ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE HYDRODYNAMIC MODEL OF SEMICONDUCTORS

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Abstract. *Degond and Markowich* [5] *discussed the existence and uniqueness of steady-state solution in subsonic case for the one-dimensional hydrodynamic model of semiconductors. In the present paper, we reconsider the existence and uniqueness of globally smooth subsonic steady-state solution, and prove its stability for small perturbation. The proof method we adopt in this paper is based on elementary energy estimates.*

1. Introduction

Since its introduction by Bløtekjær [2], the hydrodynamic model for semiconductors has recently attracted a lot of attention because of its ability to model hot electron effects which are not accounted for in the classical drift-diffusion model. For more discussion on these models in physics and engineering, and their derivation from kinetic transport equation, we refer to [23, 27, 17, 28, 29] for details.

Recently, many papers were written on the hydrodynamic model of semiconductors. For the steady-state system, Degond and Markowich [5, 6] investigated the existence and uniqueness of subsonic solutions in one dimension and, for irrational flow, in three-dimension respectively; Markowich [24] discussed the existence of subsonic solutions in two dimension. The corresponding investigations on transonic solutions in one dimension were done by Ascher, Markowich, and Schmeiser [1] and by Markowich and Pietra [26] via phase plane analysis and the representation of discontinuous solutions, and by Gamba [8] via artificial viscosity. For the Cauchy problem of unipolar time-dependent hydrodynamic systems, Luo, Natalini and Xin [19] investigated the global existence and the asymptotic behavior of smooth solutions of the hydrodynamic model for

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semiconductors and discussed their convergence to the stationary solution of the drift-diffusion equation in $\mathbb{R}^1$, but they have to assume the stationary current density J to be zero due to a technical difficulty in the analysis. The stability and, resp., instability of the steady-state solutions of the Cauchy problem for semiconductor model was analyzed by Hattori and Zhu [12] and by Hattori [11]. Stability with convergence rates of global solutions of the hydrodynamic model to the corresponding steady-state solution on a quarter plane was analyzed by Marcati and Mei [20]. The assumption of zero current density was removed therein, too. Furthermore, for the initial boundary values problem in a bounded domain, under assumption of zero-current density on the boundary, Hsiao and Yang [14] discussed the time-asymptotic convergence of the smooth solutions of the hydrodynamic model and those of the drift-diffusion model to the unique steady-state solution. Subsequent to [19, 20, 14], in this paper we are going to study the asymptotic stability of steady-state solutions for the non-zero current density case in a bounded domain, which improves the previous works [19, 20, 14].

Regarding other topics on such hydrodynamic models for semiconductors device, we note the followings. In [30], Poupaud, Rascle and Vila showed the global existence of the solutions with arbitrarily large data by using a trick concerning charge conservation. By finite-difference schemes and compensated compactness, Marcati and Natalini [21, 22] proved the existence of weak solutions and discussed the relaxation limit to the drift-diffusion model for $1 < \gamma \leq \frac{5}{3}$. K. Zhang [35, 36] discussed the existence of weak solutions and the relaxation limits for $\gamma > \frac{5}{3}$. Gasser and Natalini [10] discussed the relaxation limit for the non-isentropic hydrodynamic model. On the strip domain, the local existence of smooth solutions was proved by Zhang [34]; its large time behavior was analysed by Chen, Jerome and Zhang [3]. The existence of weak solutions was obtained by Zhang [33] via Godunov schemes and by Fang and Ito [7] via vanishing viscosity. Hsiao and K. Zhang [15, 16] discussed the relaxation limit and verified the boundary condition for weak solutions in the sense of trace. By shock capturing schemes, Chen and Wang [4] investigated the existence of weak solutions in high-dimensional compact domain with geometry-symmetry. Also, there are lots of works on numerical analysis and simulation, for instance, by Jerome and Shu [18] and references therein.

After an appropriate scaling, the one-dimensional time-dependent system in the case of one carrier type, i.e., electrons, reads

$$\rho_t + (\rho u)_x = 0, \tag{1.1}$$

$$(\rho u)_t + (\rho u^2 + p(\rho))_x = \rho\phi_x - \frac{\rho u}{\tau}, \tag{1.2}$$

$$\phi_{xx} = \rho - \mathcal{C}(x), \tag{1.3}$$

where $\rho > 0$, u and ϕ denote the electron density, velocity, and the electrostatic potential respectively. $j = \rho u$ is called the current density. $p = p(\rho)$ is the pressure-density relation which satisfies

$$\rho^2 p'(\rho) \text{ is strictly monotonically increasing from } (0,\infty) \text{ into } (0,\infty). \tag{1.4}$$

In the present paper, we assume that

$$p \in C^3(0,+\infty). \tag{1.5}$$

And $\tau = \tau(\rho, \rho u) > 0$ is the momentum relaxation time. The device domain is the x-interval $(0,1)$. $\mathcal{C} = \mathcal{C}(x) > 0$ is the doping profile which stands for the given background density of changed ions. We assume that there is a function $\mathcal{A} = \mathcal{A}(x) \in C^2(0,1)$ such that

$$\mathcal{A}(x) > 0,\ \mathcal{A}(0) = \rho_1,\ \mathcal{A}(1) = \rho_2,\ \mathcal{A}(x) - \mathcal{C}(x) \in C(0,1). \tag{1.6}$$

In the present paper, we first consider the initial boundary value problems (IBVP for simplicity) for (1.1)–(1.3) with the following initial data,

$$(\rho, j)(x, 0) = (\tilde{\rho}, \tilde{j})(x), \quad x \in (0,1). \tag{1.7}$$

and the density and potential Dirichlet boundary conditions

$$\rho(0,t) = \rho_1, \quad \rho(1,t) = \rho_2, \quad t \geq 0, \tag{1.8}$$

$$\phi(0,t) = 0, \quad \phi(1,t) = \phi_1, \quad t \geq 0. \tag{1.9}$$

This kind of boundary conditions are of importance in physics of semiconductor devices [23].

Our interest is to investigate the existence and stability of the smooth steady-state solutions of the hydrodynamic model of semiconductors, namely, the solutions of the boundary value problem (BVP) for the following system

$$j = \text{const}, \tag{1.10}$$

$$\left(\frac{j^2}{\rho} + p(\rho)\right)_x = \rho\phi_x - \frac{j}{\tau}, \tag{1.11}$$

$$\phi_{xx} = \rho - \mathcal{C}(x), \tag{1.12}$$

with boundary conditions (1.8) and (1.9).

The main results in the present paper shows that for any $J_0 \neq 0$ satisfying condition (2.5) with $|\rho_2 - \rho_1| \ll 1$, there is a $\Phi_0 > 0$ such that for any $0 < \phi_1 \leq \Phi_0$, the BVP (1.10)–(1.12) and (1.8)–(1.9) has a unique regular solution $(\rho_0, j_0, \phi_0)(x)$ with $|j_0| \leq |J_0|$, and for any small initial perturbation of (ρ_0, j_0), the global solutions (ρ, j, ϕ) of IBVP (1.1)–(1.3) and (1.7)–(1.9) exists and tends exponentially to the solution (ρ_0, j_0, ϕ_0) as $t \to +\infty$.

This paper is arranged as follows. In section 2, the existence, uniqueness and properties on the solutions to the BVP (1.10)–(1.12) and (1.8)–(1.9) are shown. The global existence and the asymptotic behavior of the solution of IBVP (1.1)–(1.3) and (1.7)–(1.9) are introduced and proved in section 3.

Notation. We make some notation for simplicity. C always denotes a positive constant. $L^2(0,1)$ is the space of square integrable real valued function defined on $[0,1]$ with the norm $||\cdot||$, and $H^k(0,1)$ denotes the usual Sobolev space with the norm $||\cdot||_k$, especially, $||\cdot||_0 = ||\cdot||$. Let T and B be a positive constant and a Banach space, respectively. $C^k(0,T;B)$ $(k \geq 0)$ denotes the space of B-valued k-times continuously differentiable functions on $[0,T]$, and $L^2(0,T;B)$ denotes the space of B-valued L^2-functions on $(0,T)$. The corresponding spaces of B-valued functions on $[0,\infty)$ are defined analogously.

2. Steady-state system

In this section, we consider the properties of the steady-state solution of the BVP (1.10)–(1.12), (1.8)–(1.9) for the hydrodynamic model of semiconductors. For simplicity, we assume $\tau = 1$ from now on.

According to those shown in [5] for subsonic solutions, the boundary data of the BVP (1.10)–(1.12) and (1.8)–(1.9) should satisfy the current-voltage relationship

$$\phi_1 = F(\rho_2, j) - F(\rho_1, j) + j\int_0^1 \frac{dx}{\rho(x)}, \tag{2.1}$$

where

$$F(\rho, j) = \frac{j^2}{2\rho^2} + h(\rho), \quad h'(\rho) = \frac{1}{\rho}p'(\rho). \tag{2.2}$$

Since, by (2.1), the case $j = 0$ yields $\phi_1 = 0$, we consider, in the present paper, the physically more interesting case $j \neq 0$.

