Solution with finite energy to a BGK system relaxing to isentropic gas dynamics

F. Berthelin[1] and F. Bouchut[2]

[1] Université d'Orléans et CNRS, UMR 6628
Département de Mathématiques
BP 6759
45067 Orléans cedex 2, France
e-mail: florent.berthelin@labomath.univ-orleans.fr

[2] Département de Mathématiques et Applications
Ecole Normale Supérieure et CNRS, UMR 8553
45, rue d'Ulm
75230 Paris cedex 05, France
e-mail: Francois.Bouchut@ens.fr

Abstract

We consider a vector kinetic BGK equation leading to isentropic gas dynamics in the relaxation limit. We prove the existence of a weak solution satisfying a kinetic entropy inequality for any initial data with finite energy.

Key-words: isentropic gas dynamics – kinetic BGK model – kinetic entropy

Mathematics Subject Classification: 35L65, 76N15, 35B35, 82C40

Work partially supported by European TMR network projects
Hyperbolic Conservation Laws contract # ERBFMRXCT960033
Kinetic theory contract # ERBFMRXCT970157

Contents

1 Introduction and main results

Relaxation models such as proposed in [6] have proved their efficiency in building numerical methods for conservation laws that are very easy to code and have nice properties. General strategies have been proposed in [9], [13], [1], [3] to build BGK approaches for systems of conservation laws. However, analysis of such models has mainly be achieved until now only when the relaxed equation is scalar (see [14]). Here we consider a BGK model for the one-dimensional system of isentropic gas dynamics

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \kappa \rho^\gamma) = 0, \end{cases} \tag{1.1}$$

with $\rho(t,x) \geq 0,\ u(t,x) \in \mathbb{R}$ and $\kappa > 0,\ 1 < \gamma < 3$. The kinetic model has been introduced in [3] and can be written

$$\partial_t f + \xi \partial_x f = \frac{M[f] - f}{\varepsilon} \quad \text{in }]0,\infty[\times \mathbb{R} \times \mathbb{R}, \tag{1.2}$$

where $f = f(t,x,\xi) \in \mathbb{R}^2$, $t \geq 0$, $x \in \mathbb{R}$, $\xi \in \mathbb{R}$,

$$f(t,x,\xi) \in D, \tag{1.3}$$

$$D = \{(f_0, f_1) \in \mathbb{R}^2,\ f_0 > 0 \text{ or } f_1 = f_0 = 0\}, \tag{1.4}$$

$$M[f](t,x,\xi) = M\left(\rho(t,x), u(t,x), \xi\right), \tag{1.5}$$

$$\rho(t,x) = \int_{\mathbb{R}} f_0(t,x,\xi)\, d\xi,\ \ \rho(t,x)u(t,x) = \int_{\mathbb{R}} f_1(t,x,\xi)\, d\xi, \tag{1.6}$$

and

$$M(\rho,u,\xi) = \Big(\chi(\rho, \xi - u), ((1-\theta)u + \theta\xi)\chi(\rho, \xi - u)\Big), \tag{1.7}$$

$$\chi(\rho,\xi) = c_{\gamma,\kappa} \left(\frac{4\gamma\kappa}{(\gamma-1)^2} \rho^{\gamma-1} - \xi^2 \right)_+^\lambda, \tag{1.8}$$

$$\theta = \frac{\gamma - 1}{2},\ \lambda = \frac{1}{\gamma - 1} - \frac{1}{2},\ c_{\gamma,\kappa} = \frac{1}{J_\lambda}\left(\frac{4\gamma\kappa}{(\gamma-1)^2}\right)^{-1/(\gamma-1)}, \tag{1.9}$$

$$J_\lambda = \int_{-1}^{1} (1 - z^2)^\lambda \, dz = \sqrt{\pi}\Gamma(\lambda + 1)/\Gamma(\lambda + 3/2). \tag{1.10}$$

The kinetic equilibrium χ in (1.8) has been introduced in [7] as a generating function for entropies, and has also been used in the stability analysis of [10]. This function is also involved in [12] for the so called kinetic formulation, which is different from the BGK equation considered here. Previous results for scalar equations can be found in [4], [11], [17] (see also [5]). The first existence result for a BGK equation was given in [15] (see also [16]).

The main result of this paper is the existence of a global solution to the system (1.2) with initial data

$$f(0, x, \xi) = f^0(x, \xi) \tag{1.11}$$

satisfying energy bounds. We recall that the energy for (1.1) is given by

$$\eta(\rho, u) = \rho u^2/2 + \frac{\kappa}{\gamma - 1}\rho^\gamma, \tag{1.12}$$

and is a mathematical entropy for (1.1). The corresponding kinetic entropy for (1.2) is given, for $f = (f_0, f_1) \in D$, by

$$\begin{aligned} H(f, \xi) &= \frac{\theta}{1-\theta}\frac{\xi^2}{2} f_0 + \frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}}\frac{f_0^{1+1/\lambda}}{1 + 1/\lambda} + \frac{1}{1-\theta}\frac{1}{2}\frac{f_1^2}{f_0} - \frac{\theta}{1-\theta}\xi f_1 \\ &= \frac{\theta}{1-\theta}\frac{1}{2}\left(\frac{f_1}{\sqrt{f_0}} - \xi\sqrt{f_0}\right)^2 + \frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}}\frac{f_0^{1+1/\lambda}}{1 + 1/\lambda} + \frac{1}{2}\frac{f_1^2}{f_0} \geq 0, \end{aligned} \tag{1.13}$$

$$H(0, \xi) = 0. \tag{1.14}$$

We have the following theorems.

Theorem 1.1 *Assume that $f^0 \in L^1(\mathbb{R}_x \times \mathbb{R}_\xi)$ satisfies*

$$f^0(x, \xi) \in D \text{ a.e. in } \mathbb{R} \times \mathbb{R}, \tag{1.15}$$

$$\iint_{\mathbb{R}\times\mathbb{R}} H(f^0(x, \xi), \xi) \, dx d\xi = C_H^0 < \infty, \tag{1.16}$$

and

$$\iint_{\mathbb{R}\times\mathbb{R}} x^2 (f^0)_0(x, \xi) \, dx d\xi = C_\Delta^0 < \infty. \tag{1.17}$$

Then there exists a solution f to (1.2)-(1.11) satisfying

$$f \in C([0, \infty[, L^1(\mathbb{R} \times \mathbb{R})), \tag{1.18}$$

$$f \in L^\infty([0, \infty[, L^1(\mathbb{R} \times \mathbb{R})), \tag{1.19}$$

$$\forall t \geq 0,\ f(t,x,\xi) \in D \ a.e.\ in\ \mathbb{R}\times\mathbb{R}, \tag{1.20}$$

$$\partial_t\Big(\int_{\mathbb{R}} f\,d\xi\Big) + \partial_x\Big(\int_{\mathbb{R}} \xi f\,d\xi\Big) = 0, \tag{1.21}$$

$$\forall t \geq 0,\ \iint\limits_{\mathbb{R}\times\mathbb{R}} f(t,x,\xi)\,dxd\xi = \iint\limits_{\mathbb{R}\times\mathbb{R}} f^0(x,\xi)\,dxd\xi, \tag{1.22}$$

$$\forall t \geq 0,\ \iint\limits_{\mathbb{R}\times\mathbb{R}} H(f(t,x,\xi),\xi)\,dxd\xi \leq C_H^0, \tag{1.23}$$

$$\forall t \geq 0,\ \iint\limits_{\mathbb{R}\times\mathbb{R}} x^2 f_0(t,x,\xi)\,dxd\xi \leq \max\left(C_\Delta^0, \frac{4\varepsilon^2}{\theta}C_H^0\right)(1+t/\varepsilon)^2. \tag{1.24}$$

Theorem 1.2 *If (1.17) is not satisfied, we have the same result except (1.24).*

To obtain this result, we have to work with the Tychonoff-Schauder theorem instead of the usual Schauder theorem. The only significant change is for compactness and will be explain in the corresponding section.

Theorem 1.3 *The solution f obtained in Theorems 1.1 or 1.2 satisfies*

$$H(f,\xi) \in C([0,\infty[, L^1(\mathbb{R}\times\mathbb{R})), \tag{1.25}$$

$$\partial_t(H(f,\xi)) + \xi\partial_x(H(f,\xi)) = \frac{1}{\varepsilon}H'(f,\xi)\cdot(M[f]-f), \tag{1.26}$$

with

$$H'(f,\xi) = \left(\frac{\theta}{1-\theta}\frac{\xi^2}{2} + \frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}}f_0^{1/\lambda} - \frac{1}{1-\theta}\frac{1}{2}\frac{f_1^2}{f_0^2}, -\frac{\theta}{1-\theta}\xi + \frac{1}{1-\theta}\frac{f_1}{f_0}\right) \tag{1.27}$$

for $f \neq 0$, and $H'(0,\xi) = (\frac{\theta}{1-\theta}\frac{\xi^2}{2}, -\frac{\theta}{1-\theta}\xi)$ by convention. Moreover,

$$H'(f,\xi)\cdot(M[f]-f) \in L^1(]0,T[\times\mathbb{R}\times\mathbb{R}) \qquad \textit{for any } T>0 \tag{1.28}$$

as well as each term from the decomposition (1.27). Besides, $\xi H(f(t,x,\xi),\xi) \in L^1(]0,T[\times B_R\times\mathbb{R})$ for any $T>0$, $R>0$,

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{\mathbb{R}} H(f(t,x,\xi),\xi)\,d\xi + \frac{\partial}{\partial x}\int_{\mathbb{R}} \xi H(f(t,x,\xi),\xi)\,d\xi \\ &= \frac{1}{\varepsilon}\int_{\mathbb{R}}\left(H'(f,\xi) - (\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1} - \frac{u^2}{2}, u)\right)\cdot(M[f]-f)\,d\xi \leq 0,\end{aligned} \tag{1.29}$$

$$\begin{aligned}&\frac{d}{dt}\iint\limits_{\mathbb{R}\times\mathbb{R}} H(f(t,x,\xi),\xi)\,dxd\xi \\ &= \frac{1}{\varepsilon}\iint\limits_{\mathbb{R}\times\mathbb{R}}\left(H'(f,\xi) - (\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1} - \frac{u^2}{2}, u)\right)\cdot(M[f]-f)\,dxd\xi \leq 0,\end{aligned} \tag{1.30}$$

and the integrand in the right-hand sides of (1.29) and (1.30) is nonpositive.

