ERROR ESTIMATES FOR APPROXIMATE SOLUTIONS FOR NONLINEAR SCALAR CONSERVATION LAWS

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ABSTRACT. This paper is devoted to analyse the errors for approximate solutions for nonlinear conservation laws. We review the recent work on error estimates for nonlinear scalar conservation laws. In particular, we show in this note how to obtain optimal pointwise-error estimates for piecewise smooth solutions with shocks.

1. INTRODUCTION

There has been an enormous amount of work related to the error estimates for approximate solutions to scalar conservation laws. The methods of analysis include matching the traveling wave solutions, [9]; matching the Green function of the linearized problem [24]; weak $W^{-1,1}$ convergence theory [34]; the Kruzkov-functional method [20]; and the energy-like method [35]. The results on error estimates include

- For BV entropy solutions, an $\mathcal{O}(\sqrt{\epsilon})$ convergence rate in L^1 obtained by Kuznetsov [20], Lucier [25], Sanders [30], Cockburn-Gremaud-Yang [6] etc;
- For BV entropy solutions, an $\mathcal{O}(\epsilon)$ convergence rate in $W^{-1,1}$ obtained by Tadmor-Nessyahu-Tassa [26, 27, 34], Liu-Wang-Warnecke [23] etc;
- For piecewise smooth solutions, an $\mathcal{O}(\epsilon)$ convergence rate in L^1 obtained by Bakhvalov [1], Harabetian [10], Teng-Tang-Zhang [43, 46, 48], Fan [8] etc;
- For piecewise smooth solutions, an $\mathcal{O}(\epsilon)$ convergence rate in smooth region of the entropy solution obtained by Goodman-Xin [9], Engquist-Sjogreen [7], Tadmor-Tang [35, 38] etc.

In this paper, we will review some known results on error estimates and will discuss some recent developments on the convergence rates. The corresponding methods in obtaining these results will be briefly illustrated.

2. L^1-CONVERGENCE RATE IN BV-SOLUTION SPACE

Consider the IVP for a scalar conservation law:

$$\begin{cases} u_t + f(u)_x = 0\,, & -\infty < x < \infty \;, t > 0\,, \\ \\ u(x,0) = u_0(x), & -\infty < x < \infty\,. \end{cases} \tag{2.1}$$

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We consider numericl approximations to weak solutions of the conservation law which are obtained by $(2k+1)$-point explicit schemes in conservation form

$$
\begin{aligned}
u_j^{n+1} &= G(u_{j-k}^n, \cdots, u_{j+k}^n) \\
&= u_j^n - \frac{\Delta t}{\Delta x}(\bar{f}_{j+1/2}^n - \bar{f}_{j-1/2}^n)
\end{aligned}
\tag{2.2}
$$

where $\bar{f}_{j+1/2}^n = \bar{f}(v_{j-k+1}^n, \cdots, v_{j+k}^n)$. Here $v_j^n = v(j\Delta x, n\Delta t)$, and $\bar{f}$ is a numerical flux function. We require the numerical flux function to be consistent with the flux $f(u)$ in the following sense: $\bar{f}(u, \cdots, u) = f(u)$.

To begin with, we consider the so-called monotone difference scheme. A conservative finite-difference scheme is monotone if G in (2.2) is a monotone function with respect to all its arguments:

$$\frac{\partial}{\partial v_i} G(v_{-k}, \cdots, v_k) \geq 0, \quad -k \leq i \leq k.$$

The first result on the convergence rate of the monotone schemes was given by Kuznetsov in 1976 [20].

Throrem 2.1. *Monotone solutions $u_{\Delta x}$ converge to BV-bounded entropy solution u in L^1 with convergence rate of order one half:*

$$\|u_{\Delta x}(\bullet, t) - u(\bullet, t)\|_{L^1} = \mathcal{O}(\sqrt{\Delta x}), \quad \Delta x \to 0.$$

