# A STRONGLY DEGENERATE CONVECTION-DIFFUSION PROBLEM MODELING CENTRIFUGATION OF FLOCCULATED SUSPENSIONS

RAIMUND BÜRGER[A] AND KENNETH HVISTENDAHL KARLSEN[B]

ABSTRACT. We prove existence and uniqueness of $BV$ entropy solutions of a strongly degenerate convection-diffusion problem modeling centrifugation of flocculated suspensions. A modification of the generalized upwind method is employed to solve the initial-boundary value problem numerically, i.e., to simulate the centrifugation process.

## 1. INTRODUCTION

We consider the quasilinear strongly degenerate convection-diffusion equation

$$u_t + f(u,x)_x = A(u)_{xx} + g(u,x),\ A(u) := \int_0^u a(s)ds;\ (x,t) \in Q_T := (x_1,x_2) \times (0,T), \quad (1)$$

where we assume that $f \geq 0$, $\operatorname{supp} f(\cdot,x) \subset [0,1]$, $a(u) = 0$ for $u \leq u_{\mathrm{c}}$ and $u \geq 1$, and $a(u) > 0$ otherwise, i.e., equation (1) is of hyperbolic type for $u \leq u_{\mathrm{c}}$ and $u \geq 1$ and of parabolic type for $u_{\mathrm{c}} < u < 1$. We assume that $f(\cdot,x)$ is continuous and piecewise differentiable with $\|\partial_u f(\cdot,x)\| \leq M$ and that $A(\cdot)$ and $g(\cdot,x)$ are Lipschitz continuous uniformly in $x$, and that $f(u,\cdot), g(u,\cdot) \in C^1(x_1,x_2)$ uniformly in $u$. In particular, the diffusion coefficient $a(\cdot)$ is allowed to be discontinuous. We consider the initial and boundary conditions

$$u(x,0) = u_0(x),\ 0 \leq u_0(x) \leq 1,\ x \in [x_1,x_2], \quad (2)$$

$$f(u,x_{\mathrm{b}}) - A(u(x_{\mathrm{b}},t))_x = 0,\ x_{\mathrm{b}} \in \{x_1,x_2\},\ t \in (0,T], \quad (3)$$

and assume that the initial function satisfies

$$u_0 \in \big\{ u \in BV(x_1,x_2) : u(x) \in [0,1];\ \exists M_0 > 0 :\ \forall \varepsilon > 0 : \mathrm{TV}_{(x_1,x_2)}\big(\partial_x A^\varepsilon(u)\big) < M_0 \big\}, \quad (4)$$

where $A^\varepsilon$ is defined in terms of a standard $C^\infty$ mollifier $\omega_\varepsilon$ with $\operatorname{supp}\omega_\varepsilon \subset (-\varepsilon,\varepsilon)$ via

$$a^\varepsilon(u) := \big((a+\varepsilon) * \omega_\varepsilon\big)(u), \quad A^\varepsilon(u) := \int_0^u a^\varepsilon(s)\,ds. \quad (5)$$

**Remark 1.** *If $A \in C^1$, then it is sufficient to assume that $\mathrm{TV}_{(x_1,x_2)}(u_0') < \infty$. Moreover, the regularity assumption on $f(u,\cdot)$ used this paper can be weakened, see* [14] *for details.*

---

[a]Institute of Mathematics A, University of Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart, Germany. E-mail: `buerger@mathematik.uni-stuttgart.de`.

[b]Department of Mathematics, University of Bergen, Johs. Brunsgt. 12, N-5008 Bergen, Norway. E-mail: `kennethk@mi.uib.no`.

The study of strongly degenerate hyperbolic-parabolic equations such as (1) is in part motivated by a recent theory of sedimentation-consolidation processes [8]. While the paper [4] is concerned with the case of settling in a gravitational field, which leads to an equation similar to (1) but without a source term, we here focus on the application of this theory to a centrifugal field in a rotating frame of reference [3]. It is the purpose of this contribution to briefly outline how the analysis of strongly degenerate convection-diffusion problems presented in detail in [4, 14] (see also [6]) can be extended to the initial-boundary value problem (IBVP) (1)–(3), and to draw attention to a new application of strongly degenerate equations. In addition, we utilize an adaptation of the generalized upwind finite difference method presented in [5] to solve the IBVP numerically, that is to simulate the centrifugation process. For an overview of the mathematical and numerical theory of (strongly) degenerate parabolic equations, we refer to the lecture notes by Espedal and Karlsen [11].

This paper is organized as follows. In Section 2 we state and comment the definition of entropy solutions of the IBVP. We then outline in Sections 3 and 4 the existence and uniqueness proofs of entropy solutions, following the vanishing viscosity method and recent ideas by Carrillo [9] and Karlsen and Risebro [14], respectively. We come back to the application to centrifugation in Section 5, in which we present a working numerical algorithm and numerical solutions for the IBVP.

## 2. Entropy solutions

Due to both the degeneracy of the diffusion coefficient $a(\cdot)$ and the nonlinearity of $f(\cdot, x)$, solutions of (1) are discontinuous and have to be defined as entropy solutions.

