Optimal Convergence Rates to Diffusion Waves for Solutions of the Hyperbolic Conservation Laws with Damping*

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Abstract

This paper is devoted to study on the asymptotic behaviors of solutions to a model of hyperbolic balance laws with damping on the quarter plane $(x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$. We show the optimal convergence rates of the solutions to their corresponding nonlinear diffusion waves, which are the solutions of the corresponding nonlinear parabolic equation given by the related Darcy's law. The optimal rates we obtained improve those in recent works on the IBVP by K. Nishihara and T. Yang [*J. Differential Equations* **156** (1999) 439-458] and by P. Marcati and M. Mei [*Quart. Appl. Math.* **56** (2000)]. The energy method with the method of Fourier transform together are efficiently used for proof.

Contents

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1 Introduction

Subsequent to the previous works by Marcati-Mei [11] and Nishihara-Yang [18], we continue to consider the following model of hyperbolic equations with damping, on the quarter plane $\mathbb{R}_+ \times \mathbb{R}_+$, ($\mathbb{R}_+ = (0, +\infty)$), given by

$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = -\alpha u, \end{cases} \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \tag{1.1}$$

which models a compressible flow with dissipative external force field in Lagrangian coordinates. The external force term $-\alpha u$ appears in the momentum equation. Here, $v > 0$ is the specific volume, u is the velocity, the pressure $p(v)$ is a smooth function of v such that $p(v) > 0$, $p'(v) < 0$, and $\alpha > 0$ is the damping constant. Such a system was firstly studied by van Duyn and Peletier [4].

For the corresponding Cauchy problem, Marcati-Milani [12] in the case of weak solutions, and Hsiao-Liu [5] and Nishihara [15] in the case of smooth solutions proved that the solutions $(v,u)(x,t)$ to the corresponding Cauchy problem of (1.1) tend time-asymptotically to the nonlinear self-similar diffusion wave solutions $(\bar{v}, \bar{u})(x,t)$ $(\bar{v}(x,t) = \phi(x/\sqrt{1+t}))$ of the porous media equation

$$\begin{cases} \bar{v}_t = -\frac{1}{\alpha} p(\bar{v})_{xx} \\ p(\bar{v})_x = -\alpha \bar{u}, \end{cases} \tag{1.2}$$

namely

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0 \\ p(\bar{v})_x = -\alpha \bar{u}. \end{cases} \tag{1.3}$$

For the related initial-boundary value problems (IBVPs), Marcati-Mei [11] and Nishihara-Yang [18] studied the convergence to diffusion waves on the quarter plane $\mathbb{R}_+ \times \mathbb{R}_+$, and Hsiao-Pan [8] on bounded domain $x \in [0,1]$. As we know, all convergence rates appeared in above-mentioned works are less sufficient. It should be significant to show the optimal decay rates for the convergence both in mathematical and physical point of view. Recently, by using the method of pointwise estimates and the approximating Green function for a parabolic equation, Nishihara-Wang-Yang [17] succeeded in obtaining the optimal rates for the Cauchy problem case, which improves the previous works by Hsiao-Liu [5] and Nishihara [15]. In this paper, we are going to study the IBVP case. The main purpose is to improve the previous convergence rates showed by Marcati-Mei [11] and Nishihara-Yang [18] to be optimal.

On the related IBVPs on $\mathbb{R}_+ \times \mathbb{R}_+$ to the damped p-system (1.1), we have two observations. Firstly, we discover that, in almost all cases studied by Marcati-Mei [11] and Nishihara-Yang [18] (but except for the case $u|_{x=0} = 0$ with $v_0(0) \neq v_+$, for the details, we refer to Section 4 below), each corresponding diffusion wave $\bar{v}(x,t)$ has a constant asymptotic state, say v_+, namely, $|\bar{v}(x,t) - v_+| \to 0$ as $t \to \infty$ for all $x \in \mathbb{R}_+$. This means that the original solution $v(x,t)$ can be expected to converge to the constant v_+ time-asymptotically, too. Thus, we can have a linearization around the constant state v_+ for our working equations, and we can get a formal expression of solution (c.f. Appendix below). Such a fact makes that our process in this paper is much simpler than that in [17]. Secondly, we observe that, in Nishihara-Yang's

Dirichlet boundary case, they used the linear diffusion wave as the asymptotic profile of (1.1), which is an approximating profile and causes that the difference between the solution of (1.1) and such a linear wave is a bit "big" in some sense, namely, the convergence rates are slower. To overcome this problem, we are going to find a much closer asymptotic profile of (1.1) which is a nonlinear diffusion wave, and then prove that the convergence rate to such a nonlinear wave is optimal. A similar consideration is also applied for Nishihara-Yang's Nuemann boundary case and Marcati-Mei's Dirichlet boundary case.

The method we adopt in this paper is the Fourier transform together with the energy method. We first reduce the fundamental solutions for the IBVPs to the corresponding linear equation, and show the energy decay rates for the fundamental solutions by the Fourier transform method. Based on it, we can see what are the optimal rates we expect for the nonlinear problems. This will be given in Appendix, the last part of the present paper. Then we apply the basic energy estimates of the linear IBVP to the nonlinear cases, and improve the previous convergence rates in [11, 17] to be optimal, which is the main part of this paper and will be carried out in Sections 2, 3 and 4.

Theoretically speaking, the approach used in [17](see also the original idea in [10]) for the Cauchy problem case should work on the IBVP case, too, but the construction of a suitable Green function for the IBVP and the estimates on such a Green function become quite complicated and difficult, because so does the Cauchy problem case in [17]. However, as we know, this method is one of the most effective methods used for proof of stability of nonliner elementary waves such as diffusion wave, viscous shock wave and rarefaction wave, etc. We note that, in the case $u|_{x=0} = 0$ with $v_0(0) \neq v_+$ studied in [18], since the corresponding diffusion wave $\bar{v}(x,t)$ does not always converge to the constant state v_+ when time t goes to infinity, our method is invalid to get much more optimal decay rates like the other cases, for example, see Corollaries 2.1–4.1 below. Herein, we get only L^2-optimal rate as $\|(v-\bar{v})(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-3/4}$. The shortcoming is due to that, our elementary solution, or say, Green function for the damped wave equation with constant coefficient $-p'(v_+)$ is not yet the real Green function for the equation with variable coefficient $-p'(\bar{v}(x,t))$, for the detail, see Section 4 below. In order to get more optimal rates, we believe that the approximating Green function method together with the method of pointwise estimate adopted in [17] will be available for the problem, but we will not try in the present paper.

For the other model equations dealing with the stability theory of diffusion waves, we refer to [1, 2, 3, 6, 7, 9, 10, 16] and the references therein.

2 Dirichlet Boundary Case (I)

In this section, we consider again the IBVP case studied by Nishihara-Yang [18]. The main goal is to improve the previous stability of diffusion wave and to show the optimal convergence rates. We consider the following IBVP to the damped p-system

$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = -\alpha u, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ (v,u)(x,0) = (v_0,u_0)(x) \to (v_+,u_+) \ \text{ as } x \to \infty \\ u(0,t) = 0. \end{cases} \tag{2.1}$$

As showed in [18], we make a heuristic analysis of the asymptotic state to the IBVP (2.1). Since $u(0,t)=0$ for all $t\geq 0$, it implies that 0 is the asymptotic state of $u(x,t)$ as $t\to\infty$. Furthermore, the second equation of (2.1) gives $v_x(0,t)=0$ and $v(+\infty,t)=v_+$, so we expect that the asymptotic profile of $v(x,t)$ is the solution $\bar v(x,t)$ of

$$\bar v_t-\kappa\bar v_{xx}=0,\quad \bar v_x(0,t)=0,\quad \bar v(+\infty,t)=v_+, \tag{2.2}$$

which solves explicitly

$$\bar v(x,t)=v_++\frac{\delta_0}{\sqrt{4\kappa\pi(t+1)}}\exp\Big(-\frac{x^2}{4\kappa(t+1)}\Big), \tag{2.3}$$

where

$$\kappa=-p'(v_+)/\alpha>0,\quad \delta_0=2\Big(\int_0^\infty(v_0(x)-v_+)dx-\frac{u_+}{\alpha}\Big). \tag{2.4}$$

Defining $\bar u(x,t):=\kappa\bar v_x(x,t)$, we get the so-called linear diffusion wave $(\bar v,\bar u)(x,t)$ which satisfies

$$\begin{cases}\bar v_t-\bar u_x=0\\ p'(v_+)\bar v_x=-\alpha\bar u\\ \bar u(0,t)=0,\quad (\bar v,\bar u)(\infty,t)=(v_+,0).\end{cases} \tag{2.5}$$

We then define the auxiliary function $(\hat v,\hat u)(x,t)$ by

$$(\hat v,\hat u)(x,t)=\Big(\frac{u_+m_0(x)}{-\alpha}e^{-\alpha t},u_+\int_0^x m_0(y)dye^{-\alpha t}\Big), \tag{2.6}$$

where m_0 is a smooth function with compact support such that $\int_0^\infty m_0(y)dy=1$ and $m_0(x)\in C_0^\infty(\mathbb{R}_+)$. Thus, $(\hat v,\hat u)(x,t)$ satisfies

$$\begin{cases}\hat v_t-\hat u_x=0\\ \hat u_t=-\alpha\hat u\\ \hat u(0,t)=0,\quad (\hat v,\hat u)(\infty,t)=(0,u_+e^{-\alpha t}).\end{cases} \tag{2.7}$$

From the first equations in (2.1), (2.5) and (2.7) respectively, and due to (2.4), it can be verified that

$$\int_0^\infty(v-\bar v-\hat v)(x,t)dx=\int_0^\infty(v_0(x)-v_+)dx-\frac{\delta_0}{2}-\frac{u_+}{\alpha}=0. \tag{2.8}$$

Thus, it is reasonable to introduce the following perturbations as our new variables

$$V(x,t):=-\int_x^\infty(v-\bar v-\hat v)(y,t)dy \tag{2.9}$$

$$z(x,t):=u(x,t)-\bar u(x,t)-\hat u(x,t). \tag{2.10}$$

We finally get a new IBVP on $(V,z)(x,t)$ as follows

$$\begin{cases}V_t-z=0\\ z_t+(p'(\bar v)V_x)_x+\alpha z=-F,\quad (x,t)\in\mathbb{R}_+\times\mathbb{R}_+\\ (V,z)|_{t=0}=(V_0,z_0)(x)\\ V|_{x=0}=0,\end{cases} \tag{2.11}$$

where

$$
\begin{aligned}
F :=& \frac{p'(v_+)}{\alpha}\bar{v}_{xt} + (p'(\bar{v}) - p'(v_+))\bar{v}_x + (p(V_x + \bar{v} + \hat{v}) - p(\bar{v}) - p'(\bar{v})V_x)_x, && (2.12)\\
V_0(x) :=& -\int_x^{\infty} [v_0(y) - \bar{v}(y,0) - \hat{v}(y,0)]dy, && (2.13)\\
z_0(x) :=& u_0(x) - \bar{u}(x,0) - \hat{u}(x,0). && (2.14)
\end{aligned}
$$

The unique global existence of solutions to the IBVP (2.11) and the convergence with some decay rates for $(V,z)(x,t)$ (namely, $(v,u)(x,t) \to (\bar{v},\bar{u})(x,t)$ as $t \to \infty$) were studied by Nishihara and Yang in [18] as follows.