Dividing (1.11) by ρ, differentiating it again, and using (1.10) and (1.12), we obtain finally

$$\left(\frac{\partial F}{\partial \rho}(\rho, j)\rho_x\right)_x + j\left(\frac{1}{\rho}\right)_x - \rho = -\mathcal{C}(x), \quad 0 < x < 1. \tag{2.3}$$

Thus, to make sure the existence of regular solutions, the subsonic condition is required to be satisfied, i.e.,

$$\frac{\partial F}{\partial \rho}(\rho, j) = -\frac{j^2}{\rho^3} + \frac{1}{\rho}p'(\rho) > 0 \Leftrightarrow \rho^2 p'(\rho) > j^2. \tag{2.4}$$

Due to (1.4), we conclude that there is a unique $\rho_m = \rho_m(j)$ such that $\frac{\partial F}{\partial \rho}(\rho, j) > 0$ for $\rho > \rho_m$. Also, by (2.4), we know that the minimal point ρ_m of $\rho \to F(\rho, j)$ is a strictly increasing function of j with $\rho_m(j = 0) = 0$. These analysis imply that the equation (2.3) is uniformly elliptic for $\rho \geq \rho^* > \rho_m$, which, by (2.4), means a fully subsonic flow $|u| < c(\rho)$. Here $c(\rho) = \sqrt{p'(\rho)}$ denotes the sound speed.

The main result in this section is the following theorem.

Theorem 2.1. *Assume* (1.4), (1.5), *and* (1.6) *hold. Let* $J_0 \neq 0$ *be such that*

$$\rho_1, \ \rho_2, \ \inf_{x\in(0,1)} \mathcal{C}(x) > \rho_m(J_0), \tag{2.5}$$

and assume $|\rho_2 - \rho_1| \ll 1$. *Then there is a constant* $\Phi_0 > 0$, *such that for all* $0 < \phi_1 \leq \Phi_0$, *the BVP* (1.10)–(1.12) *and* (1.8)–(1.9) *has a unique solution* $(\rho_0, j_0, \phi_0)(x)$, *which satisfies* $|j_0| \leq |J_0|$ *and*

$$\mathcal{C}_- \triangleq \min\{\rho_1, \ \rho_2, \ \inf_{x\in(0,1)} \mathcal{C}(x)\} \leq \rho_0(x) \leq \max\{\rho_1, \ \rho_2, \ \sup_{x\in(0,1)} \mathcal{C}(x)\} \triangleq \mathcal{C}_+, \tag{2.6}$$

$$\|\rho_0 - \mathcal{A}\|_2^2 + \|\rho_{0x}\|_1^2 \leq C_0\delta_0, \tag{2.7}$$

$$\|\phi_{0x}\|_1^2 \leq C_0\delta_0, \tag{2.8}$$

where C_0 *is a positive constant related to* $\mathcal{C}_\pm$ *and* $|j_0|$, *and*

$$\delta_0 = \max_{x\in(0,1)}\{|\mathcal{A}'(x)| + |\mathcal{A}''(x)| + |\mathcal{A}(x) - \mathcal{C}(x)|\} + (|\phi_1| + |\rho_2 - \rho_1|)(1 + |\ln \mathcal{C}_+ - \ln \mathcal{C}_+|).$$

□

Proof: We are going to prove Theorem 2.1 in the following two steps.

Step 1. The apriori estimates. Let $(\rho_0, j_0, \phi_0)(x)$ be a regular solution of the BVP (1.10)–(1.12) and (1.8)–(1.9), which is bounded and satisfies the subsonic condition (2.4). We prove that (2.7) and (2.8) hold for $(\rho_0, j_0, \phi_0)(x)$.

Set

$$\chi = \rho_0 - \mathcal{A}(x). \tag{2.9}$$

By (1.11) and (1.12), we obtain the equation for χ

$$\left(\frac{p'(\rho_0) - \frac{j_0^2}{\rho_0^2}}{\rho_0}(\mathcal{A}'(x) + \chi_x)\right)_x + \left(\frac{j_0}{\rho_0}\right)_x = \chi + \mathcal{A}(x) - \mathcal{C}(x). \tag{2.10}$$

Multiplying (2.10) with χ, integrating it over $(0,1)$, using $\chi(0) = \chi(1) = 0$ and integration by parts, one has

$$\begin{aligned}&\int_0^1 \chi^2 dx + \int_0^1 \frac{p'(\rho_0) - \frac{j_0^2}{\rho_0^2}}{\rho_0}\chi_x^2 dx \\ =& -\int_0^1 \chi(\mathcal{A} - \mathcal{C})(x)dx - \int_0^1 \frac{j_0}{\rho_0}\chi_x dx - \int_0^1 \frac{p'(\rho_0) - \frac{j_0^2}{\rho_0^2}}{\rho_0}\mathcal{A}'(x)\chi_x dx.\end{aligned} \tag{2.11}$$

Since

$$\begin{aligned}|\int_0^1 \frac{j_0}{\rho_0}\chi_x dx| \leq& |\int_0^1 \frac{j_0}{\rho_0}\rho_{0x}dx| + |\int_0^1 \frac{j_0}{\rho_0}\mathcal{A}'(x)dx| \\ \leq& (|\phi_1| + |\rho_2 - \rho_1|)(|\ln \mathcal{C}_+ - \ln \mathcal{C}_-| + \frac{1}{\mathcal{C}_-}\max_{x\in(0,1)}|\mathcal{A}'(x)|),\end{aligned} \tag{2.12}$$

it follows from (2.1), (2.11) and (2.12) that

$$\begin{aligned}&\int_0^1 \chi^2 dx + \int_0^1 \chi_x^2 dx \\ \leq& \mathrm{C}_0 \max_{x\in(0,1)}(|(\mathcal{A} - \mathcal{C})(x)| + |\mathcal{A}'(x)| + (|\phi_1| + |\rho_2 - \rho_1|)\ln \mathcal{C}_+ - \ln \mathcal{C}_-|)\end{aligned} \tag{2.13}$$

where $\mathrm{C}_0 > 0$ is a constant related to $|j_0|$ and $\mathcal{C}_\pm$.

We multiply (2.10) with χ_{xx}, integrate by parts over $(0,1)$. With $\chi(0) = \chi(1) = 0$ and (2.13), we have, similarly to (2.13),

$$\begin{aligned}&\int_0^1 \chi_x^2 dx + \int_0^1 \chi_{xx}^2 dx \\ \leq& \mathrm{C}_0 \max_{x\in(0,1)}(|(\mathcal{A} - \mathcal{C})(x)| + |\mathcal{A}'(x)| + |\mathcal{A}''(x)| + (|\phi_1| + |\rho_2 - \rho_1)\ln \mathcal{C}_+ - \ln \mathcal{C}_-|)\end{aligned} \tag{2.14}$$

The combination of (2.13) and (2.14), in view of (2.9) yields (2.7).

Multiplying (1.12) with $[\phi_0(x) - x\phi_1]$, and integrating by parts over $(0,1)$, one has

$$\int_0^1 \phi_{0x}^2 dx \leq O(1)(|\phi_1| + |\mathcal{A}(x) - \mathcal{C}(x)| + \int_0^1 \chi^2 dx), \tag{2.15}$$

which implies (2.8) in view of (2.1), (1.12), (2.9), and (2.7).

Step 2. Existence of regular solutions. Define

$$\Phi_0 = F(\rho_2, J_0) - F(\rho_1, J_0) + \frac{|J_0|}{\mathcal{C}_+}, \tag{2.16}$$

which implies in terms of (2.5) and $|\rho_2 - \rho_1| \ll 1$ that

$$\Phi_0 > 0. \tag{2.17}$$

In addition, one can verify that

$$\frac{d\Phi}{dj} =: \frac{\partial F}{\partial j}(\rho_2, j) - \frac{\partial F}{\partial j}(\rho_1, j) + \frac{1}{\mathcal{C}_+} > 0, \tag{2.18}$$

for $|j| < |J_0|$ and $|\rho_2 - \rho_1| \ll 1$.