**Definition 1.** *A function $u \in L^\infty(Q_T) \cap BV(Q_T)$ is an* entropy solution *of the IBVP* (1)–(3) *if* (a) $A(u)_x \in L^2(Q_T)$, (b) *for almost all $t \in (0,T)$, $\gamma_{x_\mathrm{b}}(f(u,\cdot) - A(u)_x) = 0$, $x_\mathrm{b} \in \{x_1, x_2\}$,* (c) *$\lim_{t\downarrow 0} u(x,t) = u_0(x)$ for almost all $x \in (x_1,x_2)$, and* (d)

$$\forall \varphi \in C_0^\infty(Q_T),\ \varphi \geq 0,\ \forall k \in \mathbb{R} : \iint_{Q_T} \Big\{ |u-k|\varphi_t \\ + \operatorname{sgn}(u-k) \Big[ \big[ f(u,x) - f(k,x) - A(u)_x \big] \varphi_x - \big[ f_x(k,x) - g(x,u) \big] \varphi \Big] \Big\}\, dt dx \geq 0. \quad (6)$$

Here $\gamma_{x_1}$ and $\gamma_{x_2}$ denote the traces with respect to $x \downarrow x_1$ and $x \uparrow x_2$, respectively. Entropy inequalities such as (6) go back to Kružkov [16] and Vol'pert [17] for first order equations and to Vol'pert and Hudjaev [18] for second order equations.

**Remark 2.** *The $BV(Q_T)$ assumption in Definition 1 is only used to ensure the existence of the traces $\gamma_{x_1}$ and $\gamma_{x_2}$. Moreover, note that we can require that the boundary conditions are satisfied in a pointwise sense almost everywhere, whereas Dirichlet boundary conditions such as those stated in* [4] *(which are not considered here) have to be treated as entropy boundary conditions* [7]*. Finally, we point out that it is at present not known whether jump conditions for hyperbolic-parabolic equations such as those by Wu and Yin* [20] *(see also* [7]*) are valid here, since these jump conditions rely on stronger regularity properties (for example, Lipschitz continuity) of the diffusion coefficient $a(\cdot)$ than is stipulated here.*

## 3. Existence of entropy solutions

Existence of entropy solutions is shown by the vanishing viscosity method. To this end, consider the regularized uniformly parabolic IBVP, in which the functions $f$, $g$ and $u_0$ have been replaced by smooth approximations that ensure compatibility conditions and existence of smooth solutions for fixed $\varepsilon > 0$, and where $A^\varepsilon$ is defined in (5):

$$u_t^\varepsilon + f^\varepsilon(u,x)_x = A^\varepsilon(u^\varepsilon)_{xx} + g^\varepsilon(u^\varepsilon,x), \quad (x,t) \in Q_T, \tag{7a}$$

$$u^\varepsilon(x,0) = u_0^\varepsilon(x),\ x \in (x_1,x_2);\ f^\varepsilon(u^\varepsilon,x_{\mathrm{b}}) - A^\varepsilon(u^\varepsilon(x_{\mathrm{b}},t))_x = 0,\ x_{\mathrm{b}} \in \{x_1,x_2\},\ t \in (0,T]. \tag{7b}$$

**Lemma 1.** *The following uniform estimates are valid for solutions $u^\varepsilon$ of the regularized IBVP* (7)*, where the constants $M_1$ to $M_3$ are independent of $\varepsilon$:*

$$\|u^\varepsilon\|_{L^\infty(Q_T)} \leq M_1;\ \|u_x^\varepsilon(\cdot,t)\|_{L^\infty(x_1,x_2)} \leq M_2 \text{ and } \|u_t^\varepsilon(\cdot,t)\|_{L^\infty(x_1,x_2)} \leq M_3\, \forall t \in (0,T]. \tag{8}$$

*Sketch of proof.* The application of the maximum principle to establish the first estimate is standard and not repeated here; we refer to the proof of Lemma 9 in [4] for details.

Defining $\operatorname{sgn}_\eta(\tau) := \operatorname{sgn}(\tau)$ if $|\tau| > \eta$, $\operatorname{sgn}_\eta(\tau) := \tau/\eta$ if $|\tau| \leq \eta$ and $|x|_\eta := \int_0^x \operatorname{sgn}_\eta(\xi)d\xi$ for $\eta > 0$, we obtain by differentiating (7a) with respect to $x$, multiplying the result by $\operatorname{sgn}_\eta(u_x^\varepsilon)$, integrating over $Q_{T_0}$, $0 < T_0 \leq T$, integrating by parts and using (7b)

$$\begin{aligned}\int_{x_1}^{x_2} \big|u_x^\varepsilon(x,T_0)\big|_\eta dx &\leq \int_{x_1}^{x_2} \big|(u_0^\varepsilon)'(x)\big|_\eta dx + \int_0^{T_0} \operatorname{sgn}_\eta\big(u_x^\varepsilon(x,t)\big)u_t^\varepsilon(x,t)\Big|_{x_1}^{x_2} dt + \iint_{Q_{T_0}} \operatorname{sgn}'_\eta(u_x^\varepsilon) \\ &\times u_{xx}^\varepsilon\big[f_u^\varepsilon(u^\varepsilon,x) - \partial_x\big(a^\varepsilon(u^\varepsilon)\big)\big]u_x^\varepsilon\, dtdx + \iint_{Q_{T_0}} \operatorname{sgn}_\eta(u_x^\varepsilon)\big(g_u^\varepsilon(u^\varepsilon,x)u_x^\varepsilon + g_x^\varepsilon(u^\varepsilon,x)\big)\, dtdx.\end{aligned} \tag{9}$$

To derive the second inequality of Lemma 1 from (9), we repeat the proof of part (a) of Lemma 11 in [4] to estimate the first three integrals of the right-hand side of inequality (9) for $\eta \to 0$. The integrand of the last term can for $\eta \to 0$ be rewritten as $|u_x^\varepsilon|g_u^\varepsilon + \operatorname{sgn}(u_x^\varepsilon)g_x^\varepsilon$. Since $g_u^\varepsilon$ and $g_x^\varepsilon$ are uniformly bounded due to our assumptions on $g$, the desired estimate on $\|u_x^\varepsilon(\cdot,t)\|_{L^\infty(x_1,x_2)}$ can be established by an application of Gronwall's lemma.

