = 1, \cdots, n). \tag{3.4}$$

For the time being it is supposed that on any given existence domain $0 \le t \le T$ with (3.2) of the C^1 solution $u = u(t,x)$ we have

$$|u(t,x)| \le \delta. \tag{3.5}$$

At the end of the proof of Lemma 3.3, we shall explain that this hypothesis is reasonable. Thus, in order to prove Theorem 1.1 we only need to establish a uniform *a priori* estimate on the C^0 norm of v and w (see (2.3)-(2.4) in [1] for the definitions of v and w) on any given existence domain $0 \le t \le T$ of the C^1 solution $u = u(t,x)$, where T satisfies (3.2).

By (3.1) and (3.5), on the existence domain $0 \le t \le T$ (where T satisfies (3.2)) of the C^1 solution $u = u(t,x)$ we have

$$0 < \lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u), \tag{3.6}$$

provided that δ is suitably small.

For any fixed $T>0$, let

$$D_-^T=\left\{(t,x)\,|\,0\le t\le T,\,x\le -t\right\},\tag{3.7}$$

$$D_0^T=\left\{(t,x)\,|\,0\le t\le T,\,-t\le x\le\left(\lambda_1\left(0\right)-\delta_0\right)t\right\},\tag{3.8}$$

$$D_+^T=\left\{(t,x)\,|\,0\le t\le T,\,x\ge\left(\lambda_n\left(0\right)+\delta_0\right)t\right\},\tag{3.9}$$

$$D^T=\left\{(t,x)\,|\,0\le t\le T,\,\left(\lambda_1\left(0\right)-\delta_0\right)t\le x\le\left(\lambda_n\left(0\right)+\delta_0\right)t\right\}\tag{3.10}$$

and for $i=1,\cdots,n$,

$$\begin{aligned}D_i^T=&\left\{(t,x)\,|\,0\le t\le T,\right.\\&\left.-\left[\delta_0+\eta\left(\lambda_i\left(0\right)-\lambda_1\left(0\right)\right)\right]t\le x-\lambda_i\left(0\right)t\le\left[\delta_0+\eta\left(\lambda_n\left(0\right)-\lambda_i\left(0\right)\right)\right]t\right\},\end{aligned}\tag{3.11}$$

where $\eta>0$ is suitably small.

Noting that $\eta>0$ is small, we observe from (3.3) that

$$D_i^T\cap D_j^T=\emptyset,\quad\forall\, i\neq j\tag{3.12}$$

and

$$\bigcup_{i=1}^{n}D_i^T\subset D^T.\tag{3.13}$$

Let

$$V\left(D_\pm^T\right)=\max_{i=1,\cdots,n}||\left(1+|x|\right)v_i\left(t,x\right)||_{L^\infty\left(D_\pm^T\right)},\tag{3.14}$$

$$V\left(D_0^T\right)=\max_{i=1,\cdots,n}||\left(1+t\right)v_i\left(t,x\right)||_{L^\infty\left(D_0^T\right)},\tag{3.15}$$

$$W\left(D_\pm^T\right)=\max_{i=1,\cdots,n}||\left(1+|x|\right)w_i\left(t,x\right)||_{L^\infty\left(D_\pm^T\right)},\tag{3.16}$$

$$W\left(D_0^T\right)=\max_{i=1,\cdots,n}||\left(1+t\right)w_i\left(t,x\right)||_{L^\infty\left(D_0^T\right)},\tag{3.17}$$

$$W_\infty^c\left(T\right)=\max_{i=1,\cdots,n}\sup_{(t,x)\in D^T\setminus D_i^T}\left(1+|x-\lambda_i\left(0\right)t|\right)|w_i\left(t,x\right)|,\tag{3.18}$$

$$W_1\left(T\right)=\max_{i=1,\cdots,n}\sup_{0\le t\le T}\int_{D_i^T(t)}|w_i\left(t,x\right)|dx\tag{3.19}$$

and

$$V_\infty\left(T\right)=\max_{i=1,\cdots,n}\sup_{\substack{0\le t\le T\\ x\in\mathbb{R}}}|v_i\left(t,x\right)|,\tag{3.20}$$

where $D_i^T(t)$ $(t \geq 0)$ denotes the t-section of D_i^T:

$$D_i^T(t) = \left\{(\tau, x) \mid \tau = t, (\tau, x) \in D_i^T\right\}. \tag{3.21}$$

Obviously, $V_\infty(T)$ is equivalent to

$$U_\infty(T) = \max_{i=1,\cdots,n} \sup_{\substack{0 \leq t \leq T \\ x \in \mathbb{R}}} |u_i(t,x)|. \tag{3.22}$$

It is easy to see that Lemma 3.1 in [1] is still valid, namely,

Lemma 3.1. For each $i = 1, \cdots, n$, on the domain $D^T \backslash D_i^T$ we have

$$ct \leq |x - \lambda_i(0)t| \leq Ct, \quad cx \leq |x - \lambda_i(0)t| \leq Cx, \tag{3.23}$$

where c and C are positive constants independent of (t,x) and T. $\square$

In the present situation, similar to Lemma 3.2 in [1] and Appendix in [2], we have

Lemma 3.2. Suppose that (3.1) holds and $A(u) \in C^2$ in a neighbourhood of $u = 0$. Suppose furthermore that $\psi(x)$ is a C^1 vector function satisfying (1.10). Then there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, on any given existence domain $0 \leq t \leq T$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (1.1) and (1.9), there exist positive constants k_1 and k_2 independent of ε and T such that the following uniform *a priori* estimates hold:

$$V\left(D_\pm^T\right),\ V\left(D_0^T\right) \leq k_1\varepsilon \tag{3.24}$$

and

$$W\left(D_\pm^T\right),\ W\left(D_0^T\right) \leq k_2\varepsilon. \quad \square \tag{3.25}$$

Remark 3.1. In Lemma 3.2 we neither require that system must be weakly linearly degenerate nor demand that T satisfies (3.2). $\square$

Proof of Lemma 3.2. For the time being it is supposed that on any given existence domain $\{0 \leq t \leq T\} \backslash D^T$ of the C^1 solution $u = u(t,x)$ we have

$$|u(t,x)| \leq \delta. \tag{3.26}$$

Completely repeating the proof of Lemma 3.2 in [1] and the discussion carried out in Appendix in [2], we can easily prove that there exist two positive constants k_1 and k_2 independent of ε and T such that (3.24) and (3.25) hold.

Finally, taking $\varepsilon_0 > 0$ suitably small, we obtain from (3.24) that

$$\sup_{(t,x)\in D_-^T\cup D_0^T\cup D_+^T} |u(t,x)| \le C_0 k_1 \varepsilon \le C_0 k_1 \varepsilon_0 \le \frac{1}{2}\delta, \tag{3.27}$$

where C_0 is a positive constant independent of ε and T. This implies the validity of hypothesis (3.26). The proof of Lemma 3.2 is finished. Q.E.D.

Lemma 3.3. Suppose that (3.1) holds and, in a neighbourhood of $u = 0$, $A(u) \in C^2$ and (1.5)-(1.6) hold. Suppose furthermore that $\psi(x)$ is a C^1 vector function satisfying (1.10). Then there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, on any given existence domain $0 \le t \le T$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (1.1) and (1.9), there exist positive constants k_i $(i = 3,4,5)$ independent of ε and T such that the following uniform *a priori* estimates hold:

$$W_\infty^c(T) \le k_3 \varepsilon, \tag{3.28}$$

$$W_1(T) \le k_4 h \tag{3.29}$$

and

$$V_\infty(T), U_\infty(T) \le k_5 h, \tag{3.30}$$

where h is a small positive constant independent of ε, and T satisfies

$$T \le \exp\left(h\varepsilon^{-1}\right). \quad \Box \tag{3.31}$$

Remark 3.2. In Lemma 3.3 we do not yet require that system must be weakly linearly degenerate. $\Box$

Proof of Lemma 3.3. This lemma will be proved in a way similar to the proof of Lemma 3.3 in [1]. In what follows we only point out the essentially different part in the proof and $\varepsilon_0 > 0$ is always supposed to be suitably small.

As in the proof of Lemma 3.3 in [1], we first estimate $\widetilde{W}_1(T)$ (see (3.47) in [1]).

In the present situation, instead of (3.54) in [1] we have

$$
\begin{aligned}
\left|q_i\left(t,\widetilde{x}_i\left(t,y\right)\right)\right|_{t=t(y)} \leq\ & \left|w_i\left(\tfrac{y}{\lambda_n(0)+\delta_0},y\right)\right| \\
& +C_1\left\{\left(W_\infty^c\left(T\right)\right)^2\int_{\frac{y}{\lambda_n(0)+\delta_0}}^{t(y)}\left(1+s\right)^{-1}\left(1+\left|\widetilde{x}_i\left(s,y\right)\right|\right)^{-1}\tfrac{\partial\widetilde{x}_i(s,y)}{\partial y}ds\right. \\
& \left.+W_\infty^c\left(T\right)\sum_{k=1}^{n}\int_{\left(s,\widetilde{x}_i(s,y)\right)\in D_k^T}\left(1+s\right)^{-1}\left|w_k\left(s,\widetilde{x}_i\left(s,y\right)\right)\right|\tfrac{\partial\widetilde{x}_i(s,y)}{\partial y}ds\right\},
\end{aligned}
\tag{3.32}
$$

henceforth $C_j\ (j=1,2,\cdots)$ will denote positive constants independent of ε and T. Then, similar to (3.57) in [1], using (3.25) we obtain from (3.32) that

$$
\begin{aligned}
\textstyle\int_0^{y_2}\left|q_i\left(t,\widetilde{x}_i\left(t,y\right)\right)\right|_{t=t(y)}dy \ \leq\ & C_2\left\{k_2\varepsilon\log(1+C_3T)+\left(W_\infty^c(T)\right)^2\log(1+T)\right. \\
& \left.\times\log(1+C_3T)+W_\infty^c(T)W_1(T)\log(1+T)\right\}.
\end{aligned}
$$

Thus, similar to (3.58) in [1], we get

$$
\widetilde{W}_1(T)\leq C_4\left\{k_2\varepsilon\log(1+T)+\left(W_\infty^c(T)\log(1+T)\right)^2+W_\infty^c(T)W_1(T)\log(1+T)\right\}. \tag{3.33}
$$

On the other hand, similar to (3.62) in [1], we have

$$
W_1(T)\leq C_5\left\{k_2\varepsilon\log(1+T)+\left(W_\infty^c(T)\log(1+T)\right)^2+W_\infty^c(T)W_1(T)\log(1+T)\right\}. \tag{3.34}
$$

We next estimate $W_\infty^c\left(T\right)$.

