ON THE UNIQUENESS AND STABILITY OF ENTROPY SOLUTIONS OF NONLINEAR DEGENERATE PARABOLIC EQUATIONS WITH ROUGH COEFFICIENTS

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ABSTRACT. We study nonlinear degenerate parabolic equations where the flux function $f(x,t,u)$ does not depend Lipschitz continuously on the spatial location x. By properly adapting the "doubling of variables" device due to Kružkov [24] and Carrillo [12], we prove a uniqueness result within the class of entropy solutions for the initial value problem. We also prove a result concerning the continuous dependence on the initial data and the flux function for degenerate parabolic equations with flux function of the form $k(x)f(u)$, where $k(x)$ is a vector-valued function and $f(u)$ is a scalar function.

1. INTRODUCTION

The main subject of this paper is uniqueness and stability properties of entropy solutions of nonlinear degenerate parabolic equations where the flux function depends explicitly on the spatial location. In particular, this paper is concerned with the case where the flux function does not depend Lipschitz continuously on the spatial variable. Our study is motivated by applications where one frequently encounters flux functions possessing minimal smoothness in the spatial variable.

The problems that we study are initial value problems of the form

$$\begin{aligned} &u_t + \operatorname{div} f(x,t,u) = \Delta A(u) + q(x,t,u), \quad (x,t) \in \Pi_T = \mathbf{R}^d \times (0,T), \\ &u(x,0) = u_0(x), \quad x \in \mathbf{R}^d, \end{aligned} \tag{1.1}$$

where $T > 0$ is fixed, $u(x,t)$ is the scalar unknown function that is sought, $f = f(x,t,u)$ is called the flux function, $A = A(u)$ is the diffusion function, and $q = (x,t,u)$ is the source term. The coefficients f, A, q of problem (1.1) are given functions satisfying certain regularity assumptions. The regularity assumptions on f, q will be given later.

For the initial value problem (1.1) to be well-posed, we must require that $A : \mathbf{R} \to \mathbf{R}$ satisfies

$$A \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}) \text{ and } A(\cdot) \text{ is nondecreasing with } A(0) = 0. \tag{1.2}$$

The second part of (1.2) implies that the nonlinear operator $u \mapsto \Delta A(u)$ is of *degenerate elliptic* type, and hence many well known nonlinear and linear partial differential equations are special cases of (1.1). In particular, the scalar conservation law ($A' \equiv 0$) is a "simple" special case. Included is also the heat equation, porous medium type equations characterized by one-point degeneracy, two-phase reservoir flow equations characterized by the two-point degeneracy, as well as *strongly* degenerate convection-diffusion equations where $A'(s) \equiv 0$ for all s in some interval $[\alpha, \beta]$. Consequently, partial differential equations of the type (1.1) model a wide variety of phenomena, ranging from porous media flow [32], via flow of glaciers [19] and sedimentation processes [9], to traffic flow [35].

We recall that if the problem (1.1) is non-degenerate (uniformly parabolic), it is well known that it admits a unique classical solution. This contrasts with the case where (1.1) is allowed to degenerate at certain points, that is, $A'(s) = 0$ for some values of s. Then solutions are not necessarily smooth (but typically continuous) and weak solutions must be sought. On the other hand, if $A'(s)$ is zero on an interval $[\alpha, \beta]$, (weak) solutions may be discontinuous and are not

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uniquely determined by their initial data. Consequently, an entropy condition must be imposed to single out the physically correct solution.

Roughly speaking, we call a function $u \in L^1 \cap L^\infty$ an *entropy solution* of the initial value problem (1.1) if

$$\begin{cases} \text{(i)} \;\; |u-c|_t + \operatorname{div}\left[\operatorname{sign}(u-c)\left(f(x,t,u)-f(x,t,c)\right)\right] - \Delta\left|A(u)-A(c)\right| \\ \qquad +\operatorname{sign}(u-c)\left(\operatorname{div} f(x,t,c) - q(x,t,u)\right) \le 0 \text{ in } \mathcal{D}' \;\forall c \in \mathbf{R}, \\ \text{(ii)} \;\; \nabla A(u) \text{ belongs to } L^2. \end{cases} \tag{1.3}$$

In addition, we require that the initial function u_0 is assumed in the strong L^1 sense. We refer to §2 for a precise definition of an entropy solution.

The mathematical (L^1/BV) theory of parabolic equations was initiated by Oleĭnik [27]. She proved well-posedness of the initial value problem in the non-degenerate case with $A(u) = u$, and showed that weak solutions are in this case classical.

In the hyperbolic case ($A' \equiv 0$) with the flux $f = f(x,t,u)$ depending (smoothly) on x and t, the notion of entropy solution was introduced independently by Kružkov [24] and Vol'pert [33] (the latter author considered the smaller BV class). These authors also proved general existence, uniqueness, and stability results for the entropy solution, see also Oleĭnik [27] for similar results in the convex case $f_{uu} \ge 0$.

In the mixed hyperbolic-parabolic case ($A' \ge 0$), the notion of entropy solution goes back to Vol'pert and Hudjaev [34], who were the first to study strongly degenerate parabolic equations. These authors showed existence of a BV entropy solution using the viscosity method and obtained some partial uniqueness results in the BV class (i.e., when the first order partial derivatives of u are finite measures). In the one-dimensional case, Wu and Yin [36] later provided a complete uniqueness proof in the BV class. Further results in the one-dimensional case were obtained by Bénilan and Touré [3, 4] using nonlinear semigroup theory.

As for the uniqueness issue in the multi-dimensional case, Brézis and Crandall [6] established uniqueness of weak solutions when $f \equiv 0$. Later, under the assumption that $A(s)$ is strictly increasing, Yin [37] showed uniqueness of weak solutions in the BV class. Bénilan and Gariepy [2] showed that the BV weak solution studied in [37] is actually a strong solution. The assumption that u_t should be a finite measure was removed in [38, 39].

An important step forward in the general case of $A(\cdot)$ being merely nondecreasing was made recently by Carrillo [12], who showed uniqueness of the entropy solution for a particular boundary value problem with the boundary condition "$A(u) = 0$". His method of proof is an elegant extension of the by now famous "doubling of variables" device introduced by Kružkov [24]. In [12], the author also showed existence of an entropy solution using the semigroup method.

In [7] (see also [29]), the uniqueness proof of Carrillo was adopted to several initial-boundary value problems arising the theory of sedimentation-consolidation processes [9], which in some cases call for the notion of an entropy boundary condition (see also [8] for the BV approach).

In the present paper we generalize Carrillo's uniqueness result [12] by showing that it holds for the Cauchy problem with a flux function $f = f(x,t,u)$ where the spatial dependence is non-smooth (non-Lipschitz). Only the case $f = f(u)$ was studied in [12]. Moreover, we also establish continuous dependence on the flux function in the case $f(x,t,u) = k(x)f(u)$.

With the assumptions on the diffusion function A already given (see (1.2)), we now present the (regularity) assumptions that are needed on the flux function f and the source term q, with the those on f being the most important ones. Concerning the source term $q : \mathbf{R}^d \times (0,T) \times \mathbf{R} \to \mathbf{R}$, we assume that $q(x,t,0) = 0 \;\forall x,t$ and

$$q(\cdot,\cdot,u) \in L^1(0,T;L^\infty(\mathbf{R}^d)) \;\forall u; \quad q(x,t,\cdot) \in \operatorname{Lip}_{\mathrm{loc}}(\mathbf{R}) \text{ uniformly in } x,t. \tag{1.4}$$

With the phrase "uniformly in x,t" in (1.4), we mean

$$|q(x,t,v) - q(x,t,u)| \le C|v-u|, \quad \forall x,t,v,u,$$

for some constant $C > 0$ (independently of x,t,v,u).

Concerning the flux function $f : \mathbf{R}^d \times [0,T] \times \mathbf{R} \to \mathbf{R}^d$, we assume without loss of generality that $f(x,t,0) = f_x(x,t,0) = 0$. Moreover, we assume that

$$f(\cdot,\cdot,u) \in L^1(0,T;W^{1,1}(\mathbf{R}^d))\ \forall u; \quad f(x,t,\cdot) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}) \text{ uniformly in } x,t; \tag{1.5}$$

$$f_x(\cdot,\cdot,u) \in L^1(0,T;L^\infty(\mathbf{R}^d))\ \forall u; \quad f_x(x,t,\cdot) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}) \text{ uniformly in } x,t, \tag{1.6}$$

where $f_x = f_x(x,t,u)$ in (1.6) denotes the function obtained by taking the divergence of the flux $f = f(x,t,u)$ with respect to the first variable. With the phrase "uniformly in x,t" in (1.5) and (1.6), we mean

$$|f(x,t,v) - f(x,t,u)|,\ |f_x(x,t,v) - f_x(x,t,u)| \le C|v-u|, \quad \forall x,t,v,u,$$

for some constant $C > 0$ (independently of x,t,v,u).

The conditions in (1.4)-(1.6) are sufficient to make sense to the notion of entropy solution (see §2). In the general case, however, we need one additional regularity assumption on the x dependency of f to get uniqueness of the entropy solution. Inspired by Capuzzo-Dolcetta and Perthame [10], we assume that

$$\big(F(x,t,v,u) - F(y,s,v,u)\big)\cdot(x-y) \ge -\gamma\,|v-u|\,|x-y|^2, \qquad \forall x,y,t,v,u, \tag{1.7}$$

for some constant $\gamma > 0$ (independent of x,t,v,u), where

$$F(x,t,v,u) := \operatorname{sign}(v-u)\,\big[f(x,t,v) - f(x,t,u)\big]. \tag{1.8}$$

Note that condition (1.7) does *not* imply that f is Lipschitz continuous in the spatial variable x. We remark that if $f = f(x,u)$ is of the form

$$f = k(x)h(u),$$

for some vector valued function $k : \mathbf{R}^d \to \mathbf{R}^d$, and a Lipschitz continuous function h, then (1.7) reduces to

$$(k(x) - k(y))\cdot(x-y) \ge -\gamma\,|x-y|^2, \qquad \forall x,y,t,v,u, \tag{1.9}$$

for some constant $\gamma > 0$ (depending also on the Lipschitz constant of h). As pointed out in [10], this condition requires a bound only on the matrix $\nabla_x k + (\nabla_x k)^{\mathrm{T}}$ (the symmetric part of the Jacobian $\nabla_x k$) and k itself need not belong to any Sobolev space. To see this, let $z = x - y$ and rewrite the left-hand side of (1.9) as follows

$$\begin{aligned}
(k(x) - k(y))\cdot(x-y) &= \int_0^1 \frac{d}{d\xi}\left[(k(y+\xi z) - k(y))\cdot z\right] d\xi \\
&= \int_0^1 \nabla_x k(y+\xi z)z\cdot z\, d\xi \\
&= \frac{1}{2}\int_0^1 \left(\nabla_x k + (\nabla_x k)^{\mathrm{T}}\right)(y+\xi z)z\cdot z\, d\xi,
\end{aligned}$$

since $\frac{1}{2}\left(\nabla_x k - (\nabla_x k)^{\mathrm{T}}\right)(y+\xi z)z\cdot z \equiv 0$.