Without loss of generality, we assume that

$$\rho_2 \geq \rho_1. \tag{2.19}$$

For any $0 < \phi_1 \leq \Phi_0$, define the operator $T : p \to P$ by solving the following linear equation with the Dirichlet boundary

$$\left(\frac{\partial F}{\partial p}(p, J)P_x\right)_x - JP_x - P = \mathcal{C},\ 0 < x < 1, \tag{2.20}$$

$$P(0) = \rho_1, \quad P(1) = \rho_2, \tag{2.21}$$

where $J = J[p]$ satisfies

$$\phi_1 = F(\rho_2, J) - F(\rho_1, J) + J\int_0^1 \frac{dx}{p(x)}, \tag{2.22}$$

and

$$J < \frac{\rho_1^2\rho_2^2}{\mathcal{C}_+(\rho_2^2 - \rho_1^2)} \quad \text{for} \quad \rho_2 > \rho_1. \tag{2.23}$$

Suppose

$$\mathcal{C}_- \leq p(x) \leq \mathcal{C}_+, \tag{2.24}$$

by (2.22), (2.18), (2.16) and (2.5), one finds that

$$|J| \leq |J_0|,\ \frac{\partial F}{\partial \rho}(p, J) > \alpha_0 > 0, \tag{2.25}$$

which implies that the equation (2.20) is elliptic. For the linear elliptic BVP (2.20)-(2.21), applying the maximum principle and using energy estimates similar to (2.13)–(2.15), one obtains

$$\mathcal{C}_- \leq P = T(p) \leq \mathcal{C}_+,\quad P = T(p) \in C^2(0,1),\quad ||P - \mathcal{A}||_2 = ||T(p) - \mathcal{A}||_2 \leq C, \tag{2.26}$$

and

$$|J[P]| \leq |J_0|,\ \frac{\partial F}{\partial \rho}(P, J[P]) > \alpha_0 > 0, \tag{2.27}$$

with α_0 a constant.

Thus, it is easily shown that the operator T is continuous and bounded in $H^2(0,1)$. Applying the Schauder's fixed-point theorm and the compact embedding of $H^2(0,1)$ into $C^1(0,1)$, one obtains the existence of a fixed point, say $\rho_0(x)$, of the operator T such that the fixed point $\rho_0 = T(\rho_0)$ satisfies (2.20) and (2.21) with $p = P = \rho_0$. Let j_0 and $\phi_0(x)$ be determined by

$$\phi_1 = F(\rho_2, j_0) - F(\rho_1, j_0) + j_0\int_0^1 \frac{dx}{\rho_0(x)}, \tag{2.28}$$

$$j_0 < \frac{\rho_1^2\rho_2^2}{\mathcal{C}_+(\rho_2^2 - \rho_1^2)} \quad \text{for} \quad \rho_2 > \rho_1, \tag{2.29}$$

and

$$\phi_{0xx} = \rho_0 - \mathcal{C}, \quad 0 < x < 1, \tag{2.30}$$

$$\phi_0(0) = 0, \ \phi_0(1) = \phi_1. \tag{2.31}$$

Thus, (ρ_0, j_0, ϕ_0) is a solution to BVP (1.10)–(1.12) and (1.8)–(1.9) satisfying $|j_0| \le |J_0|$, and (2.6)–(2.8). Apply the maximum principle and the similar argument to [5, 19], one can easily prove the uniqueness of the solution. □

3. Hydrodynamic model

In this section, we consider the stability of the steady-state solution obtained in section 2.

Set

$$\psi_0 = \tilde{\rho} - \rho_0, \quad \eta_0 = \tilde{j} - j_0. \tag{3.1}$$

The main result in this section is the followings.

Theorem 3.1. *Let $(\rho_0, j_0, \phi_0)(x)$ be the regular solution of the BVP (1.10)–(1.12) and (1.8)–(1.9) given by Theorem 2.1. Assume $(\psi_0, \eta_0) \in H^2$. Then, there is $\varepsilon_0 > 0$, such that if $\|(\phi_0, \eta_0)\| + \delta_0 \le \varepsilon_0$, the global smooth solution $(\rho, j, \phi)(x,t)$ to the IBVP (1.1)–(1.3) and (1.7)–(1.9) exists and satisfies*

$$\|(\rho - \rho_0, j - j_0, \phi - \phi_0)(\cdot, t)\|_2^2 \le O(1)\|(\psi_0, \eta_0)\|_2^2 \exp\{-\beta t\}, \quad t \ge 0 \tag{3.2}$$

with a positive constant β. □

Proof: By the standard methods, we can prove the local existence of a solution of the IBVP (1.1)–(1.3) and (1.7)–(1.9), and show its regularity due to the effect of relaxation as [13]. To show the global existence of a smooth solution, we shall establish uniform apriori estimates, i.e., Lemmas 3.2–3.4.

Let $(\rho_0, j_0, \phi_0)(x)$ be the steady-state solution to the BVP (1.10)–(1.12) and (1.8)–(1.9). For any $T > 0$, assume that $(\rho, j, \phi)(x,t)$ is the solution of the IBVP (1.1)–(1.3) and (1.7)–(1.9).

Set

$$\psi = \rho - \rho_0, \ \eta = j - j_0, \ e = \phi - \phi_0, \tag{3.3}$$

then the corresponding IBVP for (ϕ, η, e) on $(0,1) \times [0, +\infty)$ is

$$\psi_t + \eta_x = 0, \tag{3.4}$$

$$\eta_t + \left[\frac{(j_0+\eta)^2}{\rho_0+\psi} - \frac{j_0}{\rho_0} + p(\rho_0+\psi) - p(\rho_0)\right]_x = \psi\phi_{0x} + (\rho_0+\psi)e_x - \eta, \tag{3.5}$$

$$e_{xx} = \psi, \tag{3.6}$$

$$\psi(0,t) = \psi(1,t) = 0, \quad t \ge 0, \tag{3.7}$$

$$e(0,t) = e(1,t) = 0, \quad t \ge 0, \tag{3.8}$$

$$\psi(x,0) = \psi_0(x), \ \eta(x,0) = \eta_0(x), \quad x \in (0,1). \tag{3.9}$$

For $T > 0$, denote the basic space for the IBVP (3.4)–(3.9) as

$$X(T) = \{(\psi, \eta, e) \in H^2,\ 0 \leq t \leq T\}, \tag{3.10}$$

with norm given by

$$M(0,T) = \max_{0 \leq t \leq T} \|(\psi, \eta, e)(t)\|_2$$

and assume that the following assumption holds

$$N(T) = \max_{0 \leq t \leq T} \|(\psi, \eta)(t)\|_2 \ll 1. \tag{3.11}$$

It is easy to verify that under the assumption (3.11) it holds that

$$0 < \rho_- \leq \rho_0 + \phi \leq \rho_+, \quad j_- \leq j_0 + \eta \leq j_+,$$

with ρ_-, ρ_+, j_-, and j_+ constants, and the subsonic condition (2.4) holds for (ρ, j, ϕ).

Lemma 3.2. *It holds for* $(\psi, \eta, e) \in X(T)$

$$\int_0^1 e_x^2 dx \leq O(1) \int_0^1 \psi^2 dx, \quad e_x^2 \leq O(1) \int_0^1 \psi^2 dx, \tag{3.12}$$

$$\int_0^1 e_{xt}^2 dx \leq O(1) \int_0^1 \psi_t^2 dx, \quad e_{xt}^2 \leq O(1) \int_0^1 \psi_t^2 dx, \quad e_t^2 \leq O(1) \int_0^1 \psi^2 dx, \tag{3.13}$$

$$\int_0^1 \eta^2 dx \leq O(1) \left(\exp\{-c_0 t\} \int_0^1 \eta_0^2 dx + \int_0^1 (\psi_t^2 + \psi^2) dx \right), \tag{3.14}$$

$$\eta^2 \leq O(1) \left(\exp\{-c_0 t\} \int_0^1 \eta_0^2 dx + \int_0^1 (\psi_t^2 + \psi^2) dx \right), \tag{3.15}$$

$$\int_0^1 \eta_t^2 dx \leq O(1) \left(\exp\{-c_0 t\} \int_0^1 \eta_0^2 dx + \int_0^1 (\psi_t^2 + \psi_x^2 + \psi^2) dx \right), \tag{3.16}$$

provided that $N(T) + \delta_0$ *small enough. Where* $c_0 > 0$ *is a constant.* □

Proof: We first show (3.12). Multiplying (3.6) with e and integrating over $(0,1)$ yields, by (3.7) and integration by parts

$$\int_0^1 e_x^2 dx + \int_0^1 e\psi dx = 0, \tag{3.17}$$

which implies, in terms of Hölder inequality, that

$$\int_0^1 e_x^2 dx \leq \left(\int_0^1 e^2 dx \right)^{1/2} \left(\int_0^1 \psi^2 dx \right)^{1/2}. \tag{3.18}$$

Then, it follows

$$\int_0^1 e_x^2 dx \leq 2 \int_0^1 \psi^2 dx, \tag{3.19}$$

from (3.8), (3.18) and the following Poincaré inequality (noting $e(0,t) = e(1,t) = 0$)

$$\left(\int_0^1 e^2 dx \right)^{1/2} \leq 2 \left(\int_0^1 e_x^2 dx \right)^{1/2}. \tag{3.20}$$

Integrating the above inequality gives (3.20).