Let us finally mention that we can study kinetic invariant domains and establish the relaxation limit to (1.1), this is done in [2].

2 Properties of the kinetic entropy

We recall the value of the moments of M,

$$\int_{\mathbb{R}} M(\rho, u, \xi)\, d\xi = (\rho, \rho u), \tag{2.1}$$

$$\int_{\mathbb{R}} \xi M(\rho, u, \xi)\, d\xi = (\rho u, \rho u^2 + \kappa \rho^\gamma), \tag{2.2}$$

and

$$\int_{\mathbb{R}} \frac{1}{2}\xi^2 M_0(\rho, u, \xi)\, d\xi = \frac{1}{2}\rho u^2 + \frac{\kappa}{\gamma - 1}\rho^\gamma = \eta(\rho, u) \tag{2.3}$$

for every $\rho \geq 0$ and $u \in \mathbb{R}$. We have the following identities which link the energy and the kinetic entropy. They can be obtained easily.

Proposition 2.1 *The energy η of (1.12) and the kinetic entropy H of (1.13) satisfy*

$$\int_{\mathbb{R}} H(M(\rho, u, \xi), \xi)\, d\xi = \eta(\rho, u) \quad \textit{for every } \rho \geq 0,\ u \in \mathbb{R}, \tag{2.4}$$

and

$$\begin{cases} \dfrac{\partial H}{\partial f_0}(M(\rho, u, \xi), \xi) = \dfrac{\partial \eta}{\partial \rho}(\rho, u) = \dfrac{\gamma\kappa}{\gamma - 1}\rho^{\gamma-1} - u^2/2, \\ \dfrac{\partial H}{\partial f_1}(M(\rho, u, \xi), \xi) = \dfrac{\partial \eta}{\partial(\rho u)}(\rho, u) = u, \end{cases} \tag{2.5}$$

for every $\rho > 0$, $u \in \mathbb{R}$ and $\xi \in \mathbb{R}$ such that $M_0(\rho, u, \xi) > 0$.

The function H has the following properties.

Proposition 2.2 *i) H is convex with respect to f in D,*
ii) H is continuous with respect to f in $\Gamma_A = \{(f_0, f_1) \in D\ ;\ |f_1| \leq A f_0\}$ for any $A > 0$.

Proof. *i)* In $]0, \infty[\times\mathbb{R}$, it is easy to compute the hessian and to check its nonnegativity. Then, on a straight line from the origin, f_1^2/f_0 is linear, thus H is convex in D.
ii) The only difficulty is at 0. Let $(f^n)_{n\geq 0}$ be a sequence in $\Gamma_A\backslash\{0\}$ which converges to 0. Then

$$\left|\frac{{(f^n)_1}^2}{(f^n)_0}\right| \leq A^2 (f^n)_0 \underset{n\to\infty}{\longrightarrow} 0,$$

thus H is continuous at 0 in Γ_A. $\square$

Lemma 2.3 (Coerciveness) *There exist $\varepsilon_0, \varepsilon_1 > 0$ such that for any $f \in D$, $\xi \in \mathbb{R}$, we have*

$$H(f, \xi) \geq \varepsilon_0 {f_0}^{p_0} + \varepsilon_1 |f_1|^{p_1}, \tag{2.6}$$

with

$$p_0 = 1 + 1/\lambda > 1, \qquad p_1 = 2(1+\lambda)/(1+2\lambda) > 1. \tag{2.7}$$

Proof. Let $f \in D$, $f \neq 0$. From identity (1.13), we deduce that

$$H(f,\xi) \geq \frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}} \frac{f_0^{1+1/\lambda}}{1+1/\lambda} + \frac{1}{2}\frac{f_1^2}{f_0}.$$

We have $|f_1|^{p_1} = \left(\frac{|f_1|}{\sqrt{f_0}}\right)^{p_1} \left(\sqrt{f_0}\right)^{p_1}$, and using Young's inequality,

$$|f_1|^{p_1} \leq \frac{p_1}{2}\left(\frac{|f_1|}{\sqrt{f_0}}\right)^2 + \frac{p_1}{2+2/\lambda} {f_0}^{1+1/\lambda}.$$

Taking $0 < \varepsilon < 1$ such that $\varepsilon < \theta/c_{\gamma,\kappa}^{1/\lambda}$, and setting $\varepsilon_0 = \left(\theta/c_{\gamma,\kappa}^{1/\lambda} - \varepsilon\right)/(2+2/\lambda)$ and $\varepsilon_1 = \varepsilon/p_1$, we get

$$\begin{aligned} H(f,\xi) &\geq \frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}} \frac{f_0^{1+1/\lambda}}{1+1/\lambda} + \varepsilon\frac{1}{2}\frac{f_1^2}{f_0} \\ &\geq \frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}} \frac{f_0^{1+1/\lambda}}{1+1/\lambda} - \frac{\varepsilon}{2+2/\lambda} f_0^{1+1/\lambda} + \frac{\varepsilon}{p_1}|f_1|^{p_1} \\ &= \varepsilon_0 {f_0}^{p_0} + \varepsilon_1 |f_1|^{p_1}. \ \square \end{aligned}$$

The key result of this section is the following.

Proposition 2.4 (Key convexity inequality) *For any $f \in D$, $\rho \geq 0$ and $u, \xi \in \mathbb{R}$, we have*

$$H(f,\xi) \geq H(M(\rho,u,\xi),\xi) + \left(\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1} - \frac{u^2}{2}, u\right) \cdot (f - M(\rho,u,\xi)). \quad (2.8)$$

Proof. Let us consider $\psi : D \to \mathbb{R}$ defined by

$$\psi(f) = H(f,\xi) - H(M(\rho,u,\xi),\xi) - \left(\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1} - \frac{u^2}{2}, u\right) \cdot (f - M(\rho,u,\xi)),$$

which is possible because $M(\rho,u,\xi) \in D$. We have to prove that $\psi \geq 0$.
– At infinity,
by Lemma 2.3, $\psi(f) \geq \varepsilon_0 {f_0}^{p_0} + \varepsilon_1 |f_1|^{p_1} + af_0 + bf_1 + e$ with $a, b, e \in \mathbb{R}$, and since $p_0, p_1 > 1$, $\psi(f) \to \infty$ when $f_0 + |f_1|$ tends to infinity. Moreover, $\psi(f) \geq af_0 + bf_1 + e$, and we deduce that ψ is lower bounded in D.
– At the frontier $\{f_0 = 0, f_1 \neq 0\}$,
$f_1^2/f_0 \to \infty$ and $\psi(f)$ also.
– At 0,
If $M_0(\rho,u,\xi) > 0$, then by Proposition 2.2(i) and (2.5), (2.8) holds true for any $f \in D$, thus $\varliminf_{f\to 0} \psi(f) \geq 0$. On the contrary, if $M_0(\rho,u,\xi) = 0$, then

$$\psi(f) = H(f,\xi) - \left(\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1} - u^2/2, u\right) \cdot f.$$

Since $H(f,\xi) \geq 0$, this yields $\varliminf_{f\to 0} \psi(f) \geq 0$, in any case.

– Inside D

It only remains to study possible minima of ψ in $int(D)$. But ψ is smooth there, and if $f_0 > 0$,

$$\frac{\partial \psi}{\partial f_0}(f,\xi) = \frac{\theta}{1-\theta}\frac{\xi^2}{2} + \frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}} f_0^{1/\lambda} - \frac{1}{1-\theta}\frac{1}{2}\left(\frac{f_1}{f_0}\right)^2 - \frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1} + \frac{u^2}{2},$$

and

$$\frac{\partial \psi}{\partial f_1}(f,\xi) = \frac{1}{1-\theta}\frac{f_1}{f_0} - \frac{\theta}{1-\theta}\xi - u.$$

If there is a minimum of ψ in $int(D)$, at this point, its partial derivatives must vanish, thus

$$\frac{f_1}{f_0} = (1-\theta)u + \theta\xi,$$

and

$$\begin{aligned}\frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}} f_0^{1/\lambda} &= \frac{1}{1-\theta}\frac{1}{2}\left(\frac{f_1}{f_0}\right)^2 + \frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1} - \frac{u^2}{2} - \frac{\theta}{1-\theta}\frac{\xi^2}{2}\\ &= \frac{\theta}{2}\left(\frac{4\gamma\kappa}{(\gamma-1)^2}\rho^{\gamma-1} - (\xi-u)^2\right).\end{aligned}$$

We deduce that $M_0(\rho,u,\xi) > 0$ and $f = M(\rho,u,\xi)$. The value of ψ at this point is 0.