Theorem 2.1 (Nishihara-Yang [18]) *Suppose that* $v_0 - v_+ \in L^1(\mathbb{R}_+)$, $(V_0, z_0)(x) \in H^3(\mathbb{R}_+) \times H^2(\mathbb{R}_+)$ *and that* $\|v_0 - v_+\|_{L^1(\mathbb{R}_+)} + \|V_0\|_{H^3(\mathbb{R}_+)} + \|z_0\|_{H^2(\mathbb{R}_+)} + |u_+| \ll 1$. *Then there exists a unique time-global solution* $(V,z)(x,t)$ *of the IBVP* (2.11)*, which satisfies*

$$
V \in C^k(0,\infty; H^{3-k}(\mathbb{R}_+)), \quad k = 0,1,2,3, \qquad z \in C^k(0,\infty; H^{2-k}(\mathbb{R}_+)), \quad k = 0,1,2
$$

and

$$
\begin{aligned}
&\|\partial_x^k V(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-k/2}, \quad k = 0,1,2,3, && (2.15)\\
&\|\partial_x^k z(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(k+2)/2}, \quad k = 0,1,2, && (2.16)
\end{aligned}
$$

and

$$
(1+t)^2\|z_t(t)\|_{L^2(\mathbb{R}_+)} + (1+t)^{-5/2}(\|z_{xt}(t)\|_{L^2(\mathbb{R}_+)} + \|z_{tt}(t)\|_{L^2(\mathbb{R}_+)}) = O(1). \qquad (2.17)
$$

These rates obtained by Nishihara-Yang [18] are less sufficient. We want to improve them to be optimal. However, caused by $(p'(\bar{v}) - p'(v_+))\bar{v}_x$, the nonlinear term F in $L^1(\mathbb{R}_+)$ does not decay so fast as we expect. In fact, $\bar{v}(x,t)$ is not a good asymptotic profile of $v(x,t)$ but just an approximating profile as Nishihara-Yang [18] stated, and $\|(p'(\bar{v}) - p'(v_+))\bar{v}_x(t)\|_{L^1(\mathbb{R}_+)} = O(1)(1+t)^{-1}$. We are going to look for a better asymptotic profile for $v(x,t)$. Due to Darcy's law, we believe that the good profile to $v(x,t)$ is the following nonlinear parabolic solution $\bar{\bar{v}}(x,t)$

$$
\alpha\bar{\bar{v}}_t + p(\bar{\bar{v}})_{xx} = 0, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \qquad (2.18)
$$

with the boundary restrictions

$$
\bar{\bar{v}}_x|_{x=0} = 0, \quad \bar{\bar{v}}|_{x=+\infty} = v_+. \qquad (2.19)
$$

To construct such a solution, let us consider a self-similar solution $\phi(x/\sqrt{t+1})$ satisfying

$$
\alpha\phi_t + p(\phi)_{xx} = 0, \quad x \in \mathbb{R} \quad \text{and} \quad \phi|_{x=\pm\infty} = 0. \qquad (2.20)
$$

By a quite similar analysis as in [4, 5], we know that such a self-similar solution $\phi(x/\sqrt{t+1})$ exists and satisfies (2.20), and further

$$
\begin{aligned}
&\phi_x|_{x=0} = 0, && (2.21)\\
&\phi|_{t=0} = \phi(x) \in L^1, && (2.22)\\
&\|\partial_t^j \partial_x^k \phi(t)\|_{L^2} = O(1)(1+t)^{-(4j+2k+1)/4}, && (2.23)\\
&\|\phi_{xt}(t)\|_{L^1} = O(1)(1+t)^{-3/2}. && (2.24)
\end{aligned}
$$

Now we construct our diffusion wave as

$$\bar{\bar{v}}(x,t) := v_+ + \bar{\delta}_0\phi(x/\sqrt{t+1}) \quad \text{for all} \quad x \ge 0, \tag{2.25}$$

where $\bar{\delta}_0$ is a constant such that

$$\int_0^\infty [v_0(x) - v_+]dx - \bar{\delta}_0\int_0^\infty \phi(x)dx - \frac{u_+}{\alpha} = 0, \tag{2.26}$$

namely,

$$\bar{\delta}_0 := \Big(\int_0^\infty [v_0(x) - v_+]dx - \frac{u_+}{\alpha}\Big)\Big/\int_0^\infty \phi(x)dx. \tag{2.27}$$

Thus, from (2.23), (2.24) and (2.25), we get

$$\|\partial_t^j\partial_x^k(\bar{\bar{v}} - v_+)(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(4j+2k+1)/4}, \tag{2.28}$$
$$\|\bar{\bar{v}}_{xt}(t)\|_{L^1(\mathbb{R}_+)} = O(1)(1+t)^{-3/2}. \tag{2.29}$$

In particular, if

$$\int_0^\infty [v_0(x) - v_+]dx - \frac{u_+}{\alpha} = 0, \tag{2.30}$$

(2.27) implies $\bar{\delta}_0 = 0$, that is, the nonlinear diffusion wave $\bar{\bar{v}}(x,t)$ becomes the state constant v_+ ($\bar{\bar{v}}(x,t) \equiv v_+$).

Now defining

$$\bar{\bar{u}}(x,t) = -\frac{1}{\alpha}p(\bar{\bar{v}})_x, \tag{2.31}$$

we obtain the new asymptotic profile $(\bar{\bar{v}},\bar{\bar{u}})(x,t)$ to $(v,u)(x,t)$, so-called nonlinear diffusion wave, satisfying

$$\begin{cases} \bar{\bar{v}}_t - \bar{\bar{u}}_x = 0 \\ p(\bar{\bar{v}})_x = -\alpha\bar{\bar{u}} \\ \bar{\bar{u}}(0,t) = 0, \quad (\bar{\bar{v}},\bar{\bar{u}})(\infty,t) = (v_+,0). \end{cases} \tag{2.32}$$

Noting (2.26), from the first equations of (2.1), (2.7) and (2.32), we have

$$\int_0^\infty [v - \bar{\bar{v}} - \hat{v}](x,t)dx = \int_0^\infty [v_0(x) - v_+]dx - \bar{\delta}_0\int_0^\infty \phi(x)dx - \frac{u_+}{\alpha} = 0. \tag{2.33}$$

Thus, it is reasonable to let us make a perturbation to $(\bar{\bar{v}},\bar{\bar{u}})(x,t)$ as

$$\bar{\bar{V}}(x,t) := -\int_x^\infty [v(y,t) - \bar{\bar{v}}(y,t) - \hat{v}(y,t)]dy \tag{2.34}$$
$$\bar{\bar{z}}(x,t) := u(x,t) - \bar{\bar{u}}(x,t) - \hat{u}(x,t), \tag{2.35}$$

from (2.1), (2.7) and (2.32), we obtain another IBVP on $(\bar{\bar{V}}, \bar{\bar{z}})(x,t)$ as follows

$$\begin{cases} \bar{\bar{V}}_t - \bar{\bar{z}} = 0 \\ \bar{\bar{z}}_t + (p'(\bar{v})\bar{\bar{V}}_x)_x + \alpha\bar{\bar{z}} = -\bar{\bar{F}}, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ (\bar{\bar{V}}, \bar{\bar{z}})|_{t=0} = (\bar{\bar{V}}_0, \bar{\bar{z}}_0)(x) \\ \bar{\bar{V}}|_{x=0} = 0, \end{cases} \tag{2.36}$$

where

$$\bar{\bar{F}} = \frac{1}{\alpha}p(\bar{v})_{xt} + (p(\bar{\bar{V}}_x + \bar{v} + \hat{v}) - p(\bar{v}) - p'(\bar{v})\bar{\bar{V}}_x)_x. \tag{2.37}$$

In a quite similar way to [18], we can firstly prove the following stability of nonlinear diffusion waves. The proof will be omitted here.

Theorem 2.2 (The case $\bar{\delta}_0 \neq 0$) *Suppose that $v_0 - v_+ \in L^1(\mathbb{R}_+)$, $(\bar{\bar{V}}_0, \bar{\bar{z}}_0)(x) \in H^3(\mathbb{R}_+) \times H^2(\mathbb{R}_+)$ and that $\|v_0 - v_+\|_{L^1(\mathbb{R}_+)} + \|\bar{\bar{V}}_0\|_{H^3(\mathbb{R}_+)} + \|\bar{\bar{z}}_0\|_{H^2(\mathbb{R}_+)} + |u_+| \ll 1$ hold. Then there exists a unique time-global solution $(\bar{\bar{V}}, \bar{\bar{z}})(x,t)$ of the IBVP (2.36), which satisfies*

$$\bar{\bar{V}} \in C^k(0,\infty; H^{3-k}(\mathbb{R}_+)), \quad k = 0,1,2,3, \qquad \bar{\bar{z}} \in C^k(0,\infty; H^{2-k}(\mathbb{R}_+)), \quad k = 0,1,2$$

and

$$\|\partial_x^k \bar{\bar{V}}(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-k/2}, \quad k = 0,1,2,3, \tag{2.38}$$

$$\|\partial_x^k \bar{\bar{z}}(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(k+2)/2}, \quad k = 0,1,2, \tag{2.39}$$

and

$$(1+t)^2\|\bar{\bar{z}}_t(t)\|_{L^2(\mathbb{R}_+)} + (1+t)^{-5/2}(\|\bar{\bar{z}}_{xt}(t)\|_{L^2(\mathbb{R}_+)} + \|\bar{\bar{z}}_{tt}(t)\|_{L^2(\mathbb{R}_+)}) = O(1). \tag{2.40}$$

Furthermore, we are going to improve the decay rates in Theorem 2.2 to be optimal as follows.

Theorem 2.3 (The case $\bar{\delta}_0 \neq 0$) *Under the assumptions in Theorem 2.1 and $(\bar{\bar{V}}_0, \bar{\bar{z}}_0)(x) \in L^1(\mathbb{R}_+)$. Then the solution $(\bar{\bar{V}}, \bar{\bar{z}})(x,t)$ decays time-asymptotically as*

$$\|\partial_x^k \bar{\bar{V}}(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(2k+1)/4}, \qquad k = 0,1,2, \tag{2.41}$$

$$\|\bar{\bar{z}}(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-5/4}. \tag{2.42}$$

Thanks to Theorem 2.3, and using the Sobelev's inequalities $\|f\|_{L^\infty} \leq \sqrt{2}\|f\|_{L^2}^{1/2}\|f_x\|_{L^2}^{1/2}$ and $\|f\|_{L^p} \leq \|f\|_{L^\infty}^{(p-2)/p}\|f\|_{L^2}^{2/p}$ for $2 \leq p \leq +\infty$, since $\hat{v}(x,t)$ decays like $e^{-\alpha t}$, we obtain the following asymptotic convergence. The convergence rates are sharp.

Corollary 2.1 (The case $\bar{\delta}_0 \neq 0$) *Under the assumptions in Theorem 2.3, the followings hold*

$$\|\partial_x^k(v-\bar{\bar{v}})(\cdot,t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(2k+3)/4}, \quad k=0,1, \tag{2.43}$$

$$\|(v-\bar{\bar{v}})(\cdot,t)\|_{L^\infty(\mathbb{R}_+)} = O(1)(1+t)^{-1}, \tag{2.44}$$

$$\|(v-\bar{\bar{v}})(\cdot,t)\|_{L^p(\mathbb{R}_+)} = O(1)(1+t)^{-(2p-1)/(2p)}, \quad 2 \le p \le +\infty. \tag{2.45}$$

Furthermore, in the special case of (2.30), which implies $\bar{\delta}_0 = 0$, we may obtain $(\bar{\bar{v}},\bar{\bar{u}})(x,t) \equiv (v_+,0)$. This case is much simpler than the above case, and we can obtain much more optimal decay rates. Now, let

$$\tilde{V}(x,t) := -\int_x^\infty [v(y,t)-v_+-\hat{v}(y,t)]dy, \quad \tilde{z}(x,t) := u(x,t)-\hat{u}(x,t),$$

we have the following working equations

$$\begin{cases} \tilde{V}_t - \tilde{z} = 0 \\ \tilde{z}_t + p'(v_+)\tilde{V}_{xx} + \alpha\tilde{z} = -\tilde{F}, \quad (x,t)\in\mathbb{R}_+\times\mathbb{R}_+ \\ (\tilde{V},\tilde{z})|_{t=0} = \Big(-\int_x^\infty [v_0(y)-v_+-\hat{v}(y,0)]dy, u_0(x)-\hat{u}(x,0)\Big) =: (\tilde{V}_0,\tilde{z}_0)(x) \\ \tilde{V}|_{x=0} = 0, \end{cases} \tag{2.46}$$

where

$$\tilde{F} = (p(\tilde{V}_x + v_+ + \hat{v}) - p(v_+) - p'(v_+)\tilde{V}_x)_x. \tag{2.47}$$

The main results are stated as follows.