Kuznetsov's approach employes a regularized version of Kruzkov's entropy pairs:

$$\eta(u, c) = |u - c|, \quad F(u, c) = sgn(u - c)\big(f(u) - f(c)\big).$$

Here, one measures by how much the entropy dissipation rate of $u_{\Delta x}$ fails to satisfy the entropy inequality, with Kruzkov's regularized entropies. Following the general convergence result of [2], we consider a family of approximation solutions, $\{u^\epsilon\}$, which satisfies

$$\partial_t \eta(v^\epsilon, c) + \nabla_x \cdot F(v^\epsilon, c) \leq \partial_t R_0(x, t) + \nabla_x \cdot R(x, t) \tag{2.3}$$

with

$$|||R_0(x, t)||| + |||R(x, t)||| \leq Const.\epsilon\,, \tag{2.4}$$

where the norm in (2.4) is related to the Lip'-consistency condition defined in Tadmor [34]. Then the convergence rate proof can be given by the following: Using the key property of symmetry of the regularized entropy pairs, $\eta^\delta = \phi_\delta \eta, F^\delta = \phi_\delta F$, one has $\int_x \eta^\delta(u^\epsilon; u)dx \leq Const.\epsilon/\delta$. In addition, there is a regularization error, $\|\eta^\delta - \eta\|_{L^1}$, of size $\mathcal{O}(\delta)$, and an L^1 error estimate of order $\mathcal{O}(\sqrt{\epsilon})$ follows (under reasonable assumptions on the L^1-initial error with respect to BV data). It can be verified that the solutions of the monotone difference schemes satisfy (2.3)-(2.4), with $v^\epsilon = u_{\Delta x}$, which leads to the result of Theorem 2.1.

There has been a large number of paper based on the Kruzkov-functional method, which include Lucier [25], Sanders [30], Cockburn [4] for finite difference schemes; Cockburn-Coquel-LeFloch [5] for finite volume methods; Kröner-Noelle-Rokyta [19] for unstructured/irregular grids; Schroll-Tveito-Winther [32, 49], Katsoulakis-Tzavaras [18], Liu-Warnecke [22] for the relaxation approximation; Johnson-Szepessy-Hansbo [16, 17] for finite element methods; Tang-Teng [41, 47], Holden-Karlson-Risbero [13, 14], Langseth-Tveito-Winther [21] for the splitting methods.

3. L^1-convergence rate in piecewise-smooth solution space

In this section we will investigate the optimal rate of convergence in L^1 in piecewise-smooth solution space. In other words, the entropy solution u has only finitely many discontinuities, which is the generic situation, see e.g. [31, 39]. Furthermore, for ease of exposition we assume for Sections 3-5 that there is only one shock discontinuty $x = X(t)$. The extension of our analysis to finitely many shocks/rarefaction waves can be carried out by following Tadmor-Tang [35], see also the brief discussion in Section 6.

We begin by the following observations:

- If the flux function f is *linear*, then the L^1-convergence rate of order $\mathcal{O}(\sqrt{\Delta x})$ is the best possible, even if the entropy solution is piecewisely smooth. The rigouros proof of this observation is provided by Tang & Teng [42];
- If the entropy solution is in BV-solution space, then the L^1-convergence rate of order $\mathcal{O}(\sqrt{\Delta x})$ is the best possible, even if the flux function f is *nonlinear*. The rigouros proof of this observation is provided by Sabac [29].

Now the natural question is what is the optimal rate of convergence for convex conservation laws, $f'' > 0$, with finitely many shock or rarefaction discontinuities (and these are the only solutions that can be computed!)? In fact, many numerical evidences show that approximation methods with nonlinear flux in piecewise-smooth solution space are of higher convergence rate, see e.g. [11, 33]. Therefore, it is of theoretical and practical implications to break down the barrier of half-order convergence rate.

To provide a rigouros analysis for the optimal rate of convergence, we recall the earlier result of Harten-Hyman-Lax [12]:

Throrem 3.1. *Monotone scheme approximates solutions of the* viscous modified equation*:*

$$u_t + f(u)_x = \Delta t[\beta(u,\lambda)u_x]_x\,, \qquad \lambda = \Delta t/\Delta x\,; \qquad \beta(u,\lambda) \ge 0, \qquad \beta(u,\lambda) \not\equiv 0 \tag{3.1}$$

to second-order accuracy. Moreover, monotone schemes are at most first-order accurate.