On the other hand, by the integral mean value theorem, there exists a curve $x_1(t)$ satisfying $0 < x_1(t) < 1$ such that

$$e_x^2(x_1(t), t) = \int_0^1 e_x^2(x,t)dx,$$

thanks to (3.6) and (3.19), we have

$$\begin{aligned} e_x^2(x,t) = &\ e_x^2(x_1(t),t) + 2\textstyle\int_{x_1(t)}^x e_x e_{xx} dx| \\ \leq &\ \int_0^1 e_x^2 dx + 2\int_0^1 |e_x e_{xx}| dx \\ \leq &\ 2\int_0^1 e_x^2 dx + \int_0^1 e_{xx}^2 dx \\ \leq &\ O(1)\int_0^1 \psi^2 dx. \end{aligned}$$

Now, we deal with (3.13). Differentiating (3.6) with respect to t leads to

$$e_{xxt} = \psi_t. \tag{3.21}$$

Multiplying (3.21) with e_t, integrating over $(0,1)$, and noticing $e_t(0,t) = e_t(1,t) = 0$, we have, after integration by parts

$$\int_0^1 e_{xt}^2 dx + \int_0^1 e_t \psi_t dx = 0. \tag{3.22}$$

Similarly to (3.19), one has

$$\int_0^1 e_{xt}^2 dx \leq O(1)\int_0^1 \psi_t^2 dx. \tag{3.23}$$

The other estimates in (3.13) follow from (3.23), (3.21) and the following inequalities

$$e_{xt}^2 \leq 2\int_0^1 e_{xt}^2 dx + \int_0^1 e_{xxt}^2 dx, \quad e_t^2 \leq \int_0^1 e_t^2 dx + \int_0^1 e_{xt}^2 dx.$$

Finally, we estimate (3.14)–(3.16). Since it holds by (3.4) that

$$\begin{aligned} \eta^2 \ \leq &\ \int_0^1 \eta^2 dx + 2\int_0^1 |\eta_x \eta| dx \\ \leq &\ 2\int_0^1 \eta^2 dx + \int_0^1 \psi_t^2 dx, \end{aligned} \tag{3.24}$$

and, in view of (3.5), (3.12), (2.7), and (2.8), that

$$\eta_t^2 \leq O(1)\left\{\left(\frac{(j_0+\eta)^2}{\rho_0+\psi} - \frac{j_0^2}{\rho_0} + p(\rho_0+\psi) - p(\rho_0)\right)_x\right\}^2 \tag{3.25}$$

$$+ O(1)\left(\eta^2 + (\psi_x \phi_{0x})^2 + (\rho_{0x}+\psi_x)^2 e_x^2\right) \tag{3.26}$$

$$\leq O(1)\left(\eta^2 + \psi_x^2 + \psi_t^2 + \psi^2\right), \tag{3.27}$$

it is sufficient to prove (3.14). Multiplying (3.5) with η and integrating it over $(0,1)$, one has, by (3.8) and integration by parts, that

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left(\int_0^1 \eta^2 dx\right) + \int_0^1 \eta^2 dx \\ = & -[\eta \tfrac{(j_0+\eta)^2-j_0^2}{\rho_0}]\Big|_0^1 + \int_0^1 \eta(\psi\phi_{0x} + (\rho_0+\psi)e_x)dx \\ & + \int_0^1 \eta_x \left(\frac{(j_0+\eta)^2}{\rho_0+\psi} - \frac{j_0^2}{\rho_0} + p(\rho_0+\psi) - p(\rho_0)\right) dx \\ \triangleq & \; I_1 + I_2 + I_3. \end{aligned} \tag{3.28}$$

The I_1, I_2 and I_3 can be estimated as follows.

$$\begin{aligned} I_1 \le & \int_0^1 \left| \eta_x \frac{(j_0+\eta)^2 - j_0^2}{\rho_0} + 2\eta\eta_x \frac{j_0+\eta}{\rho_0} - \eta\rho_{0x}\frac{(j_0+\eta)^2-j_0^2}{\rho_0^2}\right| dx \\ \le & \; O(1)\int_0^1 |\psi_t \eta| dx + \delta_0 \int_0^1 \eta^2 dx \\ \le & \; (O(1)\delta_0 + \tfrac{1}{12})\int_0^1 \eta^2 dx + O(1)\int_0^1 \psi_t^2 dx, \end{aligned} \tag{3.29}$$

$$\begin{aligned} I_2 \le & \; \frac{1}{12}\int_0^1 \eta^2 dx + O(1)\int_0^1 (\psi^2 + e_x^2)dx \\ \le & \; \frac{1}{12}\int_0^1 \eta^2 dx + O(1)\int_0^1 \psi^2 dx, \end{aligned} \tag{3.30}$$

$$\begin{aligned} I_3 \le & \int_0^1 \left|\psi_t \left(2\frac{j_0+\theta_1\eta}{\rho_0+\theta_2\psi}\eta - \frac{(j_0+\theta_3\eta)^2}{(\rho_0+\theta_4\psi)^2}\psi\right)\right| dx \\ & + \int_0^1 |\psi_t\psi p'(\rho_0+\theta_5\psi)|dx \\ \le & \; \frac{1}{12}\int_0^1 \eta^2 dx + O(1)\int_0^1 (\psi_t^2 + \psi^2)dx, \end{aligned} \tag{3.31}$$

with $0 < \theta_i < 1$ $(i = 1,2,3,4,5)$. Here we used (3.4).

Substituting (3.29)–(3.30) into (3.28) leads to

$$\frac{d}{dt}\left(\int_0^1 \eta^2 dx\right) + \left(\frac{3}{2} - O(1)\delta_0\right)\int_0^1 \eta^2 dx \le O(1)\int_0^1 (\psi^2 + \psi^2)dx. \tag{3.32}$$

Integrating (3.32) over $[0,t]$ gives

$$\int_0^1 \eta^2 dx \le \exp\{-c_0 t\}\int_0^1 \eta_0^2 dx + O(1)(1-\exp\{-c_0 t\})\int_0^1 (\psi^2+\psi^2)dx, \tag{3.33}$$

with a constant $0 < c_0 < \frac{3}{2} - O(1)\delta_0$, which implies (3.14). □

Lemma 3.3. *It holds, for $(\psi, \eta, e) \in X(T)$,*

$$\int_0^1 (\psi_t^2 + \psi_x^2 + \psi^2)dx \le O(1)||(\psi_0,\eta_0)||_2^2 \exp\{-\beta_1 t\}, \tag{3.34}$$

$$\int_0^1 (e^2 + e_x^2 + e_{xx}^2)dx \le O(1)||(\psi_0,\eta_0)||_2^2 \exp\{-\beta_1 t\}, \tag{3.35}$$

with $\beta_1 > 0$ a constant, provided that $N(T) + \delta_0$ small enough. □

Proof: Differentiating (3.5) with respect to x and using (3.4) and (3.6), we obtain the "wave equation with friction"

$$\psi_{tt} + \psi_t + (\rho_0 + \phi_{0xx} + \psi)\psi + (\rho_0 + \psi)_x e_x - \left[\frac{(j_0+\eta)^2}{\rho_0+\psi} - \frac{j_0}{\rho_0} + p(\rho_0+\psi) - p(\rho_0)\right]_{xx} = 0. \tag{3.36}$$

Multiplying (3.36) with ψ and integrating over $(0,1)$, one has, by (3.8) and integration by parts,

$$\begin{aligned}
&\frac{d}{dt}\left(\int_0^1 \frac{1}{2}\psi^2 + \psi\psi_t dx\right) - \int_0^1 \psi_t^2 dx + \int_0^1 (\rho_0 + \phi_{0xx} + \psi)\psi^2 dx \\
&= -\int_0^1 (\rho_0+\psi)_x e_x \psi dx \\
&\quad - \int_0^1 \left(\frac{(j_0+\eta)^2}{\rho_0+\psi} - \frac{j_0^2}{\rho_0} + p(\rho_0+\psi) - p(\rho_0)\right)_x \psi_x dx \\
&\triangleq I_4 + I_5.
\end{aligned} \tag{3.37}$$

By the Hölder inequality and (3.12), I_4 can be estimated as

$$\begin{aligned}
|I_4| &\le (\delta_0 + N(T))\left(\int_0^1 e_x^2 dx\right)^{1/2}\left(\int_0^1 \psi^2 dx\right)^{1/2} \\
&\le O(1)(\delta_0 + N(T))\int_0^1 \psi^2 dx.
\end{aligned} \tag{3.38}$$