Theorem 2.4 (The case $\bar{\delta}_0 = 0$) *Suppose that $\tilde{V}_0 \in L^1(\mathbb{R}_+)\cap H^3(\mathbb{R}_+)$, $\tilde{z}_0 \in L^1(\mathbb{R}_+)\cap H^2(\mathbb{R}_+)$ and (2.30) hold, and denote by $\delta = |v_+-v_-|+|u_+-u_0(0)|$. Then there exists a constant $\varepsilon_3 > 0$ such that if $\|\tilde{V}_0\|_{H^3} + \|\tilde{z}_0\|_{H^2} + \delta \le \varepsilon_3$, then the IBVP (2.46) has a unique globally defined solution $(\tilde{V},\tilde{z})(x,t)$ satisfying*

$$\tilde{V} \in \cap_{k=0}^3 W^{k,\infty}([0,\infty); H^{3-k}(\mathbb{R}_+)), \quad \tilde{z} \in \cap_{k=0}^2 W^{k,\infty}([0,\infty); H^{2-k}(\mathbb{R}_+))$$

and

$$\|\partial_x^k \tilde{V}(\cdot,t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(2k+1)/4}, \quad k=0,1,2,3, \tag{2.48}$$

$$\|\partial_x^k \tilde{V}_t(\cdot,t)\|_{L^2(\mathbb{R}_+)} = \|\partial_x^k \tilde{z}(\cdot,t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(2k+5)/4}, \quad k=0,1. \tag{2.49}$$

From Theorem 2.4, note the inequality $\|f\|_{L^\infty} \le \sqrt{2}\|f\|_{L^2}^{1/2}\|f_x\|_{L^2}^{1/2}$ and that $(\hat{v},\hat{u})(x,t)$ decays fast enough likes $e^{-\alpha t}$, we can get the following optimal convergence rates of the solution $(v,u)(x,t)$ for the IBVP (2.1) toward the constant state $(v_+,0)$.

Corollary 2.2 (The case $\bar{\delta}_0 = 0$) *Under the assumptions in Theorem 2.4, the followings hold*

$$\|\partial_x^k (v - v_+)(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(2k+3)/4}, \quad k = 0,1,2, \tag{2.50}$$

$$\|\partial_x^k (v - v_+)(t)\|_{L^\infty(\mathbb{R}_+)} = O(1)(1+t)^{-(k+2)/2}, \quad k = 0,1, \tag{2.51}$$

$$\|(v - v_+)_t(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-7/4}, \tag{2.52}$$

$$\|\partial_x^k (v - v_+)(\cdot,t)\|_{L^p(\mathbb{R}_+)} = O(1)(1+t)^{-((k+2)p-1)/(2p)}, \quad k = 0,1, \ \text{and } 2 \le p \le +\infty. \tag{2.53}$$

Now we are going to prove Theorem 2.3 and Theorem 2.4.

Proof of Theorem 2.3. Firstly, we are going to prove the optimal decay rates for $\|\partial_x^k \bar{\bar{V}}(t)\|_{L^2}$ $(k = 0,1,2)$, namely, (2.41) for $k = 0,1,2$.

For system (2.36), by substituting the first equation $\bar{\bar{z}} = \bar{\bar{V}}_t$ into the second one, and linearizing the equation at asymptotic state v_+, we obtain

$$\begin{cases} \bar{\bar{V}}_{tt} + \alpha \bar{\bar{V}}_t - \beta \bar{\bar{V}}_{xx} = G, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ (\bar{\bar{V}}, \bar{\bar{V}}_t)(x,0) = (\bar{\bar{V}}_0, \bar{\bar{V}}_1)(x)), \quad x \in \mathbb{R}_+ \\ \bar{\bar{V}}(0,t) = 0, \end{cases} \tag{2.54}$$

where $\beta = -p'(v_+)$, $\bar{\bar{V}}_1(x) = \bar{\bar{z}}_0$ and

$$G = -\bar{\bar{F}} - ((p'(\bar{v}) - p'(v_+))\bar{\bar{V}}_x)_x. \tag{2.55}$$

According to the theory for linear IBVP obtained in Appendix, see (A.17) below, we obtain an equivalent integral equation of the IBVP (2.54) as follows

$$\begin{aligned} \bar{\bar{V}}(x,t) = & \int_0^\infty [K_0(x-y,t) - K_0(x+y,t)]\bar{\bar{V}}_0(y)dy \\ & + \int_0^\infty [K_1(x-y,t) - K_1(x+y,t)]\bar{\bar{V}}_1(y)dy \\ & + \int_0^t \int_0^\infty [K_1(x-y,t-\tau) - K_1(x+y,t-\tau)]G(y,\tau)dyd\tau, \end{aligned} \tag{2.56}$$

where $K_0(x,t)$ and $K_1(x,t)$ represented in (A.7) and (A.9) are the fundamental solutions of the linear damped wave equation $\phi_{tt} + \alpha\phi_t - \beta\phi_{xx} = 0$. For the details, we refer to Appendix.

Differentiating Eq. (2.56) k-times $(k = 0,1,2)$ with respect to x, and taking its $L^2(\mathbb{R}_+)$-norm, we obtain

$$\begin{aligned} & \|\partial_x^k \bar{\bar{V}}(t)\|_{L^2(\mathbb{R}_+)} \\ \le & \left\| \partial_x^k \int_0^\infty [K_0(x-y,t) - K_0(x+y,t)]\bar{\bar{V}}_0(y)dy \right\|_{L^2(\mathbb{R}_+)} \\ + & \left\| \partial_x^k \int_0^\infty [K_1(x-y,t) - K_1(x+y,t)]\bar{\bar{V}}_1(y)dy \right\|_{L^2(\mathbb{R}_+)} \\ + & \int_0^t \left\| \partial_x^k \int_0^\infty [K_1(x-y,t-\tau) - K_1(x+y,t-\tau)]G(y,\tau)dy \right\|_{L^2(\mathbb{R}_+)} d\tau. \end{aligned} \tag{2.57}$$

Since $\bar{\bar{V}}_0 \in L^1(\mathbb{R}_+) \cap H^3(\mathbb{R}_+)$ and $\bar{\bar{V}}_1 \in L^1(\mathbb{R}_+) \cap H^2(\mathbb{R}_+)$, due to Lemma A.2 we have

$$\begin{aligned}
&\left\|\partial_x^k \int_0^\infty [K_0(x-y,t) - K_0(x+y,t)]\bar{\bar{V}}_0(y)dy\right\|_{L^2(\mathbb{R}_+)} \\
&\leq C[\|\bar{\bar{V}}_0\|_{L^1(\mathbb{R}_+)} + \|\bar{\bar{V}}_0\|_{H^3(\mathbb{R}_+)}](1+t)^{-(2k+1)/4},
\end{aligned} \tag{2.58}$$

$$\begin{aligned}
&\left\|\partial_x^k \int_0^\infty [K_1(x-y,t) - K_1(x+y,t)]\bar{\bar{V}}_1(y)dy\right\|_{L^2(\mathbb{R}_+)} \\
&\leq C[\|\bar{\bar{V}}_1\|_{L^1(\mathbb{R}_+)} + \|\bar{\bar{V}}_1\|_{H^2(\mathbb{R}_+)}](1+t)^{-(2k+1)/4}
\end{aligned} \tag{2.59}$$

for $k = 0, 1, 2, 3$.

We are now going to estimate the last term in (2.57). We note that from (2.55) and (2.37),

$$G \sim O(1)\{\bar{\bar{v}}_x \bar{\bar{v}}_t + \bar{\bar{v}}_{xt} + ((\bar{\bar{v}} - v_+)\bar{\bar{V}}_x)_x + \hat{v} + (\bar{\bar{V}}_x^2)_x\}, \tag{2.60}$$

$$\partial_x^k G \sim O(1)\{\partial_x^k \bar{\bar{v}}_x \partial_x^k \bar{\bar{v}}_t + \partial_x^k \bar{\bar{v}}_{xt} + \partial_x^k((\bar{\bar{v}} - v_+)\bar{\bar{V}}_x)_x + \partial_x^k \hat{v} + \partial_x^k(\bar{\bar{V}}_x^2)_x\}, \tag{2.61}$$

and from (2.6), (2.28), (2.29) and (2.38), the L^1-norm for G can be estimated as follows

$$\begin{aligned}
&\|G(t)\|_{L^1(\mathbb{R}_+)} \\
&\leq C\{\|\bar{\bar{v}}_t(t)\|_{L^2(\mathbb{R}_+)}\|\bar{\bar{v}}_x(t)\|_{L^2(\mathbb{R}_+)} + \|\bar{\bar{v}}_{xt}(t)\|_{L^1(\mathbb{R}_+)} \\
&\quad + \|(\bar{\bar{v}} - v_+)(t)\|_{L^2(\mathbb{R}_+)}\|\bar{\bar{V}}_{xx}(t)\|_{L^2(\mathbb{R}_+)} + \|(\bar{\bar{v}} - v_+)_x(t)\|_{L^2(\mathbb{R}_+)}\|\bar{\bar{V}}_x(t)\|_{L^2(\mathbb{R}_+)} \\
&\quad + \|\hat{v}(t)\|_{L^1(\mathbb{R}_+)} + \|\bar{\bar{V}}_x(t)\|_{L^2(\mathbb{R}_+)}\|\bar{\bar{V}}_{xx}(t)\|_{L^2(\mathbb{R}_+)}\} \\
&\leq C\{(1+t)^{-1-\frac{1}{4}-\frac{3}{4}} + (1+t)^{-\frac{3}{2}} + (1+t)^{-\frac{1}{4}-\frac{2}{2}} \qquad (2.62)\\
&\quad + (1+t)^{-\frac{3}{4}-\frac{1}{2}} + e^{-\alpha t} + (1+t)^{-\frac{1}{2}-\frac{2}{2}}\} \\
&\leq C(1+t)^{-5/4}. \qquad (2.63)
\end{aligned}$$

Similarly, we can prove also

$$\|G(t)\|_{H^k(\mathbb{R}_+)} \leq C(1+t)^{-3/2}. \tag{2.64}$$

Thus, noting (2.63), (2.64) and $3/2 > 5/4 \geq (2k+1)/4$ for $k = 0, 1, 2$, applying Lemmas A.2 and A.3 in Appendix below, we obtain optimal rates for the last term of (2.57) as follows

$$\begin{aligned}
&\int_0^t \left\|\partial_x^k \int_0^\infty [K_1(x-y,t-\tau) - K_1(x+y,t-\tau)]G(y,\tau)dy\right\|_{L^2(\mathbb{R}_+)} d\tau \\
&\leq C\int_0^t (1+t-\tau)^{-(2k+1)/4}[\|G(\tau)\|_{L^1(\mathbb{R}_+)} + \|G(\tau)\|_{H^{k-1}(\mathbb{R}_+)}]d\tau \\
&\leq C\int_0^t (1+t-\tau)^{-(2k+1)/4}[(1+\tau)^{-5/4} + (1+\tau)^{-3/2}]d\tau \\
&\leq C(1+t)^{-(2k+1)/4}, \qquad \text{for } \ k = 0, 1, 2.
\end{aligned} \tag{2.65}$$

Applying (2.58), (2.59) and (2.65) to (2.57), we prove (2.41) for $k = 0, 1, 2$.

