Generation of new Solvers involving the Source Term for a Class of Nonhomogeneous Hyperbolic systems

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1 Introduction

We consider the class of Hyperbolic systems of the form

$$q_t \ + \ m_x \ = \ 0 \ , \tag{1}$$

$$m_t \ + \ 2u \ m_x \ + \ (c^2 - u^2) \ q_x \ = \ - \ h(q, m) \ , \tag{2}$$

for $x \in \mathbb{R}, \ t > 0$, where $u = u(q, m), \ c = c(q) > 0$ are given sufficiently regular functions. We suppose that c is an increasing function.

Many physical models can be written under this form. We have two more important cases. The case $u = \frac{m}{q}$ corresponds to a general model in fluid dynamics, where u is the fluid velocity, q is the transported quantity, c is the soundspeed and m is the momentum. This includes the Shallow Water model when $q = h$ the height of water, and $c = \sqrt{gh}$ with the gravity constant g, the isentropic hydrodynamics when $q = \rho$ the gas density, and $c = \sqrt{p'(\rho)}$ with a given pressure law $p = p(\rho)$, and some other models in elastodynamics, road traffic, etc.. The case $u = 0$ corresponds to models of damped waves propagation in a medium (telegraphist equation), Lagrangian

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fluid dynamics ($p-$systems), etc.. The general case for other functions u is of lower usefulness in practice.

The source term $h(q,m)$ corresponds to the interaction with the medium: geometry or gravity terms, friction terms, damping terms, etc... We denote

$$H_0 \;=\; \{(q,m)|\; h(q,m) = 0\;\}$$

the set where h is zero in the phase plane (q,m), and assume that we have either $H_0 = \emptyset$, or H_0 reduced to a single continuous curve. Then we denote

$$H_+ \;=\; \{(q,m)|\; h(q,m)\; >\; 0\;\}, \qquad H_- \;=\; \{(q,m)\;|\; h(q,m)\; <\; 0\;\}$$

the other parts of the phase plane.

We propose to build in the phase plane the set of the values of a possible solution of (1), (2) linking a given value $G = (q_g, m_g)$ on the left to another given value $D = (q_d, m_d)$ on the right. This is different from the usual Riemann problem, for several reasons: we cannot use the fact that the constant are solutions, since the source term is taken in account, and we do not start from initial data, which can filter some useful possible profiles of solution.

From this construction we get enough informations to develop several applications for mathematical or numerical purposes. That way, we propose a general interface solver for finite volume methods, some examples of boundary solvers for either absorbing or active boundary conditions, and some mathematical applications as the explicitation of well known problems, such as the Roll Waves phenomenons or the DamBreak Problem with friction on a flat bottom.

A local analysis in the phase plane is detailed in Section 2, using very simple mathematical arguments, and the results are applied to the generation of the interface solver in Section 3. Then some other applications, such as the boundary solver and the examples are proposed in Section 4 and 5.

2 The local analysis in the phase plane

One usually thinks to a wave as a non constant profile of the solution moving with some velocity. This means that one can (at least locally) write the space variable as a function of q for example, for a given time t. Then m

becomes a function of q, at time t. By setting $m = m(q)$ in (1) (2) one gets the linear system

$$\begin{pmatrix} 1 & m'(q) \\ m'(q) & 2um'(q) - u^2 + c^2 \end{pmatrix} \begin{pmatrix} q_t \\ q_x \end{pmatrix} = \begin{pmatrix} 0 \\ -h(q,m) \end{pmatrix}. \tag{3}$$

It is well known in the homogeneous case $h \equiv 0$ that the matrix must be a singular one and by writing the determinant to be zero, one gets the ordinary differential equation

$$m'(q) \;=\; u(q, m(q)) \pm \; c(q)$$

whose solutions correspond to the Riemann invariants.