By (3.4) and (3.14), we can estimate I_5 as

$$\begin{aligned}
I_5 = &-\int_0^1 \psi_x\left\{(p'(\rho_0) - \frac{j_0^2}{\rho_0^2})\psi\right\}_x dx \\
&-\int_0^1 \psi_x\left\{(\frac{1}{2}p''(\rho_0+\theta_6\psi) + \frac{j_0^2}{(\rho_0+\theta_7\psi)^2})\psi^2\right\}_x dx \\
&-\int_0^1 \psi_x\left\{\frac{2j_0\eta}{\rho_0} + \frac{\eta^2}{\rho_0+\theta_8\psi} - \frac{2(j_0+\theta_{10}\eta)}{(\rho_0+\theta_9\psi)^2}\psi\eta\right\}_x dx \\
\le &-\int_0^1 (p'(\rho_0) - \frac{j_0^2}{\rho_0^2})\psi_x^2 dx + O(1)(\delta_0+N(T))\int_0^1 (|\psi\psi_x| + |\eta\psi_x|)dx \\
&+\int_0^1 \left(\left|\frac{2j_0}{\rho_0}\right| + N(T)\right)|\psi_t\psi_x| dx \\
\le &-\frac{1}{2}\int_0^1 (p'(\rho_0) - \frac{j_0^2}{\rho_0^2})\psi_x^2 dx + O(1)(\delta_0+N(T))\int_0^1 (\psi_x^2 + \psi^2 + \eta^2)dx \\
&+ 2\int_0^1 \frac{j_0^2}{\rho_0^2 p'(\rho_0) - j_0^2}\psi_t^2 dx \\
\le &-\frac{1}{2}\int_0^1 (p'(\rho_0) - \frac{j_0^2}{\rho_0^2})\psi_x 2 dx + O(1)(\delta_0+N(T))\int_0^1 (\psi_x^2 + \psi^2 + \psi_t^2)dx \\
&+ 2a\int_0^1 \psi_t^2 + \exp\{-c_0 t\}\int_0^1 \eta_0^2 dx,
\end{aligned} \tag{3.39}$$

where $0<\theta_i<1$, $(i=6,7,8,9,10)$, and

$$0<a<\max_{x\in(0,1)}\frac{j_0(x)^2}{\rho_0(x)^2p'(\rho_0(x))-j_0(x)^2}. \tag{3.40}$$

Substituting (3.38) and (3.39) into (3.37), we have

$$\begin{aligned}&\frac{d}{dt}\left(\int_0^1\frac{1}{2}\psi_t^2+\psi_t\psi dx\right)-(1+2a)\int_0^1\psi_t^2dx\\&+\int_0^1(\rho_0+\phi_{0xx}+\psi)\psi^2dx+\frac{1}{2}\int_0^1(p'(\rho_0)-\frac{j_0^2}{\rho_0^2})\psi_x^2dx\\&\le O(1)(\delta_0+N(T))\int_0^1(\psi_x^2+\psi^2+\psi_t^2)dx+\exp\{-c_0t\}\int_0^1\eta_0^2dx.\end{aligned} \tag{3.41}$$

We multiply (3.36) with ψ_t and integrate over $(0,1)$. Noticing that $\psi_t(0,t)=\psi_t(1,t)=0$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\int_0^1\psi_t^2+(\rho_0+\phi_{0xx}+\psi)\psi^2dx\right)+\int_0^1(\psi_t^2-\frac{1}{2}\psi_t\psi^2)dx\\&=-\int_0^1(\rho_0+\psi)_xe_x\psi_tdx\\&\quad-\int_0^1\left(\frac{(j_0+\eta)^2}{\rho_0+\psi}-\frac{j_0^2}{\rho_0}+p(\rho_0+\psi)-p(\rho_0)\right)_x\psi_{xt}dx\\&\triangleq I_6+I_7.\end{aligned} \tag{3.42}$$

We estimate I_6 and I_7 as follows. Due to (3.12), it holds

$$|I_6|\le O(1)(\delta_0+N(T))\int_0^1(\psi^2+\psi_t^2)dx. \tag{3.43}$$

By (3.4), (3.14), and (3.16), I_7 can be estimated as

$$\begin{aligned}I_7=&-\int_0^1\psi_{xt}\Bigg((p'(\rho_0+\psi)-\frac{j_0^2+\eta}{\rho_0^2+\psi})\psi_x-2\frac{j_0+\eta}{\rho_0+\psi}\psi_t\\&\qquad+\rho_{0x}(p'(\rho_0+\psi)-p'(\rho_0)-\frac{(j_0+\eta)^2}{(\rho_0+\psi)^2}+\frac{j_0^2}{\rho_0^2})\Bigg)dx\\=&-\frac{d}{dt}\left(\frac{1}{2}\int_0^1(p'(\rho_0)-\frac{j_0^2}{\rho_0^2})\psi_x^2dx\right)\\&-\frac{d}{dt}\left(\int_0^1(\frac{1}{2}\psi_x^2+\psi_x\rho_{0x})(p'(\rho_0+\psi)-p'(\rho_0)-\frac{(j_0+\eta)^2}{(\rho_0+\psi)^2}+\frac{j_0^2}{\rho_0^2})dx\right)\\&+\int_0^1(\frac{1}{2}\psi_x^2+\psi_x\rho_{0x})(p''(\rho_0+\psi)\psi_t+\frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}\psi_t-\frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\eta_t)dx\\&-\int_0^1\psi_t^2(\frac{\eta_x}{\rho_0+\psi}-\frac{j_0+\eta}{(\rho_0+\psi)^2}(\rho_0+\psi)_x)dx\\\le&-\frac{d}{dt}\left(\frac{1}{2}\int_0^1(p'(\rho_0)-\frac{j_0^2}{\rho_0^2})\psi_x^2dx\right)\end{aligned}$$

$$
\begin{aligned}
&-\frac{d}{dt}\left(\int_0^1(\frac{1}{2}\psi_x^2+\psi_x\rho_{0x})(p'(\rho_0+\psi)-p'(\rho_0)-\frac{(j_0+\eta)^2}{(\rho_0+\psi)^2}+\frac{j_0^2}{\rho_0^2})dx\right)\\
&+O(1)(N(T)+\delta_0)\int_0^1(\psi_x^2+\psi_t^2+\eta_t^2+\eta^2)dx\\
\le&-\frac{d}{dt}\left(\frac{1}{2}\int_0^1(p'(\rho_0)-\frac{j_0^2}{\rho_0^2})\psi_x^2dx\right)\\
&-\frac{d}{dt}\left(\int_0^1(\frac{1}{2}\psi_x^2+\psi_x\rho_{0x})(p'(\rho_0+\psi)-p'(\rho_0)-\frac{(j_0+\eta)^2}{(\rho_0+\psi)^2}+\frac{j_0^2}{\rho_0^2})dx\right)\\
&+O(1)(N(T)+\delta_0)\int_0^1(\psi_x^2+\psi_t^2+\psi^2)dx+O(1)\exp\{-c_0t\}\int_0^1\eta_0^2dx. \qquad (3.44)
\end{aligned}
$$

Substituting (3.43) and (3.44) into (3.42), we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_0^1\frac{1}{2}\psi_t^2+\frac{1}{2}(p'(\rho_0)-\frac{j_0^2}{\rho_0^2})\psi_x^2+(\rho_0+\phi_{0xx}+\psi)\psi^2dx\right)\\
&+\frac{d}{dt}\left(\int_0^1(\frac{1}{2}\psi_x^2+\psi_x\rho_{0x})(p'(\rho_0+\psi)-p'(\rho_0)-\frac{(j_0+\eta)^2}{(\rho_0+\psi)^2}+\frac{j_0^2}{\rho_0^2})dx\right)\\
&+\int_0^1\psi_t^2dx\\
\le&O(1)(N(T)+\delta_0)\int_0^1(\psi_x^2+\psi_t^2+\psi^2)dx+O(1)\exp\{-c_0t\}\int_0^1\eta_0^2dx. \qquad (3.45)
\end{aligned}
$$

By $[(3.41)+2(1+2a)\times(3.45)]$, we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_0^1(1+2a)\psi_t^2+\frac{1}{2}\psi^2+\psi\psi_t+(1+2a)(p'(\rho_0)-\frac{j_0^2}{\rho_0^2})\psi_x^2dx\right)\\
&+\frac{d}{dt}\left(\int_0^1 2(1+2a)(\rho_0+\phi_{0xx}+\psi)\psi^2dx\right)\\
&+\frac{d}{dt}\left(\int_0^1(1+2a)(\psi_x^2+2\psi_x\rho_{0x})(p'(\rho_0+\psi)-p'(\rho_0)-\frac{(j_0+\eta)^2}{(\rho_0+\psi)^2}+\frac{j_0^2}{\rho_0^2})dx\right)\\
&+\int_0^1((1+2a)\psi_t^2+(\rho_0+\phi_{0xx}+\psi)\psi^2+\frac{1}{2}(p'(\rho_0)-\frac{j_0^2}{\rho_0^2})\psi_x^2)dx\\
\le&O(1)(N(T)+\delta_0)\int_0^1(\psi_x^2+\psi_t^2+\psi^2)dx+O(1)\exp\{-c_0t\}\int_0^1\eta_0^2dx. \qquad (3.46)
\end{aligned}
$$