Theorem 2.1 indicates that a monotone scheme is closely related to the viscous modified equation (3.1). Therefore, we first consider the convergence rate of viscosity approximation:

$$
\begin{aligned}
u_t^\epsilon + f(u^\epsilon)_x &= \epsilon(\mathcal{B}(u^\epsilon)u_x^\epsilon)_x\,, \qquad f'' > 0, \\
u^\epsilon(x,0) &= u_0(x), \quad -\infty < x < \infty\,,
\end{aligned}
\tag{3.2}
$$

where $\mathcal{B} > 0$, to the corresponding conservation law:

$$
\begin{aligned}
u_t + f(u)_x &= 0\,, \qquad f'' > 0, \\
u(x,0) &= u_0(x), \quad -\infty < x < \infty\,.
\end{aligned}
\tag{3.3}
$$

For piecewise smooth solutions, an $\mathcal{O}(\epsilon)$ convergence rate in smooth region of the entropy solution was obtained by Goodman & Xin [9], where a matching method was developed. The basic principle of the matching method is to construct an auxiliary continuous approximation $\widehat{u}_\epsilon$ by replacing all shock jumps in the entropy solution u at each fixed time t with their corresponding travelling wave solutions U_ϵ (see Figure 1):

$$
\widehat{u}_\epsilon = u(x,t) - H\Big(x - X(t); u_-(t), u_+(t)\Big) + U_\epsilon(x - X(t); u_-(t), u_+(t)),
$$

where

- $x = X(t)$ is the shock curve, $u_\pm(t) = u(X(t) \pm 0, t)$,
- H is the Heaviside function, and $U_\epsilon(x; u, u_+)$ is a traveling wave solution for the viscosity equations with $u_\pm$ as infinity boundary conditions.

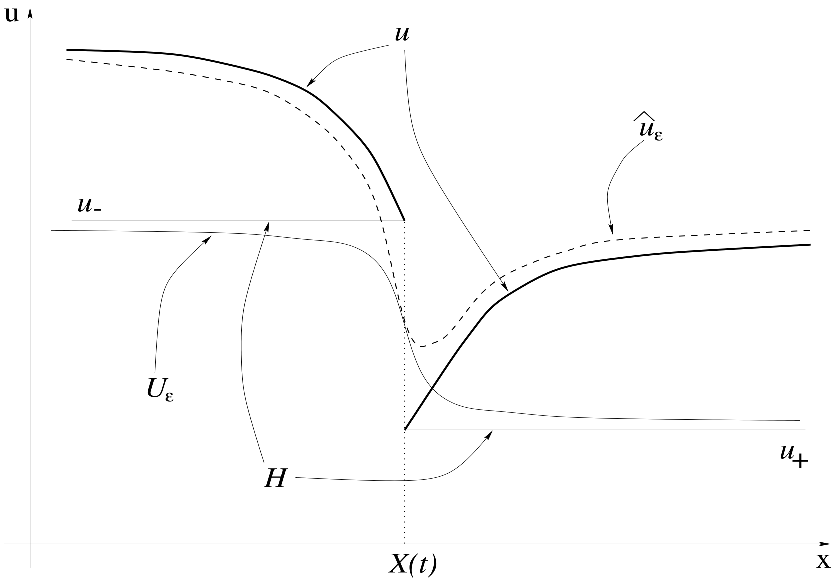

FIGURE 1. Illustration of u and $\widehat{u}_\epsilon$.

The main reason for constructing the auxiliary function $\widehat{u}_\epsilon$ is that unlike the entropy solution u it is continuous. By using the contraction property of the viscous conservation equations with source terms we can show that the L^1-error of $\widehat{u}_\epsilon - u^\epsilon$ is bounded by

$\mathcal{O}(\epsilon+\epsilon|\ln\epsilon|)$. It can also be shown that the L^1-error of $u-\widehat{u}_\epsilon$ is bounded by $\mathcal{O}(\epsilon)$ by using the properties of the travelling wave solution. The detail proof can be found in Tang-Teng [43] in which the following theorem is obtained.

Throrem 3.2. *If the entropy solution u of* (3.3) *has finitely many shock/rarefaction discontinuities, then the error between u and the viscous solution u^ϵ of* (3.2) *satisfies*

$$\|u(\cdot,t)-u^\epsilon(\cdot,t)\|_{L^1(R)}=\mathcal{O}(\epsilon|\ln\epsilon|)\,.$$

Moreover, the above error bound can be improved to $\mathcal{O}(\epsilon)$ if there is no initial central rarefaction wave and no new formed shock in u.