In [10], the authors showed the universality of (1.7) by proving that under this condition, uniqueness holds for the Kružkov-Vol'pert entropy solution of hyperbolic equations, the Crandall-Lions viscosity solution of Hamilton-Jacobi equations, and the DiPerna-Lions regularized solution of transport equations. With the present paper, we add to that list uniqueness of the entropy solution of degenerate parabolic equations. More precisely, we prove the following theorem:

Theorem 1.1 (Uniqueness)**.** *Assume that* (1.2) *and* (1.4)-(1.7) *hold. Let v,u be two entropy solutions of* (1.1) *with initial data $u_0 \in L^1(\mathbf{R}^d) \cap L^\infty(\mathbf{R}^d)$. Then $v = u$ a.e. in $\Pi_T = \mathbf{R}^d \times (0,T)$.*

By combining the arguments used in the present paper by those used in [17], Theorem 1.1 can be proved even for a large class of weakly coupled systems of degenerate parabolic equations.

We next restrict our attention to problems of the form

$$\begin{aligned}
&u_t + \operatorname{div}\big(k(x)f(u)\big) = \Delta A(u), \quad (x,t) \in \Pi_T, \\
&u(x,0) = u_0(x), \quad x \in \mathbf{R}^d,
\end{aligned} \tag{1.10}$$

where $k : \mathbf{R}^d \to \mathbf{R}^d$, $f : \mathbf{R} \to \mathbf{R}$, and $f(0) = 0$. Problems of the form (1.10) occur in several important applications. Our first result for (1.10) states that in the $L^\infty(0,T;BV(\mathbf{R}^d))$ class of entropy solutions, an L^1 contraction principle actually holds provided

$$f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}); \quad k \in W^{1,1}_{loc}(\mathbf{R}^d) \bigcap C\left(\mathbf{R}^d\right); \quad k, \mathrm{div}k \in L^\infty(\mathbf{R}^d). \tag{1.11}$$

More precisely, we prove the following theorem:

Theorem 1.2 (L^1 contraction)**.** *Assume that* (1.2) *and* (1.11) *hold. Let* $v, u \in L^\infty(0,T;BV(\mathbf{R}^d))$ *be entropy solutions of* (1.10) *with initial data* $v_0, u_0 \in L^1(\mathbf{R}^d) \cap L^\infty(\mathbf{R}^d) \cap BV(\mathbf{R}^d)$*, respectively. Then for almost all* $t \in (0,T)$*,*

$$\|v(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbf{R}^d)} \le \|v_0 - u_0\|_{L^1(\mathbf{R}^d)}.$$

In particular, there exists at most one entropy solution of the initial value problem (1.10)*.*

We remark that the existence of an $L^\infty(0,T;BV(\mathbf{R}^d))$ entropy solution of (1.10) is guaranteed if $\mathrm{div}k \in BV(\mathbf{R}^d)$. This follows from the results obtained by Karlsen and Risebro [20], who prove convergence (within the entropy solution framework) of finite difference schemes for degenerate parabolic equations with rough coefficients. For an overview of the literature on numerical methods for approximating entropy solutions of degenerate parabolic equations, we refer to the first section of [20] and the lecture notes [14] (see also the references given therein).

Let us mention that Theorem 1.2 includes the L^1 contraction property proved by Klausen and Risebro [21] for the one-dimensional scalar conservation law with a discontinuous coefficient $k(x)$. Throughout this paper the coefficient $k(x)$ is not allowed to be discontinuous. In the one-dimensional hyperbolic case ($A' \equiv 0$) with $k(x)$ depending discontinuously on x, the equation (1.1) is often written as the following 2×2 system:

$$u_t + f(k,u)_x = 0, \qquad k_t = 0. \tag{1.12}$$

If $\partial f/\partial u$ changes sign, then this system is non-strictly hyperbolic. This complicates the analysis, and in order to prove compactness of approximated solutions a singular transformation $\Psi(k,u)$ has been used by several authors [30, 16, 23, 22]. In these works convergence of the Glimm scheme and of front tracking was established in the case where k may be discontinuous. If $k \in C^2\left(\mathbf{R}^d\right)$, then convergence of the Lax-Friedrichs scheme and the upwind scheme was proved in [27]. Under weaker conditions on k ($k' \in BV$) and for f convex in u, convergence of the one-dimensional Godunov method for (1.12) (not for (1.1)) was shown by Isaacson and Temple in [18]. Recently, convergence of the one-dimensional Godunov method for (1.1) was shown by Towers [31] in the case where k is piecewise continuous. In this case, the Kružkov entropy condition (1.3) no longer applies, and in [23] a wave entropy condition analogous to the Oleĭnik entropy condition introduced in [27] was used to obtain uniqueness, see also [22]. Klausen and Risebro [21] analyzed the case of discontinuous k by "smoothing out" the coefficient k and then passing to the limit as the smoothing parameter tends to zero. In particular, they showed that the limit "entropy" solution satisfied the L^1 contraction property. We intend to study the degenerate parabolic problem (1.10) when $k(x)$ is discontinuous in future work.

Theorem 1.2 gives the desired continuous dependence on the initial data in degenerate parabolic problems of the type (1.10). Next we will establish continuous dependence also on the flux function. To this end, let us also introduce the problem

$$\begin{aligned} &v_t + \mathrm{div}\big(l(x)g(v)\big) = \Delta A(v), \quad (x,t) \in \Pi_T, \\ &v(x,0) = v_0(x), \quad x \in \mathbf{R}^d, \end{aligned} \tag{1.13}$$

where $l : \mathbf{R}^d \to \mathbf{R}^d$, $g : \mathbf{R} \to \mathbf{R}$, and $g(0) = 0$. We are interested in estimating the L^1 difference between the entropy solution v of (1.13) and the entropy solution u of (1.10). Now we assume that

$$f, g \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}); \quad k, l \in W^{1,1}(\mathbf{R}^d) \bigcap C\left(\mathbf{R}^d\right); \quad k, l, \mathrm{div}k, \mathrm{div}l \in L^\infty(\mathbf{R}^d). \tag{1.14}$$

Under these assumptions, we prove the following continuous dependence result:

Theorem 1.3 (Continuous dependence)**.** *Assume that the regularity conditions* (1.2) *and* (1.14) *hold. Let* $v, u \in L^\infty(0,T;BV(\mathbf{R}^d))$ *be entropy solutions of* (1.13), (1.10) *with initial data* $v_0, u_0 \in L^1(\mathbf{R}^d) \cap L^\infty(\mathbf{R}^d) \cap BV(\mathbf{R}^d)$, *respectively. For definiteness, let us assume that* v, u *take values in the closed interval* $I \subset \mathbf{R}$ *and that there are constants* $V_v, V_u > 0$ *such that*

$$|v(\cdot,t)|_{BV(\mathbf{R}^d)} \le V_v \ \forall t \in (0,T), \qquad |u(\cdot,t)|_{BV(\mathbf{R}^d)} \le V_u \ \forall t \in (0,T).$$

Then for almost all $t \in (0,T)$,

$$\begin{aligned}
&\|v(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbf{R}^d)} \le \|v_0 - u_0\|_{L^1(\mathbf{R}^d)} \\
&\quad + t\Big[\Big(C_1^{g,v}\|l-k\|_{L^\infty(\mathbf{R}^d)} + C_2^g|l-k|_{BV(\mathbf{R}^d)} + C_3^k\|g-f\|_{L^\infty(I)} + C_4^{k,v}\|g-f\|_{\mathrm{Lip}(I)}\Big) \\
&\qquad\quad \wedge \Big(C_1^{f,u}\|l-k\|_{L^\infty(\mathbf{R}^d)} + C_2^f|l-k|_{BV(\mathbf{R}^d)} + C_3^l\|g-f\|_{L^\infty(I)} + C_4^{l,u}\|g-f\|_{\mathrm{Lip}(I)}\Big)\Big],
\end{aligned}$$

where $C_1^{g,v} = \|g\|_{\mathrm{Lip}(I)}V_v$, $C_1^{f,u} = \|f\|_{\mathrm{Lip}(I)}V_u$, $C_2^g = \|g\|_{L^\infty(I)}$, $C_2^f = \|f\|_{L^\infty(I)}$, $C_3^k = |k|_{BV(\mathbf{R}^d)}$, $C_3^l = |l|_{BV(\mathbf{R}^d)}$, $C_4^{k,v} = \|k\|_{L^\infty(\mathbf{R}^d)}V_v$, $C_4^{l,u} = \|l\|_{L^\infty(\mathbf{R}^d)}V_u$, *and* $a \wedge b = \min(a,b)$.

We remark that Theorem 1.3 includes the continuous dependence result obtained in Klausen and Risebro [21] for the one-dimensional scalar conservation law with a discontinuous coefficient $k(x)$. Results regarding continuous dependence on the flux function in scalar conservation laws with $k(x) \equiv 1$ have been obtained by Lucier [26] and Bouchut and Perthame [5]. Finally, we mention that Cockburn and Gripenberg [13] have obtained a result regarding continuous dependence on both the flux function and the diffusion function in (1.10) when $k(x) = 1$. Their result does *not*, however, imply uniqueness of the entropy solution since their "doubling of variables" argument requires that one works with (smooth) approximate solutions. By properly combining the ideas in the present paper with those in [13], one can prove a version of Theorem 1.3 which also includes continuous dependence on the diffusion function A, see [15]. We will present the details elsewhere.

The rest of this paper is organized as follows: In the next section we introduce (precisely) the notion of entropy solution as well as stating and proving a version of an important lemma due to Carrillo [12]. Equipped with our version of Carrillo's lemma, Theorems 1.1, 1.2, and 1.3 are proved in §3, §4, and §5, respectively. Finally, in §6 (an appendix) we provide a proof of the weak chain rule needed in the proof of Carrillo's lemma.

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2. Preliminaries

We shall use the following definition of an entropy solution of (1.1):

Definition 2.1. *An entropy solution of* (1.1) *is a measurable function* $u = u(x,t)$ *satisfying:*

D.1 $u \in L^1(\Pi_T) \cap L^\infty(\Pi_T) \cap C(0,T;L^1(\mathbf{R}^d))$.