Putting all the steps together, we conclude that $\psi \geq 0$ in D. □

An important consequence of this result is the entropy minimization principle.

Proposition 2.5 *Consider $f \in L^1(\mathbb{R}_\xi)$ such that $f \in D$ a.e. and*

$$\int_{\mathbb{R}} H(f(\xi),\xi)\, d\xi < \infty.$$

Then

$$\int_{\mathbb{R}} H(M[f](\xi),\xi)\, d\xi \leq \int_{\mathbb{R}} H(f(\xi),\xi)\, d\xi.$$

Proof. Take in (2.8)

$$\rho = \int_{\mathbb{R}} f_0(\xi)\, d\xi, \qquad u = \frac{\int_{\mathbb{R}} f_1(\xi)\, d\xi}{\int_{\mathbb{R}} f_0(\xi)\, d\xi}, \qquad \text{or } u = 0 \text{ if } \rho = 0.$$

Then $M[f](\xi) = M(\rho,u,\xi)$, the terms of the right-hand side are integrable with respect to ξ, and $\int_{\mathbb{R}} M[f](\xi)\, d\xi = \int_{\mathbb{R}} f(\xi)\, d\xi$. Thus the result is obtained by integration in ξ. □

We can also deduce that the entropy dissipation in (1.29) has a sign.

Proposition 2.6 *For any $f \in D$, $\rho \geq 0$, $u, \xi \in \mathbb{R}$,*

$$\Big(H'(f,\xi) - \big(\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1} - \frac{u^2}{2}, u\big)\Big) \cdot (f - M(\rho,u,\xi)) \geq 0. \tag{2.9}$$

Proof. Since H is convex in D, continuous in the cones of Proposition 2.2 and smooth in $int(D)$, we have if $f \neq 0$

$$H(M(\rho,u,\xi),\xi) \geq H(f,\xi) + H'(f,\xi)(M(\rho,u,\xi) - f). \tag{2.10}$$

Letting $f \to 0$ on the axis $f_1 = 0$, we obtain using (1.27) that (2.10) is indeed true for any $f \in D$ with the convention

$$H'(0,\xi) = \left(\frac{\theta}{1-\theta}\frac{\xi^2}{2}, -\frac{\theta}{1-\theta}\xi\right). \tag{2.11}$$

The case $f = 0$ can also be seen directly by the definition of H. Finally, by adding (2.10) to (2.8), we get (2.9). $\square$

Let us end this part by estimates deduced from the boundedness of the kinetic entropy.

Proposition 2.7 *Consider $f \in L^1(\mathbb{R}_x \times \mathbb{R}_\xi)$ such that $f \in D$ a.e., and*

$$\iint_{\mathbb{R}\times\mathbb{R}} H(f(x,\xi),\xi)\, dxd\xi \leq C_H,$$

$$\iint_{\mathbb{R}\times\mathbb{R}} f_0(x,\xi)\, dxd\xi \leq C_0.$$

Then

$$\iint_{\mathbb{R}\times\mathbb{R}} |f_1(x,\xi)|\, dxd\xi \leq \sqrt{2C_0C_H}, \tag{2.12}$$

$$\iint_{\mathbb{R}\times\mathbb{R}} \xi^2 f_0(x,\xi)\, dxd\xi \leq \frac{4}{\theta}C_H, \tag{2.13}$$

$$\iint_{\mathbb{R}\times\mathbb{R}} |\xi||f_1(x,\xi)|\, dxd\xi \leq \sqrt{\frac{8}{\theta}}C_H, \tag{2.14}$$

$$f_i \text{ is bounded in } L^{p_i}(\mathbb{R}\times\mathbb{R}) \text{ for } i = 0,1. \tag{2.15}$$

Proof. We recall that p_0, p_1 are defined by (2.7). First, using (1.13), we have

$$\mathbb{I}_{f_0>0}\frac{f_1^2}{f_0} \leq 2H(f,\xi). \tag{2.16}$$

Then, since f_1 vanishes where $f_0 = 0$,

$$|f_1| = |\mathbb{1}_{f_0>0} f_1| = |\mathbb{1}_{f_0>0} \frac{f_1}{\sqrt{f_0}} \times \sqrt{f_0}| \leq \sqrt{2H(f,\xi) \times f_0}. \tag{2.17}$$

Thus, from the Cauchy-Schwarz inequality we obtain (2.12). Next, with (1.13) again,

$$\begin{aligned}
\xi^2 f_0 &= \left| \mathbb{1}_{f_0>0} \frac{f_1}{\sqrt{f_0}} - \xi\sqrt{f_0} - \mathbb{1}_{f_0>0} \frac{f_1}{\sqrt{f_0}} \right|^2 \\
&\leq \left(\left| \mathbb{1}_{f_0>0} \frac{f_1}{\sqrt{f_0}} - \xi\sqrt{f_0} \right| + \left| \mathbb{1}_{f_0>0} \frac{f_1}{\sqrt{f_0}} \right| \right)^2 \\
&\leq 2 \left[\left(\mathbb{1}_{f_0>0} \frac{f_1}{\sqrt{f_0}} - \xi\sqrt{f_0} \right)^2 + \mathbb{1}_{f_0>0} \frac{f_1^2}{f_0} \right] \\
&\leq 2 \left[\frac{2(1-\theta)}{\theta} H(f,\xi) + 2H(f,\xi) \right] \\
&= \frac{4}{\theta} H(f,\xi),
\end{aligned}$$

and (2.13) follows. Then, we write

$$\left| \mathbb{1}_{f_0>0} \xi f_1 \right| = \left| (\mathbb{1}_{f_0>0} f_1/\sqrt{f_0}) \times (\xi\sqrt{f_0}) \right| \leq \sqrt{2H(f,\xi)} \times \sqrt{\frac{4}{\theta} H(f,\xi)},$$

and obtain (2.14). Finally, (2.15) follows obviously from Lemma 2.3. □

3 Stability

In this section, we extend the analysis of [15] and prove the stability of an approximate solution. We shall denote by $\varphi_n(x,\xi) \to \varphi(x,\xi)$ in $L^1(\mathbb{R}_{loc} \times \mathbb{R})$ the convergence in $L^1(\omega \times \mathbb{R})$ for any $\omega \subset\subset \mathbb{R}$, and we shall use several similar conventions.
Let us first recall the classical characteristics formula for (1.2).

Lemma 3.1 *Let $h \in L^1(]0,T[, L^1(\mathbb{R} \times \mathbb{R}))$ and $f^0 \in L^1(\mathbb{R} \times \mathbb{R})$. Then there exists a unique solution*

$$f \in C([0,T], L^1(\mathbb{R} \times \mathbb{R})) \tag{3.1}$$

to the problem

$$\begin{cases} \partial_t f + \xi \partial_x f = \dfrac{h-f}{\varepsilon}, \\ f(0,x,\xi) = f^0(x,\xi). \end{cases} \tag{3.2}$$

Furthermore,

$$f(t,x,\xi) = f^0(x - t\xi, \xi) e^{-t/\varepsilon} + \frac{1}{\varepsilon} \int_0^t e^{-s/\varepsilon} h(t-s, x-s\xi, \xi)\, ds. \tag{3.3}$$

We now prove a stability result for this problem.

Proposition 3.2 *Consider $g, g_n \in L^\infty(]0,T[, L^1(\mathbb{R}\times\mathbb{R}))$ such that $g, g_n \in D$ a.e.,*

$$\iint_{\mathbb{R}\times\mathbb{R}} H(g(t,x,\xi),\xi)\,dxd\xi \le C_H, \qquad \iint_{\mathbb{R}\times\mathbb{R}} g_0(t,x,\xi)\,dxd\xi \le C_0,$$

$$\iint_{\mathbb{R}\times\mathbb{R}} H(g_n(t,x,\xi),\xi)\,dxd\xi \le C_H, \qquad \iint_{\mathbb{R}\times\mathbb{R}} (g_n)_0(t,x,\xi)\,dxd\xi \le C_0.$$

Set

$$(\rho(t,x), \rho u(t,x)) = \int_{\mathbb{R}} g(t,x,\xi)\,d\xi$$

and

$$(\rho_n(t,x), \rho_n u_n(t,x)) = \int_{\mathbb{R}} g_n(t,x,\xi)\,d\xi.$$

If $\rho_n \underset{n\to\infty}{\to} \rho$ and $\rho_n u_n \underset{n\to\infty}{\to} \rho u$ in $L^1(]0,T[\times\mathbb{R}_{loc})$, then there exists a subsequence such that $F(g_n) \underset{n\to\infty}{\to} F(g)$ in $C([0,T], L^1(\mathbb{R}_{loc}\times\mathbb{R}))$, where $F(g)$ is the solution to (3.1)-(3.2) with $h = M[g] = M(\rho,u,\xi)$ and $f^0 \in L^1(\mathbb{R}\times\mathbb{R})$ is fixed. Moreover, if $\rho_n \to \rho$ in $L^1(]0,T[\times\mathbb{R})$, then $F(g_n)\to F(g)$ in $C([0,T], L^1(\mathbb{R}\times\mathbb{R}))$.

