A DIFFERENCE SCHEME FOR CONSERVATION LAWS WITH A DISCONTINUOUS FLUX - THE NONCONVEX CASE

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Abstract. In a previous work by the author, convergence was established for a simple difference scheme approximating a scalar conservation law where the flux was concave, and had a discontinuous spatially varying coefficient. The main result of this paper is an extension of that convergence theorem to the situation where the flux may have any finite number of critical points. As in the previous paper, compactness is established by bounding the total variation of a Temple functional, but the method of achieving this bound, via discrete entropy inequalities, is new.

Key words. conservation laws, difference approximations, discontinuous coefficients, nonconvex

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1. Introduction. This paper continues the investigation, initiated in [13], of a finite difference algorithm for the scalar Cauchy problem

$$u_t + (k(x)f(u))_x = 0, \quad u(x,0) = u_0(x), \tag{1.1}$$

where $(x,t) \in \mathbf{R} \times \mathbf{R}^+$. The flux $k(x)f(u)$ has a possibly discontinuous spatial dependence through the positive coefficient $k(x)$, which is allowed to have jump discontinuities. As in [13] it will be assumed that $u_0 \in L^1 \bigcap L^\infty \bigcap BV$ and $k \in L^1_{loc} \bigcap L^\infty \bigcap BV$. Here BV denotes the space of locally integrable functions w having bounded total variation, denoted $TV(w)$. Even when $k(x)$ and the initial data $u_0(x)$ are smooth, solutions to 1.1 develop discontinuities after finite time, and so weak solutions are sought, which satisfy

$$\int_{\mathbf{R}\times\mathbf{R}^+} (\phi_t u + \phi_x k f(u)) dx dt + \int_{\mathbf{R}} \phi(x,0) u_0(x) dx = 0 \tag{1.2}$$

for every smooth test function ϕ with compact support in $\mathbf{R} \times \mathbf{R}^+$.

The scalar conservation law 1.1 can be written as a 2x2 system,

$$u_t + (kf(u))_x = 0, \quad k_t = 0. \tag{1.3}$$

In this form, it provides a simple model of a nonstrictly hyperbolic system. The eigenvalues of the system coincide wherever f' vanishes, a situation described as resonance. Resonant systems, and in particular simple models of the form 1.3, have been addressed by a number of authors [4], [5], [6], [14] in recent years.

Like [13], this paper approaches 1.1 as a scalar conservation law. The previous paper established convergence for a simple monotone scalar difference scheme, with the flux assumed to be strictly concave. It is well known that when k is constant, monotone schemes converge to the physically relevant entropy solution to the Cauchy problem, and no concavity or convexity condition on the flux is required. (See references [1], [3], [10], or the textbook [7] for the relevant theory of monotone schemes in the constant k setting.) Thus it is natural to attempt to remove the concavity condition imposed in [13]. This provides the motivation for the present paper, which extends the convergence theorem of [13] to a large class of nonconvex flux functions.

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As in the previous paper, compactness is established here by bounding the total variation of a version of the Temple singular function. However, the proof of the variation bound in [13] relied on the fact that the flux had a single extremum, and so a different approach is required. In this paper the variation bound follows from a finite number of discrete entropy inequalities, one for each maximal interval where the flux f is monotone. In order to guarantee that the Temple mapping is invertible, the flux is assumed to be nondegenerate in the sense that it has only finitely many critical points. For example, the degenerate, but possibly important, case where f is constant on some interval, is ruled out.

The construction of the numerical scheme is unchanged from [13], with the exception that the case of indefinite k is not addressed here. The spatial domain $\mathbf{R}$ is discretized into uniform cells $I_j = [x_j - \Delta x/2, x_j + \Delta x/2)$ centered at the points $x_j = j\Delta x$ for $j \in \mathbf{Z}$. Similarly, the time domain $\mathbf{R}^+$ is divided into time strips $I^n = [t^n, t^{n+1})$, where $t^n = n\Delta t$ for $n \in \mathbf{Z}^+$. Let $\chi_{j,n}$ be the characteristic function for the rectangle $I_j \times I^n$. For each mesh size $\Delta = \Delta x$, the finite difference scheme generates a piecewise constant approximation

$$u^{\Delta}(x,t) = \sum_{n\geq 0} \sum_{j=-\infty}^{+\infty} \chi_{j,n}(x,t) U_j^n.$$

Cell averages are used to discretize the initial data:

$$u^{\Delta}(x,0) = \sum_{j=-\infty}^{+\infty} \chi_j(x) U_j^0, \qquad U_j^0 = \frac{1}{\Delta x}\int_{I_j} u_0(x)dx,$$

where $\chi_j(x)$ is the characteristic function for the interval I_j. The approximations U_j^n are generated via

$$U_j^{n+1} = U_j^n - \lambda(k_{j+\frac{1}{2}} h_{j+\frac{1}{2}} - k_{j-\frac{1}{2}} h_{j-\frac{1}{2}}). \tag{1.4}$$

Here, $\lambda = \Delta t/\Delta x$, and $h_{j+\frac{1}{2}} = h(U_{j+1}^n, U_j^n)$ is the Engquist-Osher (EO) numerical flux [2], defined by

$$h(v,u) = \frac{1}{2}(f(u) + f(v)) - \frac{1}{2}\int_u^v |f'(w)|dw. \tag{1.5}$$

The EO flux is a two point monotone (nondecreasing with respect to the second argument, nonincreasing with respect to the first) numerical flux, $h(v,u)$, consistent with f, i.e., $h(u,u) = f(u)$. Employing the notational conventions $a_+ = \max(a,0)$, $a_- = \min(a,0)$, the partial derivatives of the EO flux satisfy

$$f'_-(v) = h_v(v,u) \leq 0 \leq h_u(v,u) = f'_+(u),$$

from which it is evident that $\|f'\|_\infty$ is a Lipschitz constant, with respect to both arguments, for the EO flux. In [13], convergence was established for versions of the scheme based on both the Godunov and EO numerical fluxes. For the more general nonconvex flux considered here, the author has only been able to prove convergence for the EO version. The EO flux and its associated numerical entropy flux each have a close functional relationship with both the Temple singular function, and the entropy flux associated with the convex entropy $|u-c|$. The proof of Theorem 2.5 depends on these relationships.