D.2 *For all* $c \in \mathbf{R}$ *and all non-negative test functions in* $C_0^\infty(\Pi_T)$, *the following entropy inequality holds:*

$$\begin{aligned}
\iint_{\Pi_T} \Big(&|u-c|\phi_t + \operatorname{sign}(u-c)\big(f(x,t,u) - f(x,t,c)\big)\cdot\nabla\phi + |A(u)-A(c)|\Delta\phi \\
&- \operatorname{sign}(u-c)\big(\operatorname{div} f(x,t,c) - q(x,t,u)\big)\phi\Big)\,dt\,dx \ge 0.
\end{aligned} \tag{2.1}$$

D.3 $A(u) \in L^2(0,T;H^1(\mathbf{R}^d))$.

D.4 *Essentially as* $t \downarrow 0$,

$$\int_{\mathbf{R}^d} \big|u(x,t) - u_0(x)\big|\,dx \to 0.$$

Remark 2.1. (i) Observe that when $A' \equiv 0$, (2.1) reduces to the well known entropy inequality for scalar conservation laws introduced by Kružkov [24] and Vol'pert [33].

(ii) Condition (**D**.4), i.e., that the initial datum u_0 should be taken by continuity, motivates the requirement of continuity with respect to t in condition (**D**.1).

Let u be an entropy solution. Then, since $A(u) \in H^1(\mathbf{R}^d)$ for a.e. $t \in (0,T)$, it follows from general theory of Sobolev spaces that $\nabla|A(u) - A(c)| = \operatorname{sign}(A(u) - A(c))\,\nabla A(u)$ a.e. in Π_T. Also, $\operatorname{sign}(A(u) - A(c)) = \operatorname{sign}(u-c)$ provided $A(u) \neq A(c)$. Again since $A(u) \in H^1(\mathbf{R}^d)$ for a.e. $t \in (0,T)$, it follows that $\nabla A(u) = 0$ a.e. (w.r.t. $dt\,dx$) in $\{(x,t) \in \Pi_T : A(u(x,t)) = A(c)\}$. We therefore conclude that

$$\nabla|A(u) - A(c)| = \operatorname{sign}(u-c)\,\nabla A(u) \text{ a.e. in } \Pi_T$$

and the entropy inequality (2.1) can be written equivalently as

$$\begin{aligned}\iint_{\Pi_T} \Big(&|u-c|\phi_t + \operatorname{sign}(u-c)\big[f(x,t,u) - f(x,t,c) - \nabla A(u)\big]\cdot\nabla\phi \\ &- \operatorname{sign}(u-c)\big(\operatorname{div} f(x,c) - q(x,t,u)\big)\phi\Big)\,dt\,dx \geq 0, \qquad \forall \phi \in C_0^\infty(\Pi_T).\end{aligned} \tag{2.2}$$

If we take $c > \operatorname{ess\,sup} u(x,t)$ and $c < \operatorname{ess\,inf} u(x,t)$ in (2.1), then we deduce that u satisfies

$$\iint_{\Pi_T} \Big(u\phi_t + f(x,t,u)\cdot\nabla\phi + A(u)\Delta\phi + q(x,t,u)\phi\Big)\,dt\,dx = 0, \quad \forall\phi \in C_0^\infty(\Pi_T) \tag{2.3}$$

Note that (1.5) implies

$$\|f(x,t,u)\|^2_{L^2(\Pi_T)} \leq \text{Const}\,\|u\|_{L^\infty(\Pi_T)}\,\|u\|_{L^1(\Pi_T)} < \infty, \tag{2.4}$$

so that $f(x,t,u) - \nabla A(u) \in L^2(\Pi_T;\mathbf{R}^d)$. Similarly, (1.4) implies $q(x,t,u)$ belongs to $L^2(\Pi_T)$. An integration by parts in (2.3) followed by an approximation argument will then show that the equality

$$\iint_{\Pi_T} \Big(u\phi_t + \big[f(x,t,u) - \nabla A(u)\big]\cdot\nabla\phi + q(x,t,u)\phi\Big)\,dt\,dx = 0 \tag{2.5}$$

holds for all $\phi \in L^2(0,T;H_0^1(\mathbf{R}^d)) \cap W^{1,1}(0,T;L^\infty(\mathbf{R}^d))$.

We can even go one step further. To this end, let $\langle\cdot,\cdot\rangle$ denote the usual pairing between $H^{-1}(\mathbf{R}^d)$ and $H_0^1(\mathbf{R}^d)$. From (2.5), we conclude that

$$\partial_t u \in L^2(0,T;H^{-1}(\mathbf{R}^d)),$$

so that the equality

$$-\int_0^T \langle \partial_t u, \phi\rangle\,dt + \iint_{\Pi_T} \Big(\big[f(x,t,u) - \nabla A(u)\big]\cdot\nabla\phi + q(x,t,u)\phi\Big)\,dt\,dx = 0 \tag{2.6}$$

holds for all $\phi \in L^2(0,T;H_0^1(\mathbf{R}^d)) \cap W^{1,1}(0,T;L^\infty(\mathbf{R}^d))$. The fact that an entropy solution u satisfies (2.6) will be important for the uniqueness proof.

We now set

$$\mathcal{A}_\psi(z) = \int_{z_0}^{z} \psi\big(A(r)\big)\,dr, \tag{2.7}$$

where $\psi : \mathbf{R} \to \mathbf{R}$ is a nondecreasing and Lipschitz continuous function and $z_0 \in \mathbf{R}$. Concerning the function $\mathcal{A}_\psi$, we shall need the following associated "weak chain rule" :

Lemma 2.1. *Let $u : \Pi_T \to \mathbf{R}$ be a measurable function satisfying the following four conditions:*

(a): $u \in L^1(\Pi_T) \cap L^\infty(\Pi_T) \cap C(0,T;L^1(\mathbf{R}^d))$.
(b): $u(0,\cdot) = u_0 \in L^\infty(\mathbf{R}^d) \cap L^1(\mathbf{R}^d)$.
(c): $\partial_t u \in L^2\big(0,T;H^{-1}(\mathbf{R}^d)\big)$.
(d): $A(u) \in L^2\big(0,T;H^1(\mathbf{R}^d)\big)$.

Then, for a.e. $s \in (0,T)$ *and every nonnegative* $\phi \in C_0^\infty(\mathbf{R}^d \times [0,T])$*, we have*

$$-\int_0^s \langle \partial_t u, \psi(A(u))\phi\rangle \, dt$$
$$= \int_0^s \int_{\mathbf{R}^d} \mathcal{A}_\psi(u)\phi_t \, dt\, dx + \int_{\mathbf{R}^d} \mathcal{A}_\psi(u_0)\phi(x,0)\, dx - \int_{\mathbf{R}^d} \mathcal{A}_\psi(u(x,s))\phi(x,s)\, dx.$$

Lemma 2.1 can proved more or less in the same way as the "weak chain rule" in Carrillo [12], see also Alt and Luckhaus [1] and Otto [28]. For the sake of completeness, a proof of Lemma 2.1 is given in §6 (the appendix).

In what follows, we shall frequently need a continuous approximation of sign$(\cdot)$. For $\varepsilon > 0$, set

$$\operatorname{sign}_\varepsilon(\tau) = \begin{cases} -1, & \tau < \varepsilon, \\ \tau/\varepsilon, & \varepsilon \le \tau \le \varepsilon, \\ 1 & \tau > \varepsilon. \end{cases}$$

Note that $\operatorname{sign}_\varepsilon(-r) = -\operatorname{sign}_\varepsilon(r)$ and $\operatorname{sign}'_\varepsilon(-r) = \operatorname{sign}'_\varepsilon(r)$ a.e.

We let $A^{-1} : \mathbf{R} \to \mathbf{R}$ denote the unique left-continuous function satisfying $A^{-1}(A(u)) = u$ for all $u \in \mathbf{R}$, and by E we denote the set

$$E = \Big\{ r :\ A^{-1}(\cdot) \text{ discontinuous at } r \Big\}.$$

Note that E is associated with the set of points $\big\{u :\ A'(u) = 0\big\}$ at which the operator $u \mapsto \Delta A(u)$ is degenerate elliptic.

We are now ready to state and prove the following version of an important observation made by Carrillo [12]:

Lemma 2.2 (Entropy dissipation term)**.** *Let* u *be an entropy solution of* (1.1)*. Then, for any non-negative* $\phi \in C_0^\infty(\Pi_T)$ *and* $c \in \mathbf{R}$ *such that* $A(c) \notin E$*, we have*

$$\begin{aligned}\iint_{\Pi_T} \Big(|u-c|\phi_t + \operatorname{sign}(u-c)\, \big[f(x,t,u) - f(x,t,c) - \nabla A(u)\big] \cdot \nabla\phi \\ - \operatorname{sign}(u-c)\, \big(\operatorname{div} f(x,t,c) - q(x,t,u)\big)\phi\Big)\, dt\, dx \\ = \lim_{\varepsilon \downarrow 0} \iint_{\Pi_T} \big|\nabla A(u)\big|^2 \operatorname{sign}'_\varepsilon(A(u) - A(c))\, \phi \, dt\, dx.\end{aligned} \tag{2.8}$$

Proof. The proof is similar to the proof of the corresponding result in [12]. In (2.7), introduce the function $\psi_\varepsilon(z) = \operatorname{sign}_\varepsilon(z - A(c))$ and set $z_0 = c$. Notice that the conditions of Lemma 2.1 are satisfied and hence

$$-\int_0^T \Big\langle \partial_t u, \operatorname{sign}_\varepsilon(A(u) - A(c))\,\phi \Big\rangle\, dt = \iint_{\Pi_T} \mathcal{A}_{\psi_\varepsilon}(u)\phi_t\, dt\, dx.$$

Since u satisfies (2.6) and $[\operatorname{sign}_\varepsilon(A(u) - A(c))\,\phi] \in L^2(0,T; H_0^1(\mathbf{R}))$ is a test function, we have

$$\begin{aligned}-\int_0^T \Big\langle \partial_t u, \operatorname{sign}_\varepsilon(A(u) - A(c))\,\phi \Big\rangle\, dt \\ + \iint_{\Pi_T} \Big(\big[f(x,t,u) - f(x,t,c) - \nabla A(u)\big] \cdot \nabla(\operatorname{sign}_\varepsilon(A(u) - A(c))\,\phi) \\ - \big(\operatorname{div} f(x,t,c) - q(x,t,u)\big)(\operatorname{sign}_\varepsilon(A(u) - A(c))\,\phi)\Big)\, dt\, dx = 0,\end{aligned}$$

which implies that

$$\iint_{\Pi_T} \mathcal{A}_{\psi_\varepsilon}(u)\phi_t\,dt\,dx + \iint_{\Pi_T} \Big(\big[f(x,t,u) - f(x,t,c) - \nabla A(u)\big] \cdot \nabla(\operatorname{sign}_\varepsilon(A(u)-A(c))\,\phi) \\ - \operatorname{sign}_\varepsilon(A(u)-A(c))\big(\operatorname{div} f(x,t,c) - q(x,t,u)\big)\phi\Big)\,dt\,dx = 0. \tag{2.9}$$