Here, in the non homogeneous case, the matrix must be a non singular one, and the Riemann invariants become unuseful. We can solve (3) and get the relation

$$\frac{(m'(q) - u(q, m(q)))^2 - c(q)^2}{h(q, m(q))} \; q_x \;=\; 1 \; . \tag{4}$$

Let us denote by $\psi = \psi(q)$ a function whose derivative is

$$\psi'(q) \;=\; \frac{(m'(q) - u(q, m(q)))^2 - c(q)^2}{h(q, m(q))} \quad . \tag{5}$$

By integrating with respect to x, one gets a relation of the form

$$\psi(q) \;=\; x \;-\; K(t) \; .$$

Now we derive with respect to t and get, using (5),

$$K'(t) \;=\; -\,\psi'(q)\; q_t \;=\; m'(q)\; \psi'(q)\; g_x \;=\; m'(q) \; ,$$

that we derive one time more, at first with respect to t to obtain

$$K''(t) \;=\; m''(q)\; q_t \;=\; -\; m''(q)\; m'(q)\; q_x \; ,$$

and then with respect to x, to obtain

$$0 \;=\; m''(q)\; q_x \; .$$

We have got, at least for $q_x \neq 0$,

$$m''(q) \;=\; K''(t) \;=\; 0 \quad ; \quad m'(q) \;=\; K'(t) \quad ,$$

whose solution has the form

$$m(q) \;=\; A\,q \;+\; B \;\;;\quad K(t) \;=\; A\,t \;-\; C\;, \tag{6}$$

for some constants A, B and C. This means that the set of the values of the solutions of (1) (2) in the phase plane belongs to some straight lines. It is also important to remark that this result was obtained without any explicitation of the source term $h(q, m)$. We propose to name the function ψ the inverse profile function, since we have got

$$\psi(q) \;=\; x \;-\; At \;+C\;, \tag{7}$$

which can be seen as

$$q \;=\; \psi^{-1}(x - At + C)$$

and gives the profile of q. The profile of m is obtained by writing $m = Aq+B$. We also notice that q satisfies the single wave equation

$$q_t \;+\; A\,q_x \;=\; 0\;, \tag{8}$$

saying that A stands for the wave velocity. For $A < 0$, the propagation will be leftwards and for $A > 0$ the propagation will be rightwards.

Let $M_0 = (q_0, m_0)$ a point in the phase plane, $M_0 \notin H_0$. We introduce the sets

$$T_+(M_0) \;=\; \{(q,m)|\ \exists A \in \mathbb{R}\ m = m_0 + (q - q_0)(u(q, A(q - q_0) + m_0)) + c(q))\ \},$$

$$T_-(M_0) \;=\; \{(q,m)|\ \exists A \in \mathbb{R}\ m = m_0 + (q - q_0)(u(q, A(q - q_0) + m_0)) - c(q))\ \},$$

and

$$T(M_0) \;=\; T_+(M_0) \;\cup\; T_-(M_0)\;.$$

This corresponds to the set of the points (q, m) such that $\psi'(q) = 0$ when $A = \dfrac{m - m_0}{q - q_0}$ and $B = m_0 - Aq_0$. The expression of $T(M_0)$ is far simpler in the two important cases, that is,

$$m \;=\; \frac{m_0}{q_0}\,q \;\pm\; c(q)\,\frac{q}{q_0}\,(q - q_0), \qquad for\ u = \frac{m}{q}\;, \tag{9}$$

$$m \;=\; m_0 \;\pm\; c(q)\;(q-q_0), \qquad for\; u=0\;. \tag{10}$$

Let us consider a straight line passing thru M_0. We denote its slope by A, and its equation is $m = A(q-q_0)+m_0$, that is $B = m_0 - Aq_0$. This line cuts $T(M_0)$ in one point $M \neq M_0$ belonging to $T_+(M_0)$ for $B < 0$ or to $T_-(M_0)$ for $B > 0$, with two exceptions, when $A = u_0 \pm c_0$ with $u_0 = u(q_0, m_0),\; c_0 = c(q_0)$. We denote by

$$E_-(M_0) \;=\; \{(q,m) \mid m = (u_0 - c_0)(q-q_0) + m_0\;\}$$

$$E_+(M_0) \;=\; \{(q,m) \mid m = (u_0 + c_0)(q-q_0) + m_0\;\}$$

the two corresponding lines, and call them the extremal lines associated to M_0. We can remark that these extremal lines are tangent in M_0 to the Riemann Invariants and also to $T(M_0)$.

Following any line passing thru M_0, the sign of $\psi'(q)$ is constant in a neighbourhood of M_0 which contains no point of H_0, but for the two extremal lines associated to M_0 where this sign will change by crossing M_0. We say that M_0 is left sided when $(q-q_0)\psi'(q) > 0$ along the extremal lines, ands right sided when $(q-q_0)\psi'(q) < 0$ along the associated extremal lines. Since the signs of $\psi'(q)$ and q_x are the sames, we get immediately that along a line of slope A passing thru M_0,

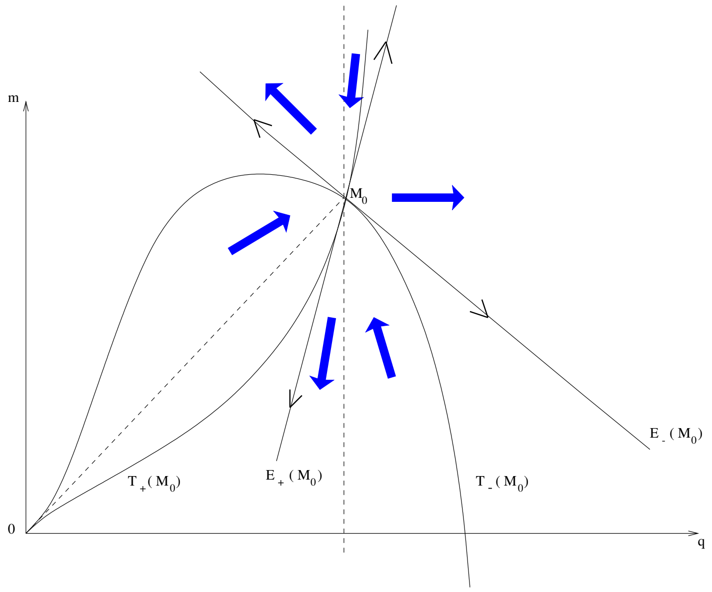
$$M_0\; left\; sided \;\Rightarrow \begin{cases} q_x > 0 & for\; u_0 - c_0 < A < u_0 + c_0, \\ \quad q_x < 0 & \qquad otherwise, \end{cases}$$

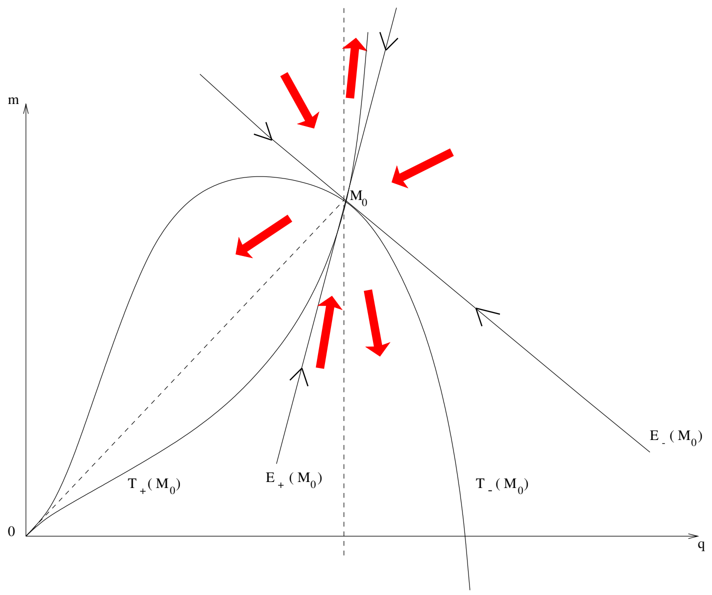
$$M_0\; right\; sided \;\Rightarrow \begin{cases} q_x < 0 & for\; u_0 - c_0 < A < u_0 + c_0, \\ \quad q_x > 0 & \qquad otherwise, \end{cases}$$

The terms *Left or Right sided* refer to the interface solver. This means that one can leave a left sided point to reach a right sided point by forwarding the x-axis. On the other side of H_0, the roles plaid by left and right are swapped since the sign of $\psi'(q)$ has changed. We have the following result, which will be used for the interface solver.

Theorem 1 *If $S \in T_+(M_0)$ then $M_0 \in E_+(S)$, and if $S \in T_-(M_0)$ then $M_0 \in E_-(S)$*

Proof: We denote $S = (q_s, m_s)$. If $S \in T_+(M_0)$, then $m_s = m_0 + (q_s - q_0)(u_s + u_s)$, where $u_s = u(q_s, m_s)$, $c_s = c(q_s)$. This also reads $m_0 = m_s + (q_0 - q_s)(u_s + c_s)$ and means $M_0 \in E_+(S)$. If $S \in T_-(M_0)$, then $m_s = m_0 + (q_s - q_0)(u_s - u_s)$, which also reads $m_0 = m_s + (q_0 - q_s)(u_s - c_s)$, that is $M_0 \in E_-(S)$.





A critical case corresponds to $M_0 \in H_0$. In this case $h(q_0, m_0) = 0$ and $\psi'(q)$ may become unbounded. When q_0 corresponds to a single root of $h(q, A(q-q_0)+m_0)$, along a line passing thru M_0, one can still integrate $\psi'(q)$ in a neighbourhood of M_0 when this line is an extremal line associated to M_0. It is possible for other lines only when $\psi'(q)$ belongs to $L^1_{loc}(q_0-\epsilon, q_0+\epsilon)$ for some $\epsilon > 0$. For example the case $\nabla h = 0$ is not applicable, thought it is often present in some models, as for the dambreak model with friction on a flat bottom. In the case where $\psi' \in L^1_{loc}(q_0-\epsilon, q_0+\epsilon)$ along a (well choosen) line, we can consider that M_0 is on each side of H_0 and it can be reached from one side as a right sided point, and left toward the other side as a left sided point. That way, H_0 can be crossed at least in one direction.

We now add some remarks about shocks and entropy, since shocks waves may occur in some cases. We denote by (q_l, m_l) and (q_r, m_r) the values on the left side and the right side of the shock respectively, and by $x = x(t)$ the path of the shock wave. The Rankine Hugoniot conditions for the shock velocity $x'(t)$ are the following

$$-x'(t)\ (q_r - q_l)\ +\ m_r - m_l\ =\ 0\ , \tag{11}$$

$$-x'(t)(m_r - m_l)\ +\ 2\,\overline{u}\,(m_r - m_l)\ +\ (c(\overline{q})^2 - \overline{u}^2)(q_r - q_l)\ =\ 0 \tag{12}$$

where $\overline{u}$ is an average value between $u_l = u(q_l, m_l)$ and $u_r = u(q_r, m_r)$, and $\overline{q}$ an average value between q_l and q_r.

The usual entropy condition in (homogeneous) fluid dynamics, that is for $u = \dfrac{m}{q}$, corresponds to impose $u_x \geq 0$ almost everywhere, and bounded everywhere. Then the case $u_x < 0$ will corresponds to a shock wave and the only stable possibility is $u_x = -\infty$. This leads to the entropy condition

$$u_l\ >\ \overline{u} >\ u_r \quad \textit{on a shock} \tag{13}$$

By eliminating $x'(t)$ between (11) and (12), we get the shock compatibility condition

$$m_r - m_l\ =\ (q_r - q_l)\ (\overline{u} \pm\ c(\overline{q}))\ . \tag{14}$$

We allways have $\overline{u} = \dfrac{u_r + u_l}{2}$ in practice, so we decide for this choice in the general case, and the formula for $\overline{q}$ can be derived from a mathematical development (for example from the weak solution definition in the conservative case) or from other physical arguments (for example from the Hugoniot experimental curves in elasticity). By writing in the fluid model ($u = \dfrac{m}{q}$, $q > 0$) that $m_r - m_l = \overline{u}(q_r - q_l) + \dfrac{q_r + q_l}{2}(u_r - u_l)$ in (14), we get

$$\frac{q_r + q_l}{2}\ (u_r - u_l) \ = \ \pm\ c(\overline{q})\ (q_r - q_l)\ , \tag{15}$$

which tell us, using (14), that $q_r - q_l > 0$ for the minus sign, $q_r - q_l < 0$ for the plus sign. We also consider that these inequality are still true for the general case.

3 The Solver

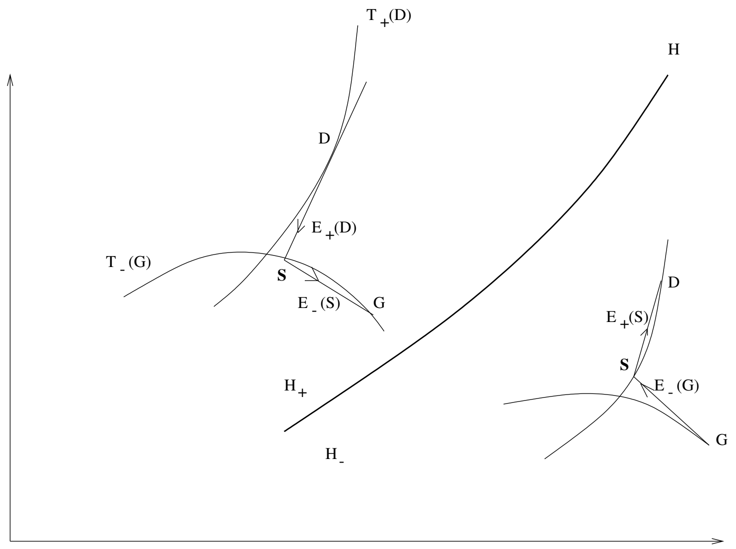
We consider two points denoted G (for left) and D (for right) in the phase plane. Our aim is to build a path linking G to D, which is compatible with the previous local analysis. We consider four cases.

3.1 Case 1: G and D are both left sided.

This case is well adapted for G, and we can leave G following the extremal line $E_-(G)$. This extremal line cuts $T_+(D)$ in one point denoted S from which we can reach D following $E_+(S)$, since $D \in E_+(S)$ from Theorem 2.1.

3.2 Case 2: G and D are both right sided

This case is well adapted for D, and we can reach D by following $E_+(D)$. Let S denote the point where $E_+(D)$ cuts $T_-(G)$. We know from Theorem 2.1 that $G \in E_-(S)$, and then a possible path consists into following $E_-(S)$ from G to S and then $E_+(D)$ from S to D.



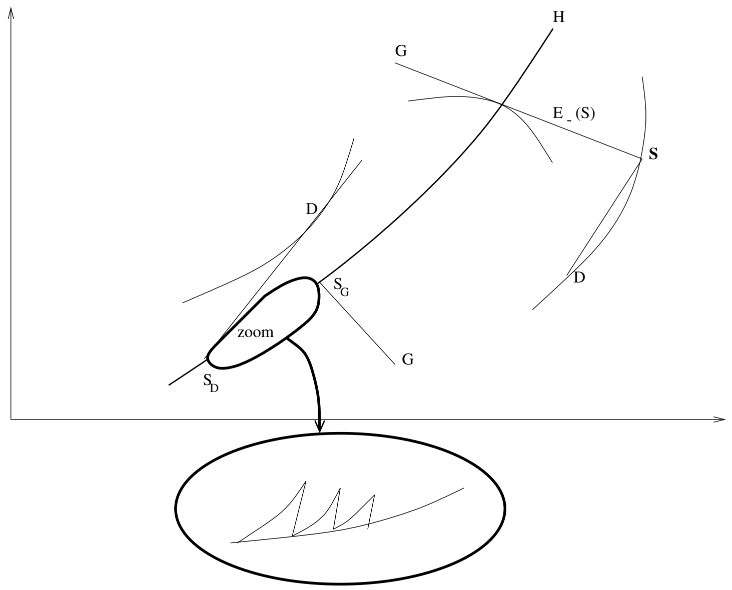
3.3 Cas 3: G is right sided and D is left sided

This case is not well adapted for each point G and D. We have to cross H_0 at some point S_0, which can be done only along an extremal line associated to S_0. From the previous local analysis, we know that S_0 can be reached from the right sided side of H_0 following any extremal line associated to S_0. One can fix S_0 in such a way that $G \in E_-(S_0)$. Now we have to reach D by starting from S_0, which are both left sided. We are in the situation of Case 1, with S_0 in place of G. We get $S \in E_-(S_0) \cap T_+(D)$.

3.4 Case 4: G is left sided and D is right sided

In this case it is easy to leave G following $E_-(G)$ and to reach D following $E_+(D)$. The problem is however: how to cross H_0. It cannot be done by jumping with a shock wave, since it is not allowed by the entropy condition (13). If we look at a point $S_0 \in H_0$, we realize that one can only leave S_0 either following $E_+(S_0)$ or $E_-(S_0)$, and never reach S_0. However S_0 *can be reached by using a shock wave.*

We propose to use creeping waves to build a solution in this case. Such a wave is defined as the limit as the amplitude tends to zero of a sequence of a succession of small shocks and regular waves along a given curve, which is here H_0. Since $c(q)$ is an increasing function of q, we know that $T_+(M)$ is convex and $T_-(M)$ is concave in a neighborhood of a given point M. From Condition (14), the shock curves are shaped the same way, and one can build a creeping wave reaching the point $S_D \in H_0 \cap E_+(D)$ following H_0 with q decreasing, or starting from $S_G \in H_0 \cap E_-(G)$ following H_0 with q increasing.



This is due to the following Proposition.

Proposition 1 *Let $S = (q_S, m_S) \in H_0$, $M_0 = (q_0, m_0) \notin H_0$ linked to S by a shock wave of velocity $\overline{u} + c(\overline{q})$, and $M = (q_S, m)$ with $m = m_0 + (u_0 + c_0)(q_S - q_0)$. Then $m > m_S$.*

If M_0 is linked to S with a shock of velocity $\overline{u} - c(\overline{q})$ and M such that $m = m_0 + (u_0 - c_0)(q_S - q_0)$, then $m < m_S$

Proof: We have in the first case

$$m_S \ = \ m_0 \ + (\overline{u} + c(\overline{q}))\ (q_S - q_0)\ ,$$

with $\overline{u}$, $\overline{q}$ computed with $(q_l, m_l) = (q_0, m_0)$ and $(q_r, m_r) = (q_S, m_S)$. We get

$$m_S - m \ = \ (q_S - q_0)\ (\overline{u} - u_0 + c(\overline{q}) - c_0)\ .$$

We have $\overline{u} < u_0$ from the entropy condition, $q_S - q_0 < 0$ and then $\overline{q} - q_0 < 0$ which gives $c(\overline{q}) - c_0 < 0$ since c is an increasing function. This leads to $m - m_S > 0$.

In the second case,

$$m_S \ = \ m_0 \ + (\overline{u} - c(\overline{q}))\ (q_S - q_0)\ ,$$

with $\overline{u}$, $\overline{q}$ computed with $(q_l, m_l) = (q_S, m_S)$ and $(q_r, m_r) = (q_0, m_0)$. We get

$$m - m_S \ = \ (q_S - q_0)\ (u_0 - \overline{u} + c(\overline{q}) - c_0)\ .$$

We have $\overline{u} > u_0$ from the entropy condition, $q_S - q_0 < 0$ and then $\overline{q} - q_0 < 0$ which gives $c(\overline{q}) - c_0 < 0$ since c is an increasing function. This leads to $m - m_S > 0$.

We have to notice that the slopes involved in the first case (sign +) corresponds to positive values in the general case, even if H_0 looks like the graph of a decreasing functions. The same way, the creeping wave of the second case (sigh -) is made of elementary waves with a negative velocity in the general case. We set

$$S \ = \ \begin{cases} S_D & if\ q(S_D) > q(S_G)\ , \\ S_G & if\ q(S_D) < q(S_G)\ . \end{cases} \tag{16}$$

That way, we have that the waves between S and D are travelling faster (and generally with a positive velocity) than the waves between G and S (whose velocity is negative in the general case).

Now, to decide where we effectively cross H_0 is of lower importance, since we may consider that we enter into H_0 in S_G, then we stick to H_0 until we reach S_D, where we loosen to reach D.

A last case corresponds to $G \in H_0$ or $D \in H_0$, which can be seen as one of the previous cases since H_0 is a part of the boundary of H_- and H_+.The idea is to make the best favorable choice, from simplicity and physical compatibility points of view.

4 Examples

We propose several examples for which the previous construction of a solution of (1) (2) is useful.

4.1 The Roll Waves

The model corresponds to the Shallow Water equations with

$$h(q,m) = gpq + k|u|u \tag{17}$$

with $u = \frac{m}{q}$, $c = \sqrt{gq}$, where g is the gravity constant, $p < 0$ is the slope and $k > 0$ is the friction coefficient. The asumption on the sigh of p is nothing else than to say on which side of the river the observer stands. A positive velocity is expected. We introduce the parameter $\lambda = \sqrt{-\frac{p}{k}}$.

A Roll Wave solution is a periodic solution made of a succession of decreasing shocks separated by increasing regular parts (see [8]). The wave velocity is claimed to be a constant and is denoted by A. The momentum $m = Aq + B$ corresponds to the water flow. We denote by q_l and q_r the values on both side of the shock, with $q_r < q_l$. The shock analysis gives

$$x'(t) \;=\; \overline{u} + \overline{q}\,\frac{u_r - u_l}{q_r - q_l} \quad ; \quad \frac{u_r - u_l}{q_r - q_l} \;=\; g\,\frac{\overline{q}}{q_r q_l} \quad with\ \overline{q} = \frac{q_l + q_r}{2}\ .$$

Since the wave velocity is a constant, we have $x'(t) = A$, and then

$$\overline{u} \;=\; A \;+\; \frac{B}{2}\left(\frac{1}{q_l} + \frac{1}{q_r}\right) \;=\; A \;+\; B\,\frac{\overline{q}}{q_l q_r} \quad with\ 0 \le \overline{u} \le A\ .$$

This leads to the expression of B

$$B \;=\; -\;\sqrt{g\ q_l\ q_r\ \overline{q}}\ . \tag{18}$$

The inverse profile functions ψ corresponds to

$$\psi'(q) \;=\; \frac{B^2 \;-\; g\ q^3}{k((Aq+B)^2 - \lambda^2 g q^3)}\ ,$$

and we see that no root of either the numerator or the denominator may appear between q_r and q_l (otherwise q_x will be zero or infinite at some point of the increasing part of the wave). Since $B^2 - gq^3 = 0$ for

$$q \;=\; q_0 \;=\; \left(\frac{B^2}{g}\right)^{\frac{1}{3}}\ ,$$

which lies between q_r and q_l, so we need q_0 to be a root of the denominator too, that is

$$(Aq_0 + B)^2 \ = \ \lambda^2 \ g \ q_0^3 \ ,$$

which gives the value of A

$$A \ = \ (1+\lambda) \ (g|B|)^{\frac{1}{3}} \ . \tag{19}$$

Now, by writing $q = \beta q_0$, with $\beta > 0$, we get

$$\psi'(q) \ = \ \frac{1}{k} \ \frac{\beta^2 + \beta + 1}{(\lambda\beta - 1)^2 - \beta} \tag{20}$$

which become infinite when

$$\beta \ = \ \frac{1 + 2\lambda \ \pm \ \sqrt{1+4\lambda}}{2\lambda^2} \quad \mathit{or} \ \lambda \ = \ \frac{1}{\beta} \ + \ \frac{1}{\sqrt{\beta}} \ . \tag{21}$$

We set

$$\beta_* \ = \ \frac{1 + 2\lambda \ + \ \sqrt{1+4\lambda}}{2\lambda^2} \ ,$$

which is the larger root. We need to have $\beta_r = \frac{q_r}{q_0} \geq \beta_*$ to avoid any root between q_r and q_l, and since $\beta_r \leq 1$, we need $\beta_* \leq 1$ too, so we have from (21) a necessary condition on λ:

$$\lambda \ \geq \ 2 \ . \tag{22}$$

From a value β_r, we get β_l by

$$\beta_l \ = \ \frac{\sqrt{\beta_r^4 + 8\beta_r} \ - \ \beta_r^2}{2 \ \beta_r} \ ,$$

and the period may be obtained by computing

$$\Delta x \ = \ \psi(q_l) - \psi(q_r) \ = \ \frac{q_0}{k} \int_{\beta_r}^{\beta_l} \frac{\xi^2 + \xi + 1}{(\lambda\xi - 1)^2 - \xi} \ d\xi.$$

This show that the use of the representation by a line $m = Aq + B$ may help to compute explicitely the solution here, and that this line is not necessary an extremal one.

4.2 The Dam Break

We consider the dam break problem on a flat dry bottom with a friction term, and propose the construction of a solution using the previous local analysis. The same problem on a slope is easier to handle. The model corresponds to the Shallow Water equations as above, with the initial condition

$$q(x,O) \;=\; \begin{cases} q_0 > 0 & for\ x < 0 \\ 0 & for\ x > 0 \end{cases} \;, \qquad u(x,0) = 0 \qquad ,$$

and a source term of the form

$$h(q,m) \;=\; h(u) \;=\; k\;|u|^{\alpha}\; sign(u) \quad with\ \alpha > 0\ .$$

We notice that a positive velocity is expected and $h(u)$ really reads $h(u) = ku^{\alpha}$. The usual choice for α is $\alpha = 2$ (Strickler friction term, see [8]) and we shall see that this choice may be a bad choice here.

We set $G = (q_0, 0),\ D = (0,0)$ which both lie in $H_0 = \Big\{(q,m)|q > 0,\ m = 0\Big\}$. The point G is impossible to leave for $\alpha \geq 1$. This leads to impose

$$0 \;\leq\; \alpha \;<\; 1\ ,$$

what hydraulics people never do. Moreover using the extremal line $E_-(G)$ corresponds to make q increasing, which is absolutely not expected too. We have to follow another line beyond $T_-(G)$, of slope A with $-c_0 < A < 0$. We denote by $S = (q_S, m_S)$ the point where this line cuts $T_-(G)$. Next we have to reach D from S following the radial line $m = u_S q$, with $u_S = \frac{m_S}{q_S}$. Since there is no restriction on the size of the flow, the maximal flow is selected, which corresponds to have the line $m = A(q - q_0)$ cutting $T_-(G)$ at the maximal flow value, that is for $q_S = \frac{3}{5}q_0$. This value is clearly different from the one obtained by the Riemann invariant, when no friction is present, which is $\frac{4}{9}q_0$. We get $m_S = \dfrac{6\sqrt{3}}{25\sqrt{5}}\, q_0\, c_0$ and $A = \dfrac{m_S}{q_S - q_0} = -\dfrac{3\sqrt{3}}{5\sqrt{5}}c_0 \;\approx -0.46$, which is far different from the soundspeed $-c_0$ found in the case without friction.

We have $B = -Aq_0$ (> 0), and one can compute q by using

$$\psi'(q)\; q_x \;=\; 1 \qquad with\ \ \psi'(q) \;=\; \frac{A^2 q_0^2 \;-\; g\; q^3}{q^{2-\alpha}\; |A|^{\alpha}\; (q_0 - q)^{\alpha}}\ .$$

This example provides two informations: the use of $E_-(S)$ for the linkage with G, instