ON SPHERICALLY SYMMETRIC SOLUTIONS OF THE COMPRESSIBLE ISENTROPIC NAVIER-STOKES EQUATIONS

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Abstract. We prove the global existence of weak solutions to the Cauchy problem for the compressible isentropic Navier-Stokes equations in $\mathbb{R}^n$ $(n = 2, 3)$ when the Cauchy data are spherically symmetric. The proof is based on the exploitation of the one-dimensional feature of symmetric solutions and use of a new property induced by the viscous flux. The present paper extends Lions' existence theorem [12] to the case $1 < \gamma < \gamma_n$ for spherically symmetric initial data, where γ is the specific heat ratio in the pressure, $\gamma_n = 3/2$ for $n = 2$ and $\gamma_n = 9/5$ for $n = 3$.

I. Introduction

We prove the global existence of spherically symmetric weak solutions to the Cauchy problem for the compressible isentropic Navier-Stokes equations in two or three space dimensions.

The spherically symmetric motion of a compressible viscous, isentropic fluid in $\mathbb{R}^n$ $(n = 2, 3)$ is described by the system of equations in Eulerian coordinates:

$$\left\{ \begin{aligned} &\rho_t + (\rho u)_x + \frac{m\rho u}{x} = 0, \\ &(\rho u)_t + \left[\rho u^2 + P(\rho)\right]_x + \frac{m\rho u^2}{x} = \mu\, u_{xx} + \mu\, m\left[\frac{u}{x}\right]_x \end{aligned} \right. \tag{1.1}$$

together with initial and boundary conditions

$$\begin{aligned} &\rho(0,x) = \rho_0(x), \quad (\rho u)(0,x) = m_0(x), \qquad x \geq 0, \\ &u(t,0) = 0, \qquad t \geq 0, \end{aligned} \tag{1.2}$$

where $m = n - 1$ $(n = 2, 3)$,

$$P(\rho) = a\,\rho^\gamma,$$

ρ, u and $P(\rho)$ are the density, the velocity and the pressure respectively, and $\gamma \geq 1$ (the specific heat ratio), $a > 0$, $\mu > 0$ are constants.

Initial boundary value problems and the initial value problem for the compressible Navier-Stokes equations (of heat-conducting flow) have been studied by a great many authors. In one dimension, it is well known that global (smooth and weak) solutions exist for large initial data and are time-asymptotically stable. In more than one dimension, Matsumura and Nishida proved the existence of global smooth solutions and obtained the decay rates of solutions for sufficiently small initial data. See [15-17], also see [4, 5, 24, 20, 7, 10] and the references cited therein on more results for small data.

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For large initial data, the global existence and large-time behavior of sloutions to the Navier-Stokes equations for compressible heat-conducting flow have been obtained in the spherically symmetric case. See for example [18, 3, 2, 6, 8], and among others. Concerning the global existence for general large initial data in general domains, little was known until Lions's work [11-13] (also see [22, 23, 25, 14] for other cases). In [12, 13] Lions used the weak convergence methods and showed the existence of global weak solutions to the Navier-Stokes for compressible isentropic flow under the assumption that the specific heat ratio γ in the pressure law $P = a\rho^\gamma$ satisfies $\gamma \geq \gamma_n$, where $\gamma_n = 3/2$ for $n = 2$ and $\gamma_n = 9/5$ for $n = 3$. Unfortunately, this assumption excludes for example the interesting case $\gamma = 1.4$ (air, et al.).

The aim of this paper is to study the existence of global weak solutions for $1 < \gamma < \gamma_n$. We will prove that if the initial data are spherically symmetric, then the Cauchy problem (i.e. the problem (1.1), (1.2)) to the compressible isentropic Navier-Stokes equations possesses a global weak solution for any $\gamma > 1$. We mention that when $\gamma = 1$, the existence of a weak solution for BV-data has been proved by Hoff [3].

Now, let us recall the definition of weak solutions of (1.1), (1.2). The notation appearing below will be defined at the end of this section.

Definition. *We call $(\rho(t,x), u(t,x))$ a global weak solution of (1.1), (1.2), if*
1) $\rho \geq 0$ a.e., and for any $T > 0$,

$$\begin{aligned}
&\rho \in L^\infty([0,T], \mathcal{L}^\gamma(\mathbb{R}^+)),\ \rho u^2 \in L^\infty([0,T], \mathcal{L}^1(\mathbb{R}^+)),\ \ u_x,\ \frac{u}{x} \in L^2([0,T], \mathcal{L}^2(\mathbb{R}^+)),\\
&\rho \in C^0([0,T], \mathcal{L}^\gamma_{\rm loc}(\mathbb{R}^+_0) - w), \quad \rho u \in C^0([0,T], \mathcal{L}^{\frac{2\gamma}{\gamma+1}}_{\rm loc}(\mathbb{R}^+_0) - w),\\
&(\rho, \rho u)(0,x) = (\rho_0, m_0)(x) \ \textit{weakly in}\ \mathcal{L}^\gamma_{\rm loc}(\mathbb{R}^+_0) \times \mathcal{L}^{\frac{2\gamma}{\gamma+1}}_{\rm loc}(\mathbb{R}^+_0).
\end{aligned} \tag{1.3}$$

2) For any $t_2 \geq t_1 \geq 0$ and any $\varphi \in C^1_0(\mathbb{R} \times \mathbb{R}^+_0)$, there holds

$$\int_0^\infty \rho\varphi x^m \,dx|_{t_1}^{t_2} - \int_0^\infty \int_0^\infty (\rho\varphi_t + \rho u \varphi_x) x^m \,dx\,dt = 0. \tag{1.4}$$

3) For any $t_2 \geq t_1 \geq 0$ and any $\varphi \in C^1_0(\mathbb{R} \times \mathbb{R}^+_0)$ satisfying $\varphi(t,0) = 0$, there holds

$$\begin{aligned}
\int_0^\infty \rho u\varphi x^m \,dx|_{t_1}^{t_2} - \int_0^\infty \int_0^\infty &\left\{\rho u\varphi_t + \rho u^2 \varphi_x + P(\rho)\left(\varphi_x + \frac{m\varphi}{x}\right)\right\} x^m \,dx\,dt\\
&= -\mu \int_{t_1}^{t_2} \int_0^\infty \left(u_x\varphi_x + \frac{mu\varphi}{x^2}\right) x^m \,dx\,dt.
\end{aligned} \tag{1.5}$$

The main result of this paper reads:

Theorem 1.1. *Let $\gamma > 1$, $0 \le \rho_0 \in \mathcal{L}^1(\mathbb{R}^+) \cap \mathcal{L}^\gamma(\mathbb{R}^+)$, $m_0 \in \mathcal{L}^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}^+)$, $m_0^2/\rho_0 \in \mathcal{L}^1(\mathbb{R}^+)$. Then there exists a global weak solution (ρ, u) of (1.1), (1.2), such that*

$$\begin{aligned}\int_0^\infty \left(\frac{\rho u^2}{2} + \psi(\rho)\right)(t,x) x^m\, dx + \mu \int_0^t \int_0^\infty \left(u_x^2 + \frac{mu^2}{x^2}\right) x^m\, dx\, dt \\ \le \int_0^\infty \left(\frac{m_0^2}{\rho_0} + \psi(\rho_0)\right)(x) x^m\, dx \qquad \forall\, t \ge 0,\end{aligned} \tag{1.6}$$

where $\psi(\rho) = \frac{1}{\gamma-1}\rho^\gamma$. Moreover, for any $T > 0$ and any $\varphi_1 \in C_0^\infty(\mathbb{R}_0^+)$ with $\varphi_1(x) = O(x^n)$ as $x \to 0$, there holds

$$\int_0^T \int_0^\infty \rho^{2\gamma}(t,x)\varphi_1^4(x)\, dx\, dt \le C.$$

If, in addition, $\gamma > n/2$, then for any $T > 0$ and any $\varphi_2 \in C_0^\infty(\mathbb{R}_0^+)$ with $\varphi_2(x) = O(x^m)$ as $x \to 0$, there holds

$$\int_0^T \int_0^\infty \rho^{\gamma+\theta}(t,x)\varphi_2(x)\, dx\, dt \le C \qquad \forall\, \theta < \frac{2\gamma}{n} - 1.$$

Remark 1.1. *(i) For the case $\lim_{x\to\infty}\rho_0(x) = \rho_\infty > 0$, $\lim_{x\to\infty} u_0(x) = u_\infty$, we could use similar cut-off function arguments to those used in [Section 5.6, 12] and obtain a similar existence theorem.*
(i) If we define $\rho(t, \vec{x}) := \rho(t,x)$, $\mathbf{U}(t,\vec{x}) := u(t,x)\vec{x}/x$ for $t \ge 0$ and $\vec{x} \in \mathbb{R}^n$, where $x = |\vec{x}|$, $\vec{x} = (x_1, \cdots, x_n)$. Then it is easy to see that $(\rho(t,\vec{x}), \mathbf{U}(t,\vec{x}))$ is a weak solution of the Cauchy problem for the compressible isentropic Navier-Stokes equations in $\mathbb{R}^n$ (cf. the proof of Theorem 5.7 in [3]).

The proof of Theorem 1.1 is based on a basic energy estimates and passing to the limit for the approximate solutions. To pass to the limit, we mainly use and adapt the idea of Lions in the study of the compressible isentropic Navier-Stokes equations [12]. As mentioned before, in Lions' arguments, $\gamma \ge \gamma_n$ has to be assumed because, in order to estimate the commutators and to exclude the concentration of mass at the origin, the higher integrabilities of ρ (for example L^2-integrability) are needed. To overcome the difficulties induced by $\gamma < \gamma_n$, here we exploit the one-dimensional feature of symmetric solutions, use integrals instead of commutators and employ a new property induced by the viscous flux: $a\rho^\gamma - \mu u_x - \mu m u/x$ (cf. Remark 3.1), which shows the $L^{2/\theta}$-integrability of $\rho^\theta - \overline{\rho^\theta}$ near the origin for some small constant $\theta > 0$ (cf. Lemma 3.2), where $\overline{\rho^\theta}$ is the weak limit of the θ-th power of the approximate density sequence. This $L^{2/\theta}$-integrability of $\rho^\theta - \overline{\rho^\theta}$, though not the L^2-integrability of ρ itself, is sufficient to conclude that the concentration of mass cannot develop at the origin.

This paper is organized as follows. In Section 2 we construct an approximate solution sequence and derive a priori estimates for the approximate solutions. In Section 3 we give the proof of Theorem 1.1.