Since $A(r) > A(c)$ if and only if $r > c$, $\operatorname{sign}_\varepsilon(A(r)-A(c)) \to 1$ as $\varepsilon \downarrow 0$ for any $r > c$. Similarly, $\operatorname{sign}_\varepsilon(A(r)-A(c)) \to -1$ as $\varepsilon \downarrow 0$ for any $r < c$. Consequently, whenever $A(c) \notin E$,

$$\mathcal{A}_{\psi_\varepsilon}(u) \to |u-c| \text{ a.e. in } \Pi_T \text{ as } \varepsilon \downarrow 0.$$

Moreover, we have $|\mathcal{A}_{\psi_\varepsilon}(u)| \le |u-c| \in L^1(\Pi_T)$, so by the Lebesgue dominated convergence theorem

$$\lim_{\varepsilon\downarrow 0} \iint_{\Pi_T} \mathcal{A}_{\psi_\varepsilon}(u)\phi_t\,dt\,dx = \iint_{\Pi_T} |u-c|\phi_t\,dt\,dx.$$

For c such that $A(c) \notin E$, we have

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \iint_{\Pi_T} \big[f(x,t,u) - f(x,t,c) - \nabla A(u)\big] \cdot \nabla\left[\operatorname{sign}_\varepsilon(A(u)-A(c))\,\phi\right]\,dt\,dx \\
&\quad= \lim_{\varepsilon\downarrow 0} \iint_{\Pi_T} \big[f(x,t,u) - f(x,t,c) - \nabla A(u)\big] \cdot \nabla \operatorname{sign}_\varepsilon(A(u)-A(c))\,\phi\,dt\,dx \\
&\qquad + \lim_{\varepsilon\downarrow 0} \iint_{\Pi_T} \operatorname{sign}_\varepsilon(A(u)-A(c))\big[f(x,t,u) - f(x,t,c) - \nabla A(u)\big] \cdot \nabla\phi\,dt\,dx \\
&\quad= \underbrace{\lim_{\varepsilon\downarrow 0} \iint_{\Pi_T} \operatorname{sign}'_\varepsilon(A(u)-A(c))\big(f(x,t,u) - f(x,t,c)\big) \cdot \nabla A(u)\phi\,dt\,dx}_{I_1} \\
&\qquad - \lim_{\varepsilon\downarrow 0} \iint_{\Pi_T} \big|\nabla A(u)\big|^2 \operatorname{sign}'_\varepsilon(A(u)-A(c))\,\phi\,dt\,dx \\
&\qquad\quad + \underbrace{\lim_{\varepsilon\downarrow 0} \iint_{\Pi_T} \operatorname{sign}_\varepsilon(A(u)-A(c))\big[f(x,t,u) - f(x,t,c) - \nabla A(u)\big] \cdot \nabla\phi\,dt\,dx}_{I_2}.
\end{aligned}$$

One can check that

$$I_1 = \lim_{\varepsilon\downarrow 0} \iint_{\Pi_T} \operatorname{div}\mathcal{Q}_\varepsilon(A(u))\phi\,dt\,dx,$$

where $\mathcal{Q}_\varepsilon$ is defined as

$$\begin{aligned}
\mathcal{Q}_\varepsilon(z) &= \int_0^z \operatorname{sign}'_\varepsilon(r - A(c))\Big(f(x,t,A^{-1}(r)) - f(x,t,c)\Big)\,dr \\
&= \frac{1}{\varepsilon}\int_{\min(z,A(c)-\varepsilon)}^{\min(z,A(c)+\varepsilon)} \Big(f(x,t,A^{-1}(r)) - f(x,t,A^{-1}(A(c)))\Big)\,dr.
\end{aligned}$$

Since $f = f(x,t,u)$ is locally Lipschitz continuous with respect to u, $\mathcal{Q}_\varepsilon(z)$ tends to zero as $\varepsilon \downarrow 0$ for all z in the image of A. Consequently, by the Lebesgue dominated convergence theorem,

$$I_1 = -\lim_{\varepsilon\downarrow 0} \iint_{\Pi_T} \mathcal{Q}_\varepsilon(A(u))\nabla\phi\,dt\,dx = 0.$$

Observe that for each $c \in \mathbf{R}$ such that $A(c) \notin E$,

$$\operatorname{sign}(u-c) = \operatorname{sign}(A(u)-A(c)) \quad \text{a.e. in } \Pi_T.$$

Therefore, from the Lebesgue bounded convergence theorem, it follows that

$$I_2 = \iint_{\Pi_T} \mathrm{sign}\,(u-c)\,\big[f(x,t,u) - f(x,t,c) - \nabla A(u)\big]\cdot\nabla\phi\,dt\,dx$$

and

$$\lim_{\varepsilon\downarrow 0}\iint_{\Pi_T} \mathrm{sign}_\varepsilon\,(A(u)-A(c))\,\big(\mathrm{div} f(x,t,c) - q(x,t,u)\big)\phi\,dt\,dx$$
$$= \iint_{\Pi_T} \mathrm{sign}\,(u-c)\,\big(\mathrm{div} f(x,t,c) - q(x,t,u)\big)\phi\,dt\,dx,$$

Therefore, letting $\varepsilon\downarrow 0$ in (2.9), we obtain the desired equality (2.8). □

3. Proof of Theorem 1.1

Equipped with the results derived in §2 (in particular Lemma 2.2), we now set out to prove Theorem 1.1 using the "doubling of variables" device, which was introduced by Kružkov [24] as a tool for proving the uniqueness (L^1 contraction property) of the entropy solution of first order hyperbolic equations. We refer to Carrillo [11, 12], Otto [28], and Cockburn and Gripenberg [13] for applications of the "doubling" device in the context of second order parabolic equations. The presentation that follows below is inspired by Carrillo [12].

Let $\phi \in C^\infty(\Pi_T\times\Pi_T)$, $\phi\geq 0$, $\phi=\phi(x,t,y,s)$, $v=v(x,t)$, and $u=u(y,s)$. We shall also need to introduce the "hyperbolic" sets

$$\mathcal{E}_v = \Big\{(x,t)\in\Pi_T : A(v(x,t))\in E\Big\},\qquad \mathcal{E}_u = \Big\{(y,s)\in\Pi_T : A(u(y,s))\in E\Big\}.$$

Observe that we have

$$\mathrm{sign}\,(v-u) = \mathrm{sign}\,(A(v)-A(u)) \tag{3.1}$$

a.e. (w.r.t. $dt\,dx\,ds\,dy$) in $\Big[\Pi_T\times(\Pi_T\backslash\mathcal{E}_v)\Big]\bigcup\Big[(\Pi_T\backslash\mathcal{E}_u)\times\Pi_T\Big]$ and

$$\nabla_x A(v) = 0 \text{ a.e. (w.r.t. } dt\,dx) \text{ in } \mathcal{E}_v,\qquad \nabla_y A(u) = 0 \text{ a.e. (w.r.t. } ds\,dy) \text{ in } \mathcal{E}_u. \tag{3.2}$$

¿From the definition of entropy solution, Lemma 2.2, and the first part of (3.2), we have

$$-\iiiint_{\Pi_T\times\Pi_T}\Big(|v-u|\phi_t + \mathrm{sign}\,(v-u)\,\big[f(x,t,v) - f(x,t,u) - \nabla_x A(v)\big]\cdot\nabla_x\phi$$
$$- \mathrm{sign}\,(v-u)\,\big(\mathrm{div}_x f(x,t,u) - q(x,t,v)\big)\phi\Big)\,dt\,dx\,ds\,dy$$
$$\leq -\lim_{\varepsilon\downarrow 0}\iiiint_{(\Pi_T\backslash\mathcal{E}_u)\times\Pi_T}\big|\nabla_x A(v)\big|^2 \mathrm{sign}'_\varepsilon\,(A(v)-A(u))\,\phi\,dt\,dx\,ds\,dy \tag{3.3}$$
$$= -\lim_{\varepsilon\downarrow 0}\iiiint_{(\Pi_T\backslash\mathcal{E}_u)\times(\Pi_T\backslash\mathcal{E}_v)}\big|\nabla_x A(v)\big|^2 \mathrm{sign}'_\varepsilon\,(A(v)-A(u))\,\phi\,dt\,dx\,ds\,dy. \tag{3.4}$$

The inequality (3.3) is obtained by using Lemma 2.2 with $v(x,t)$ where (x,t) is not in the hyperbolic set $\mathcal{E}_u$, noting that the integral over $\Pi_T\setminus\mathcal{E}_u$ is less than the integral over Π_T. Finally, (3.4) follows from (3.2).

Similarly, using Lemma 2.2 for $u = u(y, s)$, and the second part of (3.2), we find the inequality

$$
\begin{aligned}
&-\iiiint_{\Pi_T\times\Pi_T}\Big(|u-v|\phi_t + \operatorname{sign}(u-v)\left[f(y,s,u)-f(y,s,v)-\nabla_y A(u)\right]\cdot\nabla_y\phi \\
&\qquad\qquad - \operatorname{sign}(u-v)\left(\operatorname{div}_y f(y,s,v)-q(y,s,u)\right)\phi\Big)\,dt\,dx\,ds\,dy \\
&\qquad \le -\lim_{\varepsilon\downarrow 0}\iiiint_{(\Pi_T\setminus\mathcal{E}_u)\times(\Pi_T\setminus\mathcal{E}_v)} \left|\nabla_y A(u)\right|^2 \operatorname{sign}'_\varepsilon\left(A(u)-A(v)\right)\phi\,dt\,dx\,ds\,dy.
\end{aligned}
\tag{3.5}
$$

Observe that whenever $\nabla_x A(v)$ is defined,

$$
\iint_{\Pi_T}\nabla_x A(v)\cdot\nabla_y(\operatorname{sign}_\varepsilon(A(v)-A(u))\,\phi)\,ds\,dy = \nabla_x A(v)\cdot\iint_{\Pi_T}\nabla_y(\operatorname{sign}_\varepsilon(A(v)-A(u))\,\phi)\,ds\,dy = 0,
$$

or more conveniently,

$$
-\iint_{\Pi_T}\operatorname{sign}_\varepsilon(A(v)-A(u))\,\nabla_x A(v)\cdot\nabla_y\phi\;ds\,dy = \iint_{\Pi_T}\nabla_y\operatorname{sign}_\varepsilon(A(v)-A(u))\cdot\nabla_x A(v)\phi\,ds\,dy.
\tag{3.6}
$$

Similarly, for a.e. $(y, s) \in \Pi_T$,

$$
-\iint_{\Pi_T}\operatorname{sign}_\varepsilon(A(u)-A(v))\,\nabla_y A(u)\cdot\nabla_x\phi\,dt\,dx = \iint_{\Pi_T}\nabla_x\operatorname{sign}_\varepsilon(A(u)-A(v))\cdot\nabla_y A(u)\phi\,dt\,dx.
\tag{3.7}
$$