In the scheme 1.4, the coefficient k is approximated at each cell boundary, resulting in a discretized version of k,

$$k^{\Delta}(x) = \sum_{j=-\infty}^{+\infty} \chi_{j+\frac{1}{2}}(x) k_{j+\frac{1}{2}} \qquad k_{j+\frac{1}{2}} = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} k(x)dx,$$

where $\chi_{j+\frac{1}{2}}(x)$ is the characteristic function for the interval $I_{j+\frac{1}{2}} = [x_j, x_{j+1})$. The discretized version k^{Δ} is staggered with respect to that of u, resulting in a reduction in complexity, compared with the approach where the two discretizations are aligned.

The theory for the conservation law 1.1 is already well established for the case of concave f. In that setting, the authors of reference [5] proved existence and uniqueness, and studied the asymptotic behavior of the solution. A front tracking scheme was used to construct the solution, and a version of the Temple functional was employed as an analytical tool to establish compactness. The nonconvex case appears to have received less attention in the research literature. One recent contribution to the theory for the nonconvex case is the work of the same authors, in [6]. They constructed a front-tracking scheme, and again used a Temple functional, to establish existence and uniqueness for the nonconvex flux $f(u) = sin(u)$ for $u \in [-\pi, \pi]$. That problem arose as an auxiliary equation to a resonant system, which was the primary focus of their paper. Presumably, the approach in that work could be applied to other equations having a nonconvex flux.

2. Convergence. This section establishes the main result of the paper, convergence of a subsequence to a weak solution of the conservation law. As in [13], monotonicity is an important property of the scheme. A finite difference scheme such as 1.4 is monotone if

$$U_j^n \leq W_j^n, \quad \forall j \quad \Rightarrow \quad U_j^{n+1} \leq W_j^{n+1}, \quad \forall j.$$

The following fact is established in [13], and is equally valid here.

PROPOSITION 2.1. *With the CFL condition $\lambda \|k\|_{\infty} \|f'\|_{\infty} \leq 1$ the scheme 1.4 is monotone. If the initial data $u_0(x)$ lies within the interval $\Omega_0 = [0,1]$, and $f(0) = f(1) = 0$, then the computed solution will remain within Ω_0.*

Assume that $f \in C^1([0,1])$. It is clear that due to the invariance of Ω_0, there are constants $\|f\|_{\infty}$, $\|f'\|_{\infty}$, $\|h\|_{\infty}$ such that the following bounds hold uniformly in j, n, and Δ:

$$|f(U_j^n)| \leq \|f\|_{\infty}, \qquad |f'(U_j^n)| \leq \|f'\|_{\infty}, \qquad |h(U_{j+1}^n, U_j^n)| \leq \|h\|_{\infty}.$$

PROPOSITION 2.2. *With the CFL condition stated in Proposition 2.1 and the assumption that the initial data satisfies u_0, $w_0 \in L^1 \bigcap L^{\infty}$, the scheme 1.4 is L_1-contractive, i.e., the inequality*

$$\sum_j |W_j^{n+1} - U_j^{n+1}| \leq \sum_j |W_j^n - U_j^n|$$

holds for a pair of computed approximate solutions U_j^n and W_j^n generated by the scheme. The following inequality is also satisfied:

$$\sum_j |U_j^{n+1} - U_j^n| \leq \sum_j |U_j^1 - U_j^0| \leq 2\lambda \|k\|_{\infty} \|f'\|_{\infty} TV(u_0) + \lambda \|h\|_{\infty} TV(k). \tag{2.1}$$

Proof. Proposition 2.2 of [13] established all but the last inequality in 2.1. This inequality is established in the same way as the inequality 3.10 of [13] with $\|h\|_\infty$ substituted for f^* of that paper. $\square$

THEOREM 2.3. *Let $k \in L^1_{loc} \bigcap BV \bigcap L^\infty$, with $0 < \underline{k} \le k(x) \le \overline{k}$. Let $f \in C^1([0,1])$, $f(0) = f(1) = 0$, with finitely many critical points. Let $u_0 \in BV \bigcap L^1 \bigcap L^\infty$, and $u_0(x) \in [0,1]$. Let the mesh size $\Delta \to 0$ with λ fixed and satisfying the CFL condition $\lambda \|k\|_\infty \|f'\|_\infty \le 1$, resulting in the sequence of approximations u^Δ. Then, there is a weak solution u of 1.2 and a subsequence u^{Δ_i} such that $u^{\Delta_i} \to u$ a.e. and in $L^1_{loc}(\mathbf{R} \times [0,\infty))$.*

The strategy for establishing Theorem 2.3 is basically the same, with modifications where necessary, as was used to prove Theorem 3.2 of [13]. The proof of that theorem was patterned after standard compactness arguments as found in [10] and [11], with the exception that the compactness arguments were applied to the transformed quantity z, rather than the conserved quantity u. Once compactness was established for z, the variable of ultimate interest, u, was recovered by inverting the Temple functional. The same approach is used in what follows.