Noticing, for positive constants c_1, c_2, c_3, that

$$
\begin{aligned}
&c_1(\psi_t^2+\psi_x^2+\psi^2)\\
\le&(1+2a)\psi_t^2+\frac{1}{2}\psi^2+\psi\psi_t+(1+2a)(p'(\rho_0)-\frac{j_0^2}{\rho_0^2})\psi_x^2\\
&+2(1+2a)(\rho_0+\phi_{0xx}+\psi)\psi^2\\
&+(1+2a)(\psi_x^2+2\psi_x\rho_{0x})(p'(\rho_0+\psi)-p'(\rho_0)-\frac{(j_0+\eta)^2}{(\rho_0+\psi)^2}+\frac{j_0^2}{\rho_0^2})
\end{aligned}
$$

$$
\begin{aligned}
&\leq c_2^{-1}((1+2a)\psi_t^2+(\rho_0+\phi_{0xx}+\psi)\psi^2+\frac{1}{2}(p'(\rho_0)-\frac{j_0^2}{\rho_0^2})\psi_x^2)\\
&\leq c_3(\psi_t^2+\psi_x^2+\psi^2),
\end{aligned}
\tag{3.47}
$$

and integrating (3.46) over $[0,t]$, we obtain (3.34) for a constant $\beta_1>0$ from the Sobolev embedding theorem, provided that $N(T)+\delta_0$ small enough.

Thus, (3.35) follows from (3.34), (3.20), (3.12), and (3.6). □

Lemma 3.4. *For $(\psi,\eta,e)\in X(T)$, we have*

$$
\int_0^1(\psi_t^2+\psi_{xt}^2+\psi_{tt}^2+\psi_{xx}^2)dx\leq O(1)\|(\psi_0,\eta_0)\|_2^2\exp\{-\beta_2 t\},
\tag{3.48}
$$

$$
\int_0^1(e_t^2+e_{xt}^2+e_{xxt}^2)dx\leq O(1)\|(\psi_0,\eta_0)\|_2^2\exp\{-\beta_2 t\},
\tag{3.49}
$$

with $\beta_2>0$, provided that $N(T)+\delta_0+\|(\psi_0,\eta_0)\|_2$ is small enough. □

Proof: Differentiating (3.36) with respect to t leads to

$$
\begin{aligned}
&\psi_{ttt}+\psi_{tt}+(\rho_0+\phi_{0xx}+2\psi)\psi_t+(\rho_0+\psi)_x e_{xt}+e_x\psi_{xt}\\
&-\left[\frac{(j_0+\eta)^2}{(\rho_0+\psi)^2}-\frac{j_0^2}{\rho_0^2}+p(\rho_0+\psi)-p(\rho_0)\right]_{xxt}=0.
\end{aligned}
\tag{3.50}
$$

Multiplying (3.50) with ψ_t, integrating it over $(0,1)$, using $\psi_t(0,t)=\psi_t(1,t)=0$ and (3.6), we have, after integration by parts

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_0^1\frac{1}{2}\psi_t^2+\psi_t\psi_{tt}dx\right)-\int_0^1\psi_{tt}^2dx+\int_0^1(\rho_0+\phi_{0xx}+\frac{3}{2}\psi)\psi_t^2dx\\
=&-\int_0^1(\rho_0+\psi)_x e_{xt}\psi_t dx\\
&-\int_0^1\left(\frac{(j_0+\eta)^2}{\rho_0+\psi}-\frac{j_0^2}{\rho_0}+p(\rho_0+\psi)-p(\rho_0)\right)_{xt}\psi_{xt}dx\\
\triangleq& I_8+I_9.
\end{aligned}
\tag{3.51}
$$

By the Hölder inequality and (3.13), it holds

$$
\begin{aligned}
|I_8|\leq&(\delta_0+N(T))\left(\int_0^1 e_{xt}^2dx\right)^{1/2}\left(\int_0^1\psi_t^2dx\right)^{1/2}\\
\leq&O(1)(\delta_0+N(T))\int_0^1\psi_t^2dx.
\end{aligned}
\tag{3.52}
$$

By (3.4), (3.15), (3.16), and (3.34), we estimate I_9 as

$$
\begin{aligned}
I_9=-\int_0^1\Bigg[&(p'(\rho_0+\psi)-\frac{(j_0+\eta)^2}{(\rho_0+\psi)^2})\psi_{xt}-2\frac{j_0+\eta}{(\rho_0+\psi)^2}\eta_t(\rho_0+\psi)_x\\
&+2\frac{(j_0+\eta)^2}{(\rho_0+\psi)^3}\psi_t(\rho_0+\psi)_x+p''(\rho_0+\psi)\psi_t(\rho_0+\psi)_x+2\frac{j_0+\eta}{\rho_0+\psi}\eta_{xt}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\frac{\eta_t\psi_t}{\rho_0+\psi} - 2\frac{j_0+\eta}{(\rho_0+\psi)^2}\eta_x\psi_t\Big]\psi_{xt}dx \\
\leq &-\int_0^1 (p'(\rho_0) - \frac{j_0^2}{\rho_0^2})\psi_{xt}^2 dx + 2\int_0^1 \left|\frac{j_0}{\rho_0}\right| |\psi_{tt}\psi_{xt}|dx \\
&+ O(1)\int_0^1 (|\eta| + N(T))(|\psi_{tt}\psi_{xt}| + \psi_{xt}^2)dx \\
&+ O(1)(\delta_0 + N(T))\int_0^1 |\psi_{xt}|(|\eta_t| + |\psi_t| + |\psi_x|)dx \\
\leq &-\frac{1}{2}\int_0^1 (p'(\rho_0) - \frac{j_0^2}{\rho_0^2})\psi_{xt}^2 dx + 2\int_0^1 \frac{j_0^2}{\rho_0^2 p'(\rho_0) - j_0^2}\psi_{tt}^2 dx \\
&+ O(1)(||(\psi_0,\eta_0)||_2 + N(T) + \delta_0)\int_0^1 (\psi_{tt}^2 + \psi_{xt}^2)dx \\
&+ O(1)||(\psi_0,\eta_0)||_2^2(\exp\{-c_0 t\} + \exp\{-\beta_1 t\})
\end{aligned}
\tag{3.53}
$$

Substituting (3.52) and (3.53) to (3.51) leads to

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_0^1 \frac{1}{2}\psi_t^2 + \psi_t\psi_{tt}dx\right) \\
&- (1+2a)\int_0^1 \psi_{tt}^2 dx - \frac{1}{2}\int_0^1 (p'(\rho_0) - \frac{j_0^2}{\rho_0^2})\psi_{xt}^2 dx \\
&+ \int_0^1 (\rho_0 + \phi_{0xx} + \frac{3}{2}\psi)\psi_t^2 dx \\
\leq &O(1)(||(\psi_0,\eta_0)||_2 + N(T) + \delta_0)\int_0^1 (\psi_{tt}^2 + \psi_{xt}^2)dx \\
&+ O(1)||(\psi_0,\eta_0)||_2^2(\exp\{-c_0 t\} + \exp\{-\beta_1 t\}).
\end{aligned}
\tag{3.54}
$$

We multiply (3.50) by ψ_{tt} and integrate the resulting equation over $(0,1)$. In terms of $\psi_t(0,t) = \psi_t(1,t) = 0$ and integration by parts, we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_0^1 \frac{1}{2}\psi_{tt}^2 + \frac{1}{2}(\rho_0 + \phi_{0xx} + 2\psi)\psi_t^2 dx\right) \\
&+ \int_0^1 \psi_{tt}^2 dx - \int_0^1 \psi_t^3 dx \\
= &-\int_0^1 ((\rho_0+\psi)_x e_{xt} + e_x\psi_{xt})\psi_{tt}dx \\
&- \int_0^1 \left(\frac{(j_0+\eta)^2}{\rho_0+\psi} - \frac{j_0^2}{\rho_0} + p(\rho_0+\psi) - p(\rho_0)\right)_{xt}\psi_{xtt}dx \\
\triangleq &I_{10} + I_{11}.
\end{aligned}
\tag{3.55}
$$