In the present situation, instead of (3.68) in [1] we have

$$
\left|w_i\left(t_0,y\right)\right|\leq k_2\varepsilon\left(1+y\right)^{-1}\leq C_6k_2\varepsilon\left(1+t_0\right)^{-1}\leq C_7k_2\varepsilon\left(1+t\right)^{-1}, \tag{3.35}
$$

and then, instead of (3.69) in [1] we have

$$
W_\infty^c\left(T\right)\leq C_8\left\{k_2\varepsilon+\left(W_\infty^c\left(T\right)\right)^2\log(1+T)+W_\infty^c\left(T\right)\widetilde{W}_1\left(T\right)\right\}. \tag{3.36}
$$

If T satisfies (3.31) (in which h will be determined later), then it follows from (3.33)-(3.34) and (3.36) that

$$
\widetilde{W}_1(T),\ W_1(T)\leq C_9\left\{k_2h+\left(h\varepsilon^{-1}W_\infty^c(T)\right)^2+h\varepsilon^{-1}W_\infty^c(T)W_1(T)\right\} \tag{3.37}
$$

and

$$
W_\infty^c(T)\leq C_{10}\left\{k_2\varepsilon+h\varepsilon^{-1}\left(W_\infty^c(T)\right)^2+W_\infty^c(T)\widetilde{W}_1(T)\right\}, \tag{3.38}
$$

provided that $\varepsilon_0 > 0$ is suitably small.

Under the condition (3.31), we now use *continuous induction* to prove (3.28)-(3.29) and

$$\widetilde{W}_1(T) \leq k_4 h. \tag{3.39}$$

Noting (1.10), evidently we have

$$W_\infty^c(0) \leq C_{11}\varepsilon, \quad W_1(0) = \widetilde{W}_1(0) = 0. \tag{3.40}$$

Hence, by continuity there exist positive constants k_3, k_4 and h independent of ε such that (3.28)-(3.29) and (3.39) hold at least for $0 \leq T \leq \tau_0$, where τ_0 is a small positive number. Thus, in order to prove (3.28)-(3.29) and (3.39) it suffices to show that we can choose k_3, k_4 and h in such a way that for any fixed T_0 $(0 < T_0 \leq T \leq \exp(h\varepsilon^{-1}))$ such that

$$W_\infty^c(T_0) \leq 2k_3\varepsilon, \tag{3.41}$$

$$\widetilde{W}_1(T_0),\ W_1(T_0) \leq 2k_4 h, \tag{3.42}$$

we have

$$W_\infty^c(T_0) \leq k_3\varepsilon, \tag{3.43}$$

$$\widetilde{W}_1(T_0),\ W_1(T_0) \leq k_4 h. \tag{3.44}$$

Substituting (3.41)-(3.42) into the right-hand sides of (3.37) and (3.38) (in which we take $T = T_0$), we obtain

$$W_\infty^c(T_0) \leq C_{10}\varepsilon\left\{k_2 + 4k_3^2 h + 4k_3k_4h\right\} \tag{3.45}$$

and

$$\widetilde{W}_1(T_0),\ W_1(T_0) \leq C_9 h\left\{k_2 + 4k_3^2 h + 4k_3k_4h\right\}. \tag{3.46}$$

Hence, if

$$k_3 \geq 2C_{10}k_2, \quad k_4 \geq 2C_9k_2 \quad \text{and} \quad 4k_3(k_3+k_4)h \leq k_2, \tag{3.47}$$

then we get (3.43)-(3.44) immediately. This proves (3.28)-(3.29) and (3.39).

We now prove (3.30).

Similar to (3.82) in [1], we have

$$
\begin{aligned}
|u(t,x)| &\leq C_{12}k_1\varepsilon + C_{13}\left\{W_\infty^c(T) + W_1(T)\right\} \\
&\leq C_{12}k_1\varepsilon + C_{13}\left\{k_3\varepsilon + k_4 h\right\} \\
&\leq C_{14}(k_1+k_3)\varepsilon + C_{13}k_4 h \\
&\leq C_{14}(k_1+k_3)\varepsilon_0 + C_{13}k_4 h \\
&\leq 2C_{13}k_4 h,
\end{aligned}
\tag{3.48}
$$

where we have taken ε_0 so small that $C_{14}(k_1+k_3)\varepsilon_0 \leq C_{13}k_4 h$. Hence, if

$$k_5 \geq 2C_{13}k_4, \tag{3.49}$$

then we get (3.30) immediately.

Finally, if we take

$$h \leq \min\left\{(2k_5)^{-1}\delta, \quad k_2\left(4k_3\left(k_3+k_4\right)\right)^{-1}\right\}, \tag{3.50}$$

then we have

$$|u(t,x)| \leq k_5 h \leq \frac{1}{2}\delta. \tag{3.51}$$

This implies the validity of hypothesis (3.5). The proof of Lemma 3.3 is finished. Q.E.D.

Let

$$V_\infty^c(T) = \max_{i=1,\cdots,n} \sup_{(t,x)\in D^T\setminus D_i^T} \left(1+|x-\lambda_i(0)t|\right)|v_i(t,x)|, \tag{3.52}$$

$$U_\infty^c(T) = \max_{i=1,\cdots,n} \sup_{(t,x)\in D^T\setminus D_i^T} \left(1+|x-\lambda_i(0)t|\right)|u_i(t,x)|, \tag{3.53}$$

$$V_1(T) = \max_{i=1,\cdots,n} \sup_{0\leq t\leq T} \int_{D_i^T(t)} |v_i(t,x)|dx \tag{3.54}$$

and

$$W_\infty(T) = \max_{i=1,\cdots,n} \sup_{\substack{0\leq t\leq T \\ x\in \mathbb{R}}} |w_i(t,x)|. \tag{3.55}$$

We have the following lemma.

Lemma 3.4. Under the assumptions of Lemma 3.3, suppose furthermore that system (1.1) is weakly linearly degenerate. In the normalized coordinates there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, on any given existence domain $0 \leq t \leq T$ of the C^1

solution $u = u(t,x)$ to the Cauchy problem (1.1) and (1.9), there exist positive constants k_i $(i = 6, 7, 8)$ independent of ε and T such that the following uniform *a priori* estimates hold:

$$U^c_\infty(T),\ V^c_\infty(T) \le k_6\varepsilon, \tag{3.56}$$

$$V_1(T) \le k_7 h \tag{3.57}$$

and

$$W_\infty(T) \le k_8 h, \tag{3.58}$$

where h is the suitably small positive constant given in Lemma 3.3, and T satisfies

$$T \le \exp\left(h\varepsilon^{-1}\right). \quad \Box \tag{3.59}$$

Proof. This lemma will be proved in a way similar to the proof of Lemma 3.4 in [1]. In what follows we only point out the essentially different part in the proof and $\varepsilon_0 > 0$ is always supposed to be suitably small.

Similar to (3.92) in [1], noting that $h > 0$ is suitably small, we still have

$$U^c_\infty(T) \le C_{15} V^c_\infty(T). \tag{3.60}$$

As in the proof of Lemma 3.4 in [1], we first estimate $\widetilde{V}_1(T)$ (see (3.98) in [1]).

In the present situation, instead of (3.103) we have

$$\begin{aligned} |p_i(t,\widetilde{x}_i(t,y))|_{t=t(y)} \le\ & \left|v_i\left(\tfrac{y}{\lambda_n(0)+\delta_0}, y\right)\right| \\ & + C_{16}\Bigg\{W^c_\infty(T)\,V^c_\infty(T)\int_{\frac{y}{\lambda_n(0)+\delta_0}}^{t(y)} (1+s)^{-1}\left(1+|\widetilde{x}_i(s,y)|\right)^{-1}\tfrac{\partial\widetilde{x}_i(s,y)}{\partial y}ds \\ & + W^c_\infty(T)\sum_{j=1}^{n}\int_{\left(s,\widetilde{x}_i(s,y)\right)\in D_j^T}(1+s)^{-1}\left|v_j(s,\widetilde{x}_i(s,y))\right|\tfrac{\partial\widetilde{x}_i(s,y)}{\partial y}ds \\ & + V^c_\infty(T)\sum_{k=1}^{n}\int_{\left(s,\widetilde{x}_i(s,y)\right)\in D_k^T}(1+s)^{-1}\left|w_k(s,\widetilde{x}_i(s,y))\right|\tfrac{\partial\widetilde{x}_i(s,y)}{\partial y}ds\Bigg\}. \end{aligned} \tag{3.61}$$

Then, similar to (3.104) in [1], using Lemmas 3.2 and 3.3 we obtain

$$\begin{aligned} \widetilde{V}_1(T) \le\ & C_{17}\Big\{k_1\varepsilon\log(1+T) + k_3\varepsilon V^c_\infty(T)\left(\log(1+T)\right)^2 \\ & + k_3\varepsilon V_1(T)\log(1+T) + k_4 h V^c_\infty(T)\log(1+T)\Big\}. \end{aligned} \tag{3.62}$$

On the other hand, similar to (3.105)-(3.106) in [1], we have

$$\begin{aligned} V_1(T) \quad \leq \quad & C_{18}\left\{k_1\varepsilon\log(1+T)+k_3\varepsilon V_\infty^c(T)\left(\log(1+T)\right)^2\right. \\ & \left. +k_3\varepsilon V_1(T)\log(1+T)+k_4hV_\infty^c(T)\log(1+T)\right\}, \end{aligned} \tag{3.63}$$

$$V_\infty^c(T) \leq C_{19}\left\{k_1\varepsilon+k_3\varepsilon V_\infty^c(T)\log(1+T)+k_3\varepsilon\widetilde{V}_1(T)+k_4hV_\infty^c(T)\right\}. \tag{3.64}$$

Thus, in a manner similar to the proof of (3.28)-(3.29) and (3.39), we can easily prove that there exist positive constants k_6 and k_7 independent of ε and T such that (3.56)-(3.57) and

$$\widetilde{V}_1(T) \leq k_7h \tag{3.65}$$

hold, provided that $h>0$ is suitably small and T satisfies (3.59).