Proof. At first, let us check that $F(g)$ is well defined. Using Proposition 2.5, we have

$$\iint_{\mathbb{R}\times\mathbb{R}} H(M[g](t,x,\xi),\xi)\,dxd\xi \le C_H.$$

Besides,

$$\iint_{\mathbb{R}\times\mathbb{R}} M[g]_0(t,x,\xi)\,dxd\xi = \iint_{\mathbb{R}\times\mathbb{R}} g_0(t,x,\xi)\,dxd\xi \le C_0.$$

Estimate (2.12) applied to $M[g]$ gives then that

$$\iint_{\mathbb{R}\times\mathbb{R}} |M[g]_1(t,x,\xi)|\,dxd\xi \le \sqrt{2C_0C_H}.$$

Therefore, we deduce that $M[g] \in L^\infty(]0,T[, L^1(\mathbb{R}\times\mathbb{R}))$, and we have the same result for g_n with uniform bounds.
We turn now to the stability. We can find a subsequence such that $\rho_n \to \rho$ and $\rho_n u_n \to \rho u$ a.e. t,x. Set $E = \{(t,x)\in[0,T]\times\mathbb{R};\ \rho(t,x)=0\}$. In E^c, $\rho_n\to\rho$ and $u_n \to u$ a.e. t,x, thus $M[g_n]\to M[g]$ a.e. $(t,x)\in E^c$, $\xi\in\mathbb{R}$. But since

$$\iint_{(t,x)\in E, x\in B_R} \int M[g_n]_0\,dtdxd\xi = \iint_{(t,x)\in E, x\in B_R} \rho_n\,dtdx \to 0,$$

we deduce with (2.17) and Cauchy-Schwarz inequality that $M[g_n]\to 0$ in $L^1((E\times\mathbb{R})\cap\{x\in B_R\})$. Therefore, after extraction of a subsequence, we finally obtain

$$M[g_n]\to M[g] \qquad \text{a.e. in }]0,T[\times\mathbb{R}\times\mathbb{R}. \tag{3.4}$$

Besides, the estimates of Proposition 2.7 applied to $M[g_n](t,.,.)$ give that $M[g_n]$ is weakly compact in $L^1(]0,T[\times\mathbb{R}_{loc}\times\mathbb{R})$. Therefore, we are able to apply the Vitali theorem, which gives that

$$M[g_n]\to M[g] \;\text{ in } L^1(]0,T[\times\mathbb{R}_{loc}\times\mathbb{R}).$$

Now

$$\begin{aligned}
&\iint\limits_{|x|\le R,|\xi|\le S} |(F(g_n)-F(g))(t,x,\xi)|\,dxd\xi\\
\le\;& \frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon}\iint\limits_{|x|\le R,|\xi|\le S}|(M[g_n]-M[g])(t-s,x-s\xi,\xi)|\,dxd\xi\,ds\\
\le\;& \frac{1}{\varepsilon}\|M[g_n]-M[g]\|_{L^1(]0,T[\times B_{R+TS}\times B_S)}\longrightarrow 0,
\end{aligned}$$

and

$$\begin{aligned}
&\iint\limits_{|\xi|>S} |(F(g_n)-F(g))(t,x,\xi)|\,dxd\xi\\
\le\;& \frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon}\iint\limits_{|\xi|>S}\frac{|\xi|}{S}|(M[g_n]-M[g])(t-s,x-s\xi,\xi)|\,dxd\xi\,ds\\
\le\;& \frac{1}{\varepsilon S}\|\xi M[g_n]-\xi M[g]\|_{L^1(]0,T[\times\mathbb{R}\times\mathbb{R})}.
\end{aligned}$$

Since the latter norm is bounded, we deduce easily that $F(g_n)\to F(g)$ in $C([0,T],L^1(\mathbb{R}_{loc}\times\mathbb{R}))$.
Finally, if we assume moreover that $\rho_n\to\rho$ in $L^1(]0,T[\times\mathbb{R})$, then

$$\iiint\limits_{]0,T[\times\mathbb{R}\times\mathbb{R}} M[g_n]_0\,dtdxd\xi\longrightarrow \iiint\limits_{]0,T[\times\mathbb{R}\times\mathbb{R}} M[g]_0\,dtdxd\xi,$$

and since $M[g_n]_0\ge 0$, we deduce that

$$\iiint\limits_{|x|>R} M[g_n]_0\,dtdxd\xi\longrightarrow 0\quad\text{as } R\to\infty,$$

uniformly in n. With (2.17) and Cauchy-Schwarz inequality, we get also

$$\iiint\limits_{|x|>R} |M[g_n]_1|\,dtdxd\xi\longrightarrow 0\quad\text{as } R\to\infty,$$

uniformly in n, and we conclude that $M[g_n]\to M[g]$ in $L^1(]0,T[\times\mathbb{R}\times\mathbb{R})$, and therefore $F(g_n)\to F(g)$ in $C([0,T],L^1(\mathbb{R}\times\mathbb{R}))$. $\square$

4 Existence of a solution

This section is devoted to the proof of Theorem 1.1. Let $f^0 \in L^1(\mathbb{R}_x \times \mathbb{R}_\xi)$ satisfy the assumptions of Theorem 1.1, namely

$$f^0(x,\xi) \in D \; a.e. \text{ in } \mathbb{R} \times \mathbb{R}, \tag{4.1}$$

$$\iint_{\mathbb{R}\times\mathbb{R}} H(f^0(x,\xi),\xi)\,dxd\xi = C_H^0 < \infty, \tag{4.2}$$

$$\iint_{\mathbb{R}\times\mathbb{R}} x^2 (f^0)_0(x,\xi)\,dxd\xi = C_\Delta^0 < \infty, \tag{4.3}$$

and set

$$\iint_{\mathbb{R}\times\mathbb{R}} (f^0)_0(x,\xi)\,dxd\xi = C_0. \tag{4.4}$$

We take C_H and C_Δ as

$$C_H = C_H^0, \qquad C_\Delta = \max\left(C_\Delta^0; \frac{4\varepsilon^2}{\theta} C_H^0\right). \tag{4.5}$$

Define C to be the set of all $g \in L^\infty(]0,T[, L^1(\mathbb{R}\times\mathbb{R}))$ satisfying $(C1)-(C4)$ for a.e. $t \in [0,T]$, where

$$\begin{array}{ll} (C1) & g(t,x,\xi) \in D \quad a.e. \text{ in } \mathbb{R}\times\mathbb{R}, \\ (C2) & \displaystyle\iint_{\mathbb{R}\times\mathbb{R}} H(g(t,x,\xi),\xi)\,dxd\xi \le C_H, \\ (C3) & \displaystyle\iint_{\mathbb{R}\times\mathbb{R}} g_0(t,x,\xi)\,dxd\xi \le C_0, \\ (C4) & \displaystyle\iint_{\mathbb{R}\times\mathbb{R}} x^2 g_0(t,x,\xi)\,dxd\xi \le C_\Delta\,(1+t/\varepsilon)^2. \end{array} \tag{4.6}$$

Let us also introduce

$$\begin{aligned} \tilde{C} = \Big\{ & g \in C([0,T], L^1(\mathbb{R}\times\mathbb{R})) \text{ satisfying } (C5), \\ & \text{and } (C1)-(C4) \text{ for all } t \in [0,T] \Big\}, \end{aligned} \tag{4.7}$$

with

$$(C5) \quad \partial_t g + \xi \partial_x g + \frac{g}{\varepsilon} \in \frac{C}{\varepsilon}.$$

The initial data f^0 being fixed, for any $g \in C$, we denote by $F(g)$ the solution to (3.1)-(3.2) with $h = M[g]$.

Lemma 4.1 *If $g \in C$, then $M[g] \in C$ and $F(g) \in \tilde{C}$.*

Proof. Let $g \in C$. As in the proof of Proposition 3.2, $M[g] \in L^\infty(]0,T[,L^1(\mathbb{R}\times\mathbb{R}))$ and we get easily that $M[g] \in C$. Next, according to Lemma 3.1, $F(g) \in C([0,T],L^1(\mathbb{R}\times\mathbb{R}))$, and in order to prove that $F(g) \in \tilde{C}$, we need to prove $(C1)-(C5)$. Condition $(C5)$ is satisfied since $M[g] \in C$. Then, for $i = 0;1$, we have

$$F(g)_i(t,x,\xi) = (f^0)_i(x-t\xi,\xi)e^{-t/\varepsilon} + \frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon} M[g]_i(t-s,x-s\xi,\xi)\,ds. \tag{4.8}$$

Obviously, $F(g)_0 \geq 0$. Furthermore, if $F(g)_0(t,x,\xi) = 0$ then $(f^0)_0(x-t\xi,\xi) = 0$ and $\int_0^t e^{-s/\varepsilon}M[g]_0(t-s,x-s\xi,\xi)\,ds = 0$, thus $M[g]_0(t-s,x-s\xi,\xi) = 0$ *a.e.* $s \in]0,t[$. But $f^0 \in D$ and $M[g] \in D$, thus $(f^0)_1(x-t\xi,\xi) = 0$ and $M[g]_1(t-s,x-s\xi,\xi) = 0$ *a.e.* $s \in]0,t[$, and therefore $F(g)_1(t,x,\xi) = 0$. Hence $F[g] \in D$ $\forall t \in [0,T]$, *a.e.* x,ξ, and $(C1)$ is satisfied. Next, if $t > 0$, we can write

$$F(g)(t,x,\xi) = f^0(x-t\xi,\xi)e^{-t/\varepsilon} + (1-e^{-t/\varepsilon})\int_0^t M[g](t-s,x-s\xi,\xi)\frac{e^{-s/\varepsilon}\,ds}{\int_0^t e^{-\sigma/\varepsilon}\,d\sigma}, \tag{4.9}$$

and using Jensen's inequality with the convex function H, we get

$$\begin{aligned}
& H(F(g)(t,x,\xi),\xi) \\
\leq\ & H(f^0(x-t\xi,\xi),\xi)e^{-t/\varepsilon} \\
& +(1-e^{-t/\varepsilon})H\Big(\int_0^t M[g](t-s,x-s\xi,\xi)\frac{e^{-s/\varepsilon}\,ds}{\int_0^t e^{-\sigma/\varepsilon}\,d\sigma}\,,\,\xi\Big) \\
\leq\ & H(f^0(x-t\xi,\xi),\xi)e^{-t/\varepsilon} \\
& +(1-e^{-t/\varepsilon})\int_0^t H(M[g](t-s,x-s\xi,\xi),\xi)\frac{e^{-s/\varepsilon}\,ds}{\int_0^t e^{-\sigma/\varepsilon}\,d\sigma} \\
=\ & H(f^0(x-t\xi,\xi),\xi)e^{-t/\varepsilon} \\
& +\frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon}H(M[g](t-s,x-s\xi,\xi),\xi)\,ds,
\end{aligned}$$

and thus

$$\iint_{\mathbb{R}\times\mathbb{R}} H(F(g)(t,x,\xi),\xi)\,dxd\xi \leq e^{-t/\varepsilon}C_H^0 + (1-e^{-t/\varepsilon})C_H \leq C_H,$$

and $(C2)$ is satisfied. Similarly, integrating (4.8) with $i = 0$ with respect to (x,ξ) gives $(C3)$. It only remains now to prove $(C4)$. We have by (2.13)

$$\begin{aligned}
& \iint_{\mathbb{R}\times\mathbb{R}} x^2(f^0)_0(x-t\xi,\xi)\,dxd\xi \\
=\ & \iint (y+t\xi)^2(f^0)_0(y,\xi)\,dyd\xi \\
\leq\ & \left(\sqrt{\iint y^2(f^0)_0(y,\xi)\,dyd\xi} + t\sqrt{\iint \xi^2(f^0)_0(y,\xi)\,dyd\xi}\right)^2 \\
\leq\ & \left(\sqrt{C_\Delta^0} + t\sqrt{4C_H^0/\theta}\right)^2.
\end{aligned}$$

Then, since $M[g] \in C$, relation $(C4)$ gives

$$\iint_{\mathbb{R}\times\mathbb{R}} x^2 M[g]_0(s,x,\xi)\,dxd\xi \le C_\Delta \left(1+s/\varepsilon\right)^2,$$

hence

$$\begin{aligned}
& \iint_{\mathbb{R}\times\mathbb{R}} x^2 M[g]_0(t-s,x-s\xi,\xi)\,dxd\xi \\
= & \iint (y+s\xi)^2 M[g]_0(t-s,y,\xi)\,dyd\xi \\
\le & \left(\sqrt{\iint y^2 M[g]_0(t-s,y,\xi)\,dyd\xi} + s\sqrt{\iint \xi^2 M[g]_0(t-s,y,\xi)\,dyd\xi}\right)^2 \\
\le & \left(\sqrt{C_\Delta\left(1+\frac{t-s}{\varepsilon}\right)^2} + s\sqrt{4C_H/\theta}\right)^2 .
\end{aligned}$$

Therefore, with (4.5),

$$\begin{aligned}
& \iint_{\mathbb{R}\times\mathbb{R}} x^2 F(g)_0(t,x,\xi)\,dxd\xi \\
\le & \left(\sqrt{C_\Delta} + t\sqrt{4C_H/\theta}\right)^2 e^{-t/\varepsilon} \\
& + \frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon}\left(\sqrt{C_\Delta}\left(1+\frac{t-s}{\varepsilon}\right) + s\sqrt{4C_H/\theta}\right)^2 ds \\
\le & C_\Delta\left(1+\frac{t}{\varepsilon}\right)^2 e^{-t/\varepsilon} \\
& + \frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon} C_\Delta\left(1+\frac{t}{\varepsilon}\right)^2 ds \\
= & C_\Delta\left(1+\frac{t}{\varepsilon}\right)^2 .
\end{aligned}$$

This ends the proof of the lemma. $\square$

Since $\tilde{C} \subset C$, we are now able to define $F : \tilde{C} \to \tilde{C}$ and we shall use the Schauder theorem to prove Theorem 1.1.

Lemma 4.2 *The sets C and $\tilde{C}$ are convex and not empty, C is compact for the weak topology of $L^1(]0,T[\times\mathbb{R}\times\mathbb{R})$, and $\tilde{C}$ is closed in $C([0,T],L^1(\mathbb{R}\times\mathbb{R}))$.*

Proof. Since H is convex, it is obvious that C and $\tilde{C}$ are convex. Then, the constant f^0 belongs to C, and by Lemma 4.1, $F(f_0) \in \tilde{C}$. Thus C and $\tilde{C}$ are not empty. Next, from Proposition 2.7, $(C4)$, (2.17) and Dunford-Pettis' theorem, C is relatively weakly compact in $L^1(]0,T[\times\mathbb{R}\times\mathbb{R})$. Let us prove now that C is closed in weak $L^1(]0,T[\times\mathbb{R}\times\mathbb{R})$. Since C is convex, it is enough to prove that C is closed in strong $L^1(]0,T[\times\mathbb{R}\times\mathbb{R})$. Thus let $(g_n)_n$ be a sequence in C which converges to g in $L^1(]0,T[\times\mathbb{R}\times\mathbb{R})$. After extraction of a subsequence, $g_n(t,.)\to g(t,.)$ in $L^1(\mathbb{R}\times\mathbb{R})$ and a.e. for a.e. $t\in]0,T[$. First, $(g_n)_0\geq 0$ thus $g_0\geq 0$. Then, since by (2.17) $|(g_n)_1|\leq\sqrt{2H(g_n,\xi)(g_n)_0}$, we get for a fixed time t and any Borel set $\mathcal{V}$ by Fatou's lemma and Cauchy-Schwarz inequality

$$\begin{aligned}\iint\limits_{\mathcal{V}}|g_1(t,x,\xi)|\,dxd\xi &\leq \underline{\lim}\iint\limits_{\mathcal{V}}\sqrt{2H(g_n,\xi)(g_n)_0}\,dxd\xi\\ &\leq \underline{\lim}\sqrt{2C_H\iint\limits_{\mathcal{V}}(g_n)_0\,dxd\xi}\\ &= \sqrt{2C_H\iint\limits_{\mathcal{V}}g_0\,dxd\xi}.\end{aligned}$$

Taking $\mathcal{V}=\{x,\xi\ ;\ g_0(t,x,\xi)=0\}$, we obtain that $g_1(t,x,\xi)$ vanishes a.e. in $\mathcal{V}$, thus $g(t,x,\xi)\in D$ a.e. Then, we apply Fatou's lemma again, and get

$$\begin{aligned}\iint\limits_{\mathbb{R}\times\mathbb{R}}H(g(t,x,\xi),\xi)\,dxd\xi &= \iint\limits_{g_0>0}H(g(t,x,\xi),\xi)\,dxd\xi\\ &\leq \underline{\lim}\iint\limits_{g_0>0}H(g_n(t,x,\xi),\xi)\,dxd\xi\\ &\leq C_H.\end{aligned}$$

Similar applications of Fatou's lemma finally give that $g\in L^\infty(]0,T[,L^1(\mathbb{R}\times\mathbb{R}))$, and $g\in C$. The closedness of $\tilde{C}$ in $C([0,T],L^1(\mathbb{R}\times\mathbb{R}))$ is treated in a similar way, and $(C5)$ follows from the compactness of C. $\square$

Lemma 4.3 *F is continuous in $\tilde{C}$ for the topology $C([0,T],L^1(\mathbb{R}\times\mathbb{R}))$.*

Proof. Let $g_n,g\in\tilde{C}$ with $g_n\to g$ in $C([0,T],L^1(\mathbb{R}\times\mathbb{R}))$. Then, with the notations of Proposition 3.2, $\rho_n\to\rho$ and $\rho_n u_n\to\rho u$ in $C([0,T],L^1(\mathbb{R}))$. Thus Proposition 3.2 gives the existence of a subsequence such that $F(g_n)\to F(g)$ in $C([0,T],L^1(\mathbb{R}\times\mathbb{R}))$, which is enough to prove the continuity of F. $\square$

Lemma 4.4 *$F(\tilde{C})$ is relatively compact in $C([0,T],L^1(\mathbb{R}\times\mathbb{R}))$.*

Proof. Let $f_n=F(g_n)$, $g_n\in\tilde{C}$ be a sequence in $F(\tilde{C})$. Since $\tilde{C}\subset C$, by Lemma 4.2 there exist $g\in C$ and a subsequence such that $g_n\rightharpoonup g$ in weak $L^1(]0,T[\times\mathbb{R}\times\mathbb{R})$. Then with the notations of Proposition 3.2, $\rho_n\rightharpoonup\rho$, $\rho_n u_n\rightharpoonup\rho u$ in weak $L^1(]0,T[\times\mathbb{R})$. But by $(C5)$, $\partial_t g_n+\xi\partial_x g_n+g_n/\varepsilon=h_n/\varepsilon$, with $h_n\in C$. By the compactness averaging lemma of [8] and by (2.13)-(2.14), we deduce that $\int_{\mathbb{R}}g_n(t,x,\xi)\,d\xi$ is compact in $L^1_{loc}(]0,T[\times\mathbb{R})$. We conclude that $\rho_n\to\rho$ and $\rho_n u_n\to\rho u$ in $L^1(]0,T[\times\mathbb{R})$. Proposition 3.2 gives thus the existence of a subsequence such that $f_n=F(g_n)\to F(g)$ in $C([0,T],L^1(\mathbb{R}\times\mathbb{R}))$. $\square$

Proof of Theorem 1.1. We apply Schauder's theorem in $C([0,T],L^1(\mathbb{R}\times\mathbb{R}))$ to the operator $F:\tilde{C}\to\tilde{C}$. Using all the results of this section, $\tilde{C}$ is convex, closed and non empty, F is continuous $\tilde{C}\to\tilde{C}$ and $F(\tilde{C})$ is relatively compact. Thus we conclude the existence of a fixed point $f\in\tilde{C}$ verifying $F(f)=f$. This gives a solution in $[0,T]$ for any $T>0$, and by extraction of a diagonal subsequence, we obtain a solution in $[0,\infty[$. Relation (1.21) comes clearly from (1.2) because $\xi f(t,x,\xi)\in L^\infty(]0,\infty[,L^1(\mathbb{R}\times\mathbb{R}))$ by Proposition 2.7, and (1.22) is obtained by integration of (4.8). □

Proof of Theorem 1.2. If the assumption (1.17) is not satisfied, we have to use the Tychonoff-Schauder fixed point theorem in the locally convex topological vector space $C([0,T],L^1(\mathbb{R}_{loc,x}\times\mathbb{R}_\xi))$. Let us mention the changes that need to be performed to the results of Section 4. First, we have to remove condition $(C4)$ in the definitions of C and $\tilde{C}$. Then, in Lemma 4.2, C is compact for the weak topology of $L^1(]0,T[\times\mathbb{R}_{loc}\times\mathbb{R})$, and $\tilde{C}$ is closed in $C([0,T],L^1(\mathbb{R}_{loc}\times\mathbb{R})$. In Lemma 4.3, F is continuous in $\tilde{C}$ for the topology $C([0,T],L^1(\mathbb{R}_{loc}\times\mathbb{R}))$, and in Lemma 4.4, $F(\tilde{C})$ is relatively compact in $C([0,T],L^1(\mathbb{R}_{loc}\times\mathbb{R}))$. □

5 Entropy equation

In this section we prove that the entropy $H(f(t,x,\xi),\xi)$ satisfies a renormalized equation, as stated in Theorem 1.3. The difficulty is that H is not smooth at the origin, and we have to build a monotone approximation of it that is convex, smooth and that has slow growth at infinity.

5.1 Approximation of H

The most singular term in H in (1.13) is obviously

$$\Phi(f)=\mathbb{1}_{f_0>0}f_1^2/f_0,\qquad f\in D. \tag{5.1}$$

We begin by translating the singularity $f_0=0$ to $f_0=-\delta$, by defining for $\delta>0$

$$\Phi_\delta(f)=\frac{f_1^2}{f_0+\delta},\qquad f\in D, \tag{5.2}$$

which is convex and smooth in D, hence

$$\Phi_\delta(f)=\sup_{g\in D}\left(\Phi_\delta(g)+\Phi_\delta'(g)\cdot(f-g)\right),\qquad f\in D. \tag{5.3}$$

In order to obtain a linear growth at infinity, we set

$$\begin{aligned}\varphi_\delta(f)&=\sup_{g\in D_\delta}\left(\Phi_\delta(g)+\Phi_\delta'(g)\cdot(f-g)\right),\qquad f\in D,\\ D_\delta&=\left\{f\in D\ ;\ \Big|\frac{f_1}{f_0+\delta}\Big|\le c_\delta\right\},\qquad c_\delta=1/\delta^2.\end{aligned} \tag{5.4}$$

One can check that with this choice of c_δ,

$$\text{if} \quad 0<\delta' \le \delta \quad \text{then} \quad D_\delta \subset D_{\delta'}. \tag{5.5}$$

We have obviously $\varphi_\delta(f) \le \Phi_\delta(f) \le \Phi(f)$.

Lemma 5.1 *i) The value of φ_δ is*

$$\varphi_\delta(f) = \begin{cases} \dfrac{f_1^2}{f_0+\delta} & \text{if } f \in D_\delta, \\ -c_\delta^2(f_0+\delta)+2c_\delta|f_1| & \text{if } f \in D \setminus D_\delta, \end{cases} \tag{5.6}$$

ii) the function φ_δ is convex in D, nonnegative, $\varphi_\delta(0)=0$,
iii) we have $\varphi_\delta(f) \le 2c_\delta|f_1|$ for $f \in D$,
iv) if $0<\delta' \le \delta$ then $\varphi_\delta(f) \le \varphi_{\delta'}(f)$ for $f \in D$, and $\varphi_\delta(f) \uparrow \Phi(f)$ as $\delta \to 0$,
v) the function φ_δ is C^1 in $\overline{D}$ and φ'_δ is bounded in $\overline{D}$.

Proof. *i)* If $f \in D_\delta$, obviously $\varphi_\delta(f) = \Phi_\delta(f)$. If $f \in D \setminus D_\delta$,

$$\Phi_\delta(g) + \Phi'_\delta(g)\cdot(f-g) = -\frac{g_1^2}{(g_0+\delta)^2}(f_0+\delta) + 2\frac{g_1}{g_0+\delta}f_1,$$

hence

$$\varphi_\delta(f) = \sup_{-c_\delta \le x \le c_\delta} (-x^2(f_0+\delta)+2xf_1).$$

If $f_1 \ge 0$, the supremum is reached at $x=c_\delta$ because $c_\delta < f_1/(f_0+\delta)$, and if $f_1 \le 0$, it is reached at $-c_\delta$, and this gives the result.
ii) The convexity is obvious from the definition, the nonnegativity also since $0 \in D_\delta$, and $\varphi_\delta(0)$ is computed by (5.6).
iii) If $f \in D_\delta$ then $\varphi_\delta(f) = f_1^2/(f_0+\delta) \le c_\delta|f_1|$, and if $f \in D \setminus D_\delta$, then $\varphi_\delta(f) = -c_\delta^2(f_0+\delta)+2c_\delta|f_1| \le 2c_\delta|f_1|$.
iv) Let $0<\delta' \le \delta$. If $f \in D_\delta$ then $f \in D_{\delta'}$ by (5.5), thus $\varphi_\delta(f) = f_1^2/(f_0+\delta) \le f_1^2/(f_0+\delta') = \varphi_{\delta'}(f)$. If $f \in D_{\delta'} \setminus D_\delta$, then $\varphi_\delta(f) \le \Phi_\delta(f) \le \Phi_{\delta'}(f) = \varphi_{\delta'}(f)$. If $f \in D \setminus D_{\delta'}$, i.e. $|f_1| > c_{\delta'}(f_0+\delta')$, then since $c_\delta \le c_{\delta'}$,

$$\begin{aligned} \varphi_\delta(f) - \varphi_{\delta'}(f) &= -c_\delta^2(f_0+\delta) + c_{\delta'}^2(f_0+\delta') + 2(c_\delta - c_{\delta'})|f_1| \\ &\le -c_\delta^2(f_0+\delta') + c_{\delta'}^2(f_0+\delta') + 2(c_\delta - c_{\delta'})c_{\delta'}(f_0+\delta') \\ &= -(c_{\delta'}-c_\delta)^2(f_0+\delta') \\ &\le 0. \end{aligned}$$

Finally, if $f \in D$ is fixed, for small enough δ we have $f \in D_\delta$, $\varphi_\delta(f) = f_1^2/(f_0+\delta) \to \mathbb{1}_{f_0>0} f_1^2/f_0$ as $\delta \to 0$.
v) Formula (5.6) gives obviously two smooth functions. We have to check the continuity of the functions and their derivatives at the frontier where $|f_1| = c_\delta(f_0+\delta)$. The continuity of φ_δ is straightforward. For the derivatives, we have

$$\varphi'_\delta(f) = \begin{cases} \left(\dfrac{-f_1^2}{(f_0+\delta)^2}, \dfrac{2f_1}{f_0+\delta}\right), & f \in int(D_\delta), \\ (-c_\delta^2, 2c_\delta \operatorname{sgn} f_1), & f \in int(D \setminus D_\delta), \end{cases} \tag{5.7}$$

and we see that the two formula match when $|f_1| = c_\delta(f_0+\delta)$. We finally notice that $|\varphi'_\delta| \le c_\delta^2 + 2c_\delta$. $\square$

5.2 Sketch of the proof

In order to prove the equation on $H(f(t,x,\xi),\xi)$, we decompose H as

$$H(f,\xi) = \frac{\theta}{1-\theta}\frac{\xi^2}{2}f_0 - \frac{\theta}{1-\theta}\xi f_1 + \frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}}\frac{f_0^{1+1/\lambda}}{1+1/\lambda} + \frac{1}{1-\theta}\frac{1}{2}\mathbb{I}_{f_0>0}\frac{f_1^2}{f_0}, \tag{5.8}$$

and we are going to consider each term successively. A crucial argument is the following lemma.