By way of establishing notation, let Δ_+ and Δ_- denote forward and backward difference operators. For example, $\Delta_+ h_{j-\frac{1}{2}} = \Delta_- h_{j+\frac{1}{2}} = h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}$. Let $\sigma(w) = w/|w|$, for $w \neq 0$, and $\sigma(w) = 0$ if $w = 0$. Finally, let $\chi_r(w;c)$ be the characteristic function for the interval $[c, +\infty)$, and let $\chi_l(w;c)$ be the characteristic function for $(-\infty, c]$.

One other preliminary matter is the convergence of the various infinite sums over the domain $j \in \mathbf{Z}$ which will appear in what follows. Absolute convergence of these sums in each case results ultimately from the bounds $TV(k^\Delta) < \infty$ and $\sum_j |U_j^n| < \infty$. The second inequality follows from the facts the scheme is conservative, that $u_0 \in L^1$, and that $U_j^n \in [0,1]$.

As discussed in [13], the Temple functional has been used by a number of authors, (eg., [5], [8], [9], [12]) to establish compactness for resonant systems. For the case of f concave with a single critical point located at u^*, the singular function of [5], $\Psi(u,k) = (k/f(u^*)) \int_{u^*}^u |f'(w)| dw$, due originally (in a somewhat different form) to Temple, was used in [13] to prove compactness. In this paper, the following version of the Temple functional, where the normalizing constant f^* has been dropped, will be used:

$$\Psi(u,k) = k \int_0^u |f'(w)| dw.$$

It is clear that since $|f'| > 0$ a.e., $\Psi(u,k)$ is strictly increasing with respect to u.

Since f has only finitely many critical points, there are finitely many points $0 = u_0^* < u_1^* < \cdots < u_m^* < u_{m+1}^* = 1$ such that f is strictly monotone in $(u_\nu^*, u_{\nu+1}^*)$, for $0 \le \nu \le m$, and each of $u_1^*, \cdots, u_m^*$ is an extremum of f. Obviously, $\{u_1^*, \ldots, u_m^*\}$ is possibly only a subset of the critical points of f; for the purpose of this analysis, those which do not correspond to extrema are simply ignored. Without loss of generality, assume that f is strictly increasing in $(0, u_1^*)$, so that

$$u \in (u_i^*, u_{i+1}^*) \Rightarrow \sigma(f'(u)) = (-1)^i.$$

The case where f is strictly decreasing in $(0, u_1^*)$ is completely analogous.

With $\alpha = \int_0^1 |f'(w)|dw$, the following Lipschitz continuity relationships in u and k follow directly from the definition of Ψ and the assumptions about the flux f:

$$|\Psi(u,k_1) - \Psi(u,k_2)| \le \alpha |k_1 - k_2|, \tag{2.2}$$

$$|\Psi(u_1,k) - \Psi(u_2,k)| \le k\|f'\|_\infty |u_1 - u_2|. \tag{2.3}$$

Let $z^\Delta(x,t) = \Psi(u^\Delta(x,t), k^\Delta(x))$. The following time continuity estimate for z^Δ is established in [13].

Lemma 2.4. *With the assumptions stated in Theorem 2.3, there is a constant Λ, independent of the mesh size Δ, such that for $n > l \ge 0$,*

$$\int_{\mathbf{R}} |z^\Delta(x,t^n) - z^\Delta(x,t^l)|dx \le \Lambda(n-l)\Delta t.$$

As in [13], the key to the proof of convergence is a uniform bound on the spatial variation of z^Δ.

Theorem 2.5. *Let $z^n(x) = z^\Delta(x,t^n)$. With the assumptions stated in Theorem 2.3, $TV(z^n)$ is bounded uniformly, for all $n \ge 0$, and all $\Delta > 0$.*

The proof proceeds by a series of lemmas. The first four lemmas lead up to the pair of inequalities stated in Lemma 2.9, which can be viewed as a pair of discrete entropy inequalities, parameterized by c. The Temple mapping is then decomposed into components with support limited to maximal segments where f is monotone. The next two lemmas (2.10 and 2.11) then use the entropy inequalities to show that each of those components of the Temple functional has bounded variation. Only a finite set of entropy inequalities are used; specifically, those resulting from values of the parameter c located at critical points corresponding to extrema of f.

Lemma 2.6. *Let $c \in \mathbf{R}$. For the convex entropy pair $V(u) = |u-c|$, $F(u) = \sigma(u-c)(f(u)-f(c))$, and W_j computed using the EO numerical flux via*

$$W_j = U_j - \lambda k_{j+\frac12}\Delta_+ h_{j+\frac12}, \tag{2.4}$$

the following discrete entropy inequality holds

$$V(W_j) \le V(U_j) - \lambda k_{j+\frac12}\Delta_+ H_{j-\frac12}, \tag{2.5}$$

where the numerical entropy flux is defined by

$$H_{j+\frac12} = \frac12(F(U_j) + F(U_{j+1})) - \frac12 \int_{U_j}^{U_{j+1}} \sigma(f'(w))\sigma(w-c)f'(w)dw. \tag{2.6}$$