We finally estimate $W_\infty(T)$.

In the present situation, instead of (3.109) in [1] we have

$$\begin{aligned} |w_i(t,x)| \quad \leq \quad & C_{20}\left\{W\left(D_+^T\right)+\left(W_\infty^c(T)\right)^2+W_\infty^c(T)W_\infty(T)\log(1+T)\right. \\ & \left. +U_\infty(T)\left(W_\infty^c(T)\right)^2+V_\infty^c(T)\left(W_\infty(T)\right)^2\log(1+T)\right\}. \end{aligned} \tag{3.66}$$

Then, similar to (3.110) in [1], using Lemmas 3.2-3.3 and (3.56), we obtain from (3.66) that

$$W_\infty(T) \leq C_{21}\left\{k_2\varepsilon+k_3hW_\infty(T)+k_6h\left(W_\infty(T)\right)^2\right\}, \tag{3.67}$$

provided that $\varepsilon_0>0$ is suitably small.

By (3.32) in [1], there exists a positive constant k_8 independent of ε such that

$$W_\infty(\tau_0) \leq k_8h, \tag{3.68}$$

provided that $\varepsilon_0>0$ is suitably small, where $\tau_0>0$ is a small positive number. Hence in order to prove (3.58), it suffices to show that we can choose k_8 in such a way that for any fixed τ_1 $(0<\tau_1\leq T\leq\exp(h\varepsilon^{-1}))$ such that

$$W_\infty(\tau_1) \leq 2k_8h, \tag{3.69}$$

we have

$$W_\infty(\tau_1) \leq k_8h. \tag{3.70}$$

Substituting (3.69) into the right-hand side of (3.67) (in which we take $T=\tau_1$) gives

$$W_\infty(\tau_1) \leq C_{21}h\left\{1+2k_3k_8h+4k_6k_8^2h^2\right\}, \tag{3.71}$$

where we have taken ε_0 so small that $k_2\varepsilon_0 \le h$. Taking $h > 0$ so small that

$$2k_3k_8h \le 1 \quad \text{and} \quad 4k_6k_8^2h^2 \le 1, \tag{3.72}$$

we obtain from (3.71) that

$$W_\infty(\tau_1) \le 3C_{21}h. \tag{3.73}$$

Hence, if

$$k_8 \ge 3C_{21},$$

then we get (3.70) immediately. Thus (3.58) is proved. This completes the proof of Lemma 3.4. Q.E.D.

Proof of Theorem 1.1. It suffices to prove Theorem 1.1 in the normalized coordinates. Under the assumptions of Theorem 1.1, by Lemmas 3.3 and 3.4, we know that if $\varepsilon_0 > 0$ is suitably small, then for any fixed $\varepsilon \in (0, \varepsilon_0]$, on any given existence domain $0 \le t \le T$ of the C^1 solution $u = u(t, x)$ to the Cauchy problem (1.1) and (1.9), there exists a small positive constant h independent of ε such that the following uniform *a priori* estimate on the C^1 norm of the solution

$$||u\,(t,\cdot)\,||_{C^1} \triangleq ||u\,(t,\cdot)\,||_{C^0} + ||u_x\,(t,\cdot)\,||_{C^0} \le Kh \tag{3.74}$$

holds, where K is a positive constant independent of ε and T, and T satisfies

$$T \le \exp(h\varepsilon^{-1}). \tag{3.75}$$

This implies that

$$\widetilde{T}(\varepsilon) \ge \exp(h\varepsilon^{-1}). \tag{3.76}$$

Hence, taking $\kappa = h$, we get (1.17) immediately. Thus the proof of Theorem 1.1 is completed. Q.E.D.

4 Almost global existence of C^1 solution (II) — Proof of Theorem 1.2

Theorem 1.2 will be proved in a manner similar to the proof of Theorem 1.1. In what follows we only point out the essentially different part in the proof and $\varepsilon_0 > 0$ is always supposed to be suitably small.

As in Section 3, in order to prove Theorem 1.2 it suffices to establish a uniform *a priori* estimate on the C^0 norm of the C^1 solution $u = u(t,x)$ to the Cauchy problem (1.1) and (1.9) on any fixed domain $0 \leq t \leq T$ with

$$0 < T \leq \exp\left(\widetilde{\kappa}\varepsilon^{-2}\right), \tag{4.1}$$

where $\widetilde{\kappa}$ is a positive constant independent of ε and will be determined later.

For the time being it is supposed that on the existence domain of the C^1 solution $u = u\left(t,x\right)$ we have

$$|u\left(t,x\right)| \leq \delta. \tag{4.2}$$

In Remark 4.3, we shall explain that this hypothesis is reasonable. Thus, in order to prove Theorem 1.2 we only need to establish a uniform *a priori* estimate on the C^0 norm of v and w (see (2.3)-(2.4) in [1] for the definitions of v and w) on any given existence domain $0 \leq t \leq T$ of the C^1 solution $u = u\left(t,x\right)$, where T satisfies (4.1).

Instead of (3.16)-(3.18), we introduce

$$\overline{W}\left(D_{\pm}^{T}\right) = \max_{i=1,\cdots,n} || \left(1+|x|\right)^{1+\nu} w_i\left(t,x\right) ||_{L^\infty\left(D_{\pm}^{T}\right)}, \tag{4.3}$$

$$\overline{W}\left(D_{0}^{T}\right) = \max_{i=1,\cdots,n} || \left(1+t\right)^{1+\nu} w_i\left(t,x\right) ||_{L^\infty\left(D_{0}^{T}\right)} \tag{4.4}$$

and

$$\overline{W}_{\infty}^{c}\left(T\right) = \max_{i=1,\cdots,n} \sup_{(t,x)\in D^T\setminus D_i^T} \left(1+|x-\lambda_i\left(0\right)t|\right)^{1+\nu} |w_i\left(t,x\right)|. \tag{4.5}$$

Similar to Lemma 3.2 in [1] and Appendix in [2], we can easily prove the following lemma.

Lemma 4.1. Suppose that (3.1) holds and $A\left(u\right) \in C^2$ in a neighbourhood of $u = 0$. Suppose furthermore that $\psi\left(x\right)$ is a C^1 vector function satisfying (1.10) and (1.18). Then there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, on any given existence domain $0 \leq t \leq T$ of the C^1 solution $u = u\left(t,x\right)$ to the Cauchy problem (1.1) and (1.9), there exist positive constants k_1 and k_2 independent of ε and T such that the following uniform *a priori* estimates hold:

$$V\left(D_{\pm}^{T}\right),\ V\left(D_{0}^{T}\right) \leq k_1\varepsilon \tag{4.6}$$

and

$$\overline{W}\left(D_{\pm}^{T}\right),\ \overline{W}\left(D_{0}^{T}\right) \leq k_2\varepsilon. \quad \square \tag{4.7}$$

In the present situation, Lemma 3.3 in [1] is still valid and can be stated as the following.

Lemma 4.2. Suppose that (3.1) holds and, in a neighbourhood of $u = 0$, $A(u) \in C^2$ and (1.5)-(1.6) hold. Suppose furthermore that $\psi(x)$ is a C^1 vector function satisfying (1.10) and (1.18). Then there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, on any given existence domain $0 \le t \le T$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (1.1) and (1.9), there exist positive constants k_i $(i = 3, 4, 5)$ independent of ε and T such that the following uniform *a priori* estimates hold:

$$\overline{W}^c_\infty(T) \le k_3\varepsilon, \tag{4.8}$$

$$\widetilde{W}_1(T),\ W_1(T) \le k_4\varepsilon \tag{4.9}$$

and

$$V_\infty(T),\ U_\infty(T) \le k_5\varepsilon. \quad \square \tag{4.10}$$

Remark 4.1. In Lemmas 4.1 and 4.2 we neither require that system must be weakly linearly degenerate nor demand that T satisfies (4.1). $\square$

Remark 4.2. It is easy to check that Lemmas 4.1 and 4.2 are still valid in the case that $\nu > 0$. $\square$

Remark 4.3. Lemma 4.2 implies the validity of hypothesis (4.2). In fact, when $\varepsilon_0 > 0$ is suitably small, we obtain from (4.10) that

$$U_\infty(T) \le k_5\varepsilon \le k_5\varepsilon_0 \le \frac{1}{2}\delta. \tag{4.11}$$

(4.11) shows that the hypothesis (4.2) is reasonable. $\square$

For any fixed T_1 and T_2 satisfying

$$0 \le T_1 \le T_2, \tag{4.12}$$

let

$$V^c_\infty(T_1, T_2) = \max_{i=1,\cdots,n} \sup_{\substack{(t,x)\in D^{T_2}\setminus D_i^{T_2}\\ T_1\le t\le T_2}} (1 + |x - \lambda_i(0)t|)\,|v_i(t,x)|, \tag{4.13}$$

$$U_{\infty}^{c}\left(T_{1},T_{2}\right)=\max_{i=1,\cdots,n}\sup_{\substack{(t,x)\in D^{T_{2}}\setminus D_{i}^{T_{2}}\\ T_{1}\leq t\leq T_{2}}}\left(1+\left|x-\lambda_{i}\left(0\right)t\right|\right)\left|u_{i}\left(t,x\right)\right|, \tag{4.14}$$

$$V_{1}\left(T_{1},T_{2}\right)=\max_{i=1,\cdots,n}\sup_{T_{1}\leq t\leq T_{2}}\int_{D_{i}^{T}(t)}\left|v_{i}\left(t,x\right)\right|dx \tag{4.15}$$

and

$$\widetilde{V}_{1}\left(T_{1},T_{2}\right)=\max_{i=1,\cdots,n}\max_{j\neq i}\sup_{\widetilde{C}_{j}}\int_{\widetilde{C}_{j}\cap\{T_{1}\leq t\leq T_{2}\}\times R}\left|w_{i}\left(t,x\right)\right|dt, \tag{4.16}$$

where $\widetilde{C}_{j}\ (j\neq i)$ stands for any given j-th characteristic in D_{i}^{T}.