Lemma 5.2 *Assume that $g \in C([0,T], L^1_{loc}(\mathbb{R}_x \times \mathbb{R}_\xi))$ verifies*

$$g(0) \in L^1(\mathbb{R}\times\mathbb{R}), \qquad \textit{and} \qquad \partial_t g + \xi\partial_x g = h \in L^1(]0,T[\times\mathbb{R}\times\mathbb{R}). \tag{5.9}$$

Then

$$g \in C([0,T], L^1(\mathbb{R}\times\mathbb{R})), \tag{5.10}$$

$$\xi g \in L^1(]0,T[\times B_R \times \mathbb{R}), \qquad R>0, \tag{5.11}$$

$$g(t,x,\xi) = g(0,x-t\xi,\xi) + \int_0^t h(t-s,x-s\xi,\xi)\,ds, \tag{5.12}$$

$$\partial_t \int g\,d\xi + \partial_x \int \xi g\,d\xi = \int h\,d\xi \qquad in \quad]0,T[\times\mathbb{R}, \tag{5.13}$$

$$\frac{d}{dt}\iint\limits_{\mathbb{R}\times\mathbb{R}} g(t,x,\xi)\,dxd\xi = \iint\limits_{\mathbb{R}\times\mathbb{R}} h(t,x,\xi)\,dxd\xi \qquad in \quad]0,T[. \tag{5.14}$$

Proof. Define $\tilde g(t,x,\xi) = g(t,x+t\xi,\xi)$. Then $\tilde g \in C([0,T], L^1_{loc}(\mathbb{R}\times\mathbb{R}))$, and $\tilde g$ verifies $\tilde g(0) = g(0) \in L^1(\mathbb{R}\times\mathbb{R})$, $\partial_t \tilde g = h(t,x+t\xi,\xi) \in L^1(]0,T[\times\mathbb{R}\times\mathbb{R})$. Therefore,

$$\tilde g(t,x,\xi) = g(0,x,\xi) + \int_0^t h(s,x+s\xi,\xi)\,ds,$$

and we deduce (5.12) and (5.10). Then, (5.11) comes from the higher moment lemma of [15], and (5.13) follows by integration of (5.9). Finally, integrating (5.12) with respect to (x,ξ), we get for any $t\in[0,T]$

$$\iint g(t,x,\xi)\,dxd\xi = \iint g(0,x,\xi)\,dxd\xi + \int_0^t \iint\limits_{\mathbb{R}\times\mathbb{R}} h(s,x,\xi)\,dxd\xi ds, \tag{5.15}$$

which is the integral form of (5.14). □

5.3 Linear terms

We are now able to treat the first two terms in (5.8). From now on, f denotes the solution obtained in Theorems 1.1 or 1.2. We recall that the estimates of Proposition 2.7 apply to $f(t,.)$ and to $M[f](t,.)$ for any $t\geq 0$. Thus we can apply Lemma 5.2 to $\xi^2 f_0(t,x,\xi)$ and $\xi f_1(t,x,\xi)$, and we conclude that

$$\xi^2 f_0(t,x,\xi), \quad \xi f_1(t,x,\xi) \quad \in C([0,\infty[, L^1(\mathbb{R}\times\mathbb{R})), \tag{5.16}$$

and all the conclusions of Lemma 5.2 are valid.

5.4 Nonlinear terms

Let us first consider the last term in (5.8), corresponding to $\Phi(f) = 1\!\!1_{f_0>0} f_1^2/f_0$. We use the approximation $\varphi_\delta(f)$ defined in Section 5.1. Since φ_δ is C^1 and globally Lipschitz continuous, a vector adaptation of the classical renormalization property allows to multiply (1.2) by $\varphi_\delta'(f)$, which yields

$$\partial_t \varphi_\delta(f) + \xi \partial_x \varphi_\delta(f) = \varphi_\delta'(f) \frac{M[f] - f}{\varepsilon}. \tag{5.17}$$

Since $\varphi_\delta(f) \in C([0,T], L^1(\mathbb{R}\times\mathbb{R}))$, we can apply Lemma 5.2 to $\varphi_\delta(f)$. We wish now to let $\delta \to 0$ in (5.17). We notice that Φ in (5.1) is smooth outside the origin, and we define $\Phi'(0) = 0$ by convention. Thus

$$\Phi'(f) = \left(-1\!\!1_{f_0>0} \frac{f_1^2}{f_0^2}, 1\!\!1_{f_0>0} \frac{2f_1}{f_0} \right), \qquad f \in D. \tag{5.18}$$

Lemma 5.3 *The terms $\partial_0 \Phi(f) f_0$, $\partial_1 \Phi(f) f_1$, $\partial_0 \Phi(f) M_0[f]$, $\partial_1 \Phi(f) M_1[f]$ lie in $L^1(]0,T[\times\mathbb{R}\times\mathbb{R})$. The right-hand side of (5.17) is bounded in $L^1(]0,T[\times\mathbb{R}\times\mathbb{R})$ uniformly in δ, and tends to $\Phi'(f)(M[f]-f)/\varepsilon$ in $L^1(]0,T[\times\mathbb{R}\times\mathbb{R})$ as $\delta \to 0$.*

Proof. Since $0 \le \Phi(f) \le 2H(f,\xi)$ for any ξ, we have

$$\Phi(f(t,x,\xi)), \quad \Phi(M[f](t,x,\xi)) \in L^\infty(]0,\infty[, L^1(\mathbb{R}\times\mathbb{R})). \tag{5.19}$$

Then, $\partial_0 \Phi(f) f_0 = -1\!\!1_{f_0>0} f_1^2/f_0 = -\Phi(f) \in L^\infty(]0,\infty[, L^1(\mathbb{R}\times\mathbb{R}))$, $\partial_1 \Phi(f) f_1 = 1\!\!1_{f_0>0} 2 f_1^2/f_0 = 2\Phi(f) \in L^\infty(]0,\infty[, L^1(\mathbb{R}\times\mathbb{R}))$. Next, since $0 \le \varphi_\delta(f) \le \Phi(f)$, $\varphi_\delta(f)$ is bounded in $C([0,T], L^1(\mathbb{R}\times\mathbb{R}))$ independently of δ. Therefore, by (5.17) and by applying the identity (5.15) to $\varphi_\delta(f)$, we obtain that

$$\iiint_{]0,T[\times\mathbb{R}\times\mathbb{R}} \varphi_\delta'(f)\,(M[f]-f)\; dt dx d\xi \quad \text{is bounded independently of } T \text{ and } \delta. \tag{5.20}$$

Then, since φ_δ is convex in D,

$$\varphi_\delta(M[f]) - \varphi_\delta(f) - \varphi_\delta'(f)\,(M[f]-f) \ge 0. \tag{5.21}$$

According to (5.20), this function is bounded in $L^1(]0,T[\times\mathbb{R}\times\mathbb{R})$ independently of δ. Observing that with (5.7), $\varphi_\delta'(f) \to \Phi'(f)$ as $\delta \to 0$, by applying Fatou's lemma, we deduce that

$$\Phi(M[f]) - \Phi(f) - \Phi'(f)\,(M[f]-f) \quad \in L^1(]0,T[\times\mathbb{R}\times\mathbb{R}), \tag{5.22}$$

and therefore

$$\Phi'(f)\cdot M[f] = -1\!\!1_{f_0>0} \frac{f_1^2}{f_0^2} M_0[f] + 1\!\!1_{f_0>0} \frac{2f_1}{f_0} M_1[f] \in L^1(]0,T[\times\mathbb{R}\times\mathbb{R}). \tag{5.23}$$

From the inequality

$$2\left|\frac{f_1}{f_0}M_1[f]\right| \leq \frac{1}{4}\frac{f_1^2}{f_0^2}M_0[f] + 4\frac{M_1[f]^2}{M_0[f]}\mathbb{1}_{M_0[f]>0},$$

we deduce

$$\begin{aligned}\frac{f_1^2}{f_0^2}M_0[f] &= 2\frac{f_1}{f_0}M_1[f] - \left(-\frac{f_1^2}{f_0^2}M_0[f] + 2\frac{f_1}{f_0}M_1[f]\right)\\ &\leq \frac{1}{4}\frac{f_1^2}{f_0^2}M_0[f] + 4\frac{M_1[f]^2}{M_0[f]}\mathbb{1}_{M_0[f]>0} - \left(-\frac{f_1^2}{f_0^2}M_0[f] + 2\frac{f_1}{f_0}M_1[f]\right),\end{aligned}$$

and thus

$$0 \leq \mathbb{1}_{f_0>0}\frac{3}{4}\frac{f_1^2}{f_0^2}M_0[f] \leq 4\Phi(M[f]) - \Phi'(f)\cdot M[f]. \tag{5.24}$$

We conclude that

$$\partial_0\Phi(f)M_0[f], \quad \partial_1\Phi(f)M_1[f] \quad \in L^1(]0,T[\times\mathbb{R}\times\mathbb{R}). \tag{5.25}$$

Finally, we observe with (5.7) that

$$|\partial_0\varphi_\delta(f)| \leq |\partial_0\Phi(f)|, \qquad |\partial_1\varphi_\delta(f)| \leq |\partial_1\Phi(f)|, \tag{5.26}$$

and we conclude with Lebesgue's theorem the convergence of each term of the right-hand side of (5.17) in $L^1(]0,T[\times\mathbb{R}\times\mathbb{R})$ as $\delta\to 0$. □

Now, we are able to conclude. For any $t \geq 0$, $\varphi_\delta(f(t,.)) \to \Phi(f(t,.))$ in $L^1(\mathbb{R}\times\mathbb{R})$ as $\delta \to 0$ by Lebesgue's theorem. But we can write (5.12) for $\varphi_\delta(f)$,

$$\varphi_\delta(f(t,x,\xi)) = \varphi_\delta(f^0(x-t\xi,\xi)) + \int_0^t \varphi_\delta'(f)\frac{M[f]-f}{\varepsilon}(t-s,x-s\xi,\xi)\,ds.$$

By Lemma 5.3, we can pass to the limit in the right-hand side in $C([0,T],L^1(\mathbb{R}\times\mathbb{R}))$. Therefore,

$$\Phi(f(t,x,\xi)) = \Phi(f^0(x-t\xi,\xi)) + \int_0^t \Phi'(f)\frac{M[f]-f}{\varepsilon}(t-s,x-s\xi,\xi)\,ds, \tag{5.27}$$

$$\Phi(f) \in C([0,T],L^1(\mathbb{R}\times\mathbb{R})), \tag{5.28}$$

$\varphi_\delta(f) \to \Phi(f)$ in $C([0,T],L^1(\mathbb{R}\times\mathbb{R}))$, and

$$\partial_t\Phi(f) + \xi\partial_x\Phi(f) = \Phi'(f)\frac{M[f]-f}{\varepsilon}. \tag{5.29}$$

For the last term $\psi(f_0) = f_0^{1+1/\lambda}/(1+1/\lambda)$ in (5.8), we can use the approximation

$$\psi_\delta(f_0) = \begin{cases} \psi(f_0) & \text{if} \quad 0 \leq f_0 \leq 1/\delta,\\ \psi(\frac{1}{\delta}) + \psi'(\frac{1}{\delta})(f_0 - \frac{1}{\delta}) & \text{if} \quad f_0 \geq 1/\delta.\end{cases} \tag{5.30}$$

Then the argument is similar as above (even simpler, $\psi'(f_0)M_0[f] \in L^1$ is obtained directly since $f_0^{1/\lambda} \in L^{\lambda(1+1/\lambda)}$, $M_0[f] \in L^{1+1/\lambda}$), and we do not repeat the proof.

5.5 Proof of Theorem 1.3

According to the previous subsections, we can apply Lemma 5.2 to each term of the decomposition (5.8). Noticing that the convention on the value of $H'(0,\xi)$ corresponds to $\Phi'(0)=0$, we obtain (1.25)-(1.28). Moreover, $\xi H(f(t,x,\xi),\xi)\in L^1(]0,T[\times B_R\times\mathbb{R})$ and

$$\partial_t\int_{\mathbb{R}} H(f,\xi)\,d\xi+\partial_x\int_{\mathbb{R}}\xi H(f,\xi)\,d\xi=\frac{1}{\varepsilon}\int_{\mathbb{R}} H'(f,\xi)\cdot(M[f]-f)\,d\xi, \tag{5.31}$$

$$\frac{d}{dt}\iint\limits_{\mathbb{R}\times\mathbb{R}} H(f,\xi)\,dxd\xi=\frac{1}{\varepsilon}\iint\limits_{\mathbb{R}\times\mathbb{R}} H'(f,\xi)\cdot(M[f]-f)\,dxd\xi. \tag{5.32}$$

But by Proposition 2.6,

$$P(t,x,\xi)\equiv\left(H'(f,\xi)-\Big(\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1}-\frac{u^2}{2},u\Big)\right)\cdot(f-M[f])\geq 0\quad\text{a.e.}, \tag{5.33}$$

and since $\int(f-M[f])\,d\xi=0$,

$$\iiint\limits_{]0,T[\times\mathbb{R}\times\mathbb{R}} P(t,x,\xi)\,dtdxd\xi=\iiint\limits_{]0,T[\times\mathbb{R}\times\mathbb{R}} H'(f,\xi)\cdot(f-M[f])\,dtdxd\xi<\infty, \tag{5.34}$$

thus $P\in L^1(]0,T[\times\mathbb{R}\times\mathbb{R})$. Finally, since

$$\int_{\mathbb{R}} P(t,x,\xi)\,d\xi=\int_{\mathbb{R}} H'(f,\xi)\cdot(f-M[f])\,d\xi\quad\text{a.e. } t,x, \tag{5.35}$$

from (5.31) and (5.32) we obtain (1.29) and (1.30). □

Remark 5.1 It is possible to prove directly that

$$\left(\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1}-\frac{u^2}{2},u\right)\cdot(M[f]-f)\in L^\infty(]0,T[,L^1(\mathbb{R}\times\mathbb{R})). \tag{5.36}$$

Indeed by (2.4) and Proposition 2.5,

$$\frac{1}{2}\rho u^2+\frac{\kappa}{\gamma-1}\rho^\gamma=\eta(\rho,u)\leq\int H(f,\xi)\,d\xi, \tag{5.37}$$

thus

$$\int_{\mathbb{R}}\Big(\frac{1}{2}\rho u^2+\frac{\kappa}{\gamma-1}\rho^\gamma\Big)\,dx\leq C_H^0, \tag{5.38}$$

and ρu^2, $\rho^\gamma\in L^\infty(]0,\infty[,L^1(\mathbb{R}))$. Then, the inequality

$$\begin{aligned}&\Big(\iint|u(t,x)f_1(t,x,\xi)|\,dxd\xi\Big)^2\\ &\leq\Big(\iint u(t,x)^2 f_0(t,x,\xi)\,dxd\xi\Big)\times\Big(\iint\limits_{f_0(t,x,\xi)>0}\frac{f_1(t,x,\xi)^2}{f_0(t,x,\xi)}\,dxd\xi\Big)\end{aligned} \tag{5.39}$$

proves that $\|uf_1\|_{L^1(\mathbb{R}\times\mathbb{R})}\leq 2C_H^0$. A similar estimate holds for $uM_1[f]$, and the other terms in (5.36) are easy to estimate with (5.38).

References

[1] D. Aregba-Driollet, R. Natalini, Discrete kinetic schemes for multidimensional systems of conservation laws, to appear.

[2] F. Berthelin, F. Bouchut, Kinetic invariant domains and relaxation limit from a BGK model to isentropic gas dynamics, preprint.

[3] F. Bouchut, Construction of BGK models with a family of kinetic entropies for a given system of conservation laws, J. Statist. Phys. 95 (1999), 113–170.

[4] Y. Brenier, Averaged multivalued solutions for scalar conservation laws, SIAM J. Numer. Anal. 21 (1984), 1013–1037.

[5] Y. Brenier, L. Corrias, A kinetic formulation for multi-branch entropy solutions of scalar conservation laws, Ann. Inst. H. Poincaré Anal. Non Linéaire 15 (1998), 169-190.

[6] G.Q. Chen, C.D. Levermore, T.-P. Liu, Hyperbolic conservation laws with stiff relaxation terms and entropy, Comm. Pure Appl. Math. 47 (1994), 787-830.

[7] R.J. DiPerna, Convergence of the viscosity method for isentropic gas dynamics, Comm. Math. Phys. 91 (1983), 1–30.

[8] F. Golse, P.-L. Lions, B. Perthame, R. Sentis, Regularity of the moments of the solution of a transport equation, J. Funct. Anal., 76, (1988), 110–125.

[9] S. Jin, Z.-P. Xin, The relaxation schemes for systems of conservation laws in arbitrary space dimensions, Comm. Pure Appl. Math. 48 (1995), 235-276.

[10] P.-L. Lions, B. Perthame, P.E. Souganidis, Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates, Comm. Pure Appl. Math. 49 (1996), 599–638.

[11] P.-L. Lions, B. Perthame, E. Tadmor, A kinetic formulation of multidimensional scalar conservation laws and related equations, J. Amer. Math. Soc. 7 (1994), 169-191.

[12] P.-L. Lions, B. Perthame, E. Tadmor, Kinetic formulation of the isentropic gas dynamics and p-systems, Comm. Math. Phys. 163 (1994), 415–431.

[13] R. Natalini, A discrete kinetic approximation of entropy solutions to multidimensional scalar conservation laws, J. Diff. Eq. 148 (1998), 292–317.

[14] R. Natalini, Recent results on hyperbolic relaxation problems. Analysis of systems of conservation laws (Aachen, 1997), 128–198. Chapman & Hall/CRC Monogr. Surv. Pure Appl. Math., 99, Chapman & Hall/CRC, Boca Raton, FL, 1999.

[15] B. Perthame, Global existence to the BGK model of Boltzmann equation, J. Diff. Eq. 82 (1989), 191-205.

[16] B. Perthame, M. Pulvirenti, Weighted L^∞ bounds and uniqueness for the Boltzmann BGK model, Arch. Rational Mech. Anal. 125 (1993), 289-295.

[17] B. Perthame, E. Tadmor, A kinetic equation with kinetic entropy functions for scalar conservation laws, Comm. Math. Phys. 136 (1991), 501-517.

[18] D. Serre, Relaxation semi-linéaire et cinétique des systèmes de lois de conservation, Ann. Inst. H. Poincaré Anal. non linéaire 17 (2000), 169-192.