The same argument can finally be employed to extend the derivation of the estimate on $\|u_t^\varepsilon(\cdot,t)\|_{L^\infty(x_1,x_2)}$ in Lemma 11 of [4] to the present equation with source term. ■

From the estimates established in Lemma 1 we may conclude that there exists a sequence $\varepsilon = \varepsilon_n \downarrow 0$ such that the sequence of solutions $\{u^{\varepsilon_n}\}$ of the IBVP (7) converges in $L^1(Q_T)$ to a function $u \in L^\infty(Q_T) \cap BV(Q_T)$. We now have to show that $u$ is actually an entropy solution of the IBVP (1)–(3). Part (a) of Definition 1 follows from the following lemma, whose (short) proof is a straightforward extension of that of Lemma 10 in [4]:

**Lemma 2.** *The limit function $u$ of solutions $u^\varepsilon$ of* (7) *satisfies $A(u)_x \in L^2(Q_T)$.*

Finally, repeating the proofs of Lemmas 5 and 12 of [4], we can show

**Lemma 3.** *The viscosity limit function $u$ of solutions $u^\varepsilon$ of* (7) *satisfies* (6) *and the initial and boundary conditions mentioned in Definition 1.*

Summarizing, we have:

**Theorem 1.** *The IBVP* (1)–(3) *admits an entropy solution $u$.*

## 4. Uniqueness of entropy solutions

After the important work by Carrillo [9], the uniqueness proof for entropy solutions of degenerate parabolic equations has become very similar to the "doubling of variables" proof introduced by Kružkov [16] for hyperbolic equations. The key proposition allowing to apply Kružkov's "doubling device" to second order equations is the following version (see Karlsen and Risebro [14] for its proof) of an important lemma from [9], which identifies a certain entropy dissipation term (i.e., the right-hand side of (10) below).

**Lemma 4** (Carrillo's lemma)**.** *Let $u$ be an entropy solution of the IBVP* (1)–(3)*. Then, for any non-negative $\varphi \in C_0^\infty(Q_T)$ and any $k \in (u_{\mathrm{c}}, 1)$, we have*

$$\begin{aligned} \iint_{Q_T} \Big( |u-k|\varphi_t + \operatorname{sgn}(u-k)\Big[\big[f(u,x) - f(k,x) - A(u)_x\big]\varphi_x \\ - \big[f_x(k,x) - g(x,u)\big]\varphi\Big]\Big)\,dtdx = \lim_{\eta\downarrow 0} \iint_{Q_T} \big(A(u)_x\big)^2 \operatorname{sgn}_\eta'\big(A(u)-A(k)\big)\varphi\,dtdx. \end{aligned} \tag{10}$$

Equipped this lemma, one can prove the following main theorem:

**Theorem 2.** *If $v$ and $u$ are two entropy solutions of the IBVP* (1)–(3)*, then we have for any $\varphi \in C_0^\infty(Q_T)$, $\varphi \geq 0$:*

$$\begin{aligned} \iint_{Q_T} \Big( |v-u|\varphi_t + \operatorname{sgn}(v-u)\big[f(v,x) - f(u,x) - \big(A(v)_x - A(u)_y\big)\big]\varphi_x \\ + \operatorname{sgn}(v-u)\big[g(v,x) - g(x,u)\big]\varphi\Big)\,dtdx \geq 0. \end{aligned} \tag{11}$$

*Sketch of proof.* The argument given below relies on Lemma 4 and Kružkov's idea of doubling the number of dependent variables together with a penalization procedure. We let $\varphi \in C^\infty(Q_T \times Q_T)$, $\varphi \geq 0$, $\varphi = \varphi(x,t,y,s)$, $v = v(x,t)$, $u = u(y,s)$, and introduce the "hyperbolic" sets $\mathcal{E}_v = \big\{(x,t) \in Q_T : v(x,t) \leq u_{\mathrm{c}} \text{ or } v(x,t) \geq 1\big\}$ associated with $v$ and $\mathcal{E}_u = \big\{(y,s) \in Q_T : u(y,s) \leq u_{\mathrm{c}} \text{ or } u(y,s) \geq 1\big\}$ associated with $u$.

From the entropy inequality for $v(x,t)$ (with $k = u(y,t)$), the entropy inequality for $u(y,s)$ (with $k = v(x,t)$), and Lemma 4, the following inequality was derived in [14]:

$$\begin{aligned} \iiiint_{Q_T\times Q_T} \Big( |v-u|\big(\varphi_t + \varphi_s\big) + \operatorname{sgn}(v-u)\big[f(v,x) - f(u,y) - \big(A(v)_x - A(u)_y\big)\big]\big(\varphi_x + \varphi_y\big) \\ + \operatorname{sgn}(v-u)\big[g(v,x) - g(u,y)\big]\varphi\Big)\,dtdxdsdy + E_{\mathrm{Conv}} \\ \geq \lim_{\eta\downarrow 0} \iiiint_{(Q_T\setminus\mathcal{E}_u)\times(Q_T\setminus\mathcal{E}_v)} \big(A(v)_x - A(u)_y\big)^2 \operatorname{sgn}_\eta'\big(A(v)-A(u)\big)\varphi\,dtdxdsdy \geq 0, \end{aligned} \tag{12}$$

where the "error term" $E_{\mathrm{Conv}}$ takes the form

$$E_{\mathrm{Conv}} = \iiiint\limits_{Q_T\times Q_T} \mathrm{sgn}(v-u)\Big(\big[\big(f(u,y)-f(u,x)\big)\varphi\big]_x - \big[\big(f(v,x)-f(v,y)\big)\varphi\big]_y\Big)\, dtdxdsdy.$$