The above theorem incates that for convex conservation laws whose entropy solution consists of finitely many discontinuities, the L^1-error between the viscosity solution, u^ϵ, and its inviscid limit, u, is bounded by $\mathcal{O}(\epsilon|\ln\epsilon|)$. If neither central rarefaction waves nor spontaneous shocks occur, the error bound is improved to $\mathcal{O}(\epsilon)$. It is expected that the results in Theorem 3.2 can be extended to the monotone difference schemes, by following the similar proof technique used in [36, 43]. In fact, some related results have been obtained: Teng and Zhang [48] provided an $\mathcal{O}(\epsilon)$ rate for Riemann problems with finitely many shocks; Fan [8] established the L^1-convergence rate for Godunov scheme.

In Wang [50], it is shown that the $O(\epsilon|\ln\epsilon|)$ error is transient at new shock-formed time and also the $\mathcal{O}(\epsilon|\ln\epsilon|)$ L^1-error persists along centered rarefaction edges.

4. Pointwise error estimate

Introduce the weighted distance function: (i): $\phi(x)\sim|x|^\alpha, |x|\ll 1$, (ii): $x\phi'(x)>0, x\neq 0$, (iii): $\phi(x)\to 1, |x|\to\infty$, where $\alpha\geq 1$ is a finite constant. The constant α is called power of the weighted distance function ϕ, whose value is the indicator of the width of the boundary layer. In general, smaller value of α leads to sharper descrition of the boundary layer.

4.1. **Viscosity approximation.** We define the weighted error as following

$$E(x,t)=\phi(x-X(t))(u^\epsilon(x,t)-u(x,t))\,.$$

The weighted error function satisfies $E(x,t)=\mathcal{O}(\epsilon)$ for $|x-X(t)|\leq C\epsilon$. We will consider the magnitute of E in the region away from the shock wave: $|x-X(t)|\geq\mathcal{O}(\epsilon)$. To this end, we need a transport equation for the weighted error function and will use the maximum principle. It can be verified that the weighted error function satisfies a transport equation:

$$E_t+h(x,t)E_x-\epsilon E_{xx}=p(x,t)E+q(x,t)\epsilon$$

where the coefficient functions are defined by

$$h(x,t) = f'(u^\epsilon) + 2\epsilon\mathcal{B}(u^\epsilon)\frac{\phi'}{\phi}\,,$$

$$p(x,t) = \Big\{(f'(u^\epsilon) - X'(t)) + 2\epsilon\mathcal{B}(u^\epsilon)\frac{\phi'}{\phi} - \epsilon\mathcal{B}'(u^\epsilon)\Big\}\frac{\phi'}{\phi} - f''(\bullet)u_x\,,$$

$$q(x,t) = \mathcal{B}'(u^\epsilon)u_x^2\phi + \mathcal{B}(u^\epsilon)u_{xx}\phi^\delta - \mathcal{B}(u^\epsilon)\epsilon\phi''\,.$$

Since there has only one shock, the derivatives u_x, u_{xx} are bounded away from the shock curve. Therefore, it is easy to see that q is bounded by a constant (independent of ϵ) for $|x - X(t)| \geq \mathcal{O}(\epsilon)$. The key step is to upper bound the coefficient function p. In order to upper bound the coefficient function p, we use

- The Lax geometrical entropy condition, and
- A non-optimal pointwise error estimate obtained by using the so-called Lip^+-stability theory developed by Tadmor [34].

It was shown in Tadmor-Tang [35] that if $\phi(x) \sim \min(|x|, 1)$, i.e. $\alpha = 1$ then there exists a constant $D > 0$ independent of ϵ such that for $t \leq T$, the following estimate holds:

$$p(x,t) \leq Const_T\,, \qquad \text{for} \quad |x - X(t)| \geq D\epsilon\,.$$

The above results, together with the maximum principle, lead to the following pointwise error estimate (with optimal rate of convergence).

Throrem 4.1. *If the entropy solution of the convex conservation law* (3.3) *has* only one *shock discontinuity* $S(t) = \{(x,t)|x = X(t)\}$, *then the following error estimate holds:*

- *For a weighted distance function* ϕ, $\phi(x) \sim \min(|x|, 1)$,

$$|(u^\epsilon - u)(x,t)|\phi(|x - X(t)|) = \mathcal{O}(\epsilon)\,.$$

- *In particular, if* (x,t) *is away from the singular support, then*

$$|(u^\epsilon - u)(x,t)| \leq C(h)\epsilon\,, \qquad dist(x, S(t)) \geq h.$$

The above pointwise error estimates are sharp enough to recover (an almost) first-order, $\mathcal{O}(\epsilon|\ln\epsilon|)$ L^1-error estimate, see Tadmor-Tang [35].