Now, we are going to prove (2.42). We first have

$$\begin{aligned}
\bar{\bar{z}}(x,t) =&\bar{\bar{V}}_t(x,t)\\
=&\partial_t\int_0^\infty [K_0(x-y,t)-K_0(x+y,t)]\bar{\bar{V}}_0(y)dy\\
&+\partial_t\int_0^\infty [K_1(x-y,t)-K_1(x+y,t)]\bar{\bar{V}}_1(y)dy\\
&+\int_0^t\partial_t\int_0^\infty [K_1(x-y,t-\tau)-K_1(x+y,t-\tau)]G(y,\tau)dyd\tau\\
&+\int_0^\infty [K_1(x-y,0)-K_1(x+y,0)]G(y,t)dy
\end{aligned}\tag{2.66}$$

for $k=0,1$. Making use of a fashion as before, then Lemmas A.2 and A.3 help us to reach the goal

$$\begin{aligned}
&\|\bar{\bar{z}}(t)\|_{L^2(\mathbb{R}_+)} = \|\bar{\bar{V}}_t(t)\|_{L^2(\mathbb{R}_+)}\\
\le&\Big\|\ \partial_t\int_0^\infty [K_0(x-y,t)-K_0(x+y,t)]\bar{\bar{V}}_0(y)dy\Big\|_{L^2(\mathbb{R}_+)}\\
&+\Big\|\partial_t\int_0^\infty [K_1(x-y,t)-K_1(x+y,t)]\bar{\bar{V}}_1(y)dy\Big\|_{L^2(\mathbb{R}_+)}\\
&+\int_0^t\Big\|\ \partial_t\int_0^\infty [K_1(x-y,t-\tau)-K_1(x+y,t-\tau)]G(y,\tau)dy\Big\|_{L^2(\mathbb{R}_+)}d\tau\\
&+\Big\|\int_0^\infty [K_1(x-y,0)-K_1(x+y,0)]G(y,t)dy\Big\|_{L^2(\mathbb{R}_+)}\\
\le&C(\|\bar{\bar{V}}_0\|_{L^1(\mathbb{R}_+)}+\|\bar{\bar{V}}_0\|_{H^3(\mathbb{R}_+)}(1+t)^{-5/4}\\
&+C(\|\bar{\bar{V}}_1\|_{L^1(\mathbb{R}_+)}+\|\bar{\bar{V}}_1\|_{H^2(\mathbb{R}_+)}(1+t)^{-5/4}\\
&+C\int_0^t(1+t-\tau)^{-5/4}(\|G(\tau)\|_{L^1(\mathbb{R}_+)}+\|G(\tau)\|_{H^2(\mathbb{R}_+)})d\tau\\
&+C(\|G(t)\|_{L^1(\mathbb{R}_+)}+\|G(t)\|_{H^2(\mathbb{R}_+)})\\
\le&C(\|\bar{\bar{V}}_0\|_{L^1(\mathbb{R}_+)}+\|\bar{\bar{V}}_0\|_{H^3(\mathbb{R}_+)})(1+t)^{-5/4}\\
&+C(\|\bar{\bar{V}}_1\|_{L^1(\mathbb{R}_+)}+\|\bar{\bar{V}}_1\|_{H^2(\mathbb{R}_+)})(1+t)^{-5/4}\\
&+C\int_0^t(1+t-\tau)^{-5/4}((1+\tau)^{-5/4}+(1+\tau)^{-3/2})d\tau\\
&+C((1+t)^{-5/4}+(1+t)^{-3/2})\\
\le&C(1+t)^{-5/4},\quad \text{for}\ \ k=0,1.
\end{aligned}\tag{2.67}$$

This proved (2.42). □

Proof of Theorem 2.4. Firstly, plugging the first equation of (2.46) to the second one, we can have

$$\begin{cases}
\tilde{V}_{tt}+\alpha\tilde{V}_t-\beta\tilde{V}_{xx}=-\tilde{F}, & (x,t)\in\mathbb{R}_+\times\mathbb{R}_+\\
(\tilde{V},\tilde{V}_t)(x,0)=(\tilde{V}_0,\tilde{z}_0)(x)), & x\in\mathbb{R}_+\\
\tilde{V}(0,t)=0,
\end{cases}\tag{2.68}$$

where $\beta = -p'(v_+)$, which can be rewritten to the following integral form due to (A.17)

$$\begin{aligned}\tilde{V}(x,t) = &\int_0^\infty [K_0(x-y,t) - K_0(x+y,t)]\tilde{V}_0(y)dy \\ &+ \int_0^\infty [K_1(x-y,t) - K_1(x+y,t)]\tilde{z}_0(y)dy \\ &- \int_0^t \int_0^\infty [K_1(x-y,t-\tau) - K_1(x+y,t-\tau)]\tilde{F}(y,\tau)dyd\tau. \end{aligned} \tag{2.69}$$

Since $\tilde{F} \sim (\tilde{V}_x^2 + \hat{v})_x$, as done in the proof of Theorem 2.3, we can firstly calculate

$$\begin{aligned}\|\tilde{F}(t)\|_{L^1(\mathbb{R}_+)} &\le C\{\|\tilde{V}_x\tilde{V}_{xx}\|_{L^1(\mathbb{R}_+)} + \|\hat{v}_x\|_{L^1(\mathbb{R}_+)}\} \\ &\le C\{\|\tilde{V}_x\|_{L^2(\mathbb{R}_+)}\|\tilde{V}_{xx}\|_{L^2(\mathbb{R}_+)} + \|\hat{v}_x\|_{L^1(\mathbb{R}_+)}\} \\ &\le C\{(1+t)^{-\frac{1}{2}-\frac{2}{2}} + e^{-\alpha t}\} \le C(1+t)^{-\frac{3}{2}}, \end{aligned} \tag{2.70}$$

and similarly compute

$$\|\tilde{F}(t)\|_{L^2(\mathbb{R}_+)} \le C(1+t)^{-7/4}. \tag{2.71}$$

Then, by using Lemmas A.2 and A.3, in a same process in the proof to Theorem 2.3, we can prove

$$\|\partial_x^k \tilde{V}(t)\|_{L^2(\mathbb{R}_+)} \le C(1+t)^{-(2k+1)/4} \quad \text{for} \quad k = 0,1,2. \tag{2.72}$$

Now we are going to prove the other optimal rate for $k = 3$. Using the new rates

$$\|\tilde{V}_x(t)\|_{L^2(\mathbb{R}_+} \sim (1+t)^{-3/4}, \quad \|\tilde{V}_{xx}(t)\|_{L^2(\mathbb{R}_+} \sim (1+t)^{-5/4}$$

to (2.70) and (2.71) insteade of the old ones, we further improve the L^1 and L^2-rates of $\tilde{F}$ as

$$\|\tilde{F}(t)\|_{L^1(\mathbb{R}_+} \le C(1+t)^{-7/4}, \quad \|\tilde{F}(t)\|_{L^2(\mathbb{R}_+} \le C(1+t)^{-9/4}. \tag{2.73}$$

Thus, applying Lemmas A.2 and A.3, we obtain

$$\begin{aligned}&\int_0^t \Big\|\partial_x^3 \int_0^\infty [K_1(x-y,t-\tau) - K_1(x+y,t-\tau)]\tilde{F}(y,\tau)dy\Big\|_{L^2(\mathbb{R}_+)} d\tau \\ &\le C\int_x^t (1+t-\tau)^{-7/4}[\|\tilde{F}(\tau)\|_{L^1(\mathbb{R}_+)} + \|\tilde{F}(\tau)\|_{H^2(\mathbb{R}_+)}]d\tau \\ &\le C\int_0^t (1+t-\tau)^{-7/4}[(1+\tau)^{-7/4} + (1+\tau)^{-9/4}]d\tau \\ &\le C(1+t)^{-7/4}. \end{aligned} \tag{2.74}$$

Combining (2.58), (2.59) and (2.74), we prove (2.48) for $k = 3$.

Finally, we are going to prove (2.49). We first have

$$
\begin{aligned}
\partial_x^k \tilde{z}(x,t) =& \partial_x^k \tilde{V}_t(x,t) \\
=& \partial_x^k \partial_t \int_0^\infty [K_0(x-y,t) - K_0(x+y,t)]\tilde{V}_0(y)dy \\
&+ \partial_x^k \partial_t \int_0^\infty [K_1(x-y,t) - K_1(x+y,t)]\tilde{V}_1(y)dy \\
&+ \int_0^t \partial_x^k \partial_t \int_0^\infty [K_1(x-y,t-\tau) - K_1(x+y,t-\tau)]\tilde{F}(y,\tau)dyd\tau \\
&+ \partial_x^k \int_0^\infty [K_1(x-y,0) - K_1(x+y,0)]\tilde{F}(y,t)dy
\end{aligned}
\tag{2.75}
$$

for $k = 0,1$. Making use of a fashion as before, noting $7/4 \geq (2k+5)/4$ for $k = 0,1$, then Lemmas A.2 and A.3 yield

$$
\begin{aligned}
&\|\partial_x^k \tilde{z}(t)\|_{L^2(\mathbb{R}_+)} = \|\partial_x^k \tilde{V}_t(t)\|_{L^2(\mathbb{R}_+)} \\
&\leq \left\| \partial_x^k \partial_t \int_0^\infty [K_0(x-y,t) - K_0(x+y,t)]\tilde{V}_0(y)dy \right\|_{L^2(\mathbb{R}_+)} \\
&\quad + \left\| \partial_x^k \partial_t \int_0^\infty [K_1(x-y,t) - K_1(x+y,t)]\tilde{V}_1(y)dy \right\|_{L^2(\mathbb{R}_+)} \\
&\quad + \int_0^t \left\| \partial_x^k \partial_t \int_0^\infty [K_1(x-y,t-\tau) - K_1(x+y,t-\tau)]\tilde{F}(y,\tau)dy \right\|_{L^2(\mathbb{R}_+)} d\tau \\
&\quad + \left\| \partial_x^k \int_0^\infty [K_1(x-y,0) - K_1(x+y,0)]\tilde{F}(y,t)dy \right\|_{L^2(\mathbb{R}_+)} \\
&\leq C(\|\tilde{V}_0\|_{L^1(\mathbb{R}_+)} + \|\tilde{V}_0\|_{H^3(\mathbb{R}_+)}(1+t)^{-(2k+5)/4)} \\
&\quad + C(\|\tilde{V}_1\|_{L^1(\mathbb{R}_+)} + \|\tilde{V}_1\|_{H^2(\mathbb{R}_+)}(1+t)^{-(2k+5)/4} \\
&\quad + C\int_0^t (1+t-\tau)^{-(2k+5)/4}(\|\tilde{F}(\tau)\|_{L^1(\mathbb{R}_+)} + \|\tilde{F}(\tau)\|_{H^2(\mathbb{R}_+)})d\tau \\
&\quad + C(\|\tilde{F}(t)\|_{L^1(\mathbb{R}_+)} + \|\tilde{F}(t)\|_{H^2(\mathbb{R}_+)}) \\
&\leq C(\|\tilde{V}_0\|_{L^1(\mathbb{R}_+)} + \|\tilde{V}_0\|_{H^3(\mathbb{R}_+)})(1+t)^{-(2k+5)/4} \\
&\quad + C(\|\tilde{V}_1\|_{L^1(\mathbb{R}_+)} + \|\tilde{V}_1\|_{H^2(\mathbb{R}_+)})(1+t)^{-(2k+5)/4} \\
&\quad + C\int_0^t (1+t-\tau)^{-(2k+5)/4}((1+\tau)^{-7/4} + (1+\tau)^{-9/4})d\tau \\
&\quad + C((1+t)^{-7/4} + (1+t)^{-9/4}) \\
&\leq C(1+t)^{-(2k+5)/4}, \quad \text{for} \quad k = 0,1.
\end{aligned}
\tag{2.76}
$$

This proved (2.49). □

3 Dirichlet Boundary Case (II)

In this section, we investigate the IBVP case which was studied by Marcati-Mei [11] as follows

$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = -\alpha u, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ (v,u)(x,0) = (v_0,u_0)(x) \to (v_+,u_+) \ \text{ as } x \to \infty \\ v(0,t) = g(t) \to v_+ \ \text{ as } t \to \infty, \end{cases} \tag{3.1}$$

where

$$|g(t) - v_+| = O(1)|v_+ - v_+|(1+t)^{-\gamma_1}, \qquad \gamma_1 > 3/4 \tag{3.2}$$

and the compatibility condition $g(0) = v_0(0)$ holds.