Proof. The proof is an application of the numerical entropy flux constructed in the proof of Proposition 4.1 of [1]. Specifically, the numerical entropy flux

$$k_{j+\frac12}H(U_{j+1},U_j) = k_{j+\frac12}h(U_{j+1}\vee c, U_j \vee c) - k_{j+\frac12}h(U_{j+1}\wedge c, U_j\wedge c) \tag{2.7}$$

is applied to the EO flux $k_{j+\frac12}h$ consistent with the flux $k_{j+\frac12}f$. Here $a \vee b = \max(a,b)$, $a \wedge b = \min(a,b)$. Dropping the multiplier $k_{j+\frac12}$, formula 2.7 gives

$$\begin{aligned} H(U_{j+1},U_j) = {} & \frac12(f(U_j\vee c) - f(U_j\wedge c)) + \frac12(f(U_{j+1}\vee c) - f(U_{j+1}\wedge c)) \\ & - \frac12\int_{U_j\vee c}^{U_{j+1}\vee c} |f'(w)|dw + \frac12 \int_{U_j\wedge c}^{U_{j+1}\wedge c}|f'(w)|dw \end{aligned} \tag{2.8}$$

The formula 2.6 for the EO entropy flux then follows from the identities

$$f(U \vee c) - f(U \wedge c) = \sigma(U-c)(f(U)-f(c)) = F(U),$$

and

$$-\int_{U\vee c}^{V\vee c} |f'(w)|dw + \int_{U\wedge c}^{V\wedge c} |f'(w)|dw = -\int_{U}^{V} \sigma(f'(w))\sigma(w-c)f'(w)dw,$$

which are readily verified by considering the three cases corresponding to whether c is less than, greater than, or between, U and V.

□

LEMMA 2.7. *For the EO flux and its associated numerical entropy flux 2.6 consistent with entropy flux $F(u) = \sigma(u-c)(f(u)-f(c))$, the following identities are satisfied:*

$$\frac{1}{2}(\Delta_+ H_{j-\frac{1}{2}} + \Delta_+ h_{j-\frac{1}{2}}) = \int_{U_j^n}^{U_{j+1}^n} \chi_r(w;c) f_-'(w)dw + \int_{U_{j-1}^n}^{U_j^n} \chi_r(w;c) f_+'(w)dw, \tag{2.9}$$

$$\frac{1}{2}(\Delta_+ H_{j-\frac{1}{2}} - \Delta_+ h_{j-\frac{1}{2}}) = -\int_{U_j^n}^{U_{j+1}^n} \chi_l(w;c) f_-'(w)dw - \int_{U_{j-1}^n}^{U_j^n} \chi_l(w;c) f_+'(w)dw. \tag{2.10}$$

Proof. The definition of the EO flux 1.5 and its associated discrete entropy flux 2.6 yield the identities

$$\Delta_+ H_{j-\frac{1}{2}} = \int_{U_j^n}^{U_{j+1}^n} \sigma(w-c) f_-'(w)dw + \int_{U_{j-1}^n}^{U_j^n} \sigma(w-c) f_+'(w)dw, \tag{2.11}$$

$$\Delta_+ h_{j-\frac{1}{2}} = \int_{U_j^n}^{U_{j+1}^n} f_-'(w)dw + \int_{U_{j-1}^n}^{U_j^n} f_+'(w)dw. \tag{2.12}$$

Adding 2.12 to 2.11, then dividing by two, and using the identities

$$\frac{1}{2}(\sigma(w-c)+1) = \chi_r(w;c), \quad \frac{1}{2}(\sigma(w-c)-1) = -\chi_l(w;c) \tag{2.13}$$

gives 2.9. Similarly, 2.10 results by subtracting 2.12 from 2.11, then dividing by two.
□

LEMMA 2.8. *With the assumptions stated in Theorem 2.3, the following relationships hold:*

$$U_j^{n+1} = U_j^n - \lambda k_{j+\frac{1}{2}} \Delta_+ h_{j-\frac{1}{2}} + C_j^n \Delta_+ k_{j-\frac{1}{2}}, \qquad |C_j^n| \le \lambda \|h\|_\infty, \tag{2.14}$$

$$V(U_j^{n+1}) \le V(U_j^n) - \lambda k_{j+\frac{1}{2}} \Delta_+ H_{j-\frac{1}{2}} + \lambda \|h\|_\infty |\Delta_+ k_{j-\frac{1}{2}}|. \tag{2.15}$$

Proof. The equation in 2.14 is just 1.4 with $C_j^n = -\lambda h_{j-\frac{1}{2}}$. The inequality follows from

$$|C_j^n| \leq \lambda |h_{j-\frac{1}{2}}| \leq \lambda \|h\|_\infty \tag{2.16}$$

For 2.15, with W_j defined by 2.4, the discrete entropy inequality 2.5 gives

$$\begin{aligned} V(U_j^{n+1}) &\leq V(U_j^n) - \lambda k_{j+\frac{1}{2}} \Delta_+ H_{j-\frac{1}{2}} + (V(U_j^{n+1}) - V(W_j)) \\ &= V(U_j^n) - \lambda k_{j+\frac{1}{2}} \Delta_+ H_{j-\frac{1}{2}} + |U_j^{n+1} - c| - |W_j - c|. \end{aligned}$$

Inequality 2.15 then follows from

$$|U_j^{n+1} - c| - |W_j - c| \leq |U_j^{n+1} - W_j| \leq \lambda \|h\|_\infty |\Delta_+ k_{j-\frac{1}{2}}|.$$