Notation *used throughout this paper:* Let Ω be a domain in $\mathbb{R}$. Let m be an integer and let $1 \le p \le \infty$. By $W^{m,p}(\Omega)$ ($W_0^{m,p}(\Omega)$) we denote the usual Sobolev space defined over Ω. $W^{m,2}(\Omega) \equiv H^m(\Omega)$ ($W_0^{m,2}(\Omega) \equiv H_0^m(\Omega)$), $W^{0,p}(\Omega) \equiv L^p(\Omega)$ with norm $\|\cdot\|_{L^p(\Omega)}$. We define $\mathcal{L}^p(\Omega) := \{f \in L^1_{\rm loc}(\Omega);\ \int_\Omega |f(x)|^p x^m\, dx < \infty\}$ with norm $\|\cdot\|_{\mathcal{L}^p(\Omega)} := \int_\Omega |\cdot|^p x^m dx$. $\mathcal{L}^p_{\rm loc}(\Omega)$ and $\mathcal{H}^1_{\rm loc}(\Omega)$ are defined similarly to $L^p_{\rm loc}(\Omega)$ and $H^1_{\rm loc}(\Omega)$, respectively. For simplicity we also use the following abbreviations:

$$\|\cdot\|_{\mathcal{L}^p} \equiv \|\cdot\|_{\mathcal{L}^p(\mathbb{R}^+)}, \qquad \mathbb{R}^+ := (0,\infty), \quad \mathbb{R}^+_0 := [0,\infty).$$

$L^p(I,B)$ resp. $\|\cdot\|_{L^p(I,B)}$ denotes the space of all strongly measurable, pth-power integrable (essentially bounded if $p=\infty$) functions from I to B resp. its norm, $I \subset \mathbb{R}$ an interval, B a Banach space. $C^0(I, B-w)$ is the space of all functions which are in $L^\infty(I,B)$ and continuous in t with values in B endowed with the weak topology.

The same letter C (sometimes used as $C(X)$ to emphasize the dependence of C on X) will denote various positive constants which do not depend on ϵ.

II. Approximate solutions and a priori estimates

We will use the approximate solutions constructed by Hoff in [3]. First, we mollify the initial data as follows

$$\rho_0^\epsilon(x) := x^{-\frac{m}{\gamma}} \left\{ \left(x^{\frac{m}{\gamma}} \rho_0 \right) * j_{\epsilon/2} \right\}(x) + \epsilon,$$
$$m_0^\epsilon(x) := x^{-\frac{m}{2}} \sqrt{\rho_0^\epsilon} \left\{ \left(x^{\frac{m}{2}} \frac{m_0}{\sqrt{\rho_0}} \right) * j_{\epsilon/2} \right\}(x), \quad u_0^\epsilon(x) := \left\{ \frac{\chi^\epsilon m_0^\epsilon}{\rho_0^\epsilon} \right\}(x),$$

where $j_{\epsilon/2}$ is the standard Friedrichs mollifier, χ^ϵ is a $C_0^\infty(\mathbb{R}^+)$-function satisfying $\chi^\epsilon(x) = 0$ for $x \le \epsilon$ and $\chi^\epsilon(x) = 1$ for $x \ge 2\epsilon$. Then, it is not difficult to prove that for any $N > 0$,

$$\int_\epsilon^\infty \left\{ |\rho_0^\epsilon - \rho_0 - \epsilon|^\gamma + \left| \sqrt{\rho_0^\epsilon} u_0^\epsilon - \frac{m_0}{\sqrt{\rho_0}} \right|^2 \right\} x^m\, dx + \int_\epsilon^N |\rho_0^\epsilon u_0^\epsilon - m_0|^{\frac{2\gamma}{\gamma+1}} x^m dx \to 0 \tag{2.1}$$

as $\epsilon \to 0$, and moreover,

$$\begin{aligned} \int_\epsilon^\infty |\rho_0^\epsilon - \rho_0 - \epsilon| x^m dx \le{}& N^{\frac{\gamma-1}{\gamma}} \left[\int_\epsilon^N |\rho_0^\epsilon - \rho_0 - \epsilon|^\gamma x^m dx \right]^{1/\gamma} \\ &+ C \int_{N-1}^\infty \rho_0 x^m dx \to 0 \quad \text{as } \epsilon \to 0,\ N \to \infty. \end{aligned} \tag{2.2}$$

Denote

$$\psi_\epsilon(\rho) := \frac{a\rho^\gamma}{\gamma-1} - \frac{a\epsilon^{\gamma-1}\rho}{\gamma-1} - a\epsilon^{\gamma-1}\rho + a\epsilon^\gamma. \tag{2.3}$$

Then, it is easy to see by (2.1) and (2.2) that

$$\int_\epsilon^\infty \left\{ \left| \rho_0^\epsilon (u_0^\epsilon)^2 - \frac{m_0^2}{\rho_0} \right| + |\psi_\epsilon(\rho_0^\epsilon) - \psi(\rho_0)| \right\} x^m dx \quad \text{as } \epsilon \to 0. \tag{2.4}$$

Now, we consider the following approximate problem of (1.1), (1.2):

$$\begin{cases} [\rho_\epsilon]_t + [\rho_\epsilon u_\epsilon]_x + \dfrac{m\rho_\epsilon u_\epsilon}{x} = 0, \\ [\rho_\epsilon u_\epsilon]_t + \left[\rho_\epsilon (u_\epsilon)^2 + P(\rho_\epsilon)\right]_x + \dfrac{m\rho_\epsilon (u_\epsilon)^2}{x} = \mu\, [u_\epsilon]_{xx} + \mu\, m \left[\dfrac{u_\epsilon}{x}\right]_x \end{cases} \tag{2.5}$$

together with initial and boundary conditions

$$\begin{aligned} &\rho_\epsilon(0,x) = \rho_0^\epsilon(x), \quad u_\epsilon(0,x) = u_0^\epsilon(x), \qquad x \ge \epsilon, \\ &u_\epsilon(t,\epsilon) = 0, \qquad t \ge 0. \end{aligned} \tag{2.6}$$

Then, by Theorem 4.1 of [3] (also see Theorem 5.6 of [3]), the problem (2.5), (2.6) has a global weak solution $(\rho_\epsilon(t,x), u_\epsilon(t,x))$ on $[0,\infty) \times [\epsilon,\infty)$ with positive ρ_ϵ, such that $\rho_\epsilon \in C^0([0,\infty), L^\gamma_{\text{loc}}([\epsilon,\infty)))$, $u_\epsilon \in C^0([0,\infty), L^2(\epsilon,\infty)) \cap C^0((0,\infty), H_0^1(\epsilon,\infty))$, $\rho_\epsilon(t,x)$ is pointwise bounded from above, and

$$\begin{aligned} &\int_\epsilon^\infty \left(\frac{1}{2}\rho_\epsilon u_\epsilon^2 + \psi_\epsilon(\rho_\epsilon)\right)(t,x) x^m\, dx + \mu \int_0^t \int_\epsilon^\infty \left([u_\epsilon]_x^2 + m\frac{u_\epsilon^2}{x^2}\right) x^m\, dx dt \\ &\quad \le \int_\epsilon^\infty \left[\frac{1}{2}\rho_0^\epsilon (u_0^\epsilon)^2 + \psi_\epsilon(\rho_0^\epsilon)\right] x^m\, dx \le C \qquad \forall\, t \ge 0, \end{aligned} \tag{2.7}$$

where ψ_ϵ is the same as in (2.3) and we have used (2.4). Recalling the definition of ψ_ϵ, we get from (2.7) and Young's inequality that for any $t \ge 0$ and any $N > \epsilon$,

$$\int_\epsilon^N \rho_\epsilon^\gamma(t,x) x^m\, dx \le C N \epsilon^\gamma + \int_\epsilon^\infty \left[\frac{\rho_0^\epsilon (u_0^\epsilon)^2}{2} + \psi_\epsilon(\rho_0^\epsilon)\right] x^m\, dx. \tag{2.8}$$

In order to obtain Theorem 1.1 we have to prove the precompactness of the approximate solution sequence $(\rho_\epsilon, u_\epsilon)$. For this purpose, we need some higher space-time integration estimates for the density ρ_ϵ, which will be derived in the following two lemmas. For the sake of simplicity we will omit the subscript ϵ from now on until the end of this section.

Lemma 2.1. *For any $T>0$ and any $\varphi \in C_0^\infty([0,\infty))$, $\varphi(x)=O(x^n)$ as $x\to 0$, there exists a positive constant C, independent of ϵ, such that*

$$\int_0^T\int_\epsilon^\infty \rho^{2\gamma}(t,x)\varphi^4(x)\,dx\,dt \le C.$$

Proof. We multiply $(2.5)_2$ by φ and integrate over (x,∞) ($x\in[\epsilon,\infty)$) to obtain

$$\begin{aligned}&\partial_t\int_x^\infty \varphi\rho u\,dy-\int_x^\infty(\rho u^2+a\rho^\gamma)\varphi_y\,dy+\int_x^\infty\frac{m\rho u^2}{y}\varphi\,dy\\&=(a\rho^\gamma+\rho u^2)\varphi-\mu u_x\varphi-\mu m\frac{u}{x}\varphi-\mu\int_x^\infty\left[u_y+\frac{mu}{y}\right]\varphi_y\,dy,\end{aligned}\tag{2.9}$$

which yields

$$\begin{aligned}a\rho^{2\gamma}\varphi&=\mu\rho^\gamma\left(u_x+m\frac{u}{x}\right)\varphi-\rho^{1+\gamma}u^2\varphi+\rho^\gamma\partial_t\int_x^\infty\varphi\rho u\,dy\\&-\rho^\gamma\int_x^\infty(\rho u^2+a\rho^\gamma)\varphi_y dy+\rho^\gamma\int_x^\infty\frac{m\rho u^2}{y}\varphi dy+\mu\rho^\gamma\int_x^\infty\left[u_y+\frac{mu}{y}\right]\varphi_y dy.\end{aligned}\tag{2.10}$$

Mollifying the first equation of (2.5), we find that

$$\partial_t\rho^\delta+\partial_x(u\rho^\delta)+\frac{m\rho^\delta u}{x}=r^\delta,\tag{2.11}$$

where $\rho^\delta(t,x)=\rho(t,\cdot)*j_\delta$ and

$$r^\delta(t,x)=(u\rho^\delta)_x-(\rho u)_x*j_\delta+\frac{m\rho^\delta u}{x}-m\left[\frac{\rho u}{x}\right]*j_\delta.$$

By virtue of Lemma 2.3 of [11], $r^\delta\to 0$ in $L^1_{\rm loc}(\mathbb{R}^+\times\mathbb{R}^+)$. Thus, multiplying (2.11) by $\gamma(\rho^\delta)^{\gamma-1}$, we have

$$\partial_t(\rho^\delta)^\gamma+\partial_x\left(u(\rho^\delta)^\gamma\right)+\frac{m\gamma(\rho^\delta)^\gamma u}{x}=\gamma(\rho^\delta)^{\gamma-1}r^\delta+(1-\gamma)u_x(\rho^\delta)^\gamma.\tag{2.12}$$

Since $\rho\in L^\infty((0,T)\times(\epsilon,\infty))$, we get by taking $\delta\to 0$ in (2.12),

$$\partial_t\rho^\gamma+\partial_x(u\rho^\gamma)+\frac{m\gamma\rho^\gamma u}{x}=(1-\gamma)u_x\rho^\gamma\tag{2.13}$$

in $\mathcal{D}'((0,T)\times(\epsilon,\infty))$. Then

$$\begin{aligned}\rho^\gamma\partial_t\int_x^\infty\varphi\rho u\,dy&=-\partial_t\rho^\gamma\int_x^\infty\varphi\rho u\,dy+\partial_t\left[\rho^\gamma\int_x^\infty\varphi\rho u\,dy\right]\\&=\partial_t\left[\rho^\gamma\int_x^\infty\varphi\rho u\,dy\right]+\left[\partial_x(u\rho^\gamma)+(\gamma-1)\rho^\gamma u_x+\frac{m\gamma\rho^\gamma u}{x}\right]\int_x^\infty\varphi\rho u\,dy\\&=\partial_t\left[\rho^\gamma\int_x^\infty\varphi\rho u\,dy\right]+\partial_x\left[u\rho^\gamma\int_x^\infty\varphi\rho u\,dy\right]\\&\quad+\left[(\gamma-1)\rho^\gamma u_x+\frac{m\gamma\rho^\gamma u}{x}\right]\int_x^\infty\varphi\rho u\,dy+u^2\rho^{1+\gamma}\varphi.\end{aligned}\tag{2.14}$$