As showed in [11], we make a heuristic analysis of the asymptotic state to the IBVP (3.1). Since the nonlinear diffusion wave in whole space $\bar{v}(x,t) = \phi(x/\sqrt{1+t})$ of (1.2) satisfies (see [5])

$$|v_+ - \bar{v}(x,t)| \le C|v_+ - v_-|e^{-c_0\alpha\xi^2}, \quad \xi = x/\sqrt{1+t}, \quad \text{for } x \ge 0$$

for some positive constants C and c_0, let us choose the shift function $d(t)$ in $C^3(\mathbb{R}_+)$ such that

$$d(t) > 0, \ \text{for all } \ t \ge 0, \tag{3.3}$$

$$\exp\Big\{ -\alpha c_0 \Big(\frac{d(t)}{\sqrt{t+1}}\Big)^2 \Big\} \le O(1)(2+t)^{-\gamma_2}, \qquad \gamma_2 > 3/4, \tag{3.4}$$

$$d'(t)\exp\Big\{ -\alpha c_0 \Big(\frac{d(t)}{\sqrt{t+1}}\Big)^2 \Big\} \le O(1)(1+t)^{-(\gamma_2+\frac{1}{2})}\sqrt{\log(2+t)}. \tag{3.5}$$

As stated in [11], the function $d(t)$ satisfying (3.3)–(3.5) includes two kinds of important examples which are $d(t) = \sqrt{1+t}\cdot\sqrt{c\log(2+t)}$ with $c \ge \frac{\gamma_2}{\alpha c_0}$ and $d(t) = (1+t)^{\frac{1}{2}+c}$ with any $c > 0$.

Because of the second equation of (3.1), we have

$$u(x,t) \to e^{-\alpha t}u_+, \quad \text{as } x \to +\infty \tag{3.6}$$

and the implicit relation

$$u|_{x=0} = e^{-\alpha t}u_0(0) - \int_0^t e^{-\alpha(t-\tau)}p'(g(\tau))v_x(0,\tau)d\tau. \tag{3.7}$$

Let us denote $(\bar{v},\bar{u}) = (\bar{v},\bar{u})(x+d(t),t)$: *the shifted nonlinear diffusion waves of* (1.2). By (1.3) we get

$$\frac{d}{dt}\bar{v}(x+d(t),t) = d'(t)\bar{v}_x(x+d(t),t) + \bar{u}_x(x+d(t),t) \tag{3.8}$$

and by $\bar{v}(\pm\infty,t) = v_\pm$, $v_+ \ne v_-$ and $\bar{u}(\pm\infty,t) = 0$ we have

$$(\bar{v},\bar{u}) \to (v_+,0), \quad \text{as } x \to +\infty. \tag{3.9}$$

Denote

$$\begin{cases} \hat{v}(x,t) := -\frac{u_+ - u_0(0)}{\alpha} e^{-\alpha t} m_0(x) \\ \hat{u}(x,t) := e^{-\alpha t} \Big[u_+ - (u_+ - u_0(0)) \int_x^\infty m_0(y)dy \Big], \end{cases} \tag{3.10}$$

where $m_0(x) \in C_0^\infty(\mathbb{R}_+)$ satisfies $m_0 \geq 0$ for all $x \in \mathbb{R}_+$, $m_0(0) = 0$ and $\int_0^{+\infty} m_0(x)dx = 1$. We can easily check that

$$\begin{cases} \hat{v}_t - \hat{u}_x = 0 \\ \hat{u}_t = -\alpha \hat{u} \end{cases} \tag{3.11}$$

and the following boundary and limit conditions hold

$$(\hat{v}, \hat{u})|_{x=+\infty} = (0, e^{-\alpha t} u_+), \tag{3.12}$$

$$(\hat{v}, \hat{u})|_{x=0} = (0, u_0(0) e^{-\alpha t}), \tag{3.13}$$

$$|\hat{v}| \leq \alpha^{-1} |u_+ - u_0(0)| e^{-\alpha t} m_0(x). \tag{3.14}$$

Let us define the new variables

$$\begin{cases} V(x,t) := -\int_x^{+\infty} [v(y,t) - \bar{v}(y + d(t), t) - \hat{v}(y,t)]dy \\ z(x,t) := u(x,t) - \bar{u}(x + d(t), t) - \hat{u}(x,t). \end{cases} \tag{3.15}$$

From the equations (3.1), (1.3), (3.8), (3.11), and (3.15), we can reformulate the original equations (3.1) to the new one

$$\begin{cases} V_t = z + d'(t)[v_+ - \bar{v}], \\ z_t + \alpha z + (p'(\bar{v}) V_x)_x = f_1, (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ (V, V_t, z)|_{t=0} = (V_0, V_1, z_0)(x), \\ V_x|_{x=0} = G(t), \end{cases} \tag{3.16}$$

where

$$f_1 := -\frac{d}{dt} \bar{u}(x + d(t), t) - \{p(V_x + \bar{v} + \hat{v}) - p(\bar{v}) - p'(\bar{v}) V_x\}_x, \tag{3.17}$$

$$V_0(x) := -\int_x^{+\infty} [v_0(y) - \bar{v}(y + d_0, 0) - \hat{v}(y, 0)]dy, \tag{3.18}$$

$$z_0(x) := u_0(x) - \bar{u}(x + d_0, 0) - \hat{u}(x, 0), \tag{3.19}$$

$$V_1(x) := z_0(x) + d'(0)[v_+ - \bar{v}(x + d_0, 0)], \tag{3.20}$$

$$G(t) := g(t) - \bar{v}(d(t), t) = [g(t) - v_+] + [v_+ - \bar{v}(d(t), t)]. \tag{3.21}$$

Plugging the first equation of (3.16) into the second one of (3.16), we have the following Neumann type IBVP

$$\begin{cases} V_{tt} + \alpha V_t + (p'(\bar{v}) V_x)_x = F, \quad x > 0, t > 0, \\ (V, V_t)|_{t=0} = (V_0, V_1)(x), \quad x > 0, \\ V_x|_{x=0} = G(t), \quad t > 0, \end{cases} \tag{3.22}$$

where $F := f_1 + f_2$ and

$$f_2 := d''(t)(v_+ - \bar{v}) - d'(t)\frac{d}{dt}\bar{v}(x + d(t), t) + \alpha d'(t)(v_+ - \bar{v}). \tag{3.23}$$

Marcati and Mei in [11] proved the global existence and uniqueness of the solutions for (3.22) as well as the decay rates. These are stated as follows.

Theorem 3.1 (Marcati-Mei [11]) *Suppose that $\gamma_1 > 3/4$, $\gamma_2 > 3/4$, $V_0 \in H^3(\mathbb{R}_+)$, $V_1 \in H^2(\mathbb{R}_+)$ and denote by $\delta = |v_+ - v_-| + |u_+ - u_0(0)|$. Then there exists a constant $\varepsilon_1 > 0$ such that if $\|V_0\|_{H^3} + \|V_1\|_{H^2} + \delta \le \varepsilon_1$, then the IBVP (3.22) has a unique globally defined solution $V(x,t)$ satisfying*

$$V \in \cap_{k=0}^{3} W^{k,\infty}([0,\infty); H^{3-k}(\mathbb{R}_+))$$

and

$$\|\partial_x^k V(\cdot,t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-k/2}, \quad k = 0,1,2,3, \tag{3.24}$$

$$\|\partial_x^k V_t(\cdot,t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(k+2)/2}, \quad k = 0,1,2. \tag{3.25}$$

We are going to improve above rates to be optimal. We divide two cases to discuss it. One is $g(t) \not\equiv v_+$, another is $g(t) \equiv v_+$.

Case 1: $g(t) \not\equiv v_+$. Defining a new variable

$$\bar{\bar{V}}(x,t) := V(x,t) + G(t)e^{-x},$$

from Eq. (3.22), and making the linearization around v_+, we obtain a new IBVP

$$\begin{cases} \bar{\bar{V}}_{tt} + \alpha\bar{\bar{V}}_t - \beta\bar{\bar{V}}_{xx} = \bar{\bar{F}}, \quad x > 0,\ t > 0, \\ (\bar{\bar{V}}, \bar{\bar{V}}_t)|_{t=0} = (\bar{\bar{V}}_0, \bar{\bar{V}}_1)(x), \quad x > 0, \\ \bar{\bar{V}}_x|_{x=0} = 0, \quad t > 0, \end{cases} \tag{3.26}$$

where $\beta := -p'(v_+)$, $\bar{\bar{V}}_0 := V_0 + G(0)e^{-x}$, $\bar{\bar{V}}_1 := V_1 + G'(0)e^{-x}$ and

$$\bar{\bar{F}} := F + G''(t)e^{-x} + \alpha G'(t)e^{-x} + G(t)(p'(\bar{v})e^{-x})_x - ((p'(\bar{v}) - p'(v_+))\bar{\bar{V}}_x)_x.$$

Then, according to the theory of linear IBVP showed in Appendix (see (A.18) below), we rewrite (3.26) to the following integral form

$$\begin{aligned} \bar{\bar{V}}(x,t) = &\int_0^\infty [K_0(x-y,t) + K_0(x+y,t)]\bar{\bar{V}}_0(y)dy \\ &+ \int_0^\infty [K_1(x-y,t) + K_1(x+y,t)]\bar{\bar{V}}_1(y)dy \\ &+ \int_0^t \int_0^\infty [K_1(x-y,t-\tau) + K_1(x+y,t-\tau)]\bar{\bar{F}}(y,\tau)dyd\tau, \end{aligned} \tag{3.27}$$

where $K_0(x,t)$ and $K_1(x,t)$ are the fundamental solutions of $\phi_{tt} + \alpha\phi_t - \beta\phi_{xx} = 0$.

In a similar way as in last section, and noting $\|\bar{\bar{F}}(t)\|_{L^1(\mathbb{R}_+)} \le C(1+t)^{-5/4}$ and $\|\bar{\bar{F}}(t)\|_{L^2(\mathbb{R}_+)} \le C(1+t)^{-3/2}$, we can obtain the optimal decay rates as follows. The detail of proof is omitted.

Theorem 3.2 (The case $g(t) \not\equiv v_+$) *Under the assumptions in Theorem 3.1, and suppose further $V_0(x)$ and $V_1(x)$ are in $L^1(\mathbb{R}_+)$. Then*

$$\|\partial_x^k \bar{\bar{V}}(\cdot,t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(2k+1)/4}, \quad k=0,1,2, \tag{3.28}$$

$$\|\bar{\bar{V}}_t(\cdot,t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-5/4}. \tag{3.29}$$

From Theorem 3.2, and note that $(\hat{v},\hat{u})(x,t)$ decays fast enough likes $e^{-\alpha t}$, we can get the following optimal convergence rates of the solutions $(v,u)(x,t)$ for the IBVP (3.1) toward the shifted nonlinear diffusion waves $(\bar{v},\bar{u})(x+d(t),t)$.