□

LEMMA 2.9. *With the assumptions stated in Theorem 2.3, the following inequalities are valid:*

$$\begin{aligned} & k_{j+\frac{1}{2}} \int_{U_j^n}^{U_{j+1}^n} \chi_r(w;c) f'_-(w) dw + k_{j-\frac{1}{2}} \int_{U_{j-1}^n}^{U_j^n} \chi_r(w;c) f'_+(w) dw \\ & \qquad \leq \frac{1}{\lambda}(U_j^n - U_j^{n+1})_+ + (\|h\|_\infty + \|f'\|_\infty)|\Delta_+ k_{j-\frac{1}{2}}|, \end{aligned} \tag{2.17}$$

$$\begin{aligned} & -k_{j+\frac{1}{2}} \int_{U_j^n}^{U_{j+1}^n} \chi_l(w;c) f'_-(w) dw - k_{j-\frac{1}{2}} \int_{U_{j-1}^n}^{U_j^n} \chi_l(w;c) f'_+(w) dw \\ & \qquad \leq \frac{-1}{\lambda}(U_j^n - U_j^{n+1})_- + (\|h\|_\infty + \|f'\|_\infty)|\Delta_+ k_{j-\frac{1}{2}}|. \end{aligned} \tag{2.18}$$

Proof. Rearranging 2.15 and dividing by λ results in

$$\begin{aligned} k_{j+\frac{1}{2}} \Delta_+ H_{j-\frac{1}{2}} &\leq \frac{1}{\lambda}(V(U_j^n) - V(U_j^{n+1})) + \|h\|_\infty |\Delta_+ k_{j-\frac{1}{2}}| \\ &= \frac{1}{\lambda}(|U_j^n - c| - |U_j^{n+1} - c|) + \|h\|_\infty |\Delta_+ k_{j-\frac{1}{2}}| \\ &\leq \frac{1}{\lambda}|U_j^n - U_j^{n+1}| + \|h\|_\infty |\Delta_+ k_{j-\frac{1}{2}}|. \end{aligned} \tag{2.19}$$

Adding 2.19 to the following rearrangement of 2.14,

$$k_{j+\frac{1}{2}} \Delta_+ h_{j-\frac{1}{2}} = \frac{1}{\lambda}(U_j^n - U_j^{n+1}) + \frac{1}{\lambda} C_j^n \Delta_+ k_{j-\frac{1}{2}}, \tag{2.20}$$

dividing by two, then applying 2.9 yields

$$\begin{aligned} & k_{j+\frac{1}{2}} \int_{U_j^n}^{U_{j+1}^n} \chi_r(w;c) f'_-(w) dw + k_{j+\frac{1}{2}} \int_{U_{j-1}^n}^{U_j^n} \chi_r(w;c) f'_+(w) dw \\ & \qquad \leq \frac{1}{\lambda}(U_j^n - U_j^{n+1})_+ + \|h\|_\infty |\Delta_+ k_{j-\frac{1}{2}}|. \end{aligned} \tag{2.21}$$

Similarly, subtracting 2.20 from 2.19, dividing by two, then applying 2.10 gives

$$-k_{j+\frac{1}{2}} \int_{U_j^n}^{U_{j+1}^n} \chi_l(w;c) f'_-(w)dw - k_{j+\frac{1}{2}} \int_{U_{j-1}^n}^{U_j^n} \chi_l(w;c) f'_+(w)dw$$

$$\leq \frac{-1}{\lambda}(U_j^n - U_j^{n+1})_- + \|h\|_\infty |\Delta_+ k_{j-\frac{1}{2}}|. \tag{2.22}$$

The second term on the left in 2.17 results from the corresponding term in 2.21:

$$\begin{aligned} k_{j-\frac{1}{2}} \int_{U_{j-1}^n}^{U_j^n} \chi_r(w;c) f'_+(w)dw &= (k_{j+\frac{1}{2}} - \Delta_+ k_{j-\frac{1}{2}}) \int_{U_{j-1}^n}^{U_j^n} \chi_r(w;c) f'_+(w)dw \\ &\leq \; k_{j+\frac{1}{2}} \int_{U_{j-1}^n}^{U_j^n} \chi_r(w;c) f'_+(w)dw + |\Delta_+ k_{j-\frac{1}{2}}| \Big| \int_{U_{j-1}^n}^{U_j^n} |f'(w)|dw \Big|. \end{aligned}$$

Then, since U_j^n remains within the initial domain $\Omega_0 = [0,1]$, 2.17 follows from

$$\Big| \int_{U_{j-1}^n}^{U_j^n} |f'(w)|dw \Big| \leq \|f'\|_\infty |U_{j+1}^n - U_j^n| \leq \|f'\|_\infty.$$

The inequality 2.18 follows in a similar way from 2.22. ◻

Let $\Phi(u) = \Psi(u,k)/k = \int_0^u |f'(w)|dw$. For $\nu = 0, 1, \ldots, m$, let $\chi^\nu(u)$ be the characteristic function of the interval $[u_\nu^*, u_{\nu+1}^*)$. Define

$$\phi^\nu(u) = \int_0^u \chi^\nu(w)|f'(w)|dw,$$

so that Φ has the following decomposition:

$$\Phi(u) = \sum_{\nu=0}^m \phi^\nu(u).$$

Lemma 2.10. *Let E_+^i (O_+^i) denote the even (odd) integers ν with $0 \leq i \leq \nu \leq m$, and let E_-^i (O_-^i) denote the even (odd) integers ν with $0 \leq \nu < i < m$. For each $i = 1, \ldots, m$, the following pair of inequalities holds:*

$$k_{j+\frac{1}{2}} \sum_{\nu \in O_+^i} (\phi^\nu(U_j^n) - \phi^\nu(U_{j+1}^n)) + k_{j-\frac{1}{2}} \sum_{\nu \in E_+^i} (\phi^\nu(U_j^n) - \phi^\nu(U_{j-1}^n))$$

$$\leq \frac{1}{\lambda}(U_j^n - U_j^{n+1})_+ + (\|h\|_\infty + \|f'\|_\infty)|\Delta_+ k_{j-\frac{1}{2}}|, \tag{2.23}$$

$$k_{j+\frac{1}{2}} \sum_{\nu \in O_-^i} (\phi^\nu(U_{j+1}^n) - \phi^\nu(U_j^n)) + k_{j-\frac{1}{2}} \sum_{\nu \in E_-^i} (\phi^\nu(U_{j-1}^n) - \phi^\nu(U_j^n))$$

$$\leq \frac{-1}{\lambda}(U_j^n - U_j^{n+1})_- + (\|h\|_\infty + \|f'\|_\infty)|\Delta_+ k_{j-\frac{1}{2}}|. \tag{2.24}$$