Inserting (2.14) into (2.10), we find

$$
\begin{aligned}
a\rho^{2\gamma}\varphi &= \mu\rho^\gamma u_x\varphi + \mu m\frac{\rho^\gamma u\varphi}{x} + \partial_t\left[\rho^\gamma\int_x^\infty \varphi\rho u\,dy\right] + \partial_x\left[u\rho^\gamma\int_x^\infty \varphi\rho u\,dy\right]\\
&+\left[(\gamma-1)\rho^\gamma u_x + \frac{m\gamma\rho^\gamma u}{x}\right]\int_x^\infty \varphi\rho u\,dy - \rho^\gamma\int_x^\infty (\rho u^2 + a\rho^\gamma)\varphi_y\,dy\\
&+\rho^\gamma\int_x^\infty \frac{m\rho u^2}{y}\varphi\,dy + \mu\rho^\gamma\int_x^\infty u_y\varphi_y\,dy + \mu m\rho^\gamma\int_x^\infty \frac{u}{y}\varphi_y\,dy.
\end{aligned}
\tag{2.15}
$$

Now, if we multiply (2.15) by φ^3 and integrate over $(0,T)\times(\epsilon,\infty)$, we get

$$
\int_0^T\int_\epsilon^\infty \rho^{2\gamma}\varphi^4\,dx\,dt = \int_0^T\int_\epsilon^\infty \{\text{R.H.S. of (2.15)}\}\,\varphi^3 dx\,dt.
\tag{2.16}
$$

Next we estimate each term on the right hand side of (2.16). The first term on the right hand side of (2.16) can be bounded as follows using (2.7).

$$
\begin{aligned}
\int_0^T\int_\epsilon^\infty \rho^\gamma u_x\varphi^4\,dx\,dt &\le \delta\int_0^T\int_\epsilon^\infty \rho^{2\gamma}\varphi^4\,dx\,dt + \frac{1}{\delta}\int_0^T\int_\epsilon^\infty u_x^2\varphi^4\,dx\,dt\\
&\le \frac{C}{\delta} + \delta\int_0^T\int_\epsilon^\infty \rho^{2\gamma}\varphi^4\,dx\,dt. \qquad (0<\delta<1)
\end{aligned}
\tag{2.17}
$$

A similar estimate can be obtained for the term $\int_0^T\int_0^\infty \frac{\rho^\gamma u\varphi^4}{x}\,dxdt$ in the same manner.

It is easy to see that by (2.7) and (2.8),

$$
\left|\int_x^\infty \rho u\varphi\,dy\right| \le \frac{1}{2}\int_\epsilon^\infty \rho(u^2+1)\varphi\,dy\,dx \le C,
\tag{2.18}
$$

whence by (2.1),

$$
\begin{aligned}
\int_0^T\int_\epsilon^\infty \partial_t\left[\rho^\gamma\int_x^\infty \varphi\rho u\,dy\right]\varphi^3 dxdt &\le C + \int_\epsilon^\infty \rho^\gamma\varphi^3\int_x^\infty \rho u\varphi\,dydx\\
&\le C + C\int_0^\infty \varphi^3\rho^\gamma(T,x)dx \le C.
\end{aligned}
\tag{2.19}
$$

Using the boundary condition $(2.6)_2$, (2.18) and (2.7), we deduce that

$$
\begin{aligned}
\int_0^T\int_\epsilon^\infty \varphi^3\left[u\rho^\gamma\int_x^\infty \varphi\rho u dy\right]_x dxdt &= -3\int_0^T\int_\epsilon^\infty \varphi^2\varphi_x\rho^\gamma u\int_x^\infty \varphi\rho u\,dydxdt\\
&\le C\int_0^T\int_\epsilon^\infty \varphi^2|\varphi_x|\,\rho^\gamma\,|u|\,dxdt\\
&\le \delta\int_0^T\int_\epsilon^\infty \rho^{2\gamma}\varphi^4\,dx\,dt + \frac{C}{\delta}\int_0^T\int_\epsilon^\infty u^2\varphi_x^2\,dx\,dt\\
&\le \frac{C}{\delta} + \delta\int_0^T\int_\epsilon^\infty \rho^{2\gamma}\varphi^4\,dx\,dt.
\end{aligned}
\tag{2.20}
$$

The terms $\int_0^T\int_\epsilon^\infty u_x\rho^\gamma\varphi^3\int_x^\infty \varphi\rho u\,dydxdt$ and $\int_0^T\int_\epsilon^\infty \frac{\rho^\gamma u\varphi^3}{x}\int_x^\infty \varphi\rho u\,dydxdt$ can be estimated exactly in the same manner.

We employ (2.7) and (2.8) to deduce that

$$\begin{aligned}(2.21)\qquad &\int_0^T\int_\epsilon^\infty \rho^\gamma\varphi^3\int_x^\infty(\rho u^2+a\rho^\gamma)\varphi_y\,dy\,dx\,dt\\ &\quad\le C\int_0^T\int_\epsilon^\infty\rho^\gamma\varphi^3\int_x^\infty(\rho u^2+\rho^\gamma)y^m\,dy\,dx\,dt\le C.\end{aligned}$$

Analogously to (2.21), we can estimate the terms $\int_0^T\int_\epsilon^\infty\rho^\gamma\varphi^3\int_x^\infty\frac{m\rho u^2}{y}\varphi\,dydxdt$, $\int_0^T\int_\epsilon^\infty\rho^\gamma\varphi^3\int_x^\infty u_y\varphi_y\,dydxdt$ and $\int_0^T\int_\epsilon^\infty\rho^\gamma\varphi^3\int_x^\infty\frac{u}{y}\varphi_y\,dydxdt$.

Summing up (2.16), (2.17), (2.19)–(2.21) and the related estimates, we finally get

$$\int_0^T\int_\epsilon^\infty\rho^{2\gamma}\varphi^4\,dx\,dt\le C+5\delta\int_0^T\int_\epsilon^\infty\rho^{2\gamma}\varphi^4\,dx\,dt.$$

By taking $\delta=1/10$ in the above inequality, we obtain the lemma. □

Remark 2.1. *By applying (2.12), we can make some seemingly formal arguments in (2.14) rigorous. For simplicity, we will omit it here.*

In the following lemma we prove a higher space-time estimate (regularity) in $\mathbb{R}^n$ for the density around the origin.

Lemma 2.2. *Let $\gamma>n/2$. Then, for any $T>0$ and any $\psi\in C_0^\infty([0,\infty))$ with $\psi\ge0$, $\psi(x)=O(x^m)$ as $x\to0$, there is a positive constant C, independent of ϵ, such that*

$$\int_0^T\int_\epsilon^\infty\rho^{\gamma+\theta}(t,x)\psi(x)\,dxdt\le C$$

for any $0<\theta<2\gamma/n-1$.

Proof. The proof is long, we break up it into several steps. First, without loss of generality, we can assume further that $\psi(x)=x^m$ for $0\le x\le1$.

Step **1.** *A representation of $\int_0^T\int_\epsilon^\infty\rho^{\gamma+\theta}\psi\,dx\,dt$*

Let $\phi\in C_0^\infty([0,\infty))$ with $\phi=1$ on $\operatorname{supp}\psi$. Multiplying $(2.5)_2$ by ϕ and integrating then the resulting equation over (x,∞), we obtain

$$\begin{aligned}(\rho u^2+&a\rho^\gamma)\phi-\partial_t\int_x^\infty\rho u\phi\,dy-m\int_x^\infty\frac{\rho u^2}{y}\phi dy\\ &=\mu\left[u_x+m\frac{u}{x}\right]\phi+\int_x^\infty\left[\mu u_y+\mu m\frac{u}{y}-\rho u^2-a\rho^\gamma\right]\phi_y\,dy\end{aligned}$$

Now, let θ be a positive number. Multiplying the above equation by $\rho^\theta\psi$, we find

$$\begin{aligned} a\rho^{\gamma+\theta}\psi &= \left[\mu u_x + \mu m\frac{u}{x} - \rho u^2\right]\rho^\theta\psi + \rho^\theta\psi\partial_t\int_x^\infty \rho u\phi\, dy \\ &\quad + m\rho^\theta\psi\int_x^\infty \frac{\rho u^2}{y}\phi\, dy + \rho^\theta\psi\int_x^\infty\left[\mu u_y + \mu m\frac{u}{y} - \rho u^2 - a\rho^\gamma\right]\phi_y\, dy. \end{aligned} \tag{2.22}$$

When $\theta \geq 1$, we can easily show by the same arguments as in the proof of (2.13) that

$$\partial_t\rho^\theta + \partial_x(u\rho^\theta) + \frac{m\theta\rho^\theta u}{x} = (1-\theta)u_x\rho^\theta. \tag{2.23}$$

For $\theta < 1$, set $\beta_\nu(\rho) = (\rho+\nu)^\theta$. Then, multiplying (2.11) with $\beta_\nu'(\rho^\delta)$, we arrive at

$$\partial_t\beta_\nu(\rho^\delta) + \partial_x[u\beta_\nu(\rho^\delta)] + [\beta_\nu'(\rho^\delta)\rho^\delta - \beta_\nu(\rho^\delta)]u_x + \frac{m\beta_\nu'(\rho^\delta)\rho^\delta u}{x} = \beta_\nu'(\rho^\delta)r^\delta.$$

Taking $\delta \to 0$ in the above equation, one gets

$$\partial_t\beta_\nu(\rho) + \partial_x\left[u\beta_\nu(\rho)\right] + \left[\beta_\nu'(\rho)\rho - \beta_\nu(\rho)\right]u_x + \frac{m\beta_\nu'(\rho)\rho u}{x} = 0. \tag{2.24}$$

Note that

$$(\rho+\nu)^\theta - \rho^\theta = \int_0^1 \frac{\theta\nu}{(\rho+\tau\nu)^{1-\theta}}\, d\tau \leq \theta\nu^\theta\int_0^1\frac{d\tau}{\tau^{1-\theta}} \leq \nu^\theta. \tag{2.25}$$

On the other hand,

$$\begin{aligned} \beta_\nu'(\rho)\rho - \theta\rho^\theta &= \theta[(\rho+\nu)^{\theta-1}\rho - \rho^\theta] \\ &= \theta[(\rho+\nu)^{\theta-1}\rho - (\rho+\nu)^\theta] + \theta[(\rho+\nu)^\theta - \rho^\theta] \end{aligned} \tag{2.26}$$

and

$$|(\rho+\nu)^{\theta-1}\rho - (\rho+\nu)^\theta| = \nu(\rho+\nu)^{\theta-1} \leq \nu^\theta. \tag{2.27}$$

Thus, letting $\nu \to 0$ in (2.24) and utilizing (2.25)–(2.27), we find that for $\theta < 1$, (2.23) remains valid. Therefore, (2.23) holds for any $\theta > 0$.