Now using integrating (3.6), (3.1), and (3.2), we find that

$$
\begin{aligned}
&-\iiiint_{\Pi_T\times\Pi_T}\operatorname{sign}(v-u)\,\nabla_x A(v)\cdot\nabla_y\phi\,dt\,dx\,ds\,dy \\
&\quad = -\iiiint_{\Pi_T\times(\Pi_T\setminus\mathcal{E}_v)}\operatorname{sign}(A(v)-A(u))\,\nabla_x A(v)\cdot\nabla_y\phi\,dt\,dx\,ds\,dy \\
&\quad = -\lim_{\varepsilon\downarrow 0}\iiiint_{(\Pi_T\setminus\mathcal{E}_u)\times(\Pi_T\setminus\mathcal{E}_v)}\nabla_y A(u)\cdot\nabla_x A(v)\operatorname{sign}'_\varepsilon(A(v)-A(u))\,\phi\,dt\,dx\,ds\,dy.
\end{aligned}
\tag{3.8}
$$

Similarly, using (3.7), (3.1), and (3.2), we find that

$$
\begin{aligned}
&-\iiiint_{\Pi_T\times\Pi_T}\operatorname{sign}(A(u)-A(v))\,\nabla_y A(u)\cdot\nabla_x\phi\,dt\,dx\,ds\,dy \\
&\quad = -\lim_{\varepsilon\downarrow 0}\iiiint_{(\Pi_T\setminus\mathcal{E}_u)\times(\Pi_T\setminus\mathcal{E}_v)}\nabla_x A(u)\cdot\nabla_y A(v)\operatorname{sign}'_\varepsilon(A(v)-A(u))\,\phi\,dt\,dx\,ds\,dy.
\end{aligned}
\tag{3.9}
$$

Adding (3.3) and (3.8) yields

$$
\begin{aligned}
&-\iiiint_{\Pi_T\times\Pi_T}\Big(|v-u|\phi_t + \operatorname{sign}(v-u)\Big[\big(f(x,t,v)-f(x,t,u)\big)\cdot\nabla_x\phi \\
&\quad -\nabla_x A(v)\cdot\big(\nabla_x\phi+\nabla_y\phi\big)\Big] - \operatorname{sign}(v-u)\big(\operatorname{div}_x f(x,t,u)-q(x,t,v)\big)\phi\Big)\,dt\,dx\,ds\,dy \\
&\le -\lim_{\varepsilon\downarrow 0}\iiiint_{(\Pi_T\setminus\mathcal{E}_u)\times(\Pi_T\setminus\mathcal{E}_v)}\Big(\left|\nabla_x A(v)\right|^2 - \nabla_y A(u)\cdot\nabla_x A(v)\Big)\operatorname{sign}'_\varepsilon(A(v)-A(u))\,\phi\,dt\,dx\,ds\,dy.
\end{aligned}
\tag{3.10}
$$

Similarly, adding (3.5) and (3.9) yields

$$
\begin{aligned}
(3.11)\qquad & -\iiiint\limits_{\Pi_T\times\Pi_T}\Big(|u-v|\phi_s+\operatorname{sign}(u-v)\Big[\big(f(y,s,u)-f(y,s,v)\big)\cdot\nabla_y\phi \\
& -\nabla_yA(u)\cdot\big(\nabla_y\phi+\nabla_x\phi\big)\Big]-\operatorname{sign}(u-v)\big(\operatorname{div}_yf(y,s,v)-q(y,s,u)\big)\phi\Big)\,dt\,dx\,ds\,dy \\
& \le -\lim_{\varepsilon\downarrow 0}\iiiint\limits_{(\Pi_T\setminus\mathcal{E}_u)\times(\Pi_T\setminus\mathcal{E}_v)}\Big(\big|\nabla_yA(u)\big|^2-\nabla_xA(u)\cdot\nabla_yA(u)\Big)\operatorname{sign}_\varepsilon'(A(u)-A(v))\,\phi\,dt\,dx\,ds\,dy.
\end{aligned}
$$

Note that we can write

$$
\begin{aligned}
&\operatorname{sign}(v-u)\big(f(x,t,v)-f(x,t,u)\big)\cdot\nabla_x\phi-\operatorname{sign}(v-u)\operatorname{div}_xf(x,t,u)\phi \\
&\qquad=\operatorname{sign}(v-u)\big(f(x,t,v)-f(y,s,u)\big)\cdot\nabla_x\phi+\operatorname{sign}(v-u)\operatorname{div}_x\big[\big(f(y,s,u)-f(x,t,u)\big)\phi\big]
\end{aligned}
$$

and

$$
\begin{aligned}
&\operatorname{sign}(u-v)\big(f(y,s,u)-f(y,s,v)\big)\cdot\nabla_y\phi-\operatorname{sign}(u-v)\operatorname{div}_yf(y,s,v)\phi \\
&\qquad=\operatorname{sign}(v-u)\big(f(x,t,v)-f(y,s,u)\big)\cdot\nabla_y\phi-\operatorname{sign}(v-u)\operatorname{div}_y\big[\big(f(x,t,v)-f(y,s,v)\big)\phi\big].
\end{aligned}
$$

Taking these identities into account when adding (3.10) and (3.11), we get

$$
\begin{aligned}
& -\iiiint\limits_{\Pi_T\times\Pi_T}\Big(|v-u|\big(\phi_t+\phi_s\big)+I_1+I_2+I_3\Big)\,dt\,dx\,ds\,dy \\
(3.12)\qquad & \le -\lim_{\varepsilon\downarrow 0}\iiiint\limits_{(\Pi_T\setminus\mathcal{E}_u)\times(\Pi_T\setminus\mathcal{E}_v)}\big|\nabla_xA(v)-\nabla_yA(u)\big|^2\operatorname{sign}_\varepsilon'(A(v)-A(u))\,\phi\,dt\,dx\,ds\,dy \\
& \le 0,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1&=\operatorname{sign}(v-u)\Big[f(x,t,v)-f(y,s,u)-\big(\nabla_xA(v)-\nabla_yA(u)\big)\Big]\cdot\big(\nabla_x\phi+\nabla_y\phi\big) \\
I_2&=\operatorname{sign}(v-u)\Big[\operatorname{div}_x\big[\big(f(y,s,u)-f(x,t,u)\big)\phi\big]-\operatorname{div}_y\big[\big(f(x,t,v)-f(y,s,v)\big)\phi\big]\Big], \\
I_3&=\operatorname{sign}(v-u)\big(q(x,t,v)-q(y,s,u)\big)\phi.
\end{aligned}
$$

We are now on familiar ground [24, 25] and introduce a nonnegative function $\delta\in C_0^\infty(\mathbf{R})$ which satisfies

$$
\delta(\sigma)=\delta(-\sigma),\quad \delta(\sigma)\equiv 0 \text{ for } |\sigma|\ge 1,\quad \int_{\mathbf{R}}\delta(\sigma)\,d\sigma=1.
$$

For $\rho_0>0$, let

$$
\delta_{\rho_0}(\sigma)=\frac{1}{\rho_0}\delta\left(\frac{\sigma}{\rho_0}\right).
$$

Pick two (arbitrary but fixed) Lebesgue points $\nu,\tau\in(0,T)$ of $\|v(\cdot,t)-u(\cdot,t)\|_{L^1(\mathbf{R}^d)}$. For any $\alpha_0\in\big(0,\min(\nu,T-\tau)\big)$, let

$$
W_{\alpha_0}(t)=H_{\alpha_0}(t-\nu)-H_{\alpha_0}(t-\tau),\quad H_{\alpha_0}(t)=\int_{-\infty}^t\delta_{\alpha_0}(s)\,ds.
$$

Inspired by [10], we introduce a nonnegative function $\omega\in C_0^\infty(\mathbf{R}_+)$ which satisfies

$$
(3.13)\qquad \omega(z)=0 \text{ for } z\ge 1,\quad \omega'(z)\le 0 \text{ for } z\in(0,1),\quad \int_{\mathbf{R}^d}\omega\big(|z|^2\big)\,dz=1.
$$

For $\rho>0$ and $x\in\mathbf{R}^d$, let

$$
\omega_\rho(x)=\frac{1}{2\rho^d}\,\omega\left(\frac{|x|^2}{\rho^2}\right).
$$

Observe that

$$\nabla_x \omega_\rho(x-y) = \frac{1}{\rho^{d+2}}\,\omega\left(\frac{|x-y|^2}{\rho^2}\right)(x-y) = -\nabla_y \omega_\rho(x-y).$$

We now take ϕ to be of the form

$$\phi(x,t,y,s) = W_{\alpha_0}(t)\,\omega_\rho(x-y)\delta_{\rho_0}(t-s) \in C_0^\infty(\Pi_T \times \Pi_T), \tag{3.14}$$

so that the derivatives of ϕ which are singular in the limit $\rho, \rho_0 \downarrow 0$ cancel:

$$\phi_t + \phi_s = \big[\delta_{\alpha_0}(t-\nu) - \delta_{\alpha_0}(t-\tau)\big]\omega_\rho(x-y)\delta_{\rho_0}(t-s), \qquad \nabla_x\phi + \nabla_y\phi = 0. \tag{3.15}$$

Note that with this test function, $I_1 \equiv 0$ and inequality (3.12) now takes the form

$$-\iiiint_{\Pi_T\times\Pi_T} |v(x,t)-u(y,s)|\big(\phi_t+\phi_s\big)\,dt\,dx\,ds\,dy \le \iiiint_{\Pi_T\times\Pi_T}\Big(I_2+I_3\Big)\,dt\,dx\,ds\,dy. \tag{3.16}$$

Sending $\alpha_0, \rho, \rho_0 \downarrow 0$ in (3.16), by an L^1 continuity argument, we get

$$\begin{aligned}\int_{\mathbf{R}^d} |v(x,\tau)-u(x,\tau)|\,dx \\ \le \int_{\mathbf{R}^d} |v(x,\nu)-u(x,\nu)|\,dx + \lim_{\alpha,\alpha_0,\rho,\rho_0\downarrow 0}\iiiint_{\Pi_T\times\Pi_T}\Big(I_2+I_3\Big)\,dt\,dx\,ds\,dy.\end{aligned} \tag{3.17}$$

Before we continue, let us write $I_2 = I_{2,1} + I_{2,2}$, where

$$\begin{aligned} I_{2,1} &= \operatorname{sign}(v-u)\Big[\big(f(y,s,u)-f(x,t,u)\big)\cdot\nabla_x\phi - \big(f(x,t,v)-f(y,s,v)\big)\cdot\nabla_y\phi\Big], \\ I_{2,2} &= \operatorname{sign}(v-u)\,\big(\operatorname{div}_y f(y,s,v) - \operatorname{div}_x f(x,t,u)\big)\phi. \end{aligned}$$