We are now on familiar ground [16] and introduce in (12) the test function

$$\varphi(x,t,y,s) = \varphi\big(\tfrac{x+y}{2},\tfrac{t+s}{2}\big)\delta_h\big(\tfrac{x-y}{2}\big)\delta_h\big(\tfrac{t-s}{2}\big),$$

where $\{\delta_h\}_{h>0}$ is a standard regularizing sequence in $\mathbb{R}$. Observe that

$$\varphi_t+\varphi_t = \partial_t\varphi\big(\tfrac{x+y}{2},\tfrac{t+s}{2}\big)\delta_h\big(\tfrac{x-y}{2}\big)\delta_h\big(\tfrac{t-s}{2}\big), \quad \varphi_x+\varphi_y = \partial_x\varphi\big(\tfrac{x+y}{2},\tfrac{t+s}{2}\big)\delta_h\big(\tfrac{x-y}{2}\big)\delta_h\big(\tfrac{t-s}{2}\big).$$

Following [14] (see also [16] since $f(u,\cdot)$ is smooth), one can show that $\lim_{h\downarrow 0} E_{\mathrm{Conv}} = 0$. Consequently, by sending $h\downarrow 0$ in (12), we get (11). ■

Define for sufficiently small $h>0$ the functions

$$\rho_h(x) := \int_{-\infty}^{x} \delta_h(\xi)\, d\xi, \quad \mu_h(x) := 1-\rho_h(x-x_1-2h), \quad \nu_h(x) := \rho_h\big(x-(x_2-2h)\big), \tag{13}$$

where $\{\delta_h\}_{h>0}$ is again the standard regularizing sequence in $\mathbb{R}$. Concerning these functions, we have the following lemma (whose proof is easy):

**Lemma 5.** *Let $u\in L^1\big(0,T;L^\infty(x_1,x_2)\big)$. If the traces $\gamma_{x_1}u := (\gamma u)(x_1,t)$ and $\gamma_{x_2}u := (\gamma u)(x_2,t)$ exist a.e. in $(0,T)$, then we have for $\varphi\in C^\infty(Q_T)$*

$$\lim_{h\downarrow 0}\iint_{Q_T} \partial_x\Big(\varphi(x,t)\big(1-\mu_h(x)-\nu_h(x)\big)\Big)u(x,t)\, dtdx = \int_0^T \Big(\varphi(x_1,t)\gamma_{x_1}u - \varphi(x_2,t)\gamma_{x_2}u\Big)\, dt.$$

We are now in a position to deduce from (11) the following uniqueness result:

**Corollary 1.** *Let $v,u$ be entropy solutions of the IBVP (1)–(3) with initial data $v_0,u_0$, respectively. Then for all $t\in(0,T)$, $\|v(\cdot,t)-u(\cdot,t)\|_{L^1(x_1,x_2)} \le \exp\big(t\|g\|_{\mathrm{Lip}}\big)\|v_0-u_0\|_{L^1(x_1,x_2)}$. In particular, the IBVP (1)–(3) admits at most one entropy solution.*

*Proof.* In (11), we choose $\varphi(x,t) = ((1-\mu_h(x)-\nu_h(x))\chi(t)$ with $\chi\in C_0^\infty(0,T)$, $\chi\ge 0$, and $\mu_h$ and $\nu_h$ defined in (13). Note that $\varphi$ tends to $\chi(t)$ as $h\downarrow 0$. Taking the limit $h\downarrow 0$, we obtain from Lemma 5 and the boundary conditions at $x=a,b$ (see Definition 1):

$$\iint_{Q_T} |u-v|\chi'(t)\, dtdx \ge -\iint_{Q_T} \mathrm{sgn}(v-u)\big[g(v,x)-g(u,x)\big]\chi(t)\, dtdx. \tag{14}$$

From (14), it follows that

$$-\iint_{Q_T} |u-v|\chi'(t)\, dtdx \le \|g\|_{\mathrm{Lip}}\iint_{Q_T} |u-v|\chi(t)\, dtdx. \tag{15}$$

Fixing $\tau\in(0,T)$, choosing $\chi(t)$ as $\rho_h(t)-\rho_h(t-\tau)$ in (15), subsequently sending $h\downarrow 0$, and using Gronwall's lemma, we get the $L^1$ stability estimate $\|v(\cdot,\tau)-u(\cdot,\tau)\|_{L^1(x_1,x_2)} \le \exp\big(\tau\|g\|_{\mathrm{Lip}}\big)\|v(\cdot,0)-u(\cdot,0)\|_{L^1(x_1,x_2)}$. Since $\tau\in(0,T)$ was arbitrary, we are finished. ■

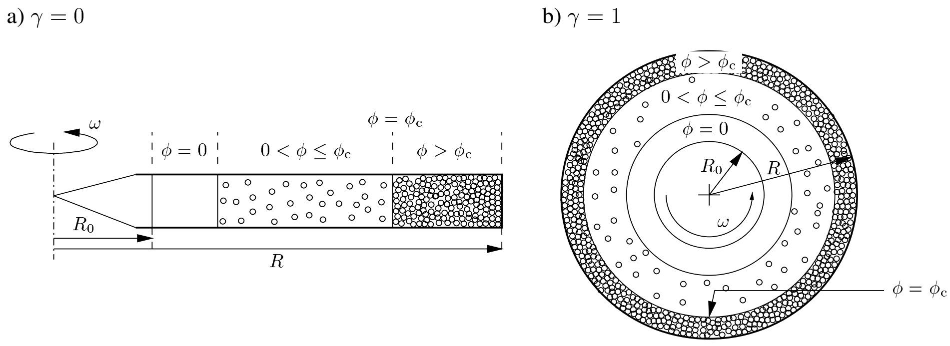


FIGURE 1. (a) Rotating tube, (b) rotating axisymmetric cylinder.