Obviously, for any fixed $T\geq 0$ we have

$$V_{\infty}^{c}\left(T\right)=V_{\infty}^{c}\left(0,T\right), \tag{4.17}$$

$$U_{\infty}^{c}\left(T\right)=U_{\infty}^{c}\left(0,T\right), \tag{4.18}$$

$$V_{1}(T)=V_{1}(0,T) \tag{4.19}$$

and

$$\widetilde{V}_{1}\left(T\right)=\widetilde{V}_{1}\left(0,T\right). \tag{4.20}$$

Moreover,

$$V_{\infty}^{c}\left(T_{2}\right)=\max\left\{V_{\infty}^{c}\left(0,T_{1}\right),V_{\infty}^{c}\left(T_{1},T_{2}\right)\right\}, \tag{4.21}$$

$$U_{\infty}^{c}\left(T_{2}\right)=\max\left\{U_{\infty}^{c}\left(0,T_{1}\right),U_{\infty}^{c}\left(T_{1},T_{2}\right)\right\}, \tag{4.22}$$

$$V_{1}(T_{2})=\max\left\{V_{1}\left(0,T_{1}\right),V_{1}\left(T_{1},T_{2}\right)\right\} \tag{4.23}$$

and

$$\widetilde{V}_{1}\left(T_{2}\right)\leq\widetilde{V}_{1}\left(0,T_{1}\right)+\widetilde{V}_{1}\left(T_{1},T_{2}\right), \tag{4.24}$$

where T_1 and T_2 satisfy (4.12).

Similar to Lemma 3.4, we have

Lemma 4.3. Under the assumptions of Lemma 4.2, suppose furthermore that system (1.1) is weakly linearly degenerate. In the normalized coordinates there exists $\varepsilon_0>0$ so small that for any fixed $\varepsilon\in(0,\varepsilon_0]$, on any given existence domain $0\leq t\leq T$ of the C^1 solution $u=u\left(t,x\right)$ to the Cauchy problem (1.1) and (1.9), there exist positive constants k_6 and k_7 independent of ε and T such that the following uniform *a priori* estimates hold:

$$U_{\infty}^{c}\left(0,T\right),\ V_{\infty}^{c}\left(0,T\right)\leq k_{6}\varepsilon, \tag{4.25}$$

$$\widetilde{V}_1(0,T),\ V_1(0,T) \le k_7\varepsilon|\log\varepsilon|, \tag{4.26}$$

where

$$T\varepsilon^3 \le 1. \quad \square \tag{4.27}$$

Proof. This lemma will be proved in a way similar to the proof of Lemma 3.4 in [1]. In what follows we only point out the essentially different part in the proof and $\varepsilon_0 > 0$ is always supposed to be suitably small.

Similar to (3.92) in [1], we still have

$$U_\infty^c(0,T) \le C_1 V_\infty^c(0,T), \tag{4.28}$$

henceforth C_j $(j = 1, 2, \cdots)$ will denote positive constants independent of ε and T.

As in the proof of Lemma 3.4 in [1], we first estimate $\widetilde{V}_1(0,T)$.

In the present situation, instead of (3.103) we have

$$\begin{aligned} |p_i(t,\widetilde{x}_i(t,y))|\big|_{t=t(y)} \le & \left|v_i\left(\tfrac{y}{\lambda_n(0)+\delta_0}, y\right)\right| \\ & + C_2\left\{\overline{W}_\infty^c(T)\,V_\infty^c(0,T)\int_{\frac{y}{\lambda_n(0)+\delta_0}}^{t(y)}(1+s)^{-(1+\nu)}\left(1+|\widetilde{x}_i(s,y)|\right)^{-1}\tfrac{\partial\widetilde{x}_i(s,y)}{\partial y}ds \right. \\ & + \overline{W}_\infty^c(T)\sum_{j=1}^n\int_{\left(s,\widetilde{x}_i(s,y)\right)\in D_j^T}(1+s)^{-(1+\nu)}|v_j(s,\widetilde{x}_i(s,y))|\tfrac{\partial\widetilde{x}_i(s,y)}{\partial y}ds + \\ & \left. + V_\infty^c(0,T)\sum_{k=1}^n\int_{\left(s,\widetilde{x}_i(s,y)\right)\in D_k^T}(1+s)^{-1}|w_k(s,\widetilde{x}_i(s,y))|\tfrac{\partial\widetilde{x}_i(s,y)}{\partial y}ds\right\}. \end{aligned} \tag{4.29}$$

Then, similar to (3.104) in [1], using Lemmas 4.1 and 4.2 we obtain

$$\begin{aligned} \widetilde{V}_1(0,T) \ \le\ & C_3\left\{k_1\varepsilon\log(1+T) + k_3\varepsilon V_\infty^c(0,T)\log(1+T)\right. \\ & \left. + k_3\varepsilon V_1(0,T) + k_4\varepsilon V_\infty^c(0,T)\log(1+T)\right\}. \end{aligned} \tag{4.30}$$

On the other hand, similar to (3.105)-(3.106) in [1], we have

$$\begin{aligned} V_1(T) \ \le\ & C_4\left\{k_1\varepsilon\log(1+T) + k_3\varepsilon V_\infty^c(0,T)\log(1+T)\right. \\ & \left. + k_3\varepsilon V_1(0,T) + k_4\varepsilon V_\infty^c(0,T)\log(1+T)\right\}, \end{aligned} \tag{4.31}$$

$$V_\infty^c(0,T) \le C_5\left\{k_1\varepsilon + k_3\varepsilon V_\infty^c(0,T) + k_3\varepsilon\widetilde{V}_1(0,T) + k_4\varepsilon V_\infty^c(0,T)\right\}. \tag{4.32}$$

If (4.27) holds, then it follows from (4.30)-(4.32) that

$$\widetilde{V}_1(0,T),\ V_1(0,T) \le C_6\left\{k_1\varepsilon|\log\varepsilon| + (k_3+k_4)\,\varepsilon|\log\varepsilon|V_\infty^c(0,T) + k_3\varepsilon V_1(0,T)\right\}, \tag{4.33}$$

and

$$V_\infty^c(0,T) \le C_7 \left\{ k_1\varepsilon + (k_3+k_4)\,\varepsilon V_\infty^c(0,T) + k_3\varepsilon \widetilde{V}_1(0,T) \right\}. \tag{4.34}$$

Thus as before[1], by the *continuous induction* we can easily prove that there exist positive constants k_6 and k_7 independent of ε and T such that (4.25) and (4.26) hold, provided that $\varepsilon_0 > 0$ is suitably small and T satisfies (4.27). The proof is finished. Q.E.D.

Lemma 4.4. Under the assumptions of Lemma 4.3, in the normalized coordinates there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, on any given existence domain $\varepsilon^{-3} \le t \le T$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (1.1) and (1.9), there exist positive constants k_8 and k_9 independent of ε and T such that the following uniform *a priori* estimates hold:

$$U_\infty^c\left(\varepsilon^{-3}, T\right),\ V_\infty^c\left(\varepsilon^{-3}, T\right) \le k_8\varepsilon, \tag{4.35}$$

$$\widetilde{V}_1\left(\varepsilon^{-3}, T\right),\ V_1\left(\varepsilon^{-3}, T\right) \le k_9\varepsilon \log(1+T), \tag{4.36}$$

where we assume that the classical solution exists on the strip: $[0, \varepsilon^{-3}] \times \mathbb{R}$, and where T satisfies

$$\varepsilon^{-3} \le T \le \exp\left(\varepsilon^{-2}\right). \quad \Box \tag{4.37}$$

Proof. This lemma will be proved in a manner similar to the proof of Lemma 4.3. In what follows we only point out the essentially different part in the proof and $\varepsilon_0 > 0$ is always supposed to be suitably small.

Similar to (4.28), we have

$$U_\infty^c\left(\varepsilon^{-3}, T\right) \le C_8 V_\infty^c\left(\varepsilon^{-3}, T\right). \tag{4.38}$$

As in the proof of Lemma 3.4 in [1], we first estimate $\widetilde{V}_1(\varepsilon^{-3}, T)$.

[1] See the proof of (3.28)-(3.29) and (3.39).

In the present situation, instead of (3.103) we have

$$
\begin{aligned}
|p_i\,(t,\widetilde{x}_i\,(t,y))|_{t=t(y)} \le & \left|v_i\left(\tfrac{y}{\lambda_n(0)+\delta_0},y\right)\right| \quad (\text{or } \; v_i\,(\varepsilon^{-3},y)) \\
& +C_9\left\{\overline{W}^c_\infty\,(T)\,V^c_\infty\,(\varepsilon^{-3},T)\int_{\varepsilon^{-3}}^{t(y)}(1+s)^{-(1+\nu)}(1+|\widetilde{x}_i\,(s,y)\,|)^{-1}\,\tfrac{\partial\widetilde{x}_i(s,y)}{\partial y}ds\right. \\
& +\overline{W}^c_\infty\,(T)\sum_{j=1}^{n}\int_{\left(s,\widetilde{x}_i(s,y)\right)\in D_j^T}(1+s)^{-(1+\nu)}\,|v_j\,(s,\widetilde{x}_i\,(s,y))\,|\tfrac{\partial\widetilde{x}_i(s,y)}{\partial y}ds \\
& \left.+\;V^c_\infty\,(\varepsilon^{-3},T)\sum_{k=1}^{n}\int_{\left(s,\widetilde{x}_i(s,y)\right)\in D_k^T}(1+s)^{-1}\,|w_k\,(s,\widetilde{x}_i\,(s,y))\,|\tfrac{\partial\widetilde{x}_i(s,y)}{\partial y}ds\right\}.
\end{aligned}
\tag{4.39}
$$

Then, similar to (4.31), using (4.6), (4.8)-(4.9) and (4.25)-(4.26) we obtain

$$
\begin{aligned}
\widetilde{V}_1\,(\varepsilon^{-3},T) \;\le\;& C_{10}\Big\{\max\left(V\left(D_+^T\right),\;V\left(D_0^T\right)\right)\;\log(1+T)+V_1\,(0,\varepsilon^{-3}) \\
& +V^c_\infty\,(0,\varepsilon^{-3})\log\left(1+\varepsilon^{-3}\right)+\overline{W}^c_\infty(T)V^c_\infty\,(\varepsilon^{-3},T)\log(1+T) \\
& +\overline{W}^c_\infty(T)V_1\,(\varepsilon^{-3},T)+\widetilde{W}(T)V^c_\infty\,(\varepsilon^{-3},T)\log(1+T)\Big\} \\
\le\;& C_{11}\left\{(k_1+k_6+k_7)\,\varepsilon\log(1+T)+k_3\varepsilon V^c_\infty\,(\varepsilon^{-3},T)\log(1+T)\right. \\
& \left.+k_3\varepsilon V_1\,(\varepsilon^{-3},T)+k_4\varepsilon V^c_\infty\,(\varepsilon^{-3},T)\log(1+T)\right\},
\end{aligned}
\tag{4.40}
$$

where we have made use of the fact that $T\ge\varepsilon^{-3}$.