With the help of (2.23) and (2.22), following a procedure similar to that used for (2.15), we obtain

$$\begin{aligned} a\rho^{\gamma+\theta}\psi &= \mu\left\{u_x + m\frac{u}{x}\right\}\rho^\theta\psi + \partial_t\left\{\rho^\theta\psi\int_x^\infty \rho u\phi dy\right\} + \partial_x\left\{u\rho^\theta\psi\int_x^\infty\rho u\phi dy\right\} \\ &\quad - \left\{u\rho^\theta\psi_x + (1-\theta)\rho^\theta u_x\psi - m\theta\rho^\theta\frac{u}{x}\psi\right\}\int_x^\infty \rho u\phi\, dy \\ &\quad + m\rho^\theta\psi\int_x^\infty\frac{\rho u^2}{y}\phi\, dy + \rho^\theta\psi\int_x^\infty\left\{\mu u_y + \mu m\frac{u}{y} - \rho u^2 - a\rho^\gamma\right\}\phi_y dy \\ &\equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned} \tag{2.28}$$

In the sequel, we derive bounds for $\int_0^T\int_\epsilon^\infty I_j\,dxdt$ $(1\le j\le 6)$ on the right-hand side of (2.28).

Step **2.** *Estimate of* $\int_0^T\int_\epsilon^\infty I_1\,dxdt$ *and* $\int_0^T\int_\epsilon^\infty I_2\,dxdt$

It follows from (2.7) and Young's inequality that

$$\begin{aligned}\left|\int_0^T\int_\epsilon^\infty \rho^\theta u_x\psi\,dx\,dt\right| &\le \delta\int_0^T\int_\epsilon^\infty \rho^{2\theta}\psi\,dx\,dt+\frac{1}{\delta}\int_0^T\int_\epsilon^\infty u_x^2\psi\,dx\,dt\\ &\le \frac{C}{\delta}+\delta\int_0^T\int_\epsilon^\infty \rho^{\gamma+\theta}\psi\,dx\,dt \qquad \text{for } \theta\le\gamma.\end{aligned}\tag{2.29}$$

The term $\int_0^T\int_\epsilon^\infty \rho^\theta u x^{-1}\psi dx\,dt$ can be bounded in the same manner.

For $x\ge 1$ we easily obtain

$$\left|\int_x^\infty \rho u\phi\,dy\right|\le\frac{1}{2}\int_1^\infty(\rho+\rho u^2)\phi x^m\,dx\le C,\tag{2.30}$$

where we have used (2.7) and (2.8). Hence,

$$\left|\int_\epsilon^\infty \rho^\theta\psi\int_x^\infty \rho u\phi\,dydx\right|\le\int_\epsilon^1\rho^\theta\psi\int_x^1\rho|u|\,dydx+C\int_\epsilon^\infty\rho^\theta\psi\,dx\tag{2.31}$$

and

$$\begin{aligned}\int_\epsilon^1\rho^\theta\psi\int_x^1\rho|u|\,dydx&=\int_\epsilon^1\rho^\theta x^m\int_x^1\rho|u|\,dydx\\ &\le\int_\epsilon^1\rho^\theta x^{\alpha_1}\int_x^1\rho|u|y^{m-\alpha_1}\,dydx,\end{aligned}\tag{2.32}$$

where $\alpha_1\le m$ is a non-negative constant which will be determined later on. From Hölder's inequality and (2.7)–(2.8) we get

$$\int_x^1\rho|u|y^{m-\alpha_1}dy\le\|\rho u^2\|^{1/2}_{\mathcal{L}^1(\epsilon,1)}\|\rho\|^{1/2}_{\mathcal{L}^\gamma(\epsilon,1)}\|y^{-\alpha_1}\|_{\mathcal{L}^q(\epsilon,1)}\le C,\tag{2.33}$$

where q, α_1 satisfy $\frac12+\frac{1}{2\gamma}+\frac1q=1$, $q\ge 2$ and $q\alpha_1-m<1$, that is, recalling $m=n-1$,

$$\alpha_1<\frac{n}{2}\left(1-\frac{1}{\gamma}\right).\tag{2.34}$$

Thus, substituting (2.33) into (2.32), we find that

$$\int_\epsilon^1\rho^\theta\psi\int_x^1\rho|u|dydx\le C\int_\epsilon^1\rho^\theta x^{\alpha_1}dx\le C\|\rho^\theta\|_{\mathcal{L}^p(\epsilon,1)}\|x^{\alpha_1-m}\|_{\mathcal{L}^q(\epsilon,1)}\le C,\tag{2.35}$$

where q, p satisfy $1/p + 1/q = 1$, $(m - \alpha_1)q < m + 1$ and $\theta p \leq \gamma$, that is,

$$\theta < \gamma - \frac{\gamma}{q} < \frac{\gamma}{n} + \frac{\gamma\alpha_1}{n} < \left(\frac{1}{n} + \frac{1}{2}\right)\gamma - \frac{1}{2}.$$

On the other hand, when $\theta < (\frac{1}{n} + \frac{1}{2})\gamma - \frac{1}{2}$, we easily deduce by (2.8) that

$$\int_\epsilon^\infty \rho^\theta \psi \, dx \leq C \int_\epsilon^\infty (1 + \rho^\gamma)\psi \, dx \leq C,$$

which together with (2.31) and (2.35) implies that for $0 < \theta < (\frac{1}{n} + \frac{1}{2})\gamma - \frac{1}{2}$,

$$\left| \int_\epsilon^\infty \rho^\theta \psi \int_x^\infty \rho u \phi \, dy dx \right| \leq C. \tag{2.36}$$

Step 3. *Estimate of $\int_0^T \int_\epsilon^\infty I_4 \, dxdt$ for $1 < \gamma \leq \frac{2n}{4-n}$*

By (2.30), we see that

$$\begin{aligned} \left| \int_0^T \int_\epsilon^\infty \rho^\theta u \psi_x \int_x^\infty \rho u \phi \, dy dx dt \right| &= \left| \int_0^T \int_\epsilon^\infty \rho u \phi \int_\epsilon^y \rho^\theta u \psi_x \, dx dy dt \right| \\ &\leq \int_0^T \int_\epsilon^1 \rho |u| \int_\epsilon^y \rho^\theta |u| x^{m-1} \, dx dy dt + C \int_0^T \int_\epsilon^\infty \rho^\theta |u| \, |\psi_x| dx dt. \end{aligned} \tag{2.37}$$

For $y \in [\epsilon, 1]$, one has by applying Hölder's inequality and (2.8) that

$$\begin{aligned} \int_\epsilon^y \rho^\theta |u| x^{m-1} dx &\leq y^{\alpha_2} \int_\epsilon^y \rho^\theta \frac{|u|}{x} x^{-\alpha_2 + m} dx \\ &\leq y^{\alpha_2} \|\rho\|^\theta_{\mathcal{L}^\gamma(\epsilon,1)} \|\frac{u}{x}\|_{\mathcal{L}^2(\epsilon,1)} \|x^{-\alpha_2}\|_{\mathcal{L}^q(\epsilon,1)} \\ &\leq C y^{\alpha_2} \|\frac{u}{x}\|_{\mathcal{L}^2(\epsilon,1)}, \end{aligned} \tag{2.38}$$

where $\theta/\gamma + 1/2 + 1/q = 1$, $\alpha_2 \geq 0$ and $q\alpha_2 - m < 1$, that is,

$$\alpha_2 < \frac{n}{q} = n\left(\frac{1}{2} - \frac{\theta}{\gamma}\right). \tag{2.39}$$

Obviously, by (2.7),

$$x^{m-1} u^2(t,x) = \int_\epsilon^x \partial_y [y^{m-1} u^2(t,y)] dy \leq C \int_\epsilon^\infty \left(u_y^2 + \frac{u^2}{y^2}\right) y^m dy \equiv A^2(t),$$

with $A(t) \in L^2(0,T)$, which results in

$$u(t,x) \le CA(t)\, x^{-(m-1)/2}, \qquad x \ge \epsilon,\ t \in [0,T]. \tag{2.40}$$

Therefore by (2.38), (2.40), (2.7) and (2.8),

$$\begin{aligned}
\int_0^T \int_\epsilon^1 \rho|u| \int_\epsilon^y \rho^\theta |u| x^{m-1} dx dy dt &\le C \int_0^T A(t) \|\frac{u}{x}\|_{\mathcal{L}^2(\epsilon,1)} \int_\epsilon^1 \rho y^{\alpha_2 - \frac{m-1}{2}} dy dt \\
&\le C \sup_{[0,T]} \int_\epsilon^1 \rho(\cdot,y) y^{\alpha_2 - \frac{m-1}{2}} dy \\
&\le C \sup_{[0,T]} \|\rho(\cdot)\|_{\mathcal{L}^\gamma(\epsilon,1)} \left[\int_\epsilon^1 y^{-\frac{(3m-2\alpha_2-1)\gamma}{2(\gamma-1)} + m} dy\right]^{\frac{(\gamma-1)}{\gamma}} \\
&\le C,
\end{aligned} \tag{2.41}$$

where α_2 satisfies$-\frac{(3m-2\alpha_2-1)\gamma}{2(\gamma-1)} + m > -1$, i.e.

$$\alpha_2 > \frac{n}{\gamma} - \frac{4-n}{2},$$

which together with (2.39) shows that θ should satisfy

$$\theta < \frac{2}{n}\gamma - 1. \tag{2.42}$$

Recalling Young's inequality and the fact that (2.42) gives $2\theta \le \gamma$, we deduce for $\theta < 2\gamma/n - 1$ that

$$\begin{aligned}
\left|\int_0^T \int_\epsilon^\infty \rho^\theta |u|\, |\psi_x| dx\, dt\right| &\le C \int_0^T \int_\epsilon^\infty \left(\frac{u^2}{x^2} + \rho^{2\theta}\right) x|\psi_x| dx\, dt \\
&\le C \int_0^T \int_\epsilon^\infty \left(\frac{u^2}{x^2} + 1 + \rho^\gamma\right) x|\psi_x| dx\, dt \\
&\le C.
\end{aligned}$$

Inserting the above inequality and (2.41) into (2.37), we obtain

$$\left|\int_0^T \int_\epsilon^\infty \rho^\theta u \psi_x \int_x^\infty \rho u \phi\, dy dx dt\right| \le C. \tag{2.43}$$

The terms $\int_0^T \int_\epsilon^\infty \rho^\theta u_x \psi \int_x^\infty \rho u \phi\, dy dx dt$ and $\int_0^T \int_\epsilon^\infty \rho^\theta u x^{-1} \psi \int_x^\infty \rho u \phi\, dy dx dt$ in I_4 can be estimated, following the same arguments as used in the derivation of (2.43).