Inserting this into (3.16), we get

$$\int_{\mathbf{R}^d} |v(x,\tau)-u(x,\tau)|\,dx \le \int_{\mathbf{R}^d} |v(x,\nu)-u(x,\nu)|\,dx + \lim_{\alpha,\alpha_0,\rho,\rho_0\downarrow 0}\Big(E_1+E_2+E_3\Big), \tag{3.18}$$

where

$$E_1 = \iiiint_{\Pi_T\times\Pi_T} I_{2,1}\,dt\,dx\,ds\,dy, \quad E_2 = \iiiint_{\Pi_T\times\Pi_T} I_{2,2}\,dt\,dx\,ds\,dy, \quad E_3 = \iiiint_{\Pi_T\times\Pi_T} I_3\,dt\,dx\,ds\,dy.$$

Using (1.6), (1.4), and an L^1 continuity argument, we get

$$\lim_{\alpha,\alpha_0,\rho,\rho_0\downarrow 0}\Big(E_2+E_3\Big) \le \text{Const}\int_\nu^\tau\int_{\mathbf{R}^d}|v(x,t)-u(x,t)|\,dt\,dx.$$

It remains to pass to the limit in E_1. The term $I_{2,1}$ can be rewritten as

$$I_{2,1} = (F(x,t,v,u) - F(y,s,v,u))\cdot\nabla_x\phi$$

where F is defined in (1.8). Sending $\alpha_0, \rho_0 \downarrow 0$ in E_2 (again using (1.5) and an L^1 continuity argument), we obtain

$$\begin{aligned}\lim_{\alpha_0,\rho_0\downarrow 0} E_1 = \int_\nu^\tau\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}\big(F(x,t,v(x,t),u(y,t)) - F(y,t,v(x,t),u(y,t))\big) \\ \cdot(x-y)\frac{1}{\rho^{d+2}}\,\omega'\left(\frac{|x-y|^2}{\rho^2}\right)\,dy\,dx\,dt\end{aligned}$$

Taking (1.7) into account, we have

$$\begin{aligned}&\big(F(x,t,v(x,t),u(y,t)) - F(y,t,v(x,t),u(y,t))\big)\cdot(x-y)\frac{1}{\rho^{d+2}}\omega'\left(\frac{|x-y|^2}{\rho^2}\right) \\ &\quad\le \gamma|v(x,t)-u(y,t)|\,\frac{|x-y|^2}{\rho^2}\,\frac{1}{\rho^d}\left|\omega'\left(\frac{|x-y|^2}{\rho^2}\right)\right| \le \gamma|v(x,t)-u(y,t)|\,\max|\omega'|\,\frac{1}{\rho^d}\mathbf{1}_{|x-y|<\rho}.\end{aligned}$$

¿From this we obtain the following estimate

$$\begin{aligned}\lim_{\alpha,\rho\downarrow}\int_\nu^\tau\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}&\Big(F(x,t,v(x,t),u(y,t))-F(y,t,v(x,t),u(y,t))\Big)\cdot(x-y)\frac{1}{\rho^{d+2}}\,\omega'\left(\frac{|x-y|^2}{\rho^2}\right)\\&\le\lim_{\rho\downarrow 0}\frac{\text{Const}}{\rho^d}\int_\nu^\tau\int_{\mathbf{R}^d}\int_{|x-y|<\rho}|v(x,t)-u(y,t)|\,dy\,dx\,dt.\\&=\text{Const}\int_\nu^\tau\int_{\mathbf{R}^d}|v(x,t)-u(x,t)|\,dx\,dt.\end{aligned}$$

Summing up, we have proved that

$$\int_{\mathbf{R}^d}|v(x,\tau)-u(x,\tau)|\,dx\le\int_{\mathbf{R}^d}|v(x,\nu)-u(x,\nu)|\,dx+C\int_\nu^\tau\int_{\mathbf{R}^d}|v(x,t)-u(x,t)|\,dx\,dt,$$

for some constant $C>0$ depending on f, q and the test function. Sending $\nu\downarrow 0$ and then using Gronwall's lemma, we get

$$\int_{\mathbf{R}^d}|v(x,\tau)-u(x,\tau)|\,dx\le e^{C\tau}\int_{\mathbf{R}^d}|v(x,0)-u(x,0)|\,dx\equiv 0. \tag{3.19}$$

Since this inequality holds for almost all $\tau\in(0,T)$, we can conclude that $v=u$ a.e. in Π_T.

Remark 3.1. If $d\le 2$ we are able to prove a slightly stronger version of Theorem 1.1; namely, we can relax the assumptions on the x dependency of f. Concretely, if $d\le 2$, let us assume that

$$f(\cdot,\cdot,u)\in L^1(0,T,W^{1,1}_{\text{loc}}(\mathbf{R}^d)). \tag{3.20}$$

To show Theorem 1.1 in this case we proceed as before up to (3.14). We then modify the definition of the test function as follows: For positive σ, we let

$$R(\sigma)=\begin{cases}1 & 0\le\sigma<1/2,\\ 1-2\sigma & 1/2\le\sigma<1,\\ 0 & 1\le\sigma,\end{cases}\qquad \tilde{R}(\sigma)=(R*\delta_\rho)(\sigma).$$

For $x\in\mathbf{R}^d$, we set $R_\alpha(x)=\tilde{R}(\alpha|x|^2)$. Then we define the test function as

$$\phi(x,t,y,s)=R_\alpha\left(\frac{x+y}{2}\right)W_{\alpha_0}(t)\,\omega_\rho(x-y)\delta_{\rho_0}(t-s). \tag{3.21}$$

Now

$$\nabla_x\phi+\nabla_y\phi=\left[\frac{\alpha(x+y)}{2}\tilde{R}'\left(\alpha\frac{|x+y|^2}{2}\right)\right]W_{\alpha_0}(t)\omega_\rho(x-y)\delta_{\rho_0}(t-s).$$

Proceeding further as before, it turns out that we must estimate a term of the type

(3.22)
$$\begin{aligned}\lim_{\alpha\downarrow 0}\Bigg[&\iint_{\Pi_T}\operatorname{sign}(v(x,t)-u(x,t))\Big(f(x,t,v(x,t))-f(x,t,u(x,t))\Big)\cdot x\alpha R'\left(\alpha|x|^2\right)W_{\alpha_0}(t)\,dtdx\\&-\iint_{\Pi_T}\operatorname{sign}(v(x,t)-u(x,t))\nabla_x\Big(A(v(x,t))-A(u(x,t))\Big)x\alpha R'\left(\alpha|x|^2\right)W_{\alpha_0}(t)\,dtdx\Bigg].\end{aligned}$$

Since $f(x,t,v)$ and $f(x,t,u)$ are bounded, the limit of the first double-integral is zero. Regarding the limit of the second double-integral, we do not have that $\nabla_x A(v)$ is bounded but only that it belongs to $L^2(\mathbf{R}^d)$. Noting that R' is zero outside $\langle 1/2,1\rangle$ and inside this interval $R'=-2$.

Hence, using Hölder's inequality, we can estimate as follows:

$$\begin{aligned}
\iint_{\Pi_T} \left|\nabla_x A(v)\cdot x\alpha R'\left(\alpha\left|x\right|^2\right)\right|\,dx &= 2\alpha \iint_{\Pi_{\alpha,T}} \left|\nabla_x A(v)\right|\left|x\right|\,dx\\
&\le 2\sqrt{\alpha}\iint_{\Pi_{\alpha,T}} \left|\nabla_x A(v)\right| 1\,dx\\
&\le 2\sqrt{\alpha}\left\|\nabla A(v)\right\|_{L^2(\Pi_{\alpha,T})}\left\|1\right\|_{L^2(\Pi_{\alpha,T})}\\
&= \mathcal{O}(1)\sqrt{\alpha}\left\|\nabla A(v)\right\|_{L^2(\Pi_{\alpha,T})}\left(\frac{1}{\sqrt{\alpha}}\right)^{d/2},
\end{aligned} \tag{3.23}$$

where $\Pi_{\alpha,T}$ denotes the set of $(x,t)\in\Pi_T$ such that $|x|\in\left[\frac{1}{2\sqrt{\alpha}},\frac{1}{\sqrt{\alpha}}\right]$. Since $\nabla A(v)$ is in $L^2\left(\Pi_T\right)$, (3.23) tends to zero when $\alpha\downarrow 0$ if $d\le 2$. Similarly, we can treat the term in (3.22) related to $A(u)$. In this way we eliminate the extra term due to the fact that $\nabla_x\phi+\nabla_y\phi\neq 0$. The remainder of the proof proceeds as before.

4. Proof of Theorem 1.2

In this section, we restrict ourselves to problems of the form (1.10), i.e., $f(x,t,u)=k(x)f(u)$ and $q(x,t,u)\equiv 0$. Let $u,v\in L^\infty(0,T;BV(\mathbf{R}^d))$ be two entropy solutions of (1.10) with initial data $u_0,v_0\in L^1(\mathbf{R}^d)\cap L^\infty(\mathbf{R}^d)\cap BV(\mathbf{R}^d)$, respectively. As before,we are interested in estimating the L^1 distance between v and u. In what follows, the test function $\phi=\phi(x,t,y,s)$ is the one defined in (3.21). Repeating everything up to (3.17), we find that

$$\int_{\mathbf{R}^d}\left|v(x,\tau)-u(x,\tau)\right|dx \le \int_{\mathbf{R}^d}\left|v(x,\nu)-u(x,\nu)\right|dx + \lim_{\alpha,\alpha_0,\rho,\rho_0\downarrow 0}\Big(E_1+E_2+E_3\Big), \tag{4.1}$$

where

$$E_1 = \iiiint_{\Pi_T\times\Pi_T} \operatorname{sign}(v-u)\,\nabla_x\phi\cdot\Big[\big(k(y)f(u)-k(x)f(u)\big)+\big(k(x)f(v)-k(y)f(v)\big)\Big]\,dt\,dx\,ds\,dy,$$

$$E_2 = \iiiint_{\Pi_T\times\Pi_T} \operatorname{sign}(v-u)\,\big(\operatorname{div}_y k(y)f(v)-\operatorname{div}_x k(x)f(u)\big)\phi.$$

$$E_3 = \iiiint_{\Pi_T\times\Pi_T} \operatorname{sign}(v-u)\left[\left(k(x)f(v)-k(y)f(u)\right)-\nabla_x\left(A(v)-A(u)\right)\right]\left(\nabla_x\phi+\nabla_y\phi\right)\,dt\,dx\,ds\,dy$$

We start by estimating E_1. To this end, introduce the function

$$F(v,u) := \operatorname{sign}(v-u)\,\big[f(v)-f(u)\big], \tag{4.2}$$

and observe that from the identity (3.15) we have

$$E_1 = \iiiint_{\Pi_T\times\Pi_T} \big(k(x)-k(y)\big)F(v,u)\cdot\nabla_x\phi\,dt\,dx\,ds\,dy.$$

To continue, we need the following simple lemma (whose easy proof can be found in, e.g., [5]):

Lemma 4.1. *Consider a function $z=z(x)$ belonging to $L^\infty(\mathbf{R}^d)\cap BV(\mathbf{R}^d)$ and let $h\in\mathrm{Lip}(I_z)$. Then $h(z)$ belongs to $L^\infty(\mathbf{R}^d)\cap BV(\mathbf{R}^d)$ and*

$$\left|\frac{\partial}{\partial x_j}h(z)\right| \le \|h\|_{\mathrm{Lip}(I_z)}\left|\frac{\partial}{\partial x_j}z\right| \quad \text{in the sense of measures, for,}\quad j=1,\dots,d,$$

where I_z denotes the interval $\left[-\|z\|_{L^\infty(\mathbf{R}^d)},\|z\|_{L^\infty(\mathbf{R}^d)}\right]$.