Corollary 3.1 (The case $g(t) \not\equiv v_+$) *Under the assumptions in Theorem 3.2, the followings hold*

$$\|\partial_x^k v(\cdot,t) - \partial_x^k \bar{v}(\cdot+d(t),t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(2k+3)/4}, \quad k=0,1, \tag{3.30}$$

$$\|v(\cdot,t) - \bar{v}(\cdot+d(t),t)\|_{L^\infty(\mathbb{R}_+)} = O(1)(1+t)^{-1}, \tag{3.31}$$

$$\|v(\cdot,t) - \bar{v}(\cdot+d(t),t)\|_{L^p(\mathbb{R}_+)} = O(1)(1+t)^{-(2p-1)/(2p)}, \quad 2 \le p \le +\infty. \tag{3.32}$$

Case 2: $g(t) \equiv v_+$. In this case, the nonlinear diffusion wave $(\bar{v},\bar{u})(x,t)$ becomes the constant state $(v_+,0)$. Thus, by defining

$$\tilde{V}(x,t) = -\int_x^\infty [v(y,t) - v_+ - \hat{v}(y,t)]dx,$$

then Eqs. (2.1) can be produced to

$$\begin{cases} \tilde{V}_{tt} + \alpha\tilde{V}_t - \beta\tilde{V}_{xx} = -(p(\tilde{V}_x + v_+ + \hat{v}) - p(v_+) - p'(v_+)\tilde{V}_x)_x =: \tilde{F}, \quad x>0,\ t>0, \\ (\tilde{V},\tilde{V}_t)|_{t=0} = \Big(-\int_x^\infty [v_0(y) - v_+ - \hat{v}(y,0)]dy, u_0(x) - \hat{u}(x,0)\Big) =: (\tilde{V}_0,\tilde{V}_1)(x), \quad x>0, \\ \tilde{V}_x|_{x=0} = 0, \quad t>0, \end{cases} \tag{3.33}$$

and which can be also rewritten as

$$\begin{aligned} \tilde{V}(x,t) = & \int_0^\infty [K_0(x-y,t) + K_0(x+y,t)]\tilde{V}_0(y)dy \\ & + \int_0^\infty [K_1(x-y,t) + K_1(x+y,t)]\tilde{V}_1(y)dy \\ & + \int_0^t \int_0^\infty [K_1(x-y,t-\tau) + K_1(x+y,t-\tau)]\tilde{F}(y,\tau)dyd\tau. \end{aligned} \tag{3.34}$$

Similarly to Theorem 2.4, we can obtain much more optimal decay rates as follows. The details of proof will be omitted too.

Theorem 3.3 (The case $g(t) \equiv v_+$) *Suppose that $\tilde{V}_0 \in L^1(\mathbb{R}_+) \cap H^3(\mathbb{R}_+)$, $\tilde{z}_0 \in L^1(\mathbb{R}_+) \cap H^2(\mathbb{R}_+)$ and denote by $\delta = |v_+ - v_-| + |u_+ - u_0(0)|$. Then there exists a constant $\varepsilon_3 > 0$ such that if $\|\tilde{V}_0\|_{H^3} + \|\tilde{z}_0\|_{H^2} + \delta \leq \varepsilon_3$, then the IBVP (3.33) has a unique globally defined solution $\tilde{V}(x,t)$ satisfying*

$$\tilde{V} \in \cap_{k=0}^{3} W^{k,\infty}([0,\infty); H^{3-k}(\mathbb{R}_+))$$

and

$$\|\partial_x^k \tilde{V}(\cdot,t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(2k+1)/4}, \quad k = 0,1,2,3, \tag{3.35}$$

$$\|\partial_x^k \tilde{V}_t(\cdot,t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(2k+5)/4}, \quad k = 0,1. \tag{3.36}$$

From Theorem 3.3, and note that $(\hat{v},\hat{u})(x,t)$ decays fast enough likes $e^{-\alpha t}$, we can get the following optimal convergence rates of the solution $(v,u)(x,t)$ for the IBVP (3.1) toward the constant state $(v_+,0)$.

Corollary 3.2 (The case $g(t) \equiv v_+$) *Under the assumptions in Theorem 3.3, the followings hold*

$$\|\partial_x^k (v - v_+)(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(2k+3)/4}, \quad k = 0,1,2, \tag{3.37}$$

$$\|\partial_x^k (v - v_+)(t)\|_{L^\infty(\mathbb{R}_+)} = O(1)(1+t)^{-(k+2)/2}, \quad k = 0,1, \tag{3.38}$$

$$\|(v - v_+)_t(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-7/4}, \tag{3.39}$$

$$\|\partial_x^k (v - v_+)(\cdot,t)\|_{L^p(\mathbb{R}_+)} = O(1)(1+t)^{-((k+2)p-1)/(2p)}, \quad k = 0,1, \ \textit{and}\ 2 \leq p \leq +\infty. \tag{3.40}$$

4 Nuemann Boundary Case

In this section, we study the following IBVP with the Neumann boundary which was analyzed firstly by Nishihara and Yang [18]

$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = -\alpha u, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ (v,u)(x,0) = (v_0,u_0)(x) \to (v_+,u_+) \ \text{ as } x \to \infty \\ u_x(0,t) = 0. \end{cases} \tag{4.1}$$

As showed in [18], heuristically, the first equation of (4.1) yields $(d/dt)v(0,t) = u_x(0,t) = 0$ and $v(0,t) = v_0(0)$ for all $t \geq 0$. Hence, we can expect that

$$(v,u)(x,t) \to (\bar{v},0)(x,t) \quad \text{as} \quad t \to \infty, \tag{4.2}$$

where $\bar{v}(x,t)$ is a diffusion wave connecting $v_0(0)$ and v_+. We are going to discuss the stability of diffusion wave in two cases: $v_0(0) = v_+$ and $v_0(0) \neq v_+$.

Case 1: $v_0(0) = v_+$. Since $v_0(0) = v_+$, it is clear that the constant state $(v_+, 0)$ is our expected asymptotic state for $(v, u)(x, t)$. Let

$$(\hat{v}, \hat{u})(x,t) = \Big(\frac{u_0(0) - u_+}{\alpha} m_0(x) e^{-\alpha t}, \Big[(u_0(0) - u_+) \int_x^\infty m_0(y)dy + u_+\Big] e^{-\alpha t}\Big) \tag{4.3}$$

and

$$(V, z)(x,t) = \Big(-\int_x^\infty (v - v_+ - \hat{v})(y,t)dy, u(x,t) - \hat{u}(x,t)\Big), \tag{4.4}$$

we get a new IBVP

$$\begin{cases} V_t - z = 0 \\ z_t + (p(V_x + v_+ + \hat{v}) - p(v_+))_x = -\alpha z \\ (V, z)|_{t=0} = (V_0, z_0)(x) \\ V_x|_{x=0} = 0, \end{cases} \tag{4.5}$$

where

$$(V_0, z_0)(x) := \Big(-\int_x^\infty (v_0(y) - v_+ - \hat{v}(y,0))dy, u_0(x) - \hat{u}(x,0)\Big).$$

For the IBVP (4.5), Nishihara and Yang [18] showed the following theorem.

Theorem 4.1 (Nishihara-Yang [18]: the case $v_0(0) = v_+$) *Suppose that $v_0 - v_+ \in L^1(\mathbb{R}_+)$, $(V_0, z_0)(x) \in H^3(\mathbb{R}_+) \times H^2(\mathbb{R}_+)$ and that $\|v_0 - v_+\|_{L^1(\mathbb{R}_+)} + \|V_0\|_{H^3(\mathbb{R}_+)} + \|z_0\|_{H^2(\mathbb{R}_+)} + |u_+ - u_0(0)| \ll 1$. Then there exists a unique time-global solution $(V, z)(x,t)$ of the IBVP (4.5), which satisfies*

$$V \in C^k(0,\infty; H^{3-k}(\mathbb{R}_+)), \quad k = 0,1,2,3, \qquad z \in C^k(0,\infty; H^{2-k}(\mathbb{R}_+)), \quad k = 0,1,2$$

and

$$\|\partial_x^k V(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-k/2}, \quad k = 0,1,2,3, \tag{4.6}$$

$$\|\partial_x^k z(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(k+2)/2}, \quad k = 0,1,2, \tag{4.7}$$

and

$$(1+t)^2\|z_t(t)\|_{L^2(\mathbb{R}_+)} + (1+t)^{-5/2}(\|z_{xt}(t)\|_{L^2(\mathbb{R}_+)} + \|z_{tt}(t)\|_{L^2(\mathbb{R}_+)}) = O(1). \tag{4.8}$$

However, the rates represented in Theorem 4.1 are less sufficient, as we know. We are going to improve these rates to be optimal.

Substituting the first equation of (4.5) into the second one, and making a linearization around v_+ for the resultant equation, we get

$$\begin{cases} V_{tt} + \alpha V_t - \beta V_{xx} = F \\ (V, V_t)|_{t=0} = (V_0, z_0)(x) \\ V_x|_{x=0} = 0, \end{cases} \tag{4.9}$$

where $\beta = -p'(v_+)$ and

$$F := -(p(V_x + \hat{v} + v_+) - p(v_+) - p'(v_+)V_x)_x.$$

Due to the linear IBVP theory in Appendix, we can rewrite (4.9) to an equivalent form of integral equation

$$\begin{aligned} V(x,t) = &\int_0^\infty [K_0(x-y,t) + K_0(x+y,t)]V_0(y)dy \\ &+ \int_0^\infty [K_1(x-y,t) + K_1(x+y,t)]V_1(y)dy \\ &+ \int_0^t \int_0^\infty [K_1(x-y,t-\tau) + K_1(x+y,t-\tau)]F(y,\tau)dyd\tau, \end{aligned} \tag{4.10}$$

where $K_0(x,t)$ and $K_1(x,t)$ are the fundamental solutions of $\phi_{tt} + \alpha\phi_t - \beta\phi_{xx} = 0$.

In a similar way as in Section 2, we can obtain the optimal decay rates as follows. Here, the detail of proof is omitted, too.

Theorem 4.2 (The case $v_0(0) = v_+$) *Suppose that $V_0 \in L^1(\mathbb{R}_+) \cap H^3(\mathbb{R}_+)$, $V_1 \in L^1(\mathbb{R}_+) \cap H^2(\mathbb{R}_+)$ and denote by $\delta = |v_+ - v_-| + |u_+ - u_0(0)|$. Then there exists a constant $\varepsilon_3 > 0$ such that if $\|V_0\|_{H^3} + \|V_1\|_{H^2} + \delta \le \varepsilon_3$, then the IBVP (4.10) has a unique globally defined solution $V(x,t)$ satisfying*

$$V \in \cap_{k=0}^3 W^{k,\infty}([0,\infty); H^{3-k}(\mathbb{R}_+))$$

and

$$\|\partial_x^k V(\cdot,t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(2k+1)/4}, \quad k = 0,1,2,3, \tag{4.11}$$

$$\|\partial_x^k V_t(\cdot,t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(2k+5)/4}, \quad k = 0,1. \tag{4.12}$$

$$\tag{4.13}$$

From Theorem 4.2, and note that $(\hat{v},\hat{u})(x,t)$ decays fast enough likes $e^{-\alpha t}$, we can get the following optimal convergence rates of the solution $(v,u)(x,t)$ for the IBVP (4.1) toward the constant state $(v_+,0)$.