Similarly, we have

$$
\begin{aligned}
V_1\,(\varepsilon^{-3},T) \;\le\;& C_{12}\left\{(k_1+k_6+k_7)\,\varepsilon\log(1+T)+k_3\varepsilon V^c_\infty\,(\varepsilon^{-3},T)\log(1+T)\right. \\
& \left.+k_3\varepsilon V_1\,(\varepsilon^{-3},T)+k_4\varepsilon V^c_\infty\,(\varepsilon^{-3},T)\log(1+T)\right\}.
\end{aligned}
\tag{4.41}
$$

On the other hand, similar to (4.32), using (4.6), (4.25) and (4.8)-(4.9) and noting that $T\ge\varepsilon^{-3}$ and $\nu\ge1$, we obtain

$$
\begin{aligned}
V^c_\infty\,(\varepsilon^{-3},T) \;\le\;& C_{13}\Big\{\max\left(V\left(D_+^T\right),\;V\left(D_0^T\right)\right)+V^c_\infty\,(0,\varepsilon^{-3})+\overline{W}^c_\infty(T)V^c_\infty\,(\varepsilon^{-3},T) \\
& +\left(1+\varepsilon^{-3}\right)^{-\nu}\overline{W}^c_\infty(T)\widetilde{V}_1\,(\varepsilon^{-3},T)+\widetilde{W}_1(T)V^c_\infty\,(\varepsilon^{-3},T)\Big\} \\
\le\;& C_{14}\left\{(k_1+k_6)\,\varepsilon+(k_3+k_4)\,\varepsilon V^c_\infty\,(\varepsilon^{-3},T)+k_3\varepsilon^4\widetilde{V}_1\,(\varepsilon^{-3},T)\right\}.
\end{aligned}
\tag{4.42}
$$

Thus, using the *continuous induction* again, when $T\le\exp\left(\varepsilon^{-2}\right)$, we can easily prove that there exist positive constants k_8 and k_9 independent of ε and T, such that (4.35) and (4.36) hold. The proof is completed. Q.E.D.

Lemma 4.5. Under the assumptions of Lemma 4.3, in the normalized coordinates there exists $\varepsilon_0>0$ so small that for any fixed $\varepsilon\in(0,\varepsilon_0]$, on any given existence domain

$0 \leq t \leq T$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (1.1) and (1.9), there exist positive constants k_{10} and k_{11} independent of ε and T such that the following uniform *a priori* estimates hold:

$$U_\infty^c(T), \; V_\infty^c(T) \leq k_{10}\varepsilon, \tag{4.43}$$

$$W_\infty(T) \leq k_{11}\varepsilon, \tag{4.44}$$

where T satisfies

$$0 \leq T \leq \exp\left(\widetilde{h}\varepsilon^{-2}\right), \tag{4.45}$$

in which $\widetilde{h} > 0$ is a small constant independent of ε. $\square$

Proof. Without loss of generality, we may suppose that $0 < \widetilde{h} \leq 1$.

Noting (4.21) and combining (4.25) and (4.35), we get (4.43) immediately.

We now estimate $W_\infty(T)$.

In the present situation, instead of (3.66) we have

$$\begin{aligned} |w_i(t,x)| \;\; \leq \;\; & C_{15}\Big\{\overline{W}\left(D_+^T\right) + + \left(\overline{W}_\infty^c(T)\right)^2 + \overline{W}_\infty^c(T)W_\infty(T) \\ & + U_\infty(T)\left(\overline{W}_\infty^c(T)\right)^2 + V_\infty^c(T)\left(W_\infty(T)\right)^2 \log(1+T)\Big\}. \end{aligned} \tag{4.46}$$

Then using (4.7), (4.8), (4.10) and (4.43), we obtain from (4.46) that

$$W_\infty(T) \leq C_{16}\left\{(k_2+k_3)\,\varepsilon + k_3\varepsilon W_\infty(T) + k_{10}\varepsilon\left(W_\infty(T)\right)^2\log(1+T)\right\}, \tag{4.47}$$

provided that $\varepsilon_0 > 0$ is suitably small.

Hence as before, noting (4.47) and using the *continuous induction* again, we can find a positive constant k_{11} and a small $\widetilde{h} > 0$ independent of ε and T such that (4.44) holds if T satisfies (4.45). The proof is completed. Q.E.D.

Proof of Theorem 1.2. Similar to the proof of Theorem 1.1, noting Lemmas 4.2 and 4.5 and taking $\widetilde{\kappa} = \widetilde{h}$, we get the conclusion of Theorem 1.2 immediately. For brevity, we omit the details. Q.E.D.

5 Asymptotic behaviour of life-span of C^1 solution — Proof of Theorem 1.3

Theorem 1.3 will be proved in a way similar to the proof of Theorem 1.2 in [1]. To do so, we need the following lemmas.

In the present situation, Lemma 3.2 is still valid. Moreover, similar to Lemma 3.3, we have

Lemma 5.1. Suppose that (3.1) holds and, in a neighbourhood of $u = 0$, $A(u) \in C^2$ and (1.5)-(1.6) hold. Suppose furthermore that $\psi(x)$ is a C^1 vector function satisfying (1.10). Then there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, on any given existence domain $0 \le t \le T$ of the C^1 solution $u = u(t, x)$ to the Cauchy problem (1.1) and (1.9), there exist positive constants k_i $(i = 3, 4, 5)$ independent of ε and T such that the following uniform *a priori* estimates hold:

$$W^c_\infty(T) \le k_3\varepsilon, \tag{5.1}$$

$$\widetilde{W}_1(T),\ W_1(T) \le k_4\varepsilon|\log\varepsilon| \tag{5.2}$$

and

$$V_\infty(T),\ U_\infty(T) \le k_5\varepsilon|\log\varepsilon|, \tag{5.3}$$

where

$$T\varepsilon^{2+\alpha} \le 1. \quad \square \tag{5.4}$$

Proof. This lemma will be proved in a way similar to the proof of Lemma 3.3. In what follows we only point out the essentially different part in the proof and $\varepsilon_0 > 0$ is always supposed to be suitably small.

Under the assumptions of Lemma 5.1, (3.33), (3.34) and (3.36) are still valid, namely, we have

$$\widetilde{W}_1(T) \le C_1\left\{k_2\varepsilon\log(1+T) + (W^c_\infty(T)\log(1+T))^2 + W^c_\infty(T)W_1(T)\log(1+T)\right\}, \tag{5.5}$$

$$W_1(T) \le C_2\left\{k_2\varepsilon\log(1+T) + (W^c_\infty(T)\log(1+T))^2 + W^c_\infty(T)W_1(T)\log(1+T)\right\} \tag{5.6}$$

and

$$W_{\infty}^{c}(T) \le C_3 \left\{ k_2 \varepsilon + (W_{\infty}^{c}(T))^2 \log(1+T) + W_{\infty}^{c}(T)\, \widetilde{W}_1(T) \right\}, \tag{5.7}$$

henceforth C_j $(j = 1, 2, 3, \cdots)$ will denote positive constants independent of ε and T.

If T satisfies (5.4), then we obtain from (5.5)-(5.7) that

$$\widetilde{W}_1(T),\ W_1(T) \le C_4 \left\{ k_2 \varepsilon |\log \varepsilon| + (W_{\infty}^{c}(T) \log \varepsilon)^2 + W_{\infty}^{c}(T) W_1(T) |\log \varepsilon| \right\} \tag{5.8}$$

and

$$W_{\infty}^{c}(T) \le C_5 \left\{ k_2 \varepsilon + (W_{\infty}^{c}(T))^2 |\log \varepsilon| + W_{\infty}^{c}(T)\, \widetilde{W}_1(T) \right\}, \tag{5.9}$$

provided that $\varepsilon_0 > 0$ is suitably small.

Thus, in a manner similar to the proof of (3.28)-(3.29) and (3.39), noting that $\varepsilon > 0$ is small, under the condition (5.4) we can easily prove (5.1)-(5.2).

On the other hand, completely repeating the procedure of proving (3.30), we get (5.3) immediately. The proof of this lemma is finished. Q.E.D.

Lemma 5.2. Under the assumptions of Theorem 1.3, in the normalized coordinates there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, on any given existence domain $0 \le t \le T$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (1.1) and (1.9), there exist positive constants k_i $(i = 6, \cdots, 9)$ independent of ε and T such that the following uniform *a priori* estimates hold:

$$U_{\infty}^{c}(T),\ V_{\infty}^{c}(T) \le k_6 \varepsilon, \tag{5.10}$$

$$\widetilde{V}_1(T),\ V_1(T) \le k_7 \varepsilon |\log \varepsilon| + k_8 (\varepsilon |\log \varepsilon|)^{2+\alpha}\, T, \tag{5.11}$$

where

$$T \varepsilon^{2+\alpha} \le 1. \tag{5.12}$$

Moreover,

$$W_{\infty}(T) \le k_9 \varepsilon, \tag{5.13}$$

where

$$T \varepsilon^{\frac{3}{4}+\alpha} \le 1. \quad \square \tag{5.14}$$

Proof. This lemma will be proved in a way similar to the proof of Lemma 4.3 in [1]. In what follows we only point out the essentially different part in the proof and $\varepsilon_0 > 0$ is always supposed to be suitably small.