Proof. For each $i = 1, \ldots, m$, substitute $c = u_i^*$ into 2.17 and 2.18. With the assumption that f is increasing in $(0, u_1^*)$, it is clear that $f'_+ = 0$ in the odd-numbered intervals, and $f'_- = 0$ in the even-numbered intervals. Thus,

$$\int_{U_j^n}^{U_{j+1}^n} \chi_r(w; u_i^*) f'_-(w)dw = \sum_{\nu \in O_+^i} \int_{U_j^n}^{U_{j+1}^n} \chi^\nu(w) f'_-(w)dw = -\sum_{\nu \in O_+^i} \Delta_+ \phi^\nu(U_j^n), \tag{2.25}$$

$$\int_{U_{j-1}^n}^{U_j^n} \chi_r(w; u_i^*) f'_+(w)dw = \sum_{\nu \in E_+^i} \int_{U_{j-1}^n}^{U_j^n} \chi^\nu(w) f'_+(w)dw = \sum_{\nu \in E_+^i} \Delta_- \phi^\nu(U_j^n), \tag{2.26}$$

$$\int_{U_j^n}^{U_{j+1}^n} \chi_l(w; u_i^*) f'_-(w)dw = \sum_{\nu \in O_-^i} \int_{U_j^n}^{U_{j+1}^n} \chi^\nu(w) f'_-(w)dw = -\sum_{\nu \in O_-^i} \Delta_+ \phi^\nu(U_j^n), \tag{2.27}$$

$$\int_{U_{j-1}^n}^{U_j^n} \chi_l(w; u_i^*) f'_+(w)dw = \sum_{\nu \in E_-^i} \int_{U_{j-1}^n}^{U_j^n} \chi^\nu(w) f'_+(w)dw = \sum_{\nu \in E_-^i} \Delta_- \phi^\nu(U_j^n). \tag{2.28}$$

The ith version of 2.17 yields the corresponding version of 2.23, using the definition of the ϕ^ν, along with the pair of identities 2.25, 2.26. The ith version of 2.24 follows in the same way from 2.18, 2.27, and 2.28. □

LEMMA 2.11. *For each $i = 0, \ldots, m$, there is a constant $E^i \geq 0$ such that*

$$\sum_j k_{j+\frac{1}{2}} |\phi^i(U_{j+1}^n) - \phi^i(U_j^n)| \leq E^i, \tag{2.29}$$

where E^i is independent of Δ and n.

Proof. Starting from the inequality 2.24, with $i = 1$, gives

$$k_{j-\frac{1}{2}}(\phi^0(U_{j-1}^n) - \phi^0(U_j^n)) \leq \frac{-1}{\lambda}(U_j^n - U_j^{n+1})_- + (\|h\|_\infty + \|f'\|_\infty)|\Delta_+ k_{j-\frac{1}{2}}|, \tag{2.30}$$

since only the $\nu = 0$ term is nonzero in this case. Since the right side of 2.30 is nonnegative, it follows that

$$\begin{aligned} -\sum_j k_{j-\frac{1}{2}}(\phi^0(U_j^n) - \phi^0(U_{j-1}^n))_- &\leq \sum_j \Big(\frac{-1}{\lambda}(U_j^n - U_j^{n+1})_- \\ &\quad + (\|h\|_\infty + \|f'\|_\infty)|\Delta_+ k_{j-\frac{1}{2}}| \Big) \\ &\leq \frac{1}{\lambda} \sum_j |U_j^n - U_j^{n+1}| + (\|h\|_\infty + \|f'\|_\infty) TV(k) \\ &\leq \frac{1}{\lambda} \sum_j |U_j^1 - U_j^0| + (\|h\|_\infty + \|f'\|_\infty) TV(k). \end{aligned} \tag{2.31}$$

Using summation by parts gives, for each ν,

$$
\begin{aligned}
\sum_j k_{j+\frac{1}{2}}|\phi^\nu(U^n_{j+1}) - \phi^\nu(U^n_j)| &= \sum_j k_{j+\frac{1}{2}}(\Delta_+\phi^\nu(U^n_j))_+ - \sum_j k_{j+\frac{1}{2}}(\Delta_+\phi^\nu(U^n_j))_- \\
&= 2\sum_j k_{j+\frac{1}{2}}(\Delta_+\phi^\nu(U^n_j))_+ - \sum_j k_{j+\frac{1}{2}}\Delta_+\phi^\nu(U^n_j) \\
&= 2\sum_j k_{j+\frac{1}{2}}(\Delta_+\phi^\nu(U^n_j))_+ + \sum_j \phi^\nu(U^n_j)\Delta_+ k_{j-\frac{1}{2}} \\
&\qquad - \sum_j \Delta_+(k_{j+\frac{1}{2}}\phi^\nu(U^n_j)) \\
&= 2\sum_j k_{j+\frac{1}{2}}(\Delta_+\phi^\nu(U^n_j))_+ + \sum_j \phi^\nu(U^n_j)\Delta_+ k_{j-\frac{1}{2}} \\
&\le 2\sum_j k_{j+\frac{1}{2}}(\Delta_+\phi^\nu(U^n_j))_+ + \alpha TV(k).
\end{aligned}
$$