**Remark 3.** *Following* [10, 13, 14]*, it is possible to prove that the unique entropy solution of the IBVP* (1)–(3) *depends continuously on the nonlinearities in the problem.*

## 5. BATCH CENTRIFUGATION OF FLOCCULATED SUSPENSIONS

5.1. **Introduction.** Extending the arguments developed in [8] for a purely gravitational force to rotating systems, Bürger and Concha [3] show that the settling of a flocculated suspension in a tube with constant cross section ($\gamma = 0$, Fig. 1 (a)) or in a rotating axisymmetric cylinder ($\gamma = 1$, Fig. 1 (b)) is governed by the field equation

$$\partial_t \phi + r^{-\gamma} \partial_r \big( f_{\mathrm{ck}}(\phi) \omega^2 r^{1+\gamma} \big) = r^{-\gamma} \partial_r \big( r^\gamma \partial_r A(\phi) \big), \tag{16}$$

in which $\phi$ is the sought volumetric solids concentration, $r$ the radius, $f_{\mathrm{ck}}(\cdot)$ the centrifugal Kynch flux density function, $\omega$ the angular velocity and $A(\cdot)$ is defined as in (1). The function $f_{\mathrm{ck}}(\cdot)$ is a nonnegative Lipschitz continuous function with support in $[0,1]$, and the diffusion coefficient is defined by $a(\phi) := f_{\mathrm{ck}}(\phi)\sigma_{\mathrm{e}}'(\phi)/(\Delta\rho\phi)$, where $\Delta\rho > 0$ is the solid-fluid mass density difference and $\sigma_{\mathrm{e}}'(\cdot)$ is the derivative of the effective solid stress function. We mention that equation (1) is based on the neglection of both the gravitational and Coriolis forces compared to the centrifugal force and refer to [3] for details on its derivation.

Equation (16) inherits its degeneracy from the constitutive assumption that $\sigma_{\mathrm{e}}(\cdot)$ vanishes while the solid flocs are not in touch with each other, i.e., while the local concentration $\phi$ does not exceed a critical value $\phi_{\mathrm{c}}$, and that $\sigma_{\mathrm{e}}'(\phi) > 0$ for $\phi_{\mathrm{c}} < \phi \le 1$. Assuming for simplicity that $\operatorname{supp} f_{\mathrm{ck}} = (0,1)$, we see that $a(\phi) = 0$ and hence (16) is of hyperbolic type for $\phi \le \phi_{\mathrm{c}}$ and $\phi \ge 1$ and that otherwise $a(\phi) > 0$, i.e., (16) is of parabolic type. Consequently, (16) is indeed strongly degenerate. The special case $a \equiv 0$ is included in our discussion and corresponds to the equation studied by Anestis and Schneider [1].

We assume that $r$ varies between an inner and outer radii $R_0 > 0$ and $R > R_0$. The solids phase velocity vanishes at $r = R_0$ and $r = R$, which implies the boundary conditions

$$\big( f_{\mathrm{ck}}(\phi)\omega^2 R_{\mathrm{b}} + \partial_r A(\phi) \big)(R_{\mathrm{b}}, t) = 0, \quad t > 0, \ R_{\mathrm{b}} \in \{R, R_0\}. \tag{17}$$

The initial condition is

$$\phi(r,0) = \phi_0(r), \quad R_0 \le r \le R. \tag{18}$$

Differentiating out the convection and diffusion terms in equation (16), we obtain in view of the model assumptions an IBVP of the type (1)–(3). The existence and uniqueness analysis therefore states that the centrifugation model admits a unique entropy solution $\phi$.

5.2. **Numerical algorithm.** To solve the IBVP given by (16), (17) and (18) numerically, we employ a modification of the generalized upwind finite difference method presented in detail in [5] for gravity settling. For an overview of numerical methods for approximating entropy solutions of degenerate parabolic equations, we refer again to [11].

Let $J, N \in \mathbb{N}$, $\Delta r := (R - R_0)/J$, $\Delta t := T/N$, $r_j := R_0 + j\Delta r$, $j = 1/2, 1, 3/2, \dots, J - 1/2, J$ and $\phi_j^n \approx \phi(r_j, n\Delta t)$. The computation starts by setting $\phi_j^0 := \phi_0(r_j)$ for $j = 0, \dots, J$. Assume then that values $\phi_j^n$, $j = 0, \dots, J$ at time level $t_n := n\Delta t$ are known. To compute the values $\phi_j^{n+1}$, we first compute the extrapolated values $\phi_j^{\mathrm{L}} := \phi_j^n - (\Delta r/2)s_j^n$ and $\phi_j^{\mathrm{R}} := \phi_j^n + (\Delta r/2)s_j^n$ for $j = 1, \dots, J-1$, where the slopes $s_j^n$ can be calculated, for example, by the minmod limiter function $M(\cdot,\cdot,\cdot)$ in the following way:

$$s_j^n = \mathrm{MM}\big(\phi_j^n - \phi_{j-1}^n, (\phi_{j+1}^n - \phi_{j-1}^n)/2, (\phi_{j+1}^n - \phi_j^n)\big)/\Delta r, \quad j = 2, \dots, J-2. \tag{19}$$

where $\mathrm{MM}(a,b,c) = \min\{a,b,c\}$ if $a,b,c > 0$, $\mathrm{MM}(a,b,c) = \max\{a,b,c\}$ if $a,b,c < 0$ and $\mathrm{MM}(a,b,c) = 0$ otherwise. Moreover we set $s_0^n = s_1^n = s_{J-1}^n = s_J^n = 0$.