Corollary 4.1 (The case $v_0(0) = v_+$) *Under the assumptions in Theorem 4.2, the followings hold*

$$\|\partial_x^k (v - v_+)(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(2k+3)/4}, \quad k = 0,1,2, \tag{4.14}$$

$$\|\partial_x^k (v - v_+)(t)\|_{L^\infty(\mathbb{R}_+)} = O(1)(1+t)^{-(k+2)/2}, \quad k = 0,1, \tag{4.15}$$

$$\|(v - v_+)_t(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-7/4}, \tag{4.16}$$

$$\|\partial_x^k (v - v_+)(\cdot,t)\|_{L^p(\mathbb{R}_+)} = O(1)(1+t)^{-((k+2)p-1)/(2p)}, \quad k = 0,1, \text{ and } 2 \le p \le +\infty. \tag{4.17}$$

Case 2: $v_0(0) \neq v_+$. As Nishihara and Yang [18] showed, in this case the asymptotic state of $(v,u)(x,t)$ is a nonlinear diffusion wave $(\bar{v},\bar{u})(x,t)$ which satisfies a nonlinear parabolic equation

$$\bar{v}_t + \frac{1}{\alpha} p(\bar{v})_{xx} = 0 \tag{4.18}$$

with a boundary restriction

$$\bar{v}|_{x=0} = v_0(0), \quad \bar{v}|_{x=0} = v_+. \tag{4.19}$$

Since $v_0(0) \neq v_+$, without loss of generality, we say, $v_+ > v_0(0) > 0$. To construct such a diffusion wave satisfying (4.18) and (4.19), let us give any constant v_- satisfying $0 < v_- < v_0(0)$, and have a self-similar solution $\phi(x/\sqrt{1+t})$ satisfying

$$\begin{cases} \phi_t + \frac{1}{\alpha} p(\phi)_{xx} = 0, \quad x \in \mathbb{R} = (-\infty, \infty) \\ \phi|_{x=\pm\infty} = v_\pm. \end{cases} \tag{4.20}$$

Thus, by defining

$$\bar{v}(x,t) := \phi(x/\sqrt{1+t}) \quad \text{for all} \quad x \geq 0, \tag{4.21}$$

we get our desired diffusion wave $\bar{v}(x,t)$ satisfying (4.18) and (4.19). Furthermore, we define

$$\bar{u}(x,t) = -\frac{1}{\alpha} p(\bar{v})_x$$

so that $\bar{u}_x(0,t) = \bar{v}_t(0,t) = 0$. Thus, we have

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0 \\ p(\bar{v})_x = -\alpha \bar{u} \\ (\bar{v}, \bar{u}_x)(0,t) = (v_0(0), 0) \\ (\bar{v}, \bar{u})(\infty, t) = (v_+, 0). \end{cases} \tag{4.22}$$

Similarly, the auxiliary function $(\hat{v},\hat{u})(x,t)$ is defined by

$$(\hat{v},\hat{u})(x,t) = \Big(\frac{u_0(0) - u_+}{\alpha} m_0(x) e^{-\alpha t}, \Big[(u_0(0) - u_+) \int_x^\infty m_0(y) dy + u_+ \Big] e^{-\alpha t} \Big),$$

where $m_0(x)$ is a $C_0^\infty(\mathbb{R}_+)$-smooth function as defined in Section 3. Hence, $(\hat{v},\hat{u})$ satisfies

$$\begin{cases} \hat{v}_t - \hat{u}_x = 0 \\ \hat{u}_t = -\alpha \hat{u} \\ (\hat{v}, \hat{u}_x)(0,t) = (u_0(0) e^{-\alpha t}, 0), \quad \hat{v}(0,t) = 0 \\ (\hat{v}, \hat{u})(\infty, t) = (0, u_+ e^{-\alpha t}). \end{cases} \tag{4.23}$$

Combining (4.1) with (4.22) and (4.23), and defining the perturbations by

$$V(x,t) = -\int_x^\infty (v - \bar{v} - \hat{v})(y,t) dy, \quad z(x,t) = (u - \bar{u} - \hat{u})(x,t),$$

we obtain a new IBVP

$$\begin{cases} V_t - z = 0 \\ z_t + (p'(\bar{v})V_x)_x + \alpha z = -F, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ (V, z)|_{t=0} = (V_0, z_0)(x) \\ V_x|_{x=0} = 0, \end{cases} \tag{4.24}$$

where

$$F := \bar{u}_t + +(p(V_x + \bar{v} + \hat{v}) - p(\bar{v}) - p'(\bar{v})V_x)_x, \tag{4.25}$$

$$V_0(x) := -\int_x^\infty [v_0(y) - \bar{v}(y,0) - \hat{v}(y,0)]dy, \tag{4.26}$$

$$z_0(x) := u_0(x) - \bar{u}(x,0) - \hat{u}(x,0). \tag{4.27}$$

The unique global existence of solutions to the IBVP (4.24) and the convergence with some decay rates for $(V,z)(x,t)$ were also studied by Nishihara and Yang in [18] as follows.

Theorem 4.3 (Nishihara-Yang [18]: the case $v_0(0) \neq v_+$) *Suppose that $v_0 - v_+ \in L^1(\mathbb{R}_+)$, $(V_0, z_0)(x) \in H^3(\mathbb{R}_+) \times H^2(\mathbb{R}_+)$ and that $\|v_0 - v_+\|_{L^1(\mathbb{R}_+)} + \|V_0\|_{H^3(\mathbb{R}_+)} + \|z_0\|_{H^2(\mathbb{R}_+)} + |u_+| \ll 1$. Then there exists a unique time-global solution $(V,z)(x,t)$ of the IBVP (4.24), which satisfies*

$$V \in C^k(0,\infty; H^{3-k}(\mathbb{R}_+)), \quad k = 0,1,2,3, \qquad z \in C^k(0,\infty; H^{2-k}(\mathbb{R}_+)), \quad k = 0,1,2$$

and

$$\|\partial_x^k V(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-k/2}, \quad k = 0,1,2,3, \tag{4.28}$$

$$\|\partial_x^k z(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(k+2)/2}, \quad k = 0,1,2, \tag{4.29}$$

and

$$(1+t)^2\|z_t(t)\|_{L^2(\mathbb{R}_+)} + (1+t)^{-5/2}(\|z_{xt}(t)\|_{L^2(\mathbb{R}_+)} + \|z_{tt}(t)\|_{L^2(\mathbb{R}_+)}) = O(1). \tag{4.30}$$

Now we are going to improve above rates to be optimal. We subsequent the first equation of (4.24) into the second one, then have

$$\begin{cases} V_{tt} + \alpha V_t - \beta V_{xx} = G, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ (V, V_t)(x,0) = (V_0, V_1)(x)), \quad x \in \mathbb{R}_+ \\ V_x(0,t) = 0, \end{cases} \tag{4.31}$$

where $\beta = -p'(v_+)$, $V_1(x) = z_0(x)$ and

$$G = -F - ((p'(\bar{\bar{v}}) - p'(v_+))\bar{\bar{V}}_x)_x. \tag{4.32}$$

According to the linear IBVP theory obtained in Appendix, see (A.18), we obtain an equivalent integral equation of the IBVP (4.31) as follows

$$
\begin{aligned}
V(x,t) = & \int_0^\infty [K_0(x-y,t) + K_0(x+y,t)]V_0(y)dy \\
& + \int_0^\infty [K_1(x-y,t) + K_1(x+y,t)]V_1(y)dy \\
& + \int_0^t \int_0^\infty [K_1(x-y,t-\tau) + K_1(x+y,t-\tau)]G(y,\tau)dyd\tau,
\end{aligned} \tag{4.33}
$$

where $K_0(x,t)$ and $K_1(x,t)$ represented in (A.7) and (A.9) are the fundamental solutions of the linear damped wave equation.

Since $\bar{v} - v_+ \in L^2(\mathbb{R}_+)$, $||(\bar{v} - v_+)(t)||_{L^2(\mathbb{R}_+)} = O(1)$, $||(\bar{v} - v_+)(t)||_{L^\infty(\mathbb{R}_+)} = O(1)$ and $||V_{xx}(t)||_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-1}$ due to (4.28), that means

$$
\begin{aligned}
&||(p'(\bar{v}) - p'(v_+))V_{xx}||_{L^1(\mathbb{R}_+)} \sim ||(\bar{v} - v_+)(t)||_{L^2(\mathbb{R}_+)} ||V_{xx}(t)||_{L^2(\mathbb{R}_+)} \sim (1+t)^{-1}, \\
&||(p'(\bar{v}) - p'(v_+))V_{xx}||_{L^2(\mathbb{R}_+)} \sim ||(\bar{v} - v_+)(t)||_{L^\infty(\mathbb{R}_+)} ||V_{xx}(t)||_{L^2(\mathbb{R}_+)} \sim (1+t)^{-1},
\end{aligned}
$$

we can prove only

$$
||G(t)||_{L^1(\mathbb{R}_+)} \le C(1+t)^{-1} \quad \text{and} \quad ||G(t)||_{L^2(\mathbb{R}_+)} \le C(1+t)^{-1}. \tag{4.34}
$$

By using the energy method as done in Section 2, we can similarly prove the following decay rates.

Theorem 4.4 (The case $v_0(0) \neq v_+$) *Under the assumptions of Theorem 4.3, and suppose that $V_0 \in L^1(\mathbb{R}_+)$, $V_1 \in L^1(\mathbb{R}_+)$. Then*

$$
||\partial_x^k V(\cdot,t)||_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(2k+1)/4}, \quad k = 0,1,2, \tag{4.35}
$$

namely,

$$
||(v - \bar{v})(\cdot,t)||_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-3/4}. \tag{4.36}
$$

Remark 4.1 Since $G(x,t)$, see (4.34), does not decay so fast that our present method is invalid to get much more optimal rates like the other cases before. The shortcoming is due to that, we use a quite non-asypmtotic state v_+ in place of the really asymptotic profile $\bar{v}(x,t)$, see the IBVP (4.31) and its equivalent integral equation (4.33). In order to show more optimal decay rates like in Corollaries 2.1–4.1, we conjecture that the approximating Green function method with the method of pointwise estimates together just as used in Nishihara-Wang-Yang [17] maybe is available for the problem, but here we will not make such an attempt. Anyway, this case is quite not complete, and we are expecting more contribution on it.

Appendix: IBVP for Linear Damped Wave Equation

In this Appendix, we reduce the fundamental solutions and their basic energy estimates for the linear damped wave equation with the null-Dirichlet or null-Nuemann boundary. Based on this, we know what the optimal rates we may obtain for the linear and nonlinear damped wave equations. In such a sense, this section is essential to above ones.

Let us consider the damped wave equation

$$\phi_{tt} + \alpha\phi_t - \beta\phi_{xx} = 0, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \tag{A.1}$$

with the initial data

$$(\phi, \phi_t)|_{t=0} = (\phi_0, \phi_1)(x), \quad x \in \mathbb{R}_+ \tag{A.2}$$

and the Dirichlet boundary condition

$$\phi|_{x=0} = 0 \tag{A.3}$$

or the Nuemann boundary condition

$$\phi_x|_{x=0} = 0, \tag{A.4}$$

where α and β are positive constants.

In order to solve the IBVP (A.1)-(A.2)-(A.3) or the IBVP (A.1)-(A.2)-(A.4), we make an odd extension

$$\psi(x,t) := \begin{cases} \phi(x,t), & x \ge 0 \\ -\phi(-x,t), & x < 0, \end{cases} \qquad \psi_i(x) := \begin{cases} \phi_i(x), & x \ge 0 \\ -\phi_i(-x), & x < 0, \end{cases} \qquad i = 0, 1$$

for the IBVP (A.1)-(A.2)-(A.3), or an even extension

$$\psi(x,t) := \begin{cases} \phi(x,t), & x \ge 0 \\ \phi(-x,t), & x < 0, \end{cases} \qquad \psi_i(x) := \begin{cases} \phi_i(x), & x \ge 0 \\ \phi_i(-x), & x < 0, \end{cases} \qquad i = 0, 1$$

for the IBVP (A.1)-(A.2)-(A.4), then we get a corresponding Cauchy problem

$$\begin{cases} \psi_{tt} + \alpha\psi_t - \beta\psi_{xx} = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+ \\ \psi|_{t=0} = \psi_0(x), \qquad \psi_t|_{t=0} = \psi_1(x). \end{cases} \tag{A.5}$$

We can represent the solution of (A.5) as

$$\psi(x,t) = K_0(t) * \psi_0 + K_1(t) * \psi_1, \tag{A.6}$$

where $K_i(x,t)$ $(i = 0, 1)$ are the fundamental solutions of (A.5), that is,

$$K_{itt} + \alpha K_{it} - \beta K_{ixx} = 0, \quad i = 0, 1 \tag{A.7}$$

with

$$\begin{cases} K_0(x,0) = \delta(x) \\ \frac{d}{dt}K_0(x,0) = 0 \end{cases} \qquad \text{and} \qquad \begin{cases} K_1(x,0) = 0 \\ \frac{d}{dt}K_1(x,0) = \delta(x), \end{cases} \tag{A.8}$$

where $\delta(x)$ is the Delta function. The mark $*$ means convolution, i.e., $K_i(t) * \psi_i = \int_{-\infty}^{\infty} K_i(x-y,t)\psi_i(y)dy$.

Let $R_i(\xi,t)$ be the Fourier transform of $K_i(x,t)$, $i=0,1$, then R_i satisfies the following ODE

$$\frac{d^2}{dt^2}R_i + \alpha\frac{d}{dt}R_i + \beta\xi^2 R_i = 0, \quad i=0,1 \tag{A.9}$$

with the initial data

$$\begin{cases} R_0(\xi,0)=1 \\ \frac{d}{dt}R_0(\xi,0)=0 \end{cases} \quad \text{and} \quad \begin{cases} R_1(\xi,0)=0 \\ \frac{d}{dt}R_1(\xi,0)=1 \end{cases} \tag{A.10}$$

respectively. Solving the above two ODEs directly, we obtain the exact solutions as

$$R_1(\xi,t) = \begin{cases} \frac{2e^{-\alpha t/2}}{\sqrt{\alpha^2-4\beta\xi^2}}\sinh\left(\frac{\sqrt{\alpha^2-4\beta\xi^2}}{2}t\right), & |\xi| < \frac{\alpha}{2\sqrt{\beta}} \\ te^{-\alpha t/2}, & |\xi| = \frac{\alpha}{2\sqrt{\beta}} \\ \frac{2e^{-\alpha t/2}}{\sqrt{4\beta\xi^2-\alpha^2}}\sin\left(\frac{\sqrt{4\beta\xi^2-\alpha^2}}{2}t\right), & |\xi| > \frac{\alpha}{2\sqrt{\beta}} \end{cases} \tag{A.11}$$

and

$$R_0(\xi,t) = \frac{\alpha}{2}R_1(\xi,t) + R_2(\xi,t) \tag{A.12}$$

where

$$R_2(\xi,t) = \begin{cases} e^{-\alpha t/2}\cosh\left(\frac{\sqrt{\alpha^2-4\beta\xi^2}}{2}t\right), & |\xi| < \frac{\alpha}{2\sqrt{\beta}} \\ e^{-\alpha t/2}, & |\xi| = \frac{\alpha}{2\sqrt{\beta}} \\ e^{-\alpha t/2}\cos\left(\frac{\sqrt{4\beta\xi^2-\alpha^2}}{2}t\right), & |\xi| > \frac{\alpha}{2\sqrt{\beta}}. \end{cases} \tag{A.13}$$

Thus, we see that the fundamental solutions $K_i(x,t)$ $(i=0,1)$ can be given by making use of the inverse Fourier transform to $R_i(\xi,t)$ $(i=0,1)$.

For the IBVP (A.1)-(A.2)-(A.3), due to the odd symmetry $\psi(-x,t) = -\psi(x,t)$ and $\psi_i(-x) = -\psi_i(x)$ $(i=0,1)$, we obtain the expression of the solution as follows, for all $x \geq 0$,

$$\begin{aligned} \phi(x,t) = \psi(x,t) &= \int_{-\infty}^{\infty} K_0(x-y,t)\psi_0(y)dy + \int_{-\infty}^{\infty} K_1(x-y,t)\psi_1(y)dy \\ &= \Big(\int_0^{\infty} + \int_{-\infty}^0\Big)K_0(x-y,t)\psi_0(y)dy \\ &\quad + \Big(\int_0^{\infty} + \int_{-\infty}^0\Big)K_1(x-y,t)\psi_1(y)dy \\ &= \int_0^{\infty}[K_0(x-y,t) - K_0(x+y,t)]\phi_0(y)dy \\ &\quad + \int_0^{\infty}[K_1(x-y,t) - K_1(x+y,t)]\phi_1(y)dy. \end{aligned} \tag{A.14}$$

Similarly, due to the even symmetry, we obtain the expression of the solution for the IBVP (A.1)-(A.2)-(A.4)

$$\phi(x,t) = \int_0^\infty [K_0(x-y,t) + K_0(x+y,t)]\phi_0(y)dy \\ + \int_0^\infty [K_1(x-y,t) + K_1(x+y,t)]\phi_1(y)dy. \tag{A.15}$$

Furthermore, for the linear equation with source term

$$\phi_{tt} + \alpha\phi_t - \beta\psi_{xx} = g(x,t), \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \tag{A.16}$$

with the initial value condition (A.2) and the Dirichlet boundary value condition (A.3) or the Nuemann boundary value condition (A.4), making an odd extension also to the source term $g(-x,t) = -g(x,t)$ for the Dirichlet boundary condition (A.3), or an even extension $g(-x,t) = g(x,t)$ for the Nuemann boundary value condition (A.4), we can conduct the expressions of solutions by the Dahamal's principle: for the IBVP (A.16)-(A.2)-(A.3)

$$\phi(x,t) = \int_0^\infty [K_0(x-y,t) - K_0(x+y,t)]\phi_0(y)dy \\ + \int_0^\infty [K_1(x-y,t) - K_1(x+y,t)]\phi_1(y)dy \\ + \int_0^t \int_0^\infty [K_1(x-y,t-\tau) - K_1(x+y,t-\tau)]g(y,\tau)dyd\tau, \tag{A.17}$$

or for the IBVP (A.16)-(A.2)-(A.4)

$$\phi(x,t) = \int_0^\infty [K_0(x-y,t) + K_0(x+y,t)]\phi_0(y)dy \\ + \int_0^\infty [K_1(x-y,t) + K_1(x+y,t)]\phi_1(y)dy \\ + \int_0^t \int_0^\infty [K_1(x-y,t-\tau) + K_1(x+y,t-\tau)]g(y,\tau)dyd\tau. \tag{A.18}$$

For the Cauchy problem case, we note that Matsumura [13] got the following energy estimates.

Lemma A.1 (Matsumura [13]) *If $g \in L^1(\mathbb{R}) \cap H^{j+k-1}(\mathbb{R})$, then*

$$\left\| \partial_t^j \partial_x^k (K_1(t) * g) \right\|_{L^2(\mathbb{R})} \le C(1+t)^{-j-\frac{2k+1}{4}} (\|g\|_{L^1(\mathbb{R})} + \|g\|_{H^{j+k-1}(\mathbb{R})}). \tag{A.19}$$

If $g \in L^1(\mathbb{R}) \cap H^{j+k}(\mathbb{R})$, then

$$\left\| \partial_t^j \partial_x^k (K_0(t) * g) \right\|_{L^2(\mathbb{R})} \le C(1+t)^{-j-\frac{2k+1}{4}} (\|g\|_{L^1(\mathbb{R})} + \|g\|_{H^{j+k}(\mathbb{R})}). \tag{A.20}$$

Based on this lemma, we may prove the energy estimates for the IBVP case.

Lemma A.2 *If $g \in L^1(\mathbb{R}_+) \cap H^{j+k-1}(\mathbb{R}_+)$, then*

$$\begin{aligned}&\left\|\partial_t^j \partial_x^k \int_0^\infty [K_1(x-y,t) - K_1(x+y,t)]g(y)dy\right\|_{L^2(\mathbb{R}_+)} \\ &\le C(1+t)^{-j-\frac{2k+1}{4}}[\|g\|_{L^1(\mathbb{R}_+)} + \|g\|_{H^{j+k-1}(\mathbb{R}_+)}],\end{aligned} \tag{A.21}$$

$$\begin{aligned}&\left\|\partial_t^j \partial_x^k \int_0^\infty [K_1(x-y,t) + K_1(x+y,t)]g(y)dy\right\|_{L^2(\mathbb{R}_+)} \\ &\le C(1+t)^{-j-\frac{2k+1}{4}}[\|g\|_{L^1(\mathbb{R}_+)} + \|g\|_{H^{j+k-1}(\mathbb{R}_+)}].\end{aligned} \tag{A.22}$$

If $g \in L^1(\mathbb{R}_+) \cap H^{j+k}(\mathbb{R}_+)$, then

$$\begin{aligned}&\left\|\partial_t^j \partial_x^k \int_0^\infty [K_0(x-y,t) - K_0(x+y,t)]g(y)dy\right\|_{L^2(\mathbb{R}_+)} \\ &\le C(1+t)^{-j-\frac{2k+1}{4}}[\|g\|_{L^1(\mathbb{R}_+)} + \|g\|_{H^{j+k}(\mathbb{R}_+)}],\end{aligned} \tag{A.23}$$

$$\begin{aligned}&\left\|\partial_t^j \partial_x^k \int_0^\infty [K_0(x-y,t) + K_0(x+y,t)]g(y)dy\right\|_{L^2(\mathbb{R}_+)} \\ &\le C(1+t)^{-j-\frac{2k+1}{4}}[\|g\|_{L^1(\mathbb{R}_+)} + \|g\|_{H^{j+k}(\mathbb{R}_+)}].\end{aligned} \tag{A.24}$$

Proof. We are firstly going to prove (A.21). Setting

$$\bar{\phi}(x,t) := \int_0^\infty [K_1(x-y,t) - K_1(x+y,t)]g(y)dy,$$

we know that $\bar{\phi}(x,t)$ is a solution of the IBVP as follows

$$\begin{cases}\bar{\phi}_{tt} + \alpha\bar{\phi}_t - \beta\bar{\phi}_{xx} = 0, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ \bar{\phi}|_{t=0} = 0, \quad \bar{\phi}_t|_{t=0} = g(x), \quad x \in \mathbb{R}_+ \\ \bar{\phi}|_{x=0} = 0.\end{cases}$$

By an odd extension to the above IBVP

$$\bar{\psi}(x,t) := \begin{cases}\bar{\phi}(x,t), & x \ge 0 \\ -\bar{\phi}(-x,t), & x < 0,\end{cases} \qquad h(x) := \begin{cases} g(x), & x \ge 0 \\ -g(-x), & x < 0,\end{cases}$$

such that we may consider its corresponding Cauchy problem for $\bar{\psi}(x,t)$ with the initial data $(0, h(x))$, and note that

$$\int_{-\infty}^\infty \bar{\psi}^2(x,t)dx = 2\int_0^\infty \bar{\phi}^2(x,t)dx, \quad \|h\|_{L^1(\mathbb{R})} = 2\|g\|_{L^1(\mathbb{R}_+)}, \quad \|h\|^2_{H^j(\mathbb{R})} = 2\|g\|^2_{H^j(\mathbb{R}_+)}$$

due to the odd symmetry, and by using Matsumura's lemma (Lemma A.1), then we obtain

$$
\begin{aligned}
&\left\|\partial_t^j\partial_x^k\int_0^\infty [K_1(x-y,t)-K_1(x+y,t)]g(y)dy\right\|_{L^2(\mathbb{R}_+)}\\
&=\left\|\partial_t^j\partial_x^k\int_{-\infty}^\infty K_1(x-y,t)h(y)dy\right\|_{L^2(\mathbb{R}_+)}\\
&=\frac{1}{\sqrt{2}}\left\|\partial_t^j\partial_x^k\int_{-\infty}^\infty K_1(x-y,t)h(y)dy\right\|_{L^2(\mathbb{R})}\\
&\le C(1+t)^{-j-\frac{2k+1}{4}}[\|h\|_{L^1(\mathbb{R})}+\|h\|_{H^{j+k-1}(\mathbb{R})}]\\
&\le C(1+t)^{-j-\frac{2k+1}{4}}[\|g\|_{L^1(\mathbb{R}_+)}+\|g\|_{H^{j+k-1}(\mathbb{R}_+)}].
\end{aligned}
\tag{A.25}
$$

Thus, we proved (A.21).

Similarly, without any difficulty we can prove (A.22), (A.23) and (A.24). Here, we omit the details. □

Finally, we list an important lemma which is effectively used for proving the decay rates as we showed before. The proof and a lot of applications of this lemma can be found in many bibliographies, for example, in [19, 13, 14, 3, 9] and the references therein.

Lemma A.3 *Let $a>0$ and $b>0$ be constants. If $\max(a,b)>1$, then*

$$
\int_0^t (1+t-s)^{-a}(1+s)^{-b}ds \le C(1+t)^{-\min(a,b)}. \tag{A.26}
$$

If $\max(a,b)=1$, then

$$
\int_0^t (1+t-s)^{-a}(1+s)^{-b}ds \le C(1+t)^{-\min(a,b)}\ln(2+t). \tag{A.27}
$$

If $\max(a,b)<1$, then

$$
\int_0^t (1+t-s)^{-a}(1+s)^{-b}ds \le C(1+t)^{1-a-b}. \tag{A.28}
$$

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