Step **4.** *Estimate of* $\int_0^T \int_\epsilon^\infty I_4\,dxdt$ *for* $\gamma > \frac{2n}{4-n}$

First we notice that the estimate (2.37) still holds. To derive bounds for the first term on the right-hand side of (2.37), we apply Hölder's inequality to arrive at

$$
\begin{aligned}
\left|\int_\epsilon^y \rho^\theta \frac{|u|}{x} x^m\,dx\right| &\le y^{\alpha_4}\left|\int_\epsilon^y \rho^\theta \frac{|u|}{x} x^{-\alpha_4} x^m\,dx\right| \\
&\le C y^{\alpha_4}\|\rho\|^\theta_{\mathcal{L}^{\gamma+\theta}(\epsilon,1)}\|\frac{u}{x}\|_{\mathcal{L}^2(\epsilon,1)}\|x^{-\alpha_4}\|_{\mathcal{L}^q(\epsilon,1)} \\
&\le C y^{\alpha_4}\|\rho\|^\theta_{\mathcal{L}^{\gamma+\theta}(\epsilon,1)}\|\frac{u}{x}\|_{\mathcal{L}^2(\epsilon,1)}, \qquad y\in[0,1],
\end{aligned}
\tag{2.44}
$$

where $\frac{\theta}{\gamma+\theta}+\frac12+\frac1q = 1$ (i.e. $q = \frac{2(\gamma+\theta)}{\gamma-\theta} > 2$) and $q\alpha_4 < m+1$, that means that α_4 should satisfy

$$
\alpha_4 < \frac{n}{2} - \frac{n\theta}{\gamma+\theta}. \tag{2.45}
$$

Similarly (also by (2.7)–(2.8)),

$$
\begin{aligned}
\int_\epsilon^1 \rho|u|\,y^{\alpha_4}dy &\le C\,A^{\frac{2}{q}}(t)\int_\epsilon^1 \rho^{\frac12-\frac1q}|u|^{1-\frac2q}\rho^{\frac12+\frac1q}y^{\alpha_4-\frac{m-1}{q}-m}y^m\,dy \\
&\le C\,A^{\frac2q}(t)\|\rho^{\frac12}u\|^{1-\frac2q}_{\mathcal{L}^2(\epsilon,1)}\|\rho\|^{\frac12+\frac1q}_{\mathcal{L}^\gamma(\epsilon,1)}\|y^{\alpha_4-\frac{m-1}{q}-m}\|_{\mathcal{L}^p(\epsilon,1)} \\
&\le C\,A^{\frac2q}(t),
\end{aligned}
\tag{2.46}
$$

where $\frac12(1-\frac2q)+\frac1\gamma(\frac12+\frac1q)+\frac1p = 1$ and $p(m+\frac{m-1}{q}-\alpha_4) < m+1$. Therefore,

$$
n-1+\frac{n-2}{q} - n\left(1-\frac1\gamma\right)\left(\frac12+\frac1q\right) < \alpha_4,
$$

which combined with (2.45) and $q = \frac{2(\gamma+\theta)}{\gamma-\theta}$ implies that $\theta < 2\gamma/n-1$. Combining (2.44) with (2.46), utilizing Young's inequality, (2.7) and recalling $\psi(x) = x^m$ for $x\in[0,1]$, we obtain under (2.42) that

$$
\int_0^T\int_\epsilon^1 \rho|u|\int_\epsilon^y \rho^\theta|u|x^{m-1}\,dxdydt \le \frac{\delta}{2}\int_0^T\int_\epsilon^1 \rho^{\gamma+\theta}\psi\,dxdt + C(\delta). \tag{2.47}
$$

From (2.40) it follows that

$$
\begin{aligned}
\int_0^T\int_\epsilon^\infty \rho^\theta|u|\,|\psi_x|dxdt &\le \int_0^T\int_\epsilon^1 \rho^\theta\frac{|u|}{x}\psi\,dxdt + C\int_0^T A(t)\int_1^\infty \rho^\theta|\psi_x|dx \\
&\le C\int_0^T\int_\epsilon^\infty\left\{\rho^{2\theta}+\frac{u^2}{x^2}\right\}\psi\,dxdt + C\int_0^T A(t)\int_1^\infty(1+\rho^\gamma)|\psi_x|dx \\
&\le \frac{\delta}{2}\int_0^T\int_\epsilon^1 \rho^{\gamma+\theta}\psi\,dxdt + C(\delta).
\end{aligned}
$$

Inserting the above estimate and (2.47) into (2.37), we conclude

$$\left|\int_0^T\int_\epsilon^\infty \rho^\theta u\psi_x \int_x^\infty \rho u\phi\, dydxdt\right| \le \delta\int_0^T\int_\epsilon^1 \rho^{\gamma+\theta}\psi dxdt + C(\delta). \tag{2.48}$$

We can use the same arguments as used in the derivation of (2.48) to bound the terms $\int_0^T\int_0^\infty \rho^\theta u_x\psi\int_x^\infty \rho u\phi\, dydxdt$ and $\int_0^T\int_0^\infty \rho^\theta u x^{-1}\psi\int_x^\infty \rho u\phi\, dydxdt$.

Step **5.** *Estimate of* $\int_0^T\int_\epsilon^\infty I_5\, dxdt$ *and* $\int_0^T\int_\epsilon^\infty I_6\, dxdt$

We make use (2.40) to arrive at

$$\begin{aligned}
&\left|\int_0^T\int_\epsilon^\infty \rho^\theta\psi\int_x^\infty \frac{\rho u^2}{y}\phi\, dydxdt\right| = \left|\int_0^T\int_\epsilon^\infty \frac{\rho u^2}{y}\phi\int_\epsilon^y \rho^\theta\psi\, dxdydt\right| \\
&\qquad \le C\int_0^T A^2(t)\int_\epsilon^\infty \frac{\rho\phi}{y^m}\int_\epsilon^y \rho^\theta\psi\, dxdydt \\
&\qquad \le C\sup_{t\in[0,T]}\int_\epsilon^1 \frac{\rho}{y^m}\int_\epsilon^y \rho^\theta x^m\, dxdy + \sup_{t\in[0,T]}\int_1^\infty \rho\phi\int_\epsilon^y \rho^\theta\psi\, dxdy.
\end{aligned} \tag{2.49}$$

Here the right hand side of (2.49) can be bounded as follows, using (2.8) and (2.42).

$$\int_1^\infty \rho\phi\int_\epsilon^y \rho^\theta\psi dx\, dy \le C\int_1^\infty \rho\phi\, dy\int_\epsilon^\infty (1+\rho^\gamma)\psi dx \le C, \tag{2.50}$$

and

$$\begin{aligned}
\int_\epsilon^1 \frac{\rho}{y^m}\int_\epsilon^y \rho^\theta x^m dx\, dy &\le \int_\epsilon^1 \rho y^{\alpha_3-m}dy\int_\epsilon^1 \rho^\theta x^{-\alpha_3+m}dx \\
&\le C\|\rho\|^{\theta+1}_{\mathcal{L}^\gamma(\epsilon,1)}\|y^{\alpha_3-2m}\|_{\mathcal{L}^{\frac{\gamma}{\gamma-1}}(\epsilon,1)}\|x^{-\alpha_3}\|_{\mathcal{L}^{\frac{\gamma}{\gamma-\theta}}(\epsilon,1)} \\
&\le C,
\end{aligned} \tag{2.51}$$

where α_3 satisfies $\frac{(\alpha_3-2m)\gamma}{\gamma-1}+m>-1$ and $-\frac{\alpha_3\gamma}{\gamma-\theta}+m>-1$, that is,

$$n-2+\frac{n}{\gamma}<\alpha_3<n-\frac{n\theta}{\gamma},$$

which gives (2.42). Therefore, under (2.42) we conclude by (2.49)–(2.51) that

$$\left|\int_0^T\int_\epsilon^\infty \rho^\theta\psi\int_x^\infty \frac{\rho u^2}{y}\phi\, dydxdt\right| \le C \tag{2.52}$$

Finally, we derive bounds for the integral of I_6. Using (2.7) and (2.8), taking into account $\phi_x(x)=0$ for $x\in[0,1]$, we easily find that

$$\begin{aligned}&\int_0^T\int_\epsilon^\infty \rho^\theta\psi\int_x^\infty\left[\rho u^2+\rho^\gamma+|u_y|+\frac{|u|}{y}\right]|\phi_y|\,dydx\\ &\leq C\int_0^T\int_\epsilon^\infty(1+\rho^\gamma)\psi dx\int_1^\infty\left[1+\rho u^2+\rho^\gamma+u_y^2+\frac{u^2}{y^2}\right]|\phi_y|\,dy\\ &\leq C.\end{aligned}\tag{2.53}$$

Integrating (2.28) over $(\epsilon,\infty)\times(0,T)$, summing up (2.28), (2.29), and (2.36), (2.43), (2.48), (2.52), (2.53) and the related estimates for the other terms of I_j $(1\leq j\leq 6)$, we obtain the lemma by choosing δ appropriately small. □

III. Proof of Theorem 1.1

In this section we extract a limiting solution (ρ,u) from the approximate solution sequence $(\rho_\epsilon,u_\epsilon)$ of (2.5), (2.6), and thus obtain a global weak solution of (1.1), (1.2).

First we extend both $u_\epsilon(t,x)$ and $\rho_\epsilon(t,x)$ to be zero for $0\leq x\leq\epsilon$. For simplicity, we still denote by $(u_\epsilon(t,x),\rho_\epsilon(t,x))$ this extension. In view of (2.7), (2.8) and Lemma 2.1, we can subtract a subsequence of $(\rho_\epsilon,u_\epsilon)$, still denoted by $(\rho_\epsilon,u_\epsilon)$, such that

$$\begin{aligned}&\rho_\epsilon\rightharpoonup\rho \text{ weak-}*\text{ in } L^\infty([0,T],\mathcal{L}^\gamma_{\text{loc}}(\mathbb{R}^+)) \text{ and weakly in } L^{2\gamma}([0,T],\mathcal{L}^{2\gamma}_{\text{loc}}(\mathbb{R}^+)),\\ &u_\epsilon\rightharpoonup u \text{ weakly in } L^2([0,T],\mathcal{H}^1_{\text{loc}}(\mathbb{R}^+)).\end{aligned}\tag{3.1}$$

Moreover, from the lower semicontinuity, (2.4), (2.7) and (2.8), we get

$$\rho\in L^\infty([0,T],\mathcal{L}^\gamma(\mathbb{R}^+))\cap L^{2\gamma}([0,T],\mathcal{L}^{2\gamma}_{\text{loc}}(\mathbb{R}^+)),\quad u_x,\ u/x\in L^2([0,T],\mathcal{L}^2(\mathbb{R}^+)).\tag{3.2}$$

In the sequel, we show that (ρ,u) obtained in (3.1) is indeed a weak solution of (1.1), (1.2). By virtue of (2.7)–(2.8), Lemma 2.1 and Hölder's inequality, $\rho_\epsilon^\theta\partial_x u_\epsilon\in L^{\frac{2\gamma}{\theta+\gamma}}([0,T],L^{\frac{2\gamma}{\theta+\gamma}}_{\text{loc}}(\mathbb{R}^+))$. So for any $0<\theta<\gamma$ we can subtract a subsequence of $\rho_\epsilon^\theta\partial_x u_\epsilon$ such that it is weak-convergent in $L^{\frac{2\gamma}{\theta+\gamma}}([0,T],L^{\frac{2\gamma}{\theta+\gamma}}_{\text{loc}}(\mathbb{R}^+))$. Similarly, $\rho_\epsilon^{\gamma+\theta}$, ρ_ϵ^γ, ρ_ϵ^θ are weak-convergent in $L^{\frac{2\gamma}{\theta+\gamma}}([0,T],L^{\frac{2\gamma}{\theta+\gamma}}_{\text{loc}}(\mathbb{R}^+))$, in $L^2([0,T],L^2_{\text{loc}}(\mathbb{R}^+))$ and in $L^{\frac{2\gamma}{\theta}}([0,T],L^{\frac{2\gamma}{\theta}}_{\text{loc}}(\mathbb{R}^+))$, respectively. For the sake of convenience we denote by $\overline{f(\rho)}$ the weak limit of $f(\rho_\epsilon)$ (in the sense of distributions) as $\epsilon\to0$. Now, we prove

Lemma 3.1. *For any $0<\theta<\gamma$, we have*

$$a\,\overline{\rho^{\gamma+\theta}}-\mu\,\mathcal{Q}=a\,\overline{\rho^\gamma}\,\overline{\rho^\theta}-\mu\,\overline{\rho^\theta}\,u_x,$$

where $\mathcal{Q}$ is the weak limit of $\rho_\epsilon^\theta \partial_x u_\epsilon$.