(2.32)

Similarly,

$$
\sum_j k_{j+\frac{1}{2}}|\phi^\nu(U^n_{j+1}) - \phi^\nu(U^n_j)| \le -2\sum_j k_{j+\frac{1}{2}}(\Delta_+\phi^\nu(U^n_j))_- + \alpha TV(k).
$$

(2.33)

From 2.31 and 2.33, it follows that

$$
\sum_j k_{j+\frac{1}{2}}|\phi^0(U^n_{j+1}) - \phi^0(U^n_j)| \le E^0,
$$

where

$$
E^0 = \frac{2}{\lambda}\sum_j |U^1_j - U^0_j| + (2\|h\|_\infty + 2\|f'\|_\infty + \alpha)TV(k).
$$

Now taking 2.24 with $i = 2$, since only contributions from $\nu = 0, 1$ are present, gives

$$
\begin{aligned}
&k_{j+\frac{1}{2}}(\phi^1(U^n_{j+1}) - \phi^1(U^n_j)) + k_{j-\frac{1}{2}}(\phi^0(U^n_{j-1}) - \phi^0(U^n_j)) \\
&\qquad\le \frac{-1}{\lambda}(U^n_j - U^{n+1}_j)_- + (\|h\|_\infty + \|f'\|_\infty)|\Delta_+ k_{j-\frac{1}{2}}|.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
k_{j+\frac{1}{2}}(\phi^1(U^n_{j+1}) - \phi^1(U^n_j))_+ &\le k_{j-\frac{1}{2}}|\phi^0(U^n_j) - \phi^0(U^n_{j-1})| + \frac{-1}{\lambda}(U^n_j - U^{n+1}_j)_- \\
&\quad + (\|h\|_\infty + \|f'\|_\infty)|\Delta_+ k_{j-\frac{1}{2}}|.
\end{aligned}
\tag{2.34}
$$

Combining 2.32 and 2.34 yields

$$
\begin{aligned}
\sum_j k_{j+\frac{1}{2}}|\phi^1(U^n_{j+1}) - \phi^1(U^n_j)| &\le 2\sum_j k_{j+\frac{1}{2}}(\Delta_+\phi^1(U^n_j))_+ + \alpha TV(k) \\
&\le 2\sum_j k_{j-\frac{1}{2}}|\Delta_+\phi^0(U^n_{j-1})| - \frac{2}{\lambda}\sum_j (U^n_j - U^{n-1}_j)_-
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\sum_j (\|h\|_\infty + \|f'\|_\infty)|\Delta_+ k_{j-\frac{1}{2}}| + \alpha TV(k) \\
&\le 2E^0 + \frac{2}{\lambda}\sum_j |U_j^n - U_j^{n+1}| \\
&+ (2\|h\|_\infty + 2\|f'\|_\infty) TV(k) + \alpha TV(k) \\
&\le 3E^0.
\end{aligned}
$$

Continuing this way inductively from left to right (increasing i) with 2.24 gives, for each $i = 0, \dots, m-1$

$$\sum_j k_{j+\frac{1}{2}} |\phi^i(U_{j+1}^n) - \phi^i(U_j^n)| \le 2(E^0 + \cdots + E^{i-1}) + E^0 \equiv E^i.$$

This leaves only ϕ^m to deal with. Starting from the inequality 2.23, with $i = m$, and proceeding as in the case where $i = 0$, gives

$$\sum_j k_{j+\frac{1}{2}} |\phi^m(U_{j+1}^n) - \phi^m(U_j^n)| \le E^0 \equiv E^m.$$

□

It is now possible to prove Theorem 2.5.

Proof. Taking into account the jumps in z^n at the cell boundaries (due to jumps in u^Δ) and the jumps at the cell centers (due to jumps in k^Δ), the total variation of $z^n(x)$ is

$$TV(z) = \sum_j |\Delta_+^u z_j^n| + \sum_j |\Delta_+^k z_{j-\frac{1}{2}}^n|, \tag{2.35}$$

where $\Delta_+^u z_j^n = \Psi(U_{j+1}^n, k_{j+\frac{1}{2}}) - \Psi(U_j^n, k_{j+\frac{1}{2}})$ and $\Delta_+^k z_{j-\frac{1}{2}} = \Psi(U_j^n, k_{j+\frac{1}{2}}) - \Psi(U_j^n, k_{j-\frac{1}{2}})$. The second sum in 2.35, due to jumps in k^Δ, is bounded by $\alpha TV(k)$, as is seen from the Lipschitz continuity estimate 2.2. For the first sum,

$$
\begin{aligned}
\sum_j |\Delta_+^u z_j^n| &= \sum_j k_{j+\frac{1}{2}} |\Phi(U_{j+1}^n) - \Phi(U_j^n)| \\
&= \sum_j k_{j+\frac{1}{2}} \Big|\sum_{i=0}^m (\phi^i(U_{j+1}^n) - \phi^i(U_j^n))\Big| \\
&\le \sum_{i=0}^m \sum_j k_{j+\frac{1}{2}} |\phi^i(U_{j+1}^n) - \phi^i(U_j^n)| \le \sum_{i=0}^m E^i.
\end{aligned}
$$

□

The following is basically a version of the Lax-Wendroff theorem. The proof in [13] is equally valid in the present situation, with the single exception that the normalizing factor f^* appearing in inequality 3.16 of that proof is not present here.

THEOREM 2.12. *Let u^Δ be a sequence of approximations computed via the scheme 1.4 which converges as in Theorem 2.3, to a function u. Then u is a weak solution of 1.1, i.e., it satisfies 1.2.*

The proof of Theorem 2.3 follows. In [13], the L^1_{loc} estimate specifically invoked the concavity assumption, and so that part of the proof below is different. The rest of the proof is very little changed.

Proof. As in [13], the computed solutions u^Δ remain within $[0,1]$, thanks to the CFL condition. Fix $B>0$. The following estimates

$$\|z^\Delta(\cdot,t)\|_\infty \le \alpha\|k\|_\infty = \alpha\overline{k}, \qquad \int_{-B}^{+B} |z^\Delta(x,t)|dx \le 2B\alpha\overline{k},$$

which hold uniformly for $t\ge 0$, $\Delta>0$, prove that for each $t\ge 0$, $z^\Delta(\cdot,t)\in L^\infty$, and $z^\Delta(\cdot,t)\in L^1_{loc}(\mathbf{R})$. Theorem 2.5 provides a uniform bound on $TV(z^\Delta(\cdot,t))$. The following time-continuity estimate is a simple consequence of Lemma 2.4:

$$\|z^\Delta(\cdot,t+\tau)-z^\Delta(\cdot,t)\|_1 \le C(|\tau|+\Delta).$$

Here the constant C is independent of Δ and t. Using the same argument as in [13], there is a subsequence denoted z^{Δ_i}, which converges in $L^1_{loc}(\mathbf{R}\times\mathbf{R}^+)\bigcap L^\infty$ and pointwise a.e. to some function $z\in L^1_{loc}(\mathbf{R}\times\mathbf{R}^+)\bigcap L^\infty$. Let $u(x,t)=\Psi^{-1}(z(x,t),k(x))$. Due to the strict monotonicity of $\Psi(\cdot,k)$, the function u is well-defined a.e., $u\in[0,1]$ a.e., and $u\in L^1_{loc}(\mathbf{R}\times\mathbf{R}^+)\bigcap L^\infty$. The proof is completed by dropping the subscript on Δ, and showing that $u^\Delta\to u$ a.e. and in $L^1_{loc}(\mathbf{R}\times\mathbf{R}^+)$. It is clear from its definition that Φ is strictly increasing and that Ψ and Φ are related via

$$\Psi^{-1}(z,k)=\Phi^{-1}(z/k).$$

Fix $T>0$. Using the identities $u=\Phi^{-1}(z/k)$, $u^\Delta=\Phi^{-1}(z^\Delta/k^\Delta)$ gives

$$\int_0^T\int_{-B}^B |u-u^\Delta|dxdt = \int_0^T\int_{-B}^B |\Phi^{-1}(z/k)-\Phi^{-1}(z^\Delta/k^\Delta)|dxdt. \tag{2.36}$$

Since $k\in BV$, $k^\Delta\to k$ pointwise a.e., and so $z^\Delta/k^\Delta\to z/k$ a.e. It follows from the continuity of Φ^{-1} that $u^\Delta\to u$ a.e. The bounded convergence theorem then guarantees that the integral in 2.36 converges to zero as Δ approaches zero. That the limit u is a weak solution of 1.1 is a consequence of Theorem 2.12. ∎

3. Conclusion. The paper has extended the convergence result of [13] to a large class of nonconvex flux functions. A number of important issues have not been addressed, and are potential areas for further investigation. For example, the related questions of uniqueness and entropy satisfaction have not been discussed at all. These are important issues for conservation laws, since the presence of discontinuities requires the satisfaction of entropy conditions to ensure that the correct, physically relevant solution is singled out. In [13], the author derived geometric entropy conditions for the limit of the scheme for the case of a concave flux. Carrying out that program for the present situation appears feasible, but is significantly more complicated, and hopefully will appear in a subsequent article.

The nondegeneracy assumption (finitely many critical points) rules out any flux that is constant over an interval, a seemingly important case. In this situation, the Temple functional would fail to be monotone, and so it would not be possible to recover the conserved quantity u from the limit function z. The question here is whether there is a real problem with convergence, or just a limitation of the analytical tool.

This paper only addressed the EO version of the scheme. A convergence proof for the Godunov version would also be of interest. Since the proof of convergence for the EO version exploited the close relationship between the EO flux and the Temple functional, it may be that a different form of the Temple functional will be required.

The convergence proofs in both [13] and the present work depended on monotonicity and L^1 contractiveness. Both of these properties are well known to imply that the scheme is limited to first order accuracy. Some numerical experiments (with a concave flux) indicate that second order schemes, constructed by applying flux corrections and minmod-type limiters to the scheme 1.4, give very satisfactory results. Due to the essential way that monotonicity was used, it seems unlikely that it will be possible to develop a convergence theory for such schemes using the approach of this paper and the previous one.

Finally, the author performed a number of numerical experiments with the flux functions $f(u) = sin(2\pi u)$ and $f(u) = sin(4\pi u)$ for $u \in [0, 1]$, with piecewise constant k, and did not detect any evidence of variation blowup of the conserved quantity u. As was observed in [13], if $TV(u)$ is bounded, as it appears to be, it should be possible to establish that bound, not just the weaker bound on $TV(z)$ that was used to prove convergence.

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