By (3.12), (3.13), and the Cauchy inequality, it is easy to verify

$$
|I_{10}| \leq \frac{1}{2}(\delta_0 + N(T))\int_0^1 (e_{xt}^2 + \psi_{tt}^2)dx + \frac{1}{2}\int_0^1 |e_x|(\psi_{tt}^2 + \psi_{xt}^2)dx
$$

$$\leq O(1)(\delta_0+N(T))\int_0^1(\psi_t^2+\psi_{tt}^2+\psi_{xt}^2)dx. \tag{3.56}$$

By (3.4), we have

$$\begin{aligned}
I_{11}=&-\int_0^1(p'(\rho_0+\psi)-\frac{(j_0+\eta)^2}{(\rho_0+\psi)^2})\psi_{xt}\psi_{xtt}dx\\
&+\int_0^1\left(\frac{2(j_0+\eta)}{\rho_0+\psi}\psi_{tt}+\frac{2\eta_t\psi_t}{\rho_0+\psi}-\frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\psi_t^2\right)\psi_{xtt}dx\\
&+\int_0^1\left(\frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\eta_t-\frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}\psi_t-p''(\rho_0+\psi)\psi_t\right)(\rho_0+\psi)_x\psi_{xtt}dx\\
\triangleq&K_1+K_2+K_3.
\end{aligned} \tag{3.57}$$

By (3.27), (3.15), and (3.34), we have

$$\begin{aligned}
K_1=&-\frac{d}{dt}\left(\int_0^1\frac{1}{2}(p'(\rho_0+\psi)-\frac{(j_0+\eta)^2}{(\rho_0+\psi)^2})\psi_{xt}^2dx\right)\\
&+\frac{1}{2}\int_0^1\psi_{xt}^2(p''(\rho_0+\psi)\psi_t+\frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}\psi_t-\frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\eta_t)dx\\
\leq&-\frac{d}{dt}\left(\int_0^1\frac{1}{2}(p'(\rho_0+\psi)-\frac{(j_0+\eta)^2}{(\rho_0+\psi)^2})\psi_{xt}^2dx\right)\\
&+O(1)(N(T)+||(\psi_0,\eta_0)||_2)\int_0^1\psi_{xt}^2dx.
\end{aligned} \tag{3.58}$$

Noticing $\psi_{tt}(0,t)=\psi_{tt}(1,t)=0$, we have, by (3.27), (3.15), (3.12), and (3.34),

$$\begin{aligned}
|K_2|=&|\int_0^1\psi_{tt}^2\left(\frac{2(j_0+\eta)}{\rho_0+\psi}\right)_xdx|+|\int_0^1\psi_{tt}\left(\frac{2\eta_t\psi_t}{\rho_0+\psi}-\frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\psi_t^2\right)_xdx|\\
\leq&O(1)\int_0^1\psi_{tt}^2(|\psi_t|+|\psi_x|+|\rho_{0x}|)dx+O(1)\int_0^1|\psi_{tt}\psi_{xt}|(|\psi_t|+|\eta_t|)dx\\
&+O(1)\int_0^1|\psi_{tt}|(\psi_t^2+|\eta_t\psi_t|+|\psi_x\psi_t|)dx\\
\leq&O(1)(N(T)+\delta_0+||(\psi_0,\eta_0)||_2)\int_0^1(\psi_{tt}^2+\psi_{xt}^2)dx\\
&+O(1)||(\psi_0,\eta_0)||_2^2\exp\{-\beta_1t\}.
\end{aligned} \tag{3.59}$$

At last, we estimate K_3 as

$$\begin{aligned}
K_3=&\frac{d}{dt}\left(\int_0^1\psi_{xt}(\rho_0+\psi)_x\left[\frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\eta_t-\frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}\psi_t-p''(\rho_0+\psi)\psi_t\right]dx\right)\\
&-\int_0^1\psi_{xt}^2\left[\frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\eta_t-\frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}\psi_t-p''(\rho_0+\psi)\psi_t\right]dx\\
&-\int_0^1\psi_{xt}(\rho_0+\psi)_x\left[\frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\eta_{tt}-4\eta_t\psi_t\frac{(j_0+\eta)}{(\rho_0+\psi)^3}+\frac{2\eta_t^2}{(\rho_0+\psi)^2}\right]dx\\
&+\int_0^1\psi_{xt}(\rho_0+\psi)_x\psi_{tt}\left[\frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}+p''(\rho_0+\psi)\right]dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 \psi_{xt}(\rho_0+\psi)_x\psi_t\left[\frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}+p''(\rho_0+\psi)\right]_t dx\\
\leq&\frac{d}{dt}\left(\int_0^1 \psi_{xt}(\rho_0+\psi)_x\left[\frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\eta_t-\frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}\psi_t-p''(\rho_0+\psi)\psi_t\right]dx\right)\\
&+O(1)(||(\psi_0,\eta_0)||_2+N(T)+\delta_0)\int_0^1(\psi_{tt}^2+\psi_{xt}^2)dx\\
&+O(1)||(\psi_0,\eta_0)||_2^2\exp\{-\beta_1 t\}+O(1)(N(T)+\delta_0)\int_0^1\eta_{tt}^2dx
\end{aligned}
\tag{3.60}
$$

Differentiating (3.5) with respect to t, and using (3.13), (3.16) and (3.34) we can estimate the last term in the right hand side of (3.60) as

$$
\begin{aligned}
\int_0^1 \eta_{tt}^2dx \leq&O(1)\int_0^1\left|\left(\frac{(j_0+\eta)^2}{\rho_0+\psi}-\frac{j_0^2}{\rho_0}+p(\rho_0+\psi)-p(\rho_0)\right)_{xt}\right|^2dx\\
&+O(1)\int_0^1(\eta_t^2+\psi_t^2+e_{xt}^2)dx\\
\leq&O(1)\int_0^1(\psi_{xt}^2+\psi_{tt}^2)dx\\
&+O(1)||(\psi_0,\eta_0)||_2^2(\exp\{-c_0t\}+\exp\{-\beta_1 t\}).
\end{aligned}
\tag{3.61}
$$

Substituting (3.61) into (3.60) leads to

$$
\begin{aligned}
K_3\leq&\frac{d}{dt}\left(\int_0^1 \psi_{xt}(\rho_0+\psi)_x\left[\frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\eta_t-\frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}\psi_t-p''(\rho_0+\psi)\psi_t\right]dx\right)\\
&+O(1)(||(\psi_0,\eta_0)||_2+N(T)+\delta_0)\int_0^1(\psi_{tt}^2+\psi_{xt}^2)dx\\
&+O(1)||(\psi_0,\eta_0)||_2^2(\exp\{-c_0t\}+\exp\{-\beta_1 t\}).
\end{aligned}
\tag{3.62}
$$

Combining (3.58), (3.59), and (3.62) with (3.57), we have

$$
\begin{aligned}
I_{11}\leq&-\frac{d}{dt}\left(\int_0^1\frac{1}{2}(p'(\rho_0)-\frac{j_0^2}{\rho_0^2})\psi_{xt}^2dx\right)\\
&-\frac{d}{dt}\left(\int_0^1\frac{1}{2}(p'(\rho_0+\psi)-p'(\rho_0)-\frac{(j_0+\eta)^2}{(\rho_0+\psi)^2}+\frac{j_0^2}{\rho_0^2})\psi_{xt}^2dx\right)\\
&+\frac{d}{dt}\left(\int_0^1 \psi_{xt}(\rho_0+\psi)_x\left[\frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\eta_t-\frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}\psi_t-p''(\rho_0+\psi)\psi_t\right]dx\right)\\
&+O(1)(||(\psi_0,\eta_0)||_2+N(T)+\delta_0)\int_0^1(\psi_{tt}^2+\psi_{xt}^2)dx\\
&+O(1)||(\psi_0,\eta_0)||_2^2(\exp\{-c_0t\}+\exp\{-\beta_1 t\}).
\end{aligned}
\tag{3.63}
$$

Then, it follows from (3.55), (3.56), (3.34) and (3.63)

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_0^1\frac{1}{2}\psi_{tt}^2+\frac{1}{2}(\rho_0+\phi_{0xx}+2\psi)\psi_t^2+\frac{1}{2}(p'(\rho_0)-\frac{j_0^2}{\rho_0^2})\psi_{xt}^2dx\right)\\
&-\frac{d}{dt}\left(\int_0^1\frac{1}{2}(p'(\rho_0+\psi)-p'(\rho_0)-\frac{(j_0+\eta)^2}{(\rho_0+\psi)^2}+\frac{j_0^2}{\rho_0^2})\psi_{xt}^2dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{d}{dt}\left(\int_0^1 \psi_{xt}(\rho_0+\psi)_x\left[\frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\eta_t-\frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}\psi_t-p''(\rho_0+\psi)\psi_t\right]dx\right)\\
&+\int_0^1 \psi_{tt}^2 dx\\
\le\,& O(1)(\|(\psi_0,\eta_0)\|_2+N(T)+\delta_0)\int_0^1(\psi_{tt}^2+\psi_{xt}^2)dx\\
&+O(1)\|(\psi_0,\eta_0)\|_2^2(\exp\{-c_0t\}+\exp\{-\beta_1 t\}).
\end{aligned}
\tag{3.64}
$$