Proof. Let $\phi \in C_0^\infty(\mathbb{R}^+)$. Multiplying $(2.5)_2$ by ϕ and integrating then the resulting equation over (ϵ, ∞), using (2.23) and following the same procedure as in the proof of (2.15) (cf. (2.28)), we find that

$$\begin{aligned}
&\rho_\epsilon^\theta \left[a\,\rho_\epsilon^\gamma - \mu\,\partial_x u_\epsilon - \mu\, m\frac{u_\epsilon}{x}\right]\phi = \partial_t \left[\rho_\epsilon^\theta \int_x^\infty \rho_\epsilon u_\epsilon \phi\, dy\right] + \partial_x\left[u_\epsilon \rho_\epsilon^\theta \int_x^\infty \rho_\epsilon u_\epsilon \phi\, dy\right] \\
&\quad + \left[m\theta\rho_\epsilon^\theta \frac{u_\epsilon}{x} - (1-\theta)\rho_\epsilon^\theta \partial_x u_\epsilon\right]\int_x^\infty \rho_\epsilon u_\epsilon \phi\, dy - \rho_\epsilon^\theta \int_x^\infty (\rho_\epsilon u_\epsilon^2 + a\rho_\epsilon^\gamma)\phi_y\, dy \\
&\quad + m\rho_\epsilon^\theta \int_x^\infty \frac{\rho_\epsilon u_\epsilon^2}{y}\phi\, dy + \mu\,\rho_\epsilon^\theta \int_x^\infty \left[\partial_y u_\epsilon + m\frac{u_\epsilon}{y}\right]\phi_y dy.
\end{aligned} \tag{3.3}$$

Let r_1, r_2 be arbitrary positive numbers with $r_1 < r_2$ and denote by I the open interval (r_1, r_2). By (2.7), (2.8) and Lemma 2.1, $u_\epsilon \in L^2([0,T], H^1(I))$ and $\rho_\epsilon \in L^\infty([0,T], L^\gamma(I)) \cap L^2([0,T], L^2(I))$, while by $(2.5)_1$ one sees that

$$\partial_t \rho_\epsilon \in L^\infty([0,T], W^{-1,\frac{2\gamma}{\gamma+1}}(I)) + L^\infty([0,T], L^{\frac{2\gamma}{\gamma+1}}(I)). \tag{3.4}$$

Hence, applying Lemma 5.1 of [12], we obtain

$$\rho_\epsilon u_\epsilon \rightharpoonup \rho u \;\text{ in } \mathcal{D}'((0,T)\times I), \tag{3.5}$$

and due to $\rho_\epsilon u_\epsilon \in L^2([0,T], L^\gamma(I))$, (3.5) in fact holds weakly in $L^2([0,T], L^\gamma(I))$. Moreover, $\int_x^\infty \rho_\epsilon u_\epsilon \phi\, dy \in L^2([0,T], W^{1,\gamma}(I))$. From (2.9) we get

$$\partial_t \left[\int_x^\infty \rho_\epsilon u_\epsilon \phi\, dy\right] \in L^\infty((0,T)\times I) + L^2((0,T)\times I) + L^\infty([0,T], L^1(I)).$$

Thus, by the classical Lions-Aubin Lemma, we obtain

$$\begin{gathered}
\int_x^\infty \rho_\epsilon u_\epsilon \phi\, dy \to \int_x^\infty \rho u \phi\, dy \text{ stronly in } L^2([0,T], L^p(I)), \\
p < \infty \text{ if } \gamma = 1, \text{ and } p = \infty \text{ if } \gamma > 1.
\end{gathered}$$

Consequently,

$$\begin{aligned}
&\rho_\epsilon^\theta \int_x^\infty \rho_\epsilon u_\epsilon \phi\, dy \rightharpoonup \overline{\rho^\theta} \int_x^\infty \rho u \phi\, dy \text{ in } \mathcal{D}'((0,T)\times I), \\
&\rho_\epsilon^\theta \partial_x u_\epsilon \int_x^\infty \rho_\epsilon u_\epsilon \phi\, dy \rightharpoonup \mathcal{Q} \int_x^\infty \rho u \phi\, dy \;\text{ in } \mathcal{D}'((0,T)\times I),
\end{aligned} \tag{3.6}$$

and due to $\rho_\epsilon^\theta \int_x^\infty \rho_\epsilon u_\epsilon \phi\, dy \in L^2((0,T)\times I)$, $(3.6)_1$ in fact holds weakly in $L^2((0,T)\times I)$.

Analogously to (3.5), we can prove that

$$\rho_\epsilon u_\epsilon^2 = \rho_\epsilon u_\epsilon \cdot u_\epsilon \rightharpoonup \rho u^2 \text{ in} \mathcal{D}'((0,T)\times I), \tag{3.7}$$

and (3.7) in fact holds weakly in $L^{\frac{4\gamma}{2\gamma+1}}([0,T], L^{\frac{4\gamma}{2\gamma+1}}(I))$ because of $\rho_\epsilon u_\epsilon^2 \in L^{\frac{4\gamma}{2\gamma+1}}([0,T], L^{\frac{4\gamma}{2\gamma+1}}(I))$, which can be shown as follows using Lemma 2.1 and (2.40):

$$\int_0^T \int_I \rho_\epsilon^{\frac{4\gamma}{2\gamma+1}} u_\epsilon^{\frac{8\gamma}{2\gamma+1}} dxdt \le C \int_0^T \int_I (\rho_\epsilon^{2\gamma} + A^2(t)\rho_\epsilon u_\epsilon^2) dxdt \le C. \tag{3.8}$$

By virtue of (3.7) and Lebesgue's dominated convergence theorem we easily see that $\int_x^\infty \frac{\rho_\epsilon u_\epsilon^2}{y}\phi\, dy \rightharpoonup \int_x^\infty \frac{\rho u^2}{y}\phi\, dy$ weakly or weak-$*$ in $L^p((0,T)\times I)$ for any $p\in(1,\infty]$, while by (2.23), (2.7), (2.8), Lemma 2.1 and (2.40), we have $\partial_t \rho_\epsilon^\theta \in L^1((0,T)\times I) + L^2([0,T], W^{-1,2\gamma/\theta}(I))$. Thus, we get in the same manner as in the proof of (3.5) that

$$\begin{aligned} &\Big\{\rho_\epsilon^\theta \int_x^\infty \frac{\rho_\epsilon u_\epsilon^2}{y}\phi\, dy,\ \rho_\epsilon^\theta \int_x^\infty \rho_\epsilon u_\epsilon^2 \phi_y\, dy,\ \rho_\epsilon^\theta \int_x^\infty \rho_\epsilon^\gamma \phi_y\, dy,\ \rho_\epsilon^\theta \int_x^\infty \partial_y u_\epsilon \phi_y\, dy, \\ &\quad \rho_\epsilon^\theta \int_x^\infty \frac{u_\epsilon}{y}\phi_y\, dy,\ \rho_\epsilon^\theta \frac{u_\epsilon}{x}\Big\} \rightharpoonup \Big\{\overline{\rho^\theta}\int_x^\infty \frac{\rho u^2}{y}\phi\, dy,\ \overline{\rho^\theta}\int_x^\infty \rho u^2 \phi_y\, dy, \\ &\quad \overline{\rho^\theta}\int_x^\infty \overline{\rho^\gamma}\phi_y\, dy,\ \overline{\rho^\theta}\int_x^\infty u_y \phi_y\, dy,\ \overline{\rho^\theta}\int_x^\infty \frac{u}{y}\phi_y\, dy,\ \overline{\rho^\theta}\frac{u}{x}\Big\} \ \text{ in } \mathcal{D}'((0,T)\times I). \end{aligned} \tag{3.9}$$

From (3.3), Lemma 2.1 and (2.40) we get $\partial_t[\rho_\epsilon^\theta \int_x^\infty \rho_\epsilon u_\epsilon \phi\, dy] \in L^1((0,T)\times I)$ $+L^2((0,T), W^{-1,2\gamma/\theta}(I))$. Thus, applying $(3.6)_1$ and Lemma 5.1 in [12], we find that

$$u_\epsilon \rho_\epsilon^\theta \int_x^\infty \rho_\epsilon u_\epsilon \phi\, dy \rightharpoonup u\,\overline{\rho^\theta}\int_x^\infty \rho u \phi\, dy \quad \text{in } \mathcal{D}'((0,T)\times I). \tag{3.10}$$

In the same way we obtain

$$\frac{u_\epsilon}{x}\rho_\epsilon^\theta \int_x^\infty \rho_\epsilon u_\epsilon \phi\, dy \rightharpoonup \frac{u}{x}\overline{\rho^\theta}\int_x^\infty \rho u \phi\, dy \quad \text{in } \mathcal{D}'((0,T)\times I). \tag{3.11}$$

Letting $\epsilon\to 0$ in (3.3), and employing (3.6) and (3.9)–(3.11), we arrive at

$$\begin{aligned} &\left[a\,\overline{\rho^{\gamma+\theta}} - \mu\,\mathcal{Q} - \mu\, m\,\overline{\rho^\theta}\frac{u}{x}\right]\phi = \left[\overline{\rho^\theta}\int_x^\infty \rho u\phi\, dy\right]_t + \left[u\,\overline{\rho^\theta}\int_x^\infty \rho u \phi\, dy\right]_x \\ &\quad + \left[m\theta\,\overline{\rho^\theta}\frac{u}{x} - (1-\theta)\mathcal{Q}\right]\int_x^\infty \rho u\phi\, dy - \overline{\rho^\theta}\int_x^\infty (\rho u^2 + a\,\overline{\rho^\gamma})\phi_y\, dy \\ &\quad + m\,\overline{\rho^\theta}\int_x^\infty \frac{\rho u^2}{y}\phi\, dy + \mu\,\overline{\rho^\theta}\int_x^\infty \left[u_y + m\frac{u}{y}\right]\phi_y dy \end{aligned} \tag{3.12}$$

in the sense of distributions. On the other hand, with the help of (3.5), (3.7) and (3.9), we have by taking $\epsilon \to 0$ in (2.5) and (2.23) that

$$\begin{cases} \partial_t \rho + (\rho u)_x + m\dfrac{\rho u}{x} = 0, \\ (\rho u)_t + \left[\rho u^2 + a\,\overline{\rho^\gamma}\right]_x + \dfrac{m\rho u^2}{x} = \mu\, u_{xx} + \mu\, m\left[\dfrac{u}{x}\right]_x, \end{cases} \tag{3.13}$$

$$\partial_t \overline{\rho^\theta} + \partial_x(u\overline{\rho^\theta}) + \frac{m\theta\overline{\rho^\theta}u}{x} = (1-\theta)\mathcal{Q} \tag{3.14}$$

in $\mathcal{D}'((0,T)\times\mathbb{R}^+)$. If we multiply the equation $(3.13)_2$ by ϕ, integrate then over (x,∞) and use (3.14), we obtain by arguments similar to those used for (3.3) (also cf. the derivation of (2.15)) that

$$\begin{aligned} \left[a\,\overline{\rho^\gamma}\,\overline{\rho^\theta} - \mu\,\overline{\rho^\theta}\,u_x - \mu\, m\,\overline{\rho^\theta}\frac{u}{x}\right]\phi &= \left[\overline{\rho^\theta}\int_x^\infty \rho u\phi\,dy\right]_t + \left[u\,\overline{\rho^\theta}\int_x^\infty \rho u\phi\,dy\right]_x \\ &+ \left[m\theta\,\overline{\rho^\theta}\frac{u}{x} - (1-\theta)\mathcal{Q}\right]\int_x^\infty \rho u\phi\,dy - \overline{\rho^\theta}\int_x^\infty (\rho u^2 + a\,\overline{\rho^\gamma})\phi_y\,dy \\ &+ m\,\overline{\rho^\theta}\int_x^\infty \frac{\rho u^2}{y}\phi\,dy + \mu\,\overline{\rho^\theta}\int_x^\infty \left[u_y + m\frac{u}{y}\right]\phi_y dy. \end{aligned} \tag{3.15}$$