In the present situation, instead of (4.17)-(4.19) in [1] we have

$$\begin{aligned}\widetilde{V}_1(T) \;\leq\;& C_6\Big\{\max\left(V\left(D_+^T\right),\ V\left(D_0^T\right)\right)\log(1+T)+W_\infty^c(T)V_\infty^c(T)\left(\log(1+T)\right)^2\\&+W_\infty^c(T)V_1(T)\log(1+T)+W_1(T)V_\infty^c(T)\log(1+T)\\&+\left(V_\infty(T)\right)^{1+\alpha}\left(W_\infty^c(T)\log(1+T)+W_1(T)\right)T\Big\},\end{aligned}\tag{5.15}$$

$$\begin{aligned}V_1(T) \;\leq\;& C_7\Big\{\max\left(V\left(D_+^T\right),\ V\left(D_0^T\right)\right)\log(1+T)+W_\infty^c(T)V_\infty^c(T)\left(\log(1+T)\right)^2\\&+W_\infty^c(T)V_1(T)\log(1+T)+W_1(T)V_\infty^c(T)\log(1+T)\\&+\left(V_\infty(T)\right)^{1+\alpha}\left(W_\infty^c(T)\log(1+T)+W_1(T)\right)T\Big\}\end{aligned}\tag{5.16}$$

and

$$\begin{aligned}V_\infty^c(T) \;\leq\;& C_8\Big\{\max\left(V\left(D_+^T\right),\ V\left(D_0^T\right)\right)+W_\infty^c(T)V_\infty^c(T)\log(1+T)\\&+W_\infty^c(T)\widetilde{V}_1(T)+\widetilde{W}_1(T)V_\infty^c(T)\Big\}.\end{aligned}\tag{5.17}$$

If T satisfies (5.12), then, using Lemmas 3.2 and 5.1 we obtain from (5.15)-(5.17) that

$$\begin{aligned}\widetilde{V}_1(T),\ V_1(T) \;\leq\;& C_9\Big\{k_1\varepsilon|\log\varepsilon|+k_3\varepsilon\left(\log\varepsilon\right)^2V_\infty^c(T)+k_3\varepsilon|\log\varepsilon|V_1(T)\\&+k_4\varepsilon\left(\log\varepsilon\right)^2V_\infty^c(T)+\left(k_3+k_4\right)k_5\left(\varepsilon|\log\varepsilon|\right)^{2+\alpha}T\Big\},\end{aligned}\tag{5.18}$$

and

$$V_\infty^c(T) \;\leq\; C_{10}\Big\{k_1\varepsilon+k_3\varepsilon|\log\varepsilon|V_\infty^c(T)+k_3\varepsilon\widetilde{V}_1(T)+k_4\varepsilon|\log\varepsilon|V_\infty^c(T)\Big\}.\tag{5.19}$$

Thus, completely repeating the procedure of proving (4.8)-(4.9) in [1], we can easily show that there exist positive constants k_6, k_7 and k_8 independent of ε and T such that (5.10) and (5.11) hold if T satisfies (5.12).

We now prove (5.13).

In the present situation, instead of (4.32) in [1] we have

$$\begin{aligned}|w_i(t,x)| \;\leq\;& C_{11}\Big\{W\left(D_+^T\right)+\left(W_\infty^c(T)\right)^2+W_\infty^c(T)W_\infty(T)\log(1+T)\\&+U_\infty(T)\left(W_\infty^c(T)\right)^2+V_\infty^c(T)\left(W_\infty(T)\right)^2\log(1+T)\\&+\left(V_\infty(T)\right)^\alpha\left(W_\infty(T)\right)^2T\Big\},\end{aligned}\tag{5.20}$$

then, similar to (4.33) in [1], using Lemmas 3.2, 5.1 and (5.10) we get

$$\begin{aligned}W_\infty(T) \;\leq\;& C_{12}\Big\{\varepsilon\left(1+|\log\varepsilon|W_\infty(T)+|\log\varepsilon|\left(W_\infty(T)\right)^2\right)\\&+\left(\varepsilon|\log\varepsilon|\right)^\alpha T\left(W_\infty(T)\right)^2\Big\},\end{aligned}\tag{5.21}$$

where $T \le \varepsilon^{2+\alpha}$.

Furthermore, we restrict T to satisfies (5.14). Then we obtain from (5.21) that

$$\begin{aligned} W_\infty(T) \ \le \ & C_{12}\left\{\varepsilon\left(1+|\log\varepsilon|W_\infty(T)+|\log\varepsilon|\left(W_\infty(T)\right)^2\right)\right. \\ & \left. +\left(\varepsilon|\log\varepsilon|\right)^\alpha \varepsilon^{-\left(\frac{3}{4}+\alpha\right)}\left(W_\infty(T)\right)^2\right\}. \end{aligned} \tag{5.22}$$

Thus as in [1], using the *continuous induction* again, we can easily prove (5.13). This completes the proof of Lemma 5.2. Q.E.D.

Remark 5.1. By Lemmas 5.1 and 5.2, when $\varepsilon_0 > 0$ is suitably small, the Cauchy problem (1.1) and (1.9) admits a unique C^1 solution $u = u(t,x)$ on $0 \le t \le \varepsilon^{-\left(\frac{3}{4}+\alpha\right)}$. Hence, we get the following lower bound on the life-span $\widetilde{T}(\varepsilon)$ of C^1 solution

$$\widetilde{T}(\varepsilon) \ge \varepsilon^{-\left(\frac{3}{4}+\alpha\right)}. \quad \square \tag{5.23}$$

Remark 5.2. Similar to the proof of (5.13) with (5.14), we can prove easily that for any fixed $\widetilde{\mu} \in (0,1)$, there exists $\varepsilon_0(\widetilde{\mu}) > 0$ so small that the Cauchy problem (1.1)and (1.9) admits a unique C^1 solution $u = u(t,x)$ on $0 \le t \le \varepsilon^{-(\widetilde{\mu}+\alpha)}$. Hence, we have

$$\widetilde{T}(\varepsilon) \ge \varepsilon^{-\left(\widetilde{\mu}+\alpha\right)}. \quad \square \tag{5.24}$$

Using Lemma 3.2, 5.1 and 5.2 and Remark 5.1, almost completely repeating the proof of Theorem 1.1 in [2], we can show Theorem 1.3 easily. In what follows we only point out the essentially different part in the proof.

Proof of Theorem 1.3. As in [2], it suffices to prove Lemma 3.1 in [2], i.e., in the normalized coordinates $\tilde{u}$ satisfying (1.16) to prove

$$\overline{\lim_{\varepsilon\to 0}}\left\{\varepsilon^{1+\alpha}\widetilde{T}(\varepsilon)\right\} \le M_0 \tag{1.25a}$$

and

$$\underline{\lim_{\varepsilon\to 0}}\left\{\varepsilon^{1+\alpha}\widetilde{T}(\varepsilon)\right\} \ge M_0, \tag{1.25b}$$

where $M_0 > 0$ is defined by (3.5) in [2]. Moreover, as in [2], we still denote $\tilde{u}$ by u.

In what follows we will directly use the notations presented in [2] and two lemmas on ordinary differential equations of Ricatti's type given in [5] (also see [2]).

(I) **Proof of (1.25a)**

By Remark 5.1, there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, the Cauchy problem (1.1) and (1.9) admits a unique C^1 solution $u = u(t, x)$ on the domain $0 \leq t \leq T_1$, where

$$T_1 \triangleq \varepsilon^{-\left(\frac{3}{4}+\alpha\right)} \leq \widetilde{T}(\varepsilon) - 1 \triangleq \overline{T}. \tag{5.25}$$

As in [2], we may suppose that

$$\widetilde{T}(\varepsilon)\varepsilon^{2+\alpha} \leq 1. \tag{5.26}$$

Thus, in what follows we only discuss the problem in the domain $0 \leq t \leq \varepsilon^{-(2+\alpha)}$.

Noting (1.10), we observe that there exist $i_0 \in J_1$ and $x_0 \in \mathbb{R}$ such that (3.19) in [2] is still valid.

When $\varepsilon_0 > 0$ is suitably small, by (5.25) we get

$$T_1 > \varepsilon^{-\alpha} \triangleq T_0 > t_0, \tag{5.27}$$

see [2] for the definition of t_0.

Noting (4.62) in [1] and (5.3), instead of (3.25) in [2] we have

$$|w_1\left(t, x_1\left(t, x_0\right)\right) - w_1\left(0, x_0\right)| \leq C_{13}\varepsilon^2 |\log \varepsilon|^\alpha, \quad \forall\, t \in [0, T_0], \tag{5.28}$$

and then, instead of (3.26) in [2] we have

$$w_1(T_0, x_1(T_0, x_0)) = \varepsilon l_1(0)\psi'(x_0) + O\left(\varepsilon^2 |\log \varepsilon|^\alpha\right). \tag{5.29}$$

Noting (5.3) again, by (3.27) in [2] we get

$$\begin{aligned} \gamma_{111}\left(u_1 e_1\right) &= a\left(\varepsilon l_1(0)\psi(x_0)\right)^\alpha + a\left[\left(u_1\left(t,\ x_1\left(t, x_0\right)\right)\right)^\alpha - \varepsilon\left(l_1(0)\psi(x_0)\right)^\alpha\right] + \\ &+ O\left(\varepsilon^{1+\alpha} |\log \varepsilon|^{1+\alpha}\right). \end{aligned} \tag{5.30}$$

where a is defined by (3.20) in [2].