Note that the function $F(v,u)$ defined in (4.2) is locally Lipschitz continuous in v and u with Lipschitz constant that of f. Now since $v(\cdot,t) \in L^\infty(\mathbf{R}^d) \cap BV(\mathbf{R}^d)$ for each t, by Lemma 4.1 $\nabla_x F(v,u)$ is a finite measure. After an integration by parts, we thus get

$$E_1 = -\iiiint\limits_{\Pi_T\times\Pi_T} \Big(\mathrm{div}_x k(x) F(v,u) + \big(k(x)-k(y)\big)\cdot\nabla_x F(v,u)\Big)\phi \, dt\, dx\, ds\, dy.$$

Since $k \in C(\mathbf{R}^d)$ and $\nabla_x F(v,u)$ is a finite measure, it follows that

$$\iiiint\limits_{\Pi_T\times\Pi_T} \big(k(x)-k(y)\big)\cdot\nabla_x F(v,u)\phi\, dt\, dx\, ds\, dy \to 0 \text{ as } \rho\downarrow 0.$$

Consequently, we end up with

$$\lim_{\alpha_0 \rho,\rho_0\downarrow 0} E_1 = -\int_\nu^T \int_{\mathbf{R}^d} \mathrm{div} k(x) F(v(x,t),u(x,t)) R_\alpha(x)\, dx\, dt.$$

Regarding E_2, since $k\in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^d)$, the usual L^1 continuity argument gives

$$\lim_{\alpha_0,\rho,\rho_0\downarrow 0} E_2 = \int_\nu^\tau \int_{\mathbf{R}^d} \mathrm{div} k(x) F(v(x,t),u(x,t)) R_\alpha(x)\, dt\, dx \equiv -\lim_{\alpha_0\rho,\rho_0\downarrow 0} E_1. \tag{4.3}$$

Finally, when estimating E_3, we have to estimate an integral of the type (3.22). Now the second integral in (3.22) can be estimated like the first, since $\nabla_x A(v)$ and $\nabla_x A(u)$ are in $L^1\left(\mathbf{R}^d\right)$ as v and u are of bounded variation. Therefore

$$\lim_{\alpha\downarrow 0} E_3 = 0.$$

¿From (4.3), we get

$$\int |v(x,\tau)-u(x,\tau)|\, dx \le \int_{\mathbf{R}^d} |v(x,\nu)-u(x,\nu)|\, dx \to \int_{\mathbf{R}^d} |v(x,0)-u(x,0)|\, dx \text{ as } \nu\downarrow 0.$$

Since $\tau\in(0,T)$ was an arbitrary Lebesgue point of $\|v(\cdot,t)-u(\cdot,t)\|_{L^1(\mathbf{R}^d)}$, we immediately obtain the L^1 contraction property claimed in Theorem 1.2.

5. Proof of Theorem 1.3

In this section, we are going to estimate the L^1 difference between the entropy solution v of (1.13) and the entropy solution u of (1.10). To do this, we proceed exactly as in the proof of Theorem 1.1. In what follows, we let $\phi = \phi(x,t,y,s)$ be an arbitrary test function on $\Pi_T\times\Pi_T$ satisfying $\nabla_x\phi = -\nabla_y\phi$.

Similarly to (3.10) and (3.11), we can derive the following integral inequalities for the entropy solutions $v=v(x,t)$ and $u=u(y,s)$ of (1.13) and (1.10):

$$\begin{aligned} -\iiiint\limits_{\Pi_T\times\Pi_T} &|v-u|\phi_t + \mathrm{sign}\,(v-u)\,\big(l(x)\big(g(v)-g(u)\big)\cdot\nabla_x\phi - \mathrm{div}_x l(x) g(u)\phi\big)\, dt\, dx\, ds\, dy \\ &\le -\lim_{\varepsilon\downarrow 0} \iiiint\limits_{(\Pi_T\setminus\mathcal{E}_u)\times(\Pi_T\setminus\mathcal{E}_v)} \Big(\left|\nabla_x A(v)\right|^2 - \nabla_y A(u)\cdot\nabla_x A(v)\Big)\mathrm{sign}'_\varepsilon\left(A(v)-A(u)\right)\phi\, dt\, dx\, ds\, dy, \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} -\iiiint\limits_{\Pi_T\times\Pi_T} &|u-v|\phi_s + \mathrm{sign}\,(u-v)\,\big(k(y)\big(f(u)-f(v)\big)\cdot\nabla_y\phi - \mathrm{div}_y k(y) f(v)\phi\big)\, dt\, dx\, ds\, dy \\ &\le -\lim_{\varepsilon\downarrow 0} \iiiint\limits_{(\Pi_T\setminus\mathcal{E}_u)\times(\Pi_T\setminus\mathcal{E}_v)} \Big(\left|\nabla_y A(u)\right|^2 - \nabla_x A(v)\cdot\nabla_y A(u)\Big)\,\mathrm{sign}'_\varepsilon\left(A(u)-A(v)\right)\phi\, dt\, dx\, ds\, dy. \end{aligned} \tag{5.2}$$

Next we write

$$\begin{aligned}
&\operatorname{sign}(v-u)\, l(x)\big(g(v)-g(u)\big)\cdot\nabla_x\phi-\operatorname{sign}(v-u)\operatorname{div}_x l(x)g(u)\phi\\
&\qquad=\operatorname{sign}(v-u)\,\big(l(x)g(v)-k(y)f(u)\big)\cdot\nabla_x\phi+\operatorname{sign}(v-u)\operatorname{div}_x\big[\big(k(y)f(u)-l(x)g(u)\big)\phi\big]
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{sign}(u-v)\, k(y)\big(f(u)-f(v)\big)\cdot\nabla_y\phi-\operatorname{sign}(u-v)\operatorname{div}_y k(y)g(v)\phi\\
&\qquad=\operatorname{sign}(v-u)\,\big(l(x)g(v)-k(y)f(u)\big)\cdot\nabla_y\phi-\operatorname{sign}(v-u)\operatorname{div}_y\big[\big(l(x)g(v)-k(y)g(v)\big)\phi\big].
\end{aligned}$$

Similarly to (3.12), by adding (3.10) and (3.11) we obtain

$$-\iiiint_{\Pi_T\times\Pi_T}\Big(|v-u|\big(\phi_t+\phi_s\big)+I_2\Big)\,dt\,dx\,ds\,dy\le 0, \tag{5.3}$$

where

$$I_2=\operatorname{sign}(v-u)\,\Big[\operatorname{div}_x\big[\big(k(y)f(u)-l(x)g(u)\big)\phi\big]-\operatorname{div}_y\big[\big(l(x)g(v)-k(y)f(v)\big)\phi\big]\Big].$$

We now specify the test function ϕ as in (3.14), so that (3.15) holds. Before we continue, let us write $I_2=I_{2,1}+I_{2,2}$ where

$$\begin{aligned}
I_{2,1}&=\operatorname{sign}(v-u)\,\Big[\big(k(y)f(u)-l(x)g(u)\big)\cdot\nabla_x\phi-\big(l(x)g(v)-k(y)f(v)\big)\cdot\nabla_y\phi\Big],\\
I_{2,2}&=\operatorname{sign}(v-u)\,\big(\operatorname{div}_y k(y)f(v)-\operatorname{div}_x l(x)g(u)\big)\phi.
\end{aligned}$$

With the test function ϕ defined in (3.14), we can send $\alpha_0,\rho,\rho_0\downarrow 0$ as usual and get

$$\int_{\mathbf{R}^d}|v(x,\tau)-u(x,\tau)|\,dx\le\int_{\mathbf{R}^d}|v(x,\nu)-u(x,\nu)|\,dx+\lim_{\alpha_0,\rho,\rho_0\downarrow 0}\Big(E_1+E_2\Big), \tag{5.4}$$

where

$$E_1=\iiiint_{\Pi_T\times\Pi_T}I_{2,1}\,dt\,dx\,ds\,dy\quad\text{and}\quad E_2=\iiiint_{\Pi_T\times\Pi_T}I_{2,2}\,dt\,dx\,ds\,dy.$$

Taking into account the identity $\nabla_y\phi=-\nabla_x\phi$, we get

$$I_{2,1}=\big(l(x)G(v,u)-k(y)F(v,u)\big)\cdot\nabla_x\phi,$$

where F is defined in (4.2) and G is defined by the same formula but with f replaced by g. Since $v(\cdot,t)\in L^\infty(\mathbf{R}^d)\cap BV(\mathbf{R}^d)$ for each t and F,G are locally Lipschitz continuous, $\nabla_x F(v,u)$ and $\nabla_x G(v,u)$ are finite measures. Therefore, after an integration by parts followed by adding and subtracting identical terms, we get

$$\begin{aligned}
E_1&=\iiiint_{\Pi_T\times\Pi_T}\Big(-\operatorname{div}_x l(x)G(v,u)-l(x)\cdot\nabla_x G(v,u)+k(y)\cdot\nabla_x F(v,u)\Big)\phi\,dt\,dx\,ds\,dy\\
&=\iiiint_{\Pi_T\times\Pi_T}\Big(-\operatorname{div}_x l(x)G(v,u)+\big(k(y)-l(x)\big)\cdot\nabla_x G(v,u)\\
&\qquad\qquad+k(y)\cdot\nabla_x\big(F(v,u)-G(v,u)\big)\Big)\phi\,dt\,dx\,ds\,dy.
\end{aligned}$$