The extrapolated values $\phi_j^{\mathrm{L}}$ and $\phi_j^{\mathrm{R}}$ appear as arguments of the numerical centrifugal Kynch flux density function $f_{\mathrm{ck}}^{\mathrm{EO}}(\cdot,\cdot)$ which, according to the Engquist-Osher scheme, is defined by $f_{\mathrm{ck}}^{\mathrm{EO}}(u,v) := f_{\mathrm{ck}}^{+}(u) + f_{\mathrm{ck}}^{-}(v)$, where $f_{\mathrm{ck}}^{+}(u) := f_{\mathrm{ck}}(0) + \int_0^u \max\{f_{\mathrm{ck}}'(s), 0\}\, ds$ and $f_{\mathrm{ck}}^{-}(v) := \int_0^v \min\{f_{\mathrm{ck}}'(s), 0\}\, ds$. The interior scheme, which approximates the field equation (16) and from which the values $\phi_1^n, \dots, \phi_{J-1}^n$ are calculated, can then be formulated as

$$\begin{aligned}\phi_j^{n+1} = \phi_j^n &- \tfrac{\omega^2 \Delta t}{r_j^\gamma \Delta r}\big[r_{j+1/2}^{1+\gamma} f_{\mathrm{ck}}^{\mathrm{EO}}(\phi_j^{\mathrm{R}}, \phi_{j+1}^{\mathrm{L}}) - r_{j-1/2}^{1+\gamma} f_{\mathrm{ck}}^{\mathrm{EO}}(\phi_{j-1}^{\mathrm{R}}, \phi_j^{\mathrm{L}})\big] \\ &+ \tfrac{\Delta t}{r_j^\gamma \Delta r^2}\big[r_{j+1/2}^{\gamma}\big(A(\phi_{j+1}^n) - A(\phi_j^n)\big) - r_{j-1/2}^{\gamma}\big(A(\phi_j^n) - A(\phi_{j-1}^n)\big)\big],\ j = 1, \dots, J-1.\end{aligned} \tag{20}$$

The boundary formulas follow by considering (20) for $j = 0$ and $j = J$ and inserting the discrete versions of the boundary conditions (17). This leads to

$$\phi_0^n = \phi_0^{n-1} - \tfrac{\omega^2 \Delta t}{R_0^\gamma \Delta r} r_{1/2}^{1+\gamma} f_{\mathrm{ck}}^{\mathrm{EO}}(\phi_0^n, \phi_1^n) + \tfrac{\Delta t}{R_0^\gamma \Delta r^2} r_{1/2}^{\gamma}\big(A(\phi_1^n) - A(\phi_0^n)\big), \tag{21}$$

$$\phi_J^n = \phi_J^{n-1} + \tfrac{\omega^2 \Delta t}{R^\gamma \Delta r} r_{J-1/2}^{1+\gamma} f_{\mathrm{ck}}^{\mathrm{EO}}(\phi_{J-1}^n, \phi_J^n) - \tfrac{\Delta t}{R^\gamma \Delta r^2} r_{J-1/2}^{\gamma}\big(A(\phi_J^n) - A(\phi_{J-1}^n)\big). \tag{22}$$

To ensure convergence of the numerical scheme to the entropy weak solution of the IBVP, the CFL stability condition $R\omega^2 \max_\phi |f_{\mathrm{ck}}'(\phi)|\,(\Delta t/\Delta r) + 2\max_\phi a(\phi)(\Delta t/\Delta r^2) \leq 1$ must be satisfied. In this work, this condition was ensured by selecting $\Delta r$ freely and determining $\Delta t$ appropriately. The accuracy was $J = 400$. For more details about the upwind method and its convergence analysis, we refer to [12, 15].

5.3. **Numerical example.** Sambuichi *et al.* [19] published centrifugation experiments with three different flocculent aqueous suspensions, namely of limestone, yeast, and clay, using a cylindrical centrifuge. For each material, the measured gravitational settling rates led to a function $f_{\mathrm{ck}}(\phi)$, and compression data determined a unique effective solid stress function $\sigma_{\mathrm{e}}(\phi)$ for each material. In this paper, we choose the published data referring

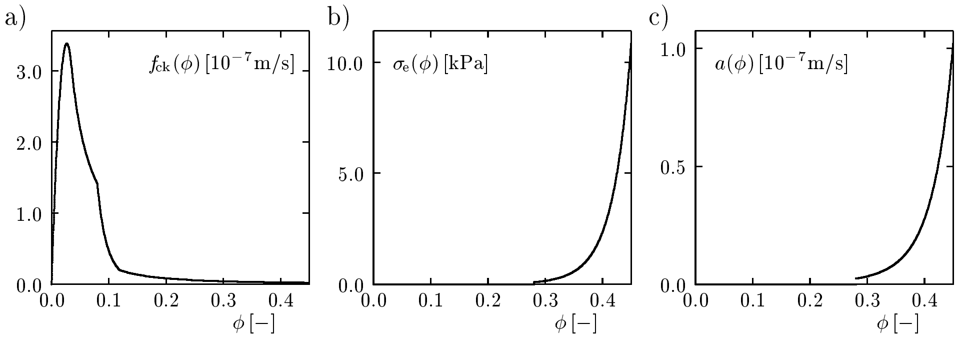


FIGURE 2. The functions (a) $f_{\mathrm{ck}}(\phi)$, (b) $\sigma_{\mathrm{e}}(\phi)$ and (c) the resulting function $a(\phi)$ used for simulating centrifugation of a limestone suspension.