It is noted that under the framework of [35], we can convert any non-optimal local estimate, such as

$$|u^\epsilon(x,t) - u(x,t)| \leq C\epsilon^\gamma\,, \qquad \text{for} \quad |x - X(t)| \geq \epsilon^\gamma \tag{4.1}$$

with $0 < \gamma < 1$, to the optimal pointwise error bound for the viscosity approximation.

4.2. **Finite-difference methods.** We follow the presentation in [28] and for simplicity we consider the three-point schemes of the following form:

$$v_j^{n+1} = v_j^n - \frac{\lambda}{2}\Big(f(v_{j+1}^n) - f(v_{j-1}^n)\Big) + \frac{1}{2}\Big(Q_{j+\frac{1}{2}}^n(v_{j+1}^n - v_j^n) - Q_{j-\frac{1}{2}}^n(v_j^n - v_{j-1}^n)\Big).$$

Thus, three-point schemes are identified solely by their numerical viscosity coefficient, $Q^n_{j+\frac{1}{2}} = Q(v^n_j, v^n_{j+1})$. The schemes of Roe, Godunov and Engquist-Osher are canonical examples of three-point monotone schemes, associated with increasing amounts of numerical viscosity coefficients.

Similar to the continuous case, we introduce some notations: Error function $e^n_j = v^n_j - u^n_j$ with $u^n_j = u(x_j, t_n)$ and Weighted error function $E^n_j = \phi^n_j \cdot e^n_j$, where $\phi^n_j = \phi(x_j - X(t_n))$. It can be shown that if (x_j, t_n) is of $\mathcal{O}(\Delta x)$ distance away from the shock curve, then the following estimate holds:

$$e^{n+1}_j = h_1 e^n_{j-1} + h_2 e^n_j + h_3 e^n_{j+1} + S^n_j + \mathcal{O}(\Delta x^2) \tag{4.2}$$

where S^n_j satisfies $|\phi^{n+1}_j S^n_j| \le C\Delta x \|E^n\|_\infty + \mathcal{O}(\Delta x^2)$. In the equation (4.2), we have used the notations $h_k = h_k(v^n_{j-1}, v^n_j, v^n_{j+1}), \tilde{h}_k = h_k(u^n_{j-1}, u^n_j, u^n_{j+1})$.

We now consider two cases. The first one is the error control in small zone near the shock. In the small zone near the shock, we need to use the fact $|\phi(x)| \sim \mathcal{O}(x)$ to control the error, since e^n_j is essentially $\mathcal{O}(1)$ in this small zone. It can be shown that if $|x_j - X(t_n)| \le \Delta x^\gamma$, with a constant $\gamma \ge 1$, then

$$|E^{n+1}_j| \le (1 + C\Delta t)\|E^n\|_\infty + \mathcal{O}(\Delta x^2)\,.$$

The second case is the error control away from the shock, namely the estimate of the discrete weighted error in the region $|x_j - X(t_n)| \ge \Delta x^\gamma$. It can be shown that for $|x_j - X(t_n)| > \Delta x^\gamma$, we have

$$h_1(u^n_{j-1}, u^n_j, u^n_{j+1}) - h_3(u^n_{j-1}, u^n_j, u^n_{j+1}) = \lambda f'(u^n_j) + \mathcal{O}(\Delta x^\gamma)\,.$$

It can be further shown that

$$\begin{aligned} |E^{n+1}_j| \;\le\; & \left(1 - \Delta t X'(t_n)\frac{(\phi')^n_j}{\phi^n_j} + C\Delta x\right)\|E^n\|_\infty \\ & + \Delta x \frac{(\phi')^n_j}{\phi^n_j}(h_1 - h_3)\|E^n\|_\infty + |\phi^{n+1}_j S^n_j| + \mathcal{O}(\Delta x^2). \end{aligned}$$

Combining the above two results gives

$$|E^{n+1}_j| \le (1 + C\Delta t)\|E^n\|_\infty + \frac{(\phi')^n_j}{\phi^n_j}\Big(f'(u^n_j) - X'(t_n)\Big)\Delta t \|E^n_j\|_\infty + \mathcal{O}(\Delta x^2).$$

Using the similar tricks as used for the viscosity approximations, we can handle the term involving $(f'(u^n_j) - X'(t_n))$ above. Finally, we can bound E^{n+1}_j in the region away from the shock curve: For $|x_j - X(t_n)| > \Delta x^\gamma$, we have

$$|E^{n+1}_j| \le (1 + C\Delta t)\|E^n\|_\infty + \mathcal{O}(\Delta x^2).$$

Combining the estimates within and away from the shock curve, we obtain Gronwall type inequality

$$\|E^{n+1}\|_\infty \le (1 + C\Delta t)\|E^n\|_\infty + C\Delta x^2\,,$$

which leads to the following result [37].