Comparing (3.12) with (3.15), we infer that $(a\,\overline{\rho^\theta}\,\overline{\rho^\gamma} - \mu\,\overline{\rho^\theta}u_x)\phi = (a\,\overline{\rho^{\gamma+\theta}} - \mu\,\mathcal{Q})\phi$, which proves the lemma. $\square$

Remark 3.1. *As noted by Hoff in [3–5], the viscous flux $F_\epsilon := a\rho_\epsilon^\gamma - \mu\,\partial_x u_\epsilon - \mu m u_\epsilon/x$ is a good term, which embodies smoothing properties of parabolic parts in the system (1.1). This can be seen by the limit $F_\epsilon\rho_\epsilon^\theta \rightharpoonup \overline{F}\,\overline{\rho^\theta}$, which follows easily from Lemma 3.1 and the proof of Lemma 3.1.*

The following lemma shows that although we do not know whether ρ is L^2-integrable for $\gamma < \gamma_n$, we can prove the $L^{2/\theta}$-integrability of $\rho^\theta - \overline{\rho^\theta}$, which suffices to exclude the concentration of mass at the origin (cf. (3.26)).

Lemma 3.2. *Let $0 < \theta < 1$ satisfy $\frac{1}{2}(1-\theta+\sqrt{1+6\theta+\theta^2}) \leq \gamma$. Then*

$$\rho^\theta - \overline{\rho^\theta} \in L^{2/\theta}([0,T], \mathcal{L}^{2/\theta}(0,1)).$$

Proof. By Lemma 3.1 we have

$$a(\overline{\rho^{\gamma+\theta}} - \overline{\rho^\gamma}\,\overline{\rho^\theta}) = \mu(\mathcal{Q} - \overline{\rho^\theta}\,u_x). \tag{3.16}$$

By virtue of convexity, $\overline{\rho^{\gamma+\theta}} \geq \overline{\rho^\gamma}^{\frac{\gamma+\theta}{\gamma}}$, $\overline{\rho^\gamma} \geq \rho^\gamma$ and $\rho^\theta \geq \overline{\rho^\theta}$. Hence,

$$
\begin{aligned}
\overline{\rho^{\gamma+\theta}} - \overline{\rho^\gamma}\,\overline{\rho^\theta} &\geq \overline{\rho^\gamma}^{\frac{\gamma+\theta}{\gamma}} - \overline{\rho^\gamma}\,\overline{\rho^\theta} \\
&= \overline{\rho^\gamma}\left(\overline{\rho^\gamma}^{\frac{\theta}{\gamma}} - \overline{\rho^\theta}\right) \\
&\geq \rho^\gamma(\rho^\theta - \overline{\rho^\theta}) \geq 0.
\end{aligned}
\tag{3.17}
$$

It is easy to see that by (2.7) and (2.8), $\rho_\epsilon^\theta \partial_x u_\epsilon$, $\rho^\theta u_x \in L^{\frac{2\gamma}{\gamma+2\theta}}([0,T], \mathcal{L}^{\frac{2\gamma}{\gamma+2\theta}}(0,1))$. Therefore, $\mathcal{Q}$ and $\overline{\rho^\theta} u_x \in L^{\frac{2\gamma}{\gamma+2\theta}}([0,T], \mathcal{L}^{\frac{2\gamma}{\gamma+2\theta}}(0,1))$, which together with (3.16) and (3.17) gives

$$
\rho^\gamma(\rho^\theta - \overline{\rho^\theta}) \in L^{\frac{2\gamma}{\gamma+2\theta}}([0,T], \mathcal{L}^{\frac{2\gamma}{\gamma+2\theta}}(0,1)).
\tag{3.18}
$$

Recalling $2 - \frac{2\gamma\theta}{\gamma+2\theta} \leq \frac{2\gamma^2}{\gamma+2\theta}$ when θ satisfies $\frac{1}{2}(1-\theta+\sqrt{1+6\theta+\theta^2}) \leq \gamma$, using (3.18) and Young's inequality, we thus obtain

$$
\begin{aligned}
\left(\rho^\theta - \overline{\rho^\theta}\right)^{\frac{2}{\theta}} &= \left(\rho^\theta - \overline{\rho^\theta}\right)^{\frac{2\gamma}{\gamma+2\theta}} \left(\rho^\theta - \overline{\rho^\theta}\right)^{\frac{2}{\theta} - \frac{2\gamma}{\gamma+2\theta}} \\
&\leq C\left(\rho^\theta - \overline{\rho^\theta}\right)^{\frac{2\gamma}{\gamma+2\theta}} \rho^{2-\frac{2\gamma\theta}{\gamma+2\theta}} \\
&\leq C\left(\rho^\theta - \overline{\rho^\theta}\right)^{\frac{2\gamma}{\gamma+2\theta}} \left(1 + \rho^{\frac{2\gamma^2}{\gamma+2\theta}}\right) \\
&\leq C\left\{1 + \rho^\gamma + \left(\rho^\theta - \overline{\rho^\theta}\right)^{\frac{2\gamma}{\gamma+2\theta}} \rho^{\frac{2\gamma^2}{\gamma+2\theta}}\right\} \in L^1([0,T], \mathcal{L}^1(0,1)),
\end{aligned}
$$

which proves the lemma. $\square$

Lemma 3.3. *The eatimate (1.6) in Theorem 1.1 holds for (ρ, u), and*

$$
\rho, \;\; \overline{\rho^\theta}^{\frac{1}{\theta}} \in L^\infty([0,T], \mathcal{L}^1(\mathbb{R}^+)).
$$

Proof. Taking into account that $a\rho_\epsilon^\gamma/(\gamma-1) \leq \psi(\rho_\epsilon) + C\epsilon^\gamma(1+\rho_\epsilon^\gamma)$, we use (3.7), (3.1), and (2.7), (2.4) and (2.8) to deduce that any δ, $N > 0$,

$$
\begin{aligned}
&\int_\delta^N [\rho u^2 + \psi(\rho)](t,x) x^m \, dx + \mu \int_0^T \int_\delta^N \left[u_x^2 + m\frac{u^2}{x^2}\right] y^m \, dx dt \\
&\leq \liminf_{\epsilon \to 0} \left\{ \int_\delta^N \left[\rho_\epsilon u_\epsilon^2 + \frac{\rho_\epsilon^\gamma}{\gamma - 1})\right] x^m \, dx + \mu \int_0^T \int_\delta^N \left[(u_\epsilon)_x^2 + m\frac{u_\epsilon^2}{x^2}\right] y^m \, dx dt \right\} \\
&\leq \int_0^\infty \left[\frac{m_0^2}{\rho_0} + \psi(\rho_0)\right] x^m \, dx + \liminf_{\epsilon \to 0} \left\{ C\epsilon^\gamma \int_\delta^N (1+\rho_\epsilon^\gamma) x^m \, dx \right\} \\
&= \int_0^\infty \left[\frac{m_0^2}{\rho_0} + \psi(\rho_0)\right] x^m \, dx,
\end{aligned}
$$

which, by taking $\delta \to 0$ and $N \to \infty$, gives (1.6).

Let $\phi_k \in C_0^\infty(\mathbb{R}^+)$ satisfying $\phi(x) = 0$ for $x \le 1$ or $x \ge k+1$ and $\phi_k(x) = 1$ for $2 \le x \le k$. Using (3.4) and (2.8), applying Lemma C.1 of [11, Appendix C], we obtain $\rho \in C^0([0,T], L^\gamma_{\rm loc}(\mathbb{R}^+) - w)$. Hence, if we multiply $(3.13)_1$ by $x^m\phi_k^2$, integrate the resulting equation over $(0,t)\times\mathbb{R}^+$ and integrate by parts, we find that

$$\begin{aligned}\int_0^\infty \rho(t,x)\phi_k^2 x^m\,dx &\le \int_0^\infty \rho_0\phi_k^2 x^m\,dx + C\int_0^t\int_0^\infty \phi_k|\partial_x\phi_k|\,\rho\,|u|x^m\,dxds\\ &\le C + \int_0^t\int_0^\infty \rho\phi_k^2 x^m\,dxds + C\int_0^t\int_1^\infty \rho u^2 x^m\,dxds\\ &\le C + \int_0^t\int_0^\infty \rho\phi_k^2 x^m\,dxds,\end{aligned}$$

which, by applying Gronwall's inequality, implies $\int_0^\infty \rho\phi_k^2 x^m dx \le C$, where C is a positive constant independent of k. So, letting $k\to\infty$, we get $\int_2^\infty \rho(t,x)x^m dx \le C$ for any $t\in[0,T]$. On the other hand, $\int_0^2 \rho(t,x)x^m dx \le C\int_0^2(1+\rho^\gamma)x^m dx \le C$. Thus, $\rho \in L^\infty([0,T], L^1(\mathbb{R}^+))$. By convexity we have $\overline{\rho^\theta}^{\frac{1}{\theta}} \le \rho$, which proves the lemma. $\square$

Now, we are in a position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Using the equations (2.5), recalling (3.8) and (3.4), we have $[\rho_\epsilon]_t \in L^\infty([0,T], W^{-1,\frac{2\gamma}{\gamma+1}}_{\rm loc}(\mathbb{R}^+))$, $[\rho_\epsilon u_\epsilon]_t \in L^{\frac{4\gamma}{2\gamma+1}}([0,T], W^{-1,\frac{4\gamma}{2\gamma+1}}_{\rm loc}(\mathbb{R}^+))$. So, we obtain by Lemma C.1 in [11, Appendix C] that

$$\begin{aligned}&\lim_{\epsilon\to0}\int_0^\infty \rho_\epsilon(t,x)\phi(x)x^m\,dx = \int_0^\infty \rho(t,x)\phi(x)x^m\,dx \in C^0([0,T]),\\ &\lim_{\epsilon\to0}\int_0^\infty (\rho_\epsilon u_\epsilon)(t,x)\phi(x)x^m\,dx = \int_0^\infty (\rho u)(t,x)\phi(x)x^m\,dx \in C^0([0,T])\end{aligned}\tag{3.19}$$

for any $\phi \in C_0^\infty(\mathbb{R}^+)$. By a density argument we find that (3.19) holds for any $\phi \in L^{\frac{2\gamma}{\gamma-1}}(\mathbb{R}^+)$ with $\operatorname{supp}\phi \subset\subset \mathbb{R}_0^+$. Therefore, $(1.3)_2$ and $(1.3)_3$ are satisfied by taking into account (2.1). Thus, in view of (3.1), (3.2), Lemmas 2.1, 2.2 and 3.3, we see that to complete the proof of Theorem 1.1, there remains only to prove that (1.4), (1.5) hold for (ρ,u).