to a limestone suspension (see [3] for the case of a clay suspension). Sambuichi *et al.* [19] approximated the measured gravity settling rates for various initial concentrations by three different connecting straight segments in a logarithmic plot, which yields the function

$$f_{\mathrm{ck}}(\phi) = \begin{cases} (-47.923\phi^2 + 2.5474\phi) \times 10^{-5}\,\mathrm{m/s} & \text{for } 0 \le \phi \le 0.035, \\ 1.3580 \times 10^{-8}\phi^{-0.92775}\,\mathrm{m/s} & \text{for } 0.035 < \phi \le 0.08, \\ 5.6319 \times 10^{-13}\phi^{-4.9228}\,\mathrm{m/s} & \text{for } 0.08 < \phi \le 0.119, \\ 5.9735 \times 10^{-10}\phi^{-1.65}\,\mathrm{m/s} & \text{for } 0.119 < \phi \le \phi_{\max} = 0.45, \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

It should be pointed out that being cut at $\phi_{\max}$, this function does *not* satisfy all assumptions on $f$ stated in Sect. 1. However, due to the presence of the diffusion term, solution values are bounded away from $\phi_{\max}$, so the numerical results presented below would look the same if the jump of the function $f_{\mathrm{ck}}$ at $\phi = \phi_{\max}$ had been smoothed out in order to produce an example in which these assumptions are precisely satisfied.

The solid pressure relationship suggested in [19] can be converted into the function

$$\sigma_{\mathrm{e}}(\phi) = 0 \text{ for } \phi \le \phi_{\mathrm{c}} := 0.28, \quad \sigma_{\mathrm{e}}(\phi) = 0.30184(1-\phi)^{-17.544}\,\mathrm{Pa} \text{ for } \phi > \phi_{\mathrm{c}}. \tag{24}$$

The density difference for this material was $\Delta\rho = 1710\,\mathrm{kg/m^3}$. The functions $f_{\mathrm{ck}}$ and $\sigma_{\mathrm{e}}$ given by (23) and (24) and the resulting diffusion coefficient $a(\cdot)$ are plotted in Figure 2.

Figure 3 shows numerical solutions of the phenomenological model calculated with the functions (23) and (24) in the case of a rotating cylindrical vessel ($\gamma = 1$). The left column of Figure 3 shows numerical settling plots, i.e., diagrams of iso-concentration lines for selected values of $\phi$, and the right column displays concentration profiles at selected times. The parameters and the data that differ in the three cases considered, viz. $\phi_0 = 0.111$ and $\omega = 146.4\,\mathrm{rad/s}$; $\phi_0 = 0.138$ and $\omega = 146.4\,\mathrm{rad/s}$; $\phi_0 = 0.138$ and $\omega = 104.9\,\mathrm{rad/s}$, were chosen in such a way that the simulated supernate-suspension interfaces could be compared with measurements by Sambuichi *et al.* [19], which are shown as open circles ($\circ$). Figure 3 thus illustrates the different effects of initial concentration and angular velocity on the dynamics of the centrifugation process.