Throrem 4.2. *Assume that the entropy solution has* only one *shock discontinuity* $S(t) = \{(x,t)|x = X(t)\}$. *Let* $v^\Delta(x,t)$ *be the piecewise linear interpolant of the grid solution. Then*

$$
\begin{aligned}
&|(v^\Delta - u)(x,t)|\phi(x - X(t)) = \mathcal{O}(\Delta x)\,,\\
&|(v^\Delta - u)(x,t)| \le C(h)\Delta x\,, \qquad dist(x, S(t)) \ge h.
\end{aligned}
$$

5. Regularity of the Approximate Solution and Superconvergence

The results list above and summarized in Sections 1 and 2 are concerned with the convergence of the approximate solution itself, essentially nothing is obtained for its derivative. In this section, we will investigate the convergence of the first derivative of the approximate solutions. By properly choosing a weighted function, we will obtain an $\mathcal{O}(\epsilon)$-bound for $u^\epsilon - u$ in a weighted $W^{1,1}$ space.

5.1. **Viscosity methods.** By properly choosing a weighted function as introduced in the last section, we will obtain an $\mathcal{O}(\epsilon)$-bound for $u^\epsilon - u$ in a weighted $W^{1,1}$ space.

Throrem 5.1. *Assume that the entropy solutions of the scalar conservation laws satisfy a weak pointwise error estimate* (4.1)*.Then there exists a constant* C *independent of* ϵ *such that*

$$
\int_{\mathbf{R}} \phi(x - X(t))\Big(|e(x,t)| + |e_x(x,t)|\Big)dx \le C\epsilon\,. \tag{5.1}
$$

The proof of the above theorem is based on the standard energy-type method and can be found in Tang-Teng [44]. For convex conservation laws, the assumptions of the weak pointwise-error estimate (4.1) is verified in for example Tadmor [34]. Therefore, one immediate application of our $W^{1,1}$-convergence theory is that for convex conservation laws we indeed have $W^{1,1}$-error bounds for the approximate solutions to conservation laws. Furthernore, the result (5.1) implies that

$$
\|u_x^\epsilon(\cdot,t) - u_x(\cdot,t)\|_{L^1(\mathbf{R}\setminus[X(t)-h,X(t)+h])} \le C(h)\epsilon\,,
$$

for any given $h > 0$. As a consequence of the above inequality, the $W^{1,1}$-estimate recovers the $\mathcal{O}(\epsilon)$-pointwise error bound obtained in the last section, i.e. Theorem 4.1. The detailed proof of this consequence can be also found in [44].

The results in Theorem 5.1 suggest that if a weak pointwise-error estimate can be established for the *nonconvex* conservation laws, then the optimal pointwise-error bounds can be obtained for the *nonconvex* case. So far, almost no pointwise-error estimates have been obtained for nonconvex conservation laws.

5.2. **Superconvergence of Finite-Difference Schemes.** Assume that the numerical scheme (2.2) is monotone with *smooth* numerical flux. Then the results of Theorem 5.1 can be extended to the monotone schemes. Furthermore, the $\mathcal{O}(\Delta x)$ pointwise-error bounds for numerical solutions away from the shock curves can be established. The detail of the proofs can be found in Tang-Teng [45].

6. Extension to finitelt many shocks

The idea described in Sections 3-5 can be easily extended to the case that the entropy solution of (3.3) has finitely many shocks $S(t) := \{(x,t) \,|\, x = X_k(t)\}$. In this general case, we define the weighted distance function as

$$\rho(x,t) = \prod_{k=1}^{K} \phi\Big(|x - X_k(t)|\Big).$$

The pointwise error estimates silimar to those in the theorems in Sections 3-5 can be obtained by considering the weighted error function $E(x,t) = (u^\epsilon(x,t) - u(x,t))\,\rho(x,t)$.

Acknowledgment. The research was supported by NSERC Canada, Hong Kong Baptist University and Hong Kong Research Grants Council.

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