Let $\theta\in(0,1)$ satisfy the condition of Lemma 3.2. First notice that by convexity, $\overline{\rho^{\gamma+\theta}}^{\frac{\gamma}{\gamma+\theta}} \ge \overline{\rho^\gamma}$ and $\overline{\rho^{\gamma+\theta}}^{\frac{\theta}{\gamma+\theta}} \ge \overline{\rho^\theta}$. Hence, we deduce from Lemma 3.1 that

$$\mathcal{Q} \ge \overline{\rho^\theta}\,u_x.\tag{3.20}$$

Using the first equation of (3.13), we get, analogously to (2.13) (also see the proof of (2.23)), that

$$\partial_t\rho^\theta + \partial_x(u\rho^\theta) + \frac{m\theta\rho^\theta u}{x} = (1-\theta)u_x\rho^\theta.\tag{3.21}$$

Thus, subtracting (3.14) from (3.21) and employing (3.20), we get

$$\partial_t(\rho^\theta - \overline{\rho^\theta})) + \partial_x(u(\rho^\theta - \overline{\rho^\theta})) + \frac{m\theta u}{x}(\rho^\theta - \overline{\rho^\theta})) \le (1-\theta)(\rho^\theta - \overline{\rho^\theta})u_x.$$

Now, we multiply the above inequality by $x^{m\theta}$ to get

$$\partial_t f + \partial_x(uf) \le (1-\theta) f u_x, \tag{3.22}$$

where $f \equiv f(t,x) := x^{m\theta}(\rho^\theta - \overline{\rho^\theta})$. Note that $f \ge 0$ by convexity.

Next, we claim that

$$\partial_t f^{1/\theta} + \partial_x(u f^{1/\theta}) \le 0 \qquad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^+). \tag{3.23}$$

In fact, applying the mollifier to (3.22), we have (cf. (2.11))

$$\partial_t f_\delta + \partial_x(u f_\delta) \le (1-\theta)u_x f_\delta + r_\delta, \tag{3.24}$$

where $f_\delta(t,x) = f(t,\cdot) * j_\delta$ and $r_\delta(t,x) = [uf_\delta]_x - [uf]_x * j_\delta + (1-\theta)[(u_x f) * j_\delta - u_x f_\delta]$. By Lemma 2.1, $f \in L^{2\gamma/\theta}_{\text{loc}}([0,T]\times\mathbb{R}^+)$. So, we apply Lemma 2.3 of [11] to conclude that $r_\delta \to 0$ in $L^q_{\text{loc}}([0,T]\times\mathbb{R}^+)$ as $\delta \to 0$, with $\frac{1}{q} = \frac{1}{2} + \frac{\theta}{2\gamma}$. Now, multiplying (3.24) by $\frac{1}{\theta} f_\delta^{1/\theta - 1}$, we infer that

$$\partial_t f_\delta^{1/\theta} + \partial_x(u f_\delta^{1/\theta}) \le \frac{1}{\theta} r_\delta f_\delta^{1/\theta-1},$$

which, by taking $\delta \to 0$ and noticing that $f^{1/\theta-1} \in L^{\frac{2\gamma}{1-\theta}}_{\text{loc}}([0,T]\times\mathbb{R}^+)$, yields (3.23).

Using the equations (3.14) and (3.21), recalling $u_x f^{1/\theta}$, $f^{1/\theta-1}x^{m\theta}(\rho^\theta u_x - \mathcal{Q}) \in L^{\frac{2\gamma}{\gamma+1}}_{\text{loc}}([0,T]\times\mathbb{R}^+)$ because of $\rho^\theta u_x$, $\mathcal{Q} \in L^{\frac{2\gamma}{\gamma+\theta}}_{\text{loc}}([0,T]\times\mathbb{R}^+)$, we have similarly to (3.23) that

$$\partial_t f^{1/\theta} + \partial_x(u f^{1/\theta}) = (1-\theta^{-1})\left(u_x f^{1/\theta} - f^{1/\theta-1}x^{m\theta}(\rho^\theta u_x - \mathcal{Q})\right),$$

which implies $\partial_t f^{1/\theta} \in L^{\frac{2\gamma}{\gamma+1}}([0,T], W^{-1,\frac{2\gamma}{\gamma+1}}_{\text{loc}}(\mathbb{R}^+))$. Hence, Lemma C.1 of [11, Appendix C] yields

$$f^{1/\theta} \in C^0([0,T], L^{2\gamma}_{\text{loc}}(\mathbb{R}^+) - w). \tag{3.25}$$

Now, take $\phi_\epsilon \in C_0^\infty(\mathbb{R}^+)$ satisfying

$$\phi_\epsilon(x) = \begin{cases} 0, & x \le \epsilon/2 \text{ or } x \ge \epsilon^{-1}+1, \\ 1, & \epsilon \le x \le \epsilon^{-1}, \end{cases}$$

and $|\partial_x \phi_\epsilon(x)| \le C\epsilon^{-1}$ for $x \le \epsilon$ and $|\partial_x \phi_\epsilon(x)| \le C$ for $x \ge \epsilon^{-1}$. Noting that $f(0,x) = 0$, we multiply (3.23) with ϕ_ϵ, integrate it over $(0,t)\times(0,\infty)$ and utilize (3.25) to find that

$$\int_0^\infty f^{1/\theta}(t,x)\phi_\epsilon(x)dx \le \frac{C}{\epsilon}\int_0^t \int_0^\epsilon |u| f^{1/\theta}\, dxds + C\int_0^t \int_{1/\epsilon}^\infty |u| f^{1/\theta}\, dxds. \tag{3.26}$$

From Lemmas 3.2 and 3.3, (2.40) and Cauchy-Schwarz's inequality it follows that $x^{-1}uf^{1/\theta} \in L^1((0,T)\times(0,1))$, $uf^{1/\theta} \in L^1((0,T)\times(1,\infty))$. Hence, letting $\epsilon \to 0$ in (3.26), we obtain for any $T > 0$ that $f^{1/\theta} = 0$ a.e. on $(0,T)\times\mathbb{R}_0^+$. This implies

$$\rho^\theta(t,x) = \overline{\rho^\theta}(t,x), \qquad \text{a.e. } (t,x) \in \mathbb{R}^+\times\mathbb{R}^+.$$

Hence by the convexity, the Young measure associated with $\{\rho_\epsilon(t,x)\}$ is the Dirac mass (see [21], [9], [19]), and by Proposition 3.1.7 in [9] and Lemma 2.1, we have

$$\rho_\epsilon \to \rho \quad \text{strongly in } L^p_{\mathrm{loc}}(\mathbb{R}_0^+\times\mathbb{R}^+) \qquad \forall\, p < 2\gamma. \tag{3.27}$$

Noticing that $\rho^\epsilon, \rho \in L^\infty([0,T], \mathcal{L}^\gamma(0,N))$ and $\rho^\epsilon u^\epsilon, \rho u \in L^\infty([0,T], \mathcal{L}^{\frac{2\gamma}{\gamma+1}}(0,N))$ uniformly in ϵ for any $N > 0$, we have by (3.27), (3.5) and Young's inequality that for any $N > 0$,

$$\rho^\epsilon \to \rho \text{ in } L^1([0,T], \mathcal{L}^1(0,N)), \quad \rho^\epsilon u^\epsilon \rightharpoonup \rho u \text{ in } L^\infty([0,T], \mathcal{L}^{\frac{2\gamma}{\gamma+1}}(0,N)). \tag{3.28}$$

Since the weak solution $(\rho^\epsilon, u^\epsilon)$ of (2.5), (2.6) satisfies (1.4) in $(0,\infty)\times(\epsilon,\infty)$ (also see [3]), we infer, recalling $\rho^\epsilon(t,x) = 0$ for $x \le \epsilon$, that

$$\int_0^\infty \rho^\epsilon \varphi x^m\, dx\Big|_{t_1}^{t_2} - \int_0^\infty \int_0^\infty (\rho^\epsilon \varphi_t + \rho^\epsilon u^\epsilon \varphi_x) x^m\, dx\, dt = 0 \tag{3.29}$$

for any $\varphi \in C_0^1(\mathbb{R}\times\mathbb{R}_0^+)$. We take $\epsilon \to 0$ in (3.29), and use (3.28) and (3.19) to see that (1.4) is satisfied.

Next, we use the method of shielding test functions (cf. [3, 1]) to show that (ρ, u) satisfies (1.4) and (1.5). To do this, we take a cut-off function $\chi^h \in C_0^\infty(\mathbb{R}^+)$ satisfying $\chi^h(x) = 0$ for $0 \le x \le h$ and $\chi^h(x) = 1$ for $x \ge 2h$. Recalling $(1.3)_2$, we multiply $(3.13)_2$ by $\chi^h(x)\varphi(t,x)x^m$ in $L^2((t_1,t_2)\times\mathbb{R}^+)$, where $\varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}_0^+)$ with $\varphi(t,0) = 0$, integrate by parts and use (3.27) to deduce that

$$\begin{aligned}
&\int_0^\infty \rho u \chi^h \varphi x^m dx\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_0^\infty \left[\rho u \varphi_t + \rho u^2 \varphi_x + a\rho^\gamma\left(\varphi_x + \frac{m\varphi}{x}\right)\right]\chi^h x^m dxdt \\
&\qquad - \int_{t_1}^{t_2}\int_0^\infty (\rho u^2 + a\rho^\gamma)\varphi \partial_x \chi^h x^m\, dxdt \\
&\qquad = -\mu \int_{t_1}^{t_2}\int_0^\infty \left\{\left(u_x\varphi_x + \frac{mu\varphi}{x^2}\right)\chi^h + u_x \varphi \partial_x \chi^h\right\} x^m\, dxdt.
\end{aligned} \tag{3.30}$$

Since $\varphi(t,0)=0$, $|\varphi(t,x)\partial_x\chi^h(x)|\le Ch^{-1}|\varphi(t,x)|\le C$ for any $0\le x\le 2h$ and any t. Thus, with the help of Lebesgue's dominated convergence theorem and (1.6), we find that

$$\begin{aligned}&\int_{t_1}^{t_2}\int_0^\infty(\rho u^2+\rho^\gamma+|u_x|)|\varphi\partial_x\chi^h|x^m\,dxdt\\ &\qquad\le C\int_{t_1}^{t_2}\int_h^{2h}(\rho u^2+\rho^\gamma+|u_x|)x^m\,dxdt\to 0,\quad \text{as } h\to 0.\end{aligned}$$

Taking $h\to 0$ in (3.30) and employing the above estimate, we see that (ρ,u) satisfies (1.5) for any $\varphi\in C_0^\infty(\mathbb{R}\times\mathbb{R}_0^+)$ with $\varphi(t,0)=0$. Moreover, a density argument shows that (1.5) still holds for any $\varphi\in C_0^1(\mathbb{R}\times\mathbb{R}_0^+)$ with $\varphi(t,0)=0$. This completes the proof of Theorem 1.1. $\square$

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