Moreover, similar to (3.25) in [2], using (5.10) we obtain

$$\begin{aligned} |\gamma_{111}\left(u\right) - \gamma_{111}\left(u_1 e_1\right)| &\leq C_{14}(1+t)^{-1} V_\infty^c\left(\overline{T}\right) \\ &\leq C_{15}\varepsilon(1+t)^{-1} \\ &\leq C_{16}\varepsilon\left(1+\varepsilon^{-\alpha}\right)^{-1} \\ &\leq C_{16}\varepsilon^{1+\alpha}, \quad \forall\, t \in [T_0, \overline{T}]. \end{aligned} \tag{5.31}$$

On the other hand, in the present situation, instead of (4.47) in [1] we have

$$u(t,x),\ v(t,x) = O\left(\varepsilon \left|\log \varepsilon\right|\right), \tag{5.32}$$

moreover, similar to (4.55) in [1], using (5.32) we get

$$\left|u_1(t,x_1(t,x_0)) - \varepsilon l_1(0)\psi(x_0)\right| \le C_{17}\varepsilon^2 \left|\log \varepsilon\right|^2. \tag{5.33}$$

Hence, noting (5.33), instead of (3.29) in [2] we obtain from (5.30)-(5.31) that

$$a_0(t) = \gamma_{111}(u(t,x_1(t,x_0))) = a\left(\varepsilon l_1(0)\psi(x_0)\right)^{\alpha} + O\left(\varepsilon^{1+\alpha}\left|\log \varepsilon\right|^{1+\alpha}\right), \quad \forall\, t \in [T_0, \overline{T}], \tag{5.34}$$

where $a_0(t)$ is defined by the first equality of (3.23) in [2]. Therefore, for $\varepsilon > 0$ suitably small, noting (3.21) in [2] we get

$$a_0(t) \ge \frac{1}{2} b\varepsilon^{\alpha} > 0, \quad \forall\, t \in [T_0, \overline{T}], \tag{5.35}$$

where b is defined by (3.21) in [2].

Similar to (3.31)-(3.34) in [2], we have

$$\int_{T_0}^{\overline{T}} |a_1(t)|dt \le C_{18} W_\infty^c(\overline{T}) \int_{T_0}^{\overline{T}} (1+t)^{-1} dt \le C_{19}\varepsilon \left|\log \varepsilon\right|, \tag{5.36}$$

$$\int_{T_0}^{\overline{T}} |a_2(t)|dt \le C_{20}\left(W_\infty^c(\overline{T})\right)^2 \int_{T_0}^{\overline{T}} (1+t)^{-2} dt \le C_{21}\varepsilon^2, \tag{5.37}$$

$$K \le \int_{T_0}^{\overline{T}} |a_2(t)|dt \cdot \exp\left(\int_{T_0}^{\overline{T}} |a_1(t)|dt\right) \le C_{22}\varepsilon^2, \tag{5.38}$$

and

$$w_1(T_0, x_1(T_0,x_0)) > C_{22}\varepsilon^2 \ge K, \tag{5.39}$$

provided that $\varepsilon_0 > 0$ is suitably small, where $a_1(t), a_2(t)$ and K are defined by (3.23) and (3.33) in [2] respectively. To get (5.39), we have made use of the assumption (3.21) in [2].

Applying Lemma 2.1 in [2] and completely repeating the rest of the proof of (3.16a) in [2], we get (1.25a) immediately.

(II) **Proof of (1.25b)**

Similar to (3.41) in [2], noting (5.3) we have

$$||u(t,x)||_{C^0[0,T]\times R} \le C_{23}\varepsilon \left|\log \varepsilon\right|, \tag{5.40}$$

where T satisfies

$$0 < T \leq M\varepsilon^{-(1+\alpha)}, \tag{5.41}$$

in which M stands for any given positive constant with $M < M_0$.

As in [2], we now estimate $\widetilde{U}_1(T)$ (see (3.46) in [2] for the definition of $\widetilde{U}_1(T)$).

Similar to (3.52) in [2], we have

$$\begin{aligned} \int_{\widetilde{C}_j} |u_i(t,x)|dt \;\; &\leq \;\; C_{24}\left(\widetilde{V}_1(T) + V_\infty^c(T)\log(1+T)\right) \\ &\quad + C_{25}\left(\widetilde{U}_1(T) + V_\infty^c(T)\log(1+T)\right)V_\infty(T). \end{aligned} \tag{5.42}$$

Noting (5.41) and using (5.3) and (5.10)-(5.11), we obtain from (5.42) that

$$\begin{aligned} \widetilde{U}_1(T) \;\; &\leq \;\; C_{26}\left\{k_7\varepsilon\,|\log\varepsilon| + k_8\left(\varepsilon\,|\log\varepsilon|\right)^{2+\alpha}\varepsilon^{-(1+\alpha)} + k_6\varepsilon\,|\log\varepsilon| \right. \\ &\quad \left. + k_5\varepsilon\,|\log\varepsilon|\left(\widetilde{U}_1(T) + k_6\varepsilon\,|\log\varepsilon|\right)\right\} \\ &\leq \;\; C_{27}\left\{\varepsilon\,|\log\varepsilon|^{2+\alpha} + \varepsilon\,|\log\varepsilon|\,\widetilde{U}_1(T)\right\}, \end{aligned} \tag{5.43}$$

provided that $\varepsilon_0 > 0$ is suitably small. Therefore, when $\varepsilon_0 > 0$ is suitably small, from (5.43) we get

$$\widetilde{U}_1(T) \leq C_{28}\varepsilon\,|\log\varepsilon|^{2+\alpha}. \tag{5.44}$$

Noting (5.34), corresponding to (3.43) in [2] we have

$$a_0(t) = a\left(\varepsilon l_1(0)\psi(y)\right)^\alpha + \left(\gamma_{111}(u) - \gamma_{111}(u_1e_1)\right) + O\left(\left(\varepsilon\,|\log\varepsilon|\right)^{1+\alpha}\right), \tag{5.45}$$

and then similar to (3.44) in [2], we have

$$\begin{aligned} w_1(0,y)\int_0^T a_0^+(t)dt \;\; &\leq \;\; \left[(l_1(0)\psi'(y))^+\varepsilon + C_{29}\varepsilon^2\right]\int_0^T \left[(a(l_1(0)\psi(y))^\alpha)^+\varepsilon^\alpha \right. \\ &\quad \left. + |\gamma_{111}(u) - \gamma_{111}(u_1e_1)| + C_{30}\left(\varepsilon\,|\log\varepsilon|\right)^{1+\alpha}\right]dt, \end{aligned} \tag{5.46}$$

where $f^+ = \max(f,0)$ and henceforth C_j $(j = 29, 30, \cdots)$ will denote positive constant independent of ε, T and y.

Similar to (3.48) in [2], we have

$$\begin{aligned} w_1(0,y)\int_0^T a_0^+(t)dt \;\; &\leq \;\; \left(a\left(l_1(0)\psi(y)\right)^\alpha l_1(0)\psi'(y)\right)^+ M + C_{31}\varepsilon\,|\log\varepsilon|^{1+\alpha} \\ &\quad + C_{32}\varepsilon\left\{\left(V(D_+^T) + V(D_0^T) + V(D_-^T)\right)\log(1+T) \right. \\ &\quad \left. + \widetilde{U}_1(T) + V_\infty^c(T)\log(1+T)\right\}. \end{aligned} \tag{5.47}$$

Noting (5.41) and using Lemma 3.2, (5.10) and (5.44), we obtain from (5.47) that

$$w_1(0,y)\int_0^T a_0^+(t) \le \left(a\left(l_1(0)\psi(y)\right)^\alpha l_1(0)\psi'(y)\right)^+ M + C_{33}\varepsilon\left|\log\varepsilon\right|^{1+\alpha}, \tag{5.48}$$

provided that $\varepsilon_0 > 0$ is suitably small.

Moreover, similar to (3.57)-(3.60) in [2], we have

$$\begin{aligned}\int_0^T |a_1(t)|dt &\le C_{34}\left\{\left(W(D_+^T)+W(D_0^T)+W(D_-^T)+W_\infty^c(T)\right)\log(1+T)+\widetilde{W}_1(T)\right\}\\ &\le C_{35}\varepsilon\left|\log\varepsilon\right|,\end{aligned} \tag{5.49}$$

$$\begin{aligned}\int_0^T |a_2(t)|dt &\le C_{36}\left\{\left(W(D_+^T)+W(D_0^T)+W(D_-^T)\right)^2+W_\infty^c(T)\widetilde{W}_1(T)+(W_\infty^c(T))^2\right\}\\ &\le C_{37}\varepsilon^2\left|\log\varepsilon\right|,\end{aligned} \tag{5.50}$$

$$K \le \int_0^T |a_2(t)|dt\cdot\exp\left(\int_0^T |a_1(t)|dt\right) \le C_{38}\varepsilon^2\left|\log\varepsilon\right| \tag{5.51}$$

and

$$\int_0^T |a_0(t)|dt \le C_{39}\varepsilon^{-1}, \tag{5.52}$$

provided that $\varepsilon_0 > 0$ is suitably small.

Thus, noting (5.48), instead of (3.61) in [2] we have

$$\begin{aligned}(w_1(0,y)+K)\int_0^T a_0^+(t)dt &\le \tfrac{M}{M_0}+C_{33}\varepsilon\left|\log\varepsilon\right|^{1+\alpha}+K\int_0^T |a_0(t)|dt\\ &\le \tfrac{M}{M_0}+C_{40}\varepsilon\left|\log\varepsilon\right|^{1+\alpha}<1,\end{aligned} \tag{5.53}$$

provided that $\varepsilon_0 > 0$ is suitably small.

Therefore using (5.49)-(5.52), completely repeating the rest of the proof of (3.16b) in [2], we obtain (1.25b) immediately. This completes the proof of Theorem 1.3. Q.E.D.

6 Breakdown of C^1 solution — Proof of Theorem 1.4 and Theorem 1.5

Theorem 1.4 and Theorem 1.5 will be proved in a way similar to the proof of Theorem 1.2 and Theorem 1.3 in [2]. As before, here we only point out the essentially different part in the proof.

We still use the normalized coordinates u as in Section 5. Let (t^*, x^*) be the starting point of the singularity of the C^1 solution $u = u(t, x)$ to the Cauchy problem (1.1) and (1.9). By Theorem 1.3 we have

$$\frac{1}{2}M_0\varepsilon^{-(1+\alpha)} < t^* < 2M_0\varepsilon^{-(1+\alpha)}, \tag{6.1}$$

where M_0 is given by (3.5) in [2]. On the domain $[0, t^*) \times \mathbb{R}$ the Cauchy problem (1.1) and (1.9) admits a unique C^1 solution $u = u(t, x)$, and by Lemma 5.1 we have

$$\|u(t, x)\|_{C^0([0,t^*)\times \mathbb{R})} \le K_1\varepsilon|\log\varepsilon|, \tag{6.2}$$

where K_1 is a positive constant independent of ε.

As in [2], let $\xi = x_i(s, y_i)$ be the i-th characteristc passing through any given point (t, x) on the domain $[0, t^*) \times \mathbb{R}$, where (s, ξ) denote the coordinates of variable point of this characteristic and y_i stands for the x-coordinate of the intersection point of this characterisitc with the x-axis.

Similar to Lemma 4.1 in [2], we have

Lemma 6.1. For $i = 1, \cdots, n$ and for any given point (t, x) on the domain $[0, t^*) \times \mathbb{R}$ we have

$$\left| w_i(t, x)\frac{\partial x_i(t, y_i)}{\partial y_i} \right| \le K_2\varepsilon \exp\left(|\log\varepsilon|^\alpha\right), \tag{6.3}$$

provided that $\varepsilon > 0$ is suitably small, where K_2 is a positive constant independent of i, t, y_i (or x) and ε. $\square$

Proof. As in the proof of Lemma 4.1 in [2], we consider the following two (only two) possibilities: Case I

$$a_i(\widetilde{\psi}_i(y_i))^\alpha \widetilde{\psi}_i'(y_i) < \frac{1}{4M_0} \tag{6.4}$$

and Case II

$$a_i(\widetilde{\psi}_i(y_i))^\alpha \widetilde{\psi}_i'(y_i) \in \left[\frac{1}{4M_0}, \frac{1}{M_0}\right], \tag{6.5}$$

where

$$a_i = -\frac{1}{\alpha!}\frac{\partial^{1+\alpha}\lambda_i}{\partial u_i^{1+\alpha}}(0), \tag{6.6}$$

and $\widetilde{\psi}_i(x)$ is defined by (4.7) in [2].

Case I. In the present situation, Remark 3.1 in [2] is still valid. Hence, similar to (4.14) in [2], we have

$$|w_i(t, x_i(t, y_i))| \le C_1\varepsilon, \quad \forall t \in [0, t^*), \tag{6.7}$$

where y_i belongs to Case I, here and hereafter C_j $(j = 1, 2, \cdots)$ denote positive constants independent of i, t, y_i and ε.

Noting (6.2) and (6.7), instead of (4.29) in [2] we have

$$|P_3(\tau)| \le C_2|u_i|^\alpha |w_i| \le C_3\varepsilon^{1+\alpha} |\log\varepsilon|^\alpha, \quad \forall\, \tau \in [0, t^*), \tag{6.8}$$

and then, instead of (4.30) in [2] we obtain

$$\int_0^s |P_3(\tau)|d\tau \le 2C_3M_0|\log\varepsilon|^\alpha, \quad \forall\, \tau \in [0, t^*). \tag{6.9}$$

On the other hand, similar to (4.31) and (4.34) in [2], we have

$$\begin{aligned} \textstyle\int_0^s |P_1(\tau)|d\tau \;&\le\; C_4\left\{\widetilde{W}_1(s) + \left(W_\infty^c(s) + W\left(D_\pm^s\right) + W\left(D_0^s\right)\right)\textstyle\int_0^s(1+\tau)^{-1}d\tau\right\} \\ &\le\; C_5\left\{\widetilde{W}_1(s) + \left(W_\infty^c(s) + W\left(D_\pm^s\right) + W\left(D_0^s\right)\right)\log(1+t^*)\right\}, \\ &\qquad\qquad \forall\, \tau \in [0, t^*) \end{aligned} \tag{6.10}$$

and

$$\begin{aligned} \textstyle\int_0^s |P_2(\tau)|d\tau \;&\le\; C_6\varepsilon\left\{\widetilde{U}_1(s) + \left(V_\infty^c(s) + V\left(D_\pm^s\right) + V\left(D_0^s\right)\right)\textstyle\int_0^s(1+\tau)^{-1}d\tau\right\} \\ &\le\; C_7\varepsilon\left\{\widetilde{U}_1(s) + \left(V_\infty^c(s) + V\left(D_\pm^s\right) + V\left(D_0^s\right)\right)\log(1+t^*)\right\}, \\ &\qquad\qquad \forall\, \tau \in [0, t^*) \end{aligned} \tag{6.11}$$

and then, noting (6.1) and using Lemma 3.2, (5.1)-(5.2), (5.10) and (5.44), we obtain from (6.10)-(6.11) that

$$\int_0^s |P_1(\tau)|d\tau \le C_8\varepsilon|\log\varepsilon|, \quad \forall\, s \in [0, t^*) \tag{6.12}$$

and

$$\int_0^s |P_2(\tau)|d\tau \le C_9\varepsilon^2|\log\varepsilon|^{2+\alpha}, \quad \forall\, s \in [0, t^*). \tag{6.13}$$

Similar to (4.38) in [2], combining (6.9) and (6.12)-(6.13), we obtain that for any fixed y_i belonging to Case I, there exists a positive constant K_3 independent of i, t, y_i and ε such that

$$0 < \frac{\partial x_i(t, y_i)}{\partial y_i} \le K_3 \exp\left(|\log\varepsilon|^\alpha\right), \quad \forall\, t \in [0, t^*), \tag{6.14}$$

provided that $\varepsilon > 0$ is suitably small.

Noting (6.7) and (6.14), we get (6.3) immediately in Case I.

Case II. As in [2], we assume that

$$\widetilde{\psi}_i'(y_i) > 0 \quad \text{and} \quad a_i(\widetilde{\psi}_i(y_i))^\alpha > 0. \tag{6.15}$$

By Lemma 5.2 and Remark 5.1 we have

$$|w_j(t,x)| \le C_{10}\varepsilon, \quad \forall\ (t,x) \in [0,T_1] \times I\!\!R, \quad \forall\ j = 1, \cdots, n, \tag{6.16}$$

where T_1 is defined by (5.25).

Moreover, similar to (4.62) in [1], noting (5.25) and using (5.1), (5.3), (5.10) and (6.16) we have

$$\begin{aligned} |w_i\,(t, x_i\,(t, y_i)) - \varepsilon l_i(0)\widetilde{\psi}_i'(y_i)| \ &\le \ C_{11}\,\{\varepsilon^2 + W_\infty\,(t)\,W_\infty^c\,(t)\log(1+t) \\ &\quad + (W_\infty\,(t))^2\,V_\infty^c\,(t)\log(1+t) + (V_\infty\,(t))^\alpha\,(W_\infty\,(t))^2\,t\Big\} \\ &\le \ C_{12}\,\{\varepsilon^2 + \varepsilon^2|\log\varepsilon| + \varepsilon^{2+\alpha}|\log\varepsilon|^\alpha t\} \\ &\le \ C_{13}\Big\{\varepsilon^2 + \varepsilon^2|\log\varepsilon| + \varepsilon^{\frac{5}{4}}|\log\varepsilon|^\alpha\Big\}\Big\} \\ &\le \ C_{14}\varepsilon^{\frac{9}{8}}, \quad \forall\ t \in [0,T_1], \end{aligned} \tag{6.17}$$

provided that $\varepsilon > 0$ is suitably small. Then we have

$$w_i(t, x_i(t,y_i)) \ge \frac{1}{2}\varepsilon\widetilde{\psi}_i'(y_i) > 0, \quad \forall\ t \in [0,T_1]. \tag{6.18}$$

On the other hand, as in [1], $w_i(t, x_i(t,y_i))$ is a strictly increasing function of t for $t \ge T_1$, then

$$w_i(t, x_i(t,y_i)) \ge \frac{1}{2}\varepsilon\widetilde{\psi}_i'(y_i) > 0, \quad \forall\ t \in [T_1, t^*). \tag{6.19}$$

Furthermore, for any $j \ne i$, using Lemma 5.1, for $\varepsilon > 0$ suitably small we have

$$\begin{aligned} |w_j\,(t, x_i\,(t,y_i))\,| \ &\le \ C_{15}\,(1+t)^{-1}\,W_\infty^c\,(t) \\ &\le \ C_{16}\,(1+T_1)^{-1}\,\varepsilon \le C_{17}\varepsilon^{\frac{7}{4}}, \quad \forall\ t \in [T_1, t^*). \end{aligned} \tag{6.20}$$

Since the C^1 norm of $\psi(x)$ is bounded, using the first inequality in (6.15) and noting (6.5), for any $j \ne i$ we obtain that for $\varepsilon > 0$ suitably small,

$$|w_j(t, x_i(t,y_i))| < w_i(t, x_i(t,y_i)), \quad \forall\ t \in [T_1, t^*). \tag{6.21}$$

Noting (6.16), completely repeating the proof of Case I, we can prove easily

$$0 < \frac{\partial x_i(t,y_i)}{\partial y_i} \le K_4 \exp\left(|\log\varepsilon|^\alpha\right), \quad \forall\, t \in [0, T_1], \tag{6.22}$$

where K_4 is a positive constant independent of i, t, y_i and ε. Moreover,

$$0 < \frac{\partial x_i(t,y_i)}{\partial y_i}, \quad \forall\, t \in [0, t^*). \tag{6.23}$$

Similar to (4.50) in [2], we have

$$Q_i(t) \le Q_i(T_1) + C_{18} W_\infty^c(t) \int_{T_1}^t (1+\tau)^{-1} Q_i(\tau) d\tau, \quad \forall\, t \in [T_1, t^*), \tag{6.24}$$

and then, noting (6.1) and using Lemma 5.1, for $\varepsilon > 0$ suitably small we get

$$Q_i(t) \le C_{19} Q_i(T_1), \quad \forall\, t \in [T_1, t^*), \tag{6.25}$$

where

$$Q_i(t) = |w_i(t, x_i(t,y_i))| \frac{\partial x_i(t,y_i)}{\partial y_i}. \tag{6.26}$$

Thus, noting (6.16) and (6.22), we obtain

$$Q_i(t) \le C_{20} \varepsilon \exp\left(|\log\varepsilon|^\alpha\right), \quad \forall\, t \in [0, t^*), \tag{6.27}$$

i.e.,

$$|w_i(t, x_i(t,y_i))| \frac{\partial x_i(t,y_i)}{\partial y_i} \le C_{20} \varepsilon \exp\left(|\log\varepsilon|^\alpha\right), \quad \forall\, t \in [0, t^*). \tag{6.28}$$

This proves (6.3) in Case II. The proof of Lemma 6.1 is finished. Q.E.D.

Using Lemma 6.1, completely repeating the proof of Theorem 1.2 and Theorem 1.3 in [2], we get the conclusion of Theorem 1.4 and Theorem 1.5 immediately.

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