By $[(3.54)+2(1+2a)\times(3.64)+(3.16)+(3.34)]$, we have, due to (3.34),

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_0^1 \frac{1}{2}\psi_t^2+\psi_t\psi_{tt}+(1+2a)\psi_{tt}^2dx\right)\\
&+\frac{d}{dt}\left(\int_0^1(1+2a)(\rho_0+\phi_{0xx}+2\psi)\psi_t^2+(1+2a)(p'(\rho_0)-\frac{j_0^2}{\rho_0^2})\psi_{xt}^2dx\right)\\
&-\frac{d}{dt}\left(\int_0^1(1+2a)(p'(\rho_0+\psi)-p'(\rho_0)-\frac{(j_0+\eta)^2}{(\rho_0+\psi)^2}+\frac{j_0^2}{\rho_0^2})\psi_{xt}^2dx\right)\\
&+\frac{d}{dt}\left(\int_0^1 2(1+2a)\psi_{xt}(\rho_0+\psi)_x\left[\frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\eta_t-\frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}\psi_t-p''(\rho_0+\psi)\psi_t\right]dx\right)\\
&+(1+2a)\int_0^1\psi_{tt}^2dx+\frac{1}{2}\int_0^1(p'(\rho_0)-\frac{j_0^2}{\rho_0^2})\psi_{xt}^2dx+\int_0^1(\psi_t^2+\eta_t^2)dx\\
\le\,& O(1)(\|(\psi_0,\eta_0)\|_2+N(T)+\delta_0)\int_0^1(\psi_{tt}^2+\psi_{xt}^2)dx\\
&+O(1)\|(\psi_0,\eta_0)\|_2^2(\exp\{-c_0t\}+\exp\{-\beta_1 t\}).
\end{aligned}
\tag{3.65}
$$

By (3.36), it holds

$$
\int_0^1 \psi_{xx}^2dx\le O(1)\int_0^1(\psi_{tt}^2+\psi_t^2+\psi_x^2+\psi_{xt}^2+e_x^2+\eta^2)dx. \tag{3.66}
$$

Then, integration of (3.65) over $(0,t)$, in terms of (3.16), (3.34), similar inequalities to (3.47), and the Sobolev embedding theorem, lead to (3.48).

The estimate (3.49) follows from (3.48) and (3.13). □

References

[1] U Ascher, P. A. Markowich and C. Schmeiser, A phase plane analysis of transonic solutions for the hydrodynamic semiconductor model, *Math. Models Meth. Appl. Sci.* **1**, 347–376 (1991).

[2] K. Bløtekjær, Transport equations for electrons in two-valley semiconductors, *IEEE Trans. Electron Devices* ED-17, (1970) 38–47.

[3] G. Chen, J. Jerome, and B. Zhang, Particle hydrodynamic moment models in biology and microelectronics: singular relaxation limits, preprint.

[4] G. Chen and D. Wang, Convergence of shock schemes for the compressible Euler-Poisson equations, *Comm. Math. Phys.* **179** (1996), 333-364.

[5] P. Degond and P. A. Markowich, On a one-dimensional steady-state hydrodynamic model, *Appl. Math. Lett.* **3** (1990), 25–29.

[6] P. Degond and P. A. Markowich, A steady-state potential flow model for semiconductors, *Ann. Math. Pure Appl.* **IV** (1993), 87-98.

[7] W. Fang and K. Ito, Steady-state solutions of a one-dimensional hydrodynamic model for semiconductors, *J. Differential Equations*, **133** (1997), 224–244.

[8] I. Gamba, Stationary transonic solutions of a one-dimensional hydrodynamic model for semiconductor, it Comm. Partial Diff. Equns **17** (3& 4) (1992), 553-577.

[9] I. Gamba and C. S. Morawetz, A viscous approximation for a 2-D steady semiconductor or transonic gas dynamic flow: existence theorem for potential flow, *Comm. Pure Appl. Math.* **49** (1996), 999–1049.

[10] I. Gasser and R. Natalini, The energy transport and the drift diffusion equations as relaxation limits of the hydrodynamic model for semiconductors, *Quart. Appl. Math.* **57** (1996), 269–282.

[11] H. Hattori, Stability and instability of steady-state solutions for a hydrodynamic model of semiconductors, *Proc. Roy. Soc. Edinburgh A* **127** (1997), 781–796.

[12] H. Hattori and C. Zhu, Asymptotic behavior of the solutions to a non-isentropic hydrodynamic model of semiconductors, *J. Differential Equations* **144** (1998), 353–389.

[13] L. Hsiao and H. L. Li, Shock reflection for the damped p-system, preprint.

[14] L. Hsiao and T. Yang, Asymptotic of initial boundary value problems for hydrodynamic and drift diffusion models for semiconductors, preprint.

[15] L. Hsiao and K. Zhang, The relaxation of the hydrodynamic model for semiconductors to drift diffusion equations, *J. Differential Equations*, to appear.

[16] L. Hsiao and K. Zhang, The global weak solution and relaxation limits of the initial boundary value problem to the bipolar hydrodynamic model for semiconductors, *Math. Models Methods Appl. Sci.* to appear.

[17] J. Jerome, *Analysis of charge transport: a mathematical study of semiconductor devices*, Springer-Verlag, Heidelberg (1996).

[18] J. Jerome and C. Shu, Energy models for one-carrier transport in semiconductor devices, accepted for publication.

[19] T. Luo, R. Natalini and Z. Xin, Large time behavior of the solutions to a hydrodynamic model for semiconductors, *SIAM J. Appl. Math.* **59** (1998), 810-830.

[20] P. Marcati and M. Mei, Asymptotic convergence to steady-state solutions of the initial boundary value problem to a hydrodynamic model for semiconductors, preprint.

[21] P. Marcati and R. Natalini, Weak solutions to a hydrodynamic model for semiconductors: the Cauchy problem, *Proc. Royal Soc. Edinburgh A* **A: 125** (1995), 115–131.

[22] P. Marcati and R. Natalini, Weak solutions to a hydrodynamic model for semiconductors and relaxation to the drift-diffusion equation, *Arch. Rational Mech. Anal.* **129** (1995), 129–145.

[23] P. A. Markowich, *The Steady-State Semiconductor Device Equations*, Springer, Wien, New York, (1986).

[24] P. A. Markowich, On steady-state Euler-Poisson model for semiconductors, *Z. Ang. Math. Phys.*, **62** (1991), 389–407.

[25] P. A. Markowich and C. Schmeiser, The drift-diffusion limit for electron-phonon interaction in semiconductors, *Math. Models Methods Appl. Sci.* **7** (1997), 707–729.

[26] P. A. Markowich and P. Pietra, A non-isentropic Euler-Poisson model for a collisionless plasma, *Math. Methods Appl. Sci.* **16** (1993), 409–442.

[27] P. A. Markowich, C. Ringhofer and C. Schmeiser, *Semiconductor Equations*, Springer, Wien, New York, (1989).

[28] F. Poupaud, On a system of nonlinear Boltzmann equations of semiconductor physics, *SIAM J. Appl. Math.* **50** (1990), 1593–1606.

[29] F. Poupaud, Diffusion approximation of the linear semiconductor Boltzmann equation: analysis of boundary layer, *Asymptotic analysis* **4** (1991), 293–317.

[30] F. Poupaud, M. Rascle and J.-P. Vila, global solutions to the isothermal Euler-Poisson system with arbitrarily large data, *J. Differential Equations*, **123** (1995), 93–121.

[31] S. Selberherr, *Analysis and Simulation of Semiconductor Device Equations*, Springer, Wien, New York, (1984).

[32] L. Yeh, Subsonic solutions of hydrodynamic model for semiconductors, *Math. Methods Appl. Sci.* **20** (1997), 1389–1410.

[33] B. Zhang, Convergence of the Godunov scheme for a simplified one-dimensional hydrodynamic model for semiconductor devices, *Comm. Math. Phys.* **157** (1993), 1–22.

[34] B. Zhang, On a local existence theorem for a simplified one-dimensional hydrodynamic model for semiconductor devices, *SIAM J. Math. Anal.* **25** (1994), 941–947.

[35] K. Zhang, Global weak solutions of the Cauchy problem to a hydrodynamic model for semiconductors, *J. Partial Differential Equations*, (1999).

[36] K. Zhang, Zero relaxation limit of global weak solutions of the Cauchy problem to a hydrodynamic model for semiconductors, preprint.