By adding and subtracting identical terms, we obtain

$$\begin{aligned}
-\operatorname{div}_x l(x)G(v,u)\phi+I_{2,2}&=\operatorname{sign}(v-u)\operatorname{div}_y k(y)f(v)-\operatorname{sign}(v-u)\operatorname{div}_x l(x)g(v)\phi\\
&=\operatorname{sign}(v-u)\,\big[\operatorname{div}_y k(y)\big(f(v)-g(v)\big)-\big(\operatorname{div}_y k(y)-\operatorname{div}_x l(x)\big)g(v)\big]\phi.
\end{aligned}$$

Adding E_1 and E_2, we thus get

(5.5)
$$
\begin{aligned}
E_1 + E_2 = \iiiint_{\Pi_T\times\Pi_T} \Big(& \operatorname{sign}(v-u) \left[\operatorname{div}_y k(y)\big(f(v)-g(v)\big) - \big(\operatorname{div}_y k(y) - \operatorname{div}_x l(x)\big)g(v)\right] \\
& + \big(k(y)-l(x)\big)\cdot \nabla_x G(v,u) + k(y)\cdot\nabla_x\big(F(v,u)-G(v,u)\big)\Big)\phi\,dt\,dx\,ds\,dy.
\end{aligned}
$$

Observe that by Lemma 4.1 we have

$$
\begin{aligned}
\left|\frac{\partial}{\partial x_j}G(v,u)\right| &\le \|g\|_{\mathrm{Lip}(I)}\left|\frac{\partial}{\partial x_j}v(x,t)\right|, \\
\left|\frac{\partial}{\partial x_j}\big(F(v,u)-G(v,u)\big)\right| &\le \|f-g\|_{\mathrm{Lip}(I)}\left|\frac{\partial}{\partial x_j}v(x,t)\right|,
\end{aligned}
\qquad \text{for} \quad j=1,\dots,d. \tag{5.6}
$$

Equipped with (5.6) and (1.14), we send $\alpha_0,\rho,\rho_0\downarrow 0$ in (5.5) to obtain

$$
\begin{aligned}
\lim_{\alpha_0,\rho,\rho_0\downarrow 0}(E_1+E_2) \le \int_\nu^\tau\int_{\mathbf{R}^d}\Big(&\left|\operatorname{div}k(x)\right| \|f-g\|_{L^\infty(I)} + \left|\operatorname{div}(k(x)-l(x))\right| \|g\|_{L^\infty(I)} \\
+ \|k-l\|_{L^\infty(\mathbf{R}^d)}\|g\|_{\mathrm{Lip}(I)}\sum_{j=1}^d\left|\frac{\partial}{\partial x_j}v(x,t)\right| &+ \|k\|_{L^\infty(\mathbf{R}^d)}\|f-g\|_{\mathrm{Lip}(I)}\sum_{j=1}^d\left|\frac{\partial}{\partial x_j}v(x,t)\right|\Big)\,dx\,dt.
\end{aligned}
$$

In view of (5.4), the following continuous dependence estimate now follows

$$
\begin{aligned}
\int_{\mathbf{R}^d}|v(x,\tau)-u(x,\tau)|\,dx &\le \int_{\mathbf{R}^d}|v(x,\nu)-u(x,\nu)|\,dx \\
&+ \tau\Big(\|g\|_{\mathrm{Lip}(I)}\sup_{t\in(0,T)}|v(\cdot,t)|_{BV(\mathbf{R}^d)}\|k-l\|_{L^\infty(\mathbf{R}^d)} + \|g\|_{L^\infty(I)}|k-l|_{BV(\mathbf{R}^d)} \\
&\quad + |k|_{BV(\mathbf{R}^d)}\|f-g\|_{L^\infty(I)} + \|k\|_{L^\infty(\mathbf{R}^d)}\sup_{t\in(0,T)}|v(\cdot,t)|_{BV(\mathbf{R}^d)}\|f-g\|_{\mathrm{Lip}(I)}\Big).
\end{aligned}
$$

Sending $\nu\downarrow 0$ and using symmetry, we finally conclude that Theorem 1.3 holds.

6. Appendix (proof of Lemma 2.1)

In this appendix, we give a proof of Lemma 2.1. The proof follows Carrillo [12], but see also Alt and Luckhaus [1] and Otto [28]. Note that $\mathcal{A}_\psi$ is a nonnegative and convex function. Convexity implies that for a.e. $(x,t)\in\Pi_T$, we have

$$
\mathcal{A}_\psi\big(u(x,t)\big) - \mathcal{A}_\psi\big(u(x,t-\tau)\big) \le \big(u(x,t)-u(x,t-\tau)\big)\psi\big(A(u(x,t))\big),
$$

where we define $u(t)=u_0$ for $t\in(-\tau,0)$. In the sequel let $\phi\in C_0^\infty(\mathbf{R}^d\times[0,T])$. Multiplying the above inequality by $\phi(x,t)$ yields

$$
\begin{aligned}
&\mathcal{A}_\psi\big(u(x,t)\big)\phi(x,t) - \mathcal{A}_\psi\big(u(x,t-\tau)\big)\phi(x,t-\tau) + \mathcal{A}_\psi\big(u(x,t-\tau)\big)\big(\phi(x,t-\tau)-\phi(x,t)\big) \\
&\quad = \mathcal{A}_\psi\big(u(x,t)\big)\phi(x,t) - \mathcal{A}_\psi\big(u(x,t-\tau)\big)\phi(x,t) \\
&\quad \le \big(u(x,t)-u(x,t-\tau)\big)\psi\big(A(u(x,t))\big)\phi(x,t),
\end{aligned}
\tag{6.1}
$$

where we define $\phi(x,t)=\phi(x,0)$ for $t<0$. Note that $\mathcal{A}_\psi(u_0)\in L^1(\mathbf{R}^d)$ and $\mathcal{A}_\psi(u)\in L^\infty\big(0,T;L^1(\mathbf{R}^d)\big)$. Dividing (6.1) by τ and integrating over $\mathbf{R}^d\times(0,s)$, we get

$$
\begin{aligned}
&\frac{1}{\tau}\int_{s-\tau}^s\int_{\mathbf{R}^d}\mathcal{A}_\psi\big(u(x,t)\big)\phi(x,t)\,dx\,dt - \frac{1}{\tau}\int_0^\tau\int_{\mathbf{R}^d}\mathcal{A}_\psi\big(u_0(x)\big)\phi(x,0)\,dx\,dt \\
&\qquad + \frac{1}{\tau}\int_0^s\int_{\mathbf{R}^d}\mathcal{A}_\psi\big(u(x,t-\tau)\big)\big(\phi(x,t-\tau)-\phi(x,t)\big)\,dx\,dt \\
&\qquad\qquad \le \frac{1}{\tau}\int_0^s\int_{\mathbf{R}^d}\big(u(x,t)-u(x,t-\tau)\big)\psi\big(A(u(x,t))\big)\phi(x,t)\,dx\,dt.
\end{aligned}
\tag{6.2}
$$

Since $\phi \in C_0^\infty(\mathbf{R}^d \times [0,T])$ and $A(u) \in L^2(0,T;H^1(\mathbf{R}^d))$, we have $\psi(A(u))\phi \in L^2\big(0,T;H_0^1(\mathbf{R}^d)\big)$. Therefore, exploiting that $u \in C\big(0,T;L^1(\mathbf{R}^d)\big)$ and $\partial_t u \in L^2\big(0,T;H^{-1}(\mathbf{R}^d)\big)$, we can let $\tau \downarrow 0$ in (6.2) and obtain

$$\int_{\mathbf{R}^d} \mathcal{A}_\psi\big(u(x,s)\big)\phi(x,s)\,dx - \int_{\mathbf{R}^d} \mathcal{A}_\psi(u_0)\phi(x,0)\,dx - \int_0^s \int_{\mathbf{R}^d} \mathcal{A}_\psi(u)\phi_t\,dx\,dt \le \int_0^s \langle \partial_t u, \psi\big(A(u)\big)\phi\rangle\,dt,$$

for a.e. $s \in (0,T)$. Convexity implies also that for a.e. $(x,t) \in \Pi_T$ and $t > \tau$, we have

$$\mathcal{A}_\psi\big(u(x,t)\big) - \mathcal{A}_\psi\big(u(x,t-\tau)\big) \ge \big(u(x,t) - u(x,t-\tau)\big)\psi\big(A(u(x,t-\tau))\big).$$

Multiplying this inequality by $\phi(x,t-\tau)$ yields

$$\begin{aligned} &\mathcal{A}_\psi\big(u(x,t)\big)\phi(x,t) - \mathcal{A}_\psi\big(u(x,t-\tau)\big)\phi(x,t-\tau) + \mathcal{A}_\psi\big(u(x,t)\big)\big(\phi(x,t-\tau) - \phi(x,t)\big) \\ &\qquad = \mathcal{A}_\psi\big(u(x,t)\big)\phi(x,t-\tau) - \mathcal{A}_\psi\big(u(x,t-\tau)\big)\phi(x,t-\tau) \\ &\qquad \ge \big(u(x,t) - u(x,t-\tau)\big)\psi\big(A(u(x,t-\tau))\big)\phi(x,t-\tau). \end{aligned} \tag{6.3}$$

After dividing (6.3) by τ and integrating over $\mathbf{R}^d \times (\tau,s)$, we obtain

$$\begin{aligned} &\frac{1}{\tau}\int_{s-\tau}^s \int_{\mathbf{R}^d} \mathcal{A}_\psi\big(u(x,t)\big)\phi(x,t)\,dx\,dt - \frac{1}{\tau}\int_0^\tau \int_{\mathbf{R}^d} \mathcal{A}_\psi\big(u(x,t)\big)\phi(x,t)\,dx\,dt \\ &\qquad + \frac{1}{\tau}\int_{\mathbf{R}^d}\int_\tau^s \mathcal{A}_\psi\big(u(x,t)\big)\big(\phi(x,t-\tau) - \phi(x,t)\big)\,dx\,dt \\ &\qquad\qquad \ge \frac{1}{\tau}\int_\tau^s \int_{\mathbf{R}^d} \big(u(x,t) - u(x,t-\tau)\big)\psi\big(A(u(x,t-\tau))\big)\phi(x,t-\tau)\,dx\,dt. \end{aligned} \tag{6.4}$$

Finally, similarly to (6.2), letting $\tau \downarrow 0$ in (6.4), we get, for a.e. $s \in (0,T)$,

$$\int_{\mathbf{R}^d} \mathcal{A}_\psi\big(u(x,s)\big)\phi(x,s)\,dx - \int_{\mathbf{R}^d} \mathcal{A}_\psi(u_0)\phi(x,0)\,dx - \int_0^s \int_{\mathbf{R}^d} \mathcal{A}_\psi(u)\phi_t\,dx\,dt \ge \int_0^s \langle \partial_t u, \psi\big(A(u)\big)\phi\rangle\,dt.$$

This concludes the proof of the Lemma 2.1.

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