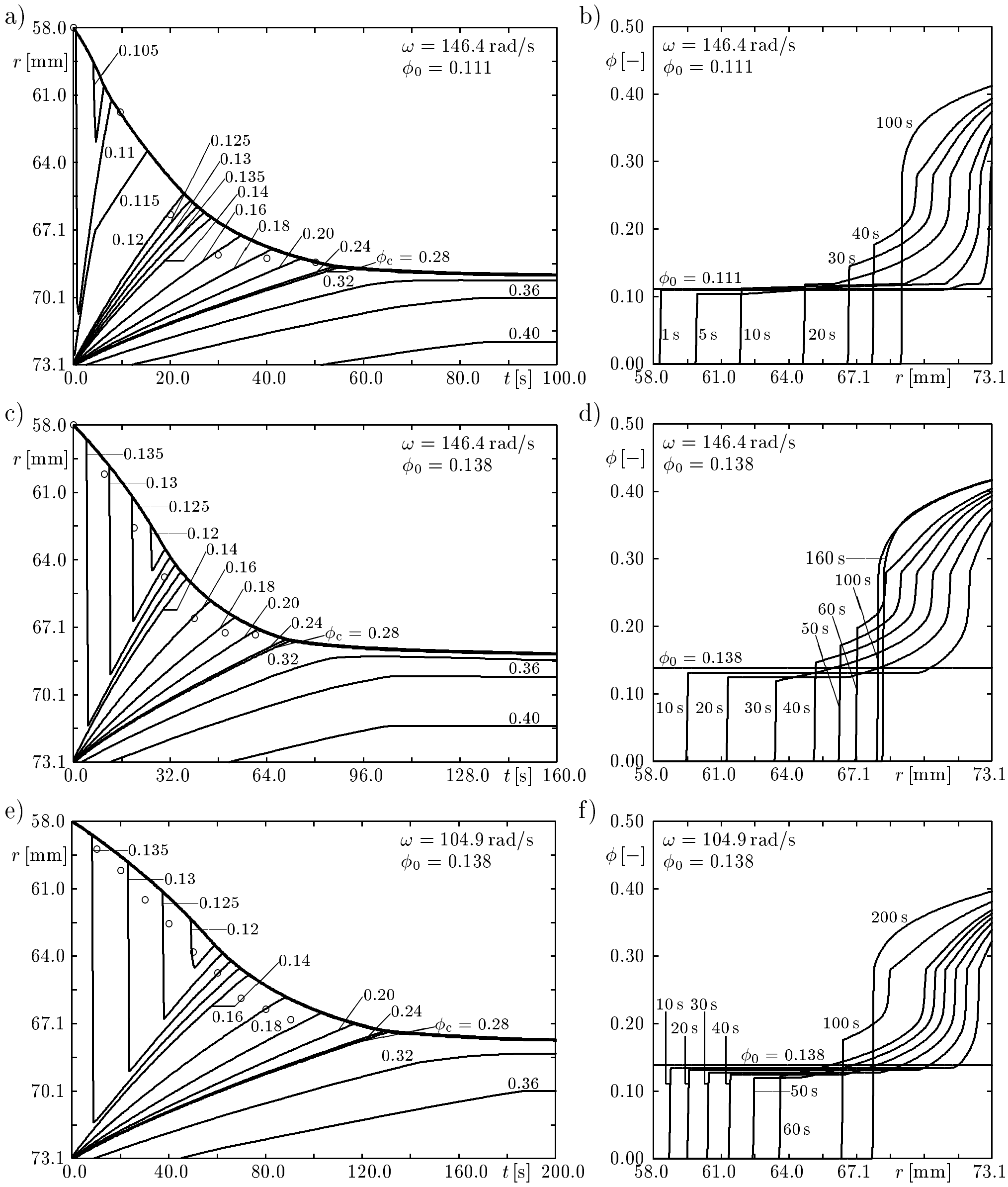


FIGURE 3. Numerical simulation of the centrifugation of a flocculated suspension.

While in the compression zone, where $\phi > \phi_c$ is valid and hence (16) is parabolic, the solutions are similar to those of the pure gravity case [2, 5], there are some distinctive features visible in the hindered settling zone ($\phi \leq \phi_c$) where (16) is hyperbolic, due to the rotating frame of reference. Most notably, the vertical iso-concentration lines indicate that the concentration of the bulk suspension is a (decreasing) function of time, and the supernate-suspension interface has a curved trajectory. These properties have previously been found by Anestis and Schneider [1], who determined exact solutions to the centrifugation model under the assumption that $\sigma_e \equiv 0$, i.e. $A \equiv 0$, using the method of characteristics. Of course, in the centrifugal case (in contrast to the gravitational) characteristics are *not* iso-concentration lines, see [1, 3].

## References


[1] G. Anestis and W. Schneider, Application of the theory of kinematic waves to the centrifugation of suspensions. Ing.-Arch. **53** (1983), 399–407.

[2] R. Bürger and F. Concha, Mathematical model and numerical simulation of the settling of flocculated suspensions. Int. J. Multiphase Flow **24** (1998), 1005–1023.

[3] R. Bürger and F. Concha, Settling velocities of particulate systems: 12. Batch centrifugation of flocculated suspensions. Preprint 2000/11, SFB 404, University of Stuttgart.

[4] R. Bürger, S. Evje, and K.H. Karlsen, On strongly degenerate convection-diffusion problems modeling sedimentation-consolidation processes. J. Math. Anal. Appl., to appear.

[5] R. Bürger and K.H. Karlsen, On some upwind difference schemes for the phenomenological sedimentation-consolidation model. Preprint 2000/05, SFB 404, University of Stuttgart.

[6] R. Bürger and W.L. Wendland, Existence, uniqueness and stability of generalized solutions of an initial-boundary value problem for a degenerating quasilinear parabolic equation. J. Math. Anal. Appl. **218** (1998), 207–239.

[7] R. Bürger and W.L. Wendland, Entropy boundary and jump conditions in the theory of sedimentation with compression, Math. Meth. Appl. Sci. **21** (1998), 865–882.

[8] R. Bürger, W.L. Wendland and F. Concha, Model equations for gravitational sedimentation-consolidation processes. Z. Angew. Math. Mech. **80** (2000), 79–92.

[9] J. Carrillo, Entropy solutions for nonlinear degenerate problems, Arch. Rat. Mech. Anal. **147** (1999), 269–361.

[10] B. Cockburn and G. Gripenberg, Continuous dependence on the nonlinearities of solutions of degenerate parabolic equations. *J. Diff. Eqns.* **151** (1999), 231–251.

[11] M. S. Espedal and K. H. Karlsen. Numerical solution of reservoir flow models based on large time step operator splitting algorithms. In A. Fasano and H. van Duijn, editors, *Filtration in Porous Media and Industrial Applications*, Lecture Notes in Mathematics. Springer. To appear.

[12] S. Evje and K. H. Karlsen. Monotone difference approximations of $BV$ solutions to degenerate convection-diffusion equations. *SIAM J. Numer. Anal.* To appear.

[13] S. Evje, K. H. Karlsen, and N. H. Risebro. A continuous dependence result for nonlinear degenerate parabolic equations with spatially dependent flux function. Preprint, Department of Mathematics, University of Bergen, 2000.

[14] K.H. Karlsen and N.H. Risebro, On the uniqueness and stability of entropy solutions of nonlinear degenerate parabolic equations with rough coefficients. Preprint, University of Bergen, 2000.

[15] K. H. Karlsen and N. H. Risebro. Convergence of finite difference schemes for viscous and inviscid conservation laws with rough coefficients. Preprint, 2000.

[16] S.N. Kružkov, First order quasilinear equations in several independent space variables. Math. USSR Sb. **10** (1970), 217–243.

[17] A.I. Vol'pert, The spaces $BV$ and quasilinear equations. Math. USSR Sb. **2** (1967), 225–267.

[18] A.I. Vol'pert and S.I. Hudjaev, Cauchy's problem for degenerate second order parabolic equations, Math. USSR Sb. **7** (1969), 365–387.

[19] M. Sambuichi, H. Nakakura and K. Osasa, Zone settling of concentrated slurries in a centrifugal field. J. Chem. Engrg. Japan **24** (1991), 489–494.

[20] Z. Wu and J. Yin, Some properties of functions in $BV_x$ and their applications to the uniqueness of solutions for degenerate quasilinear parabolic equations, Northeast. Math. J. **5** (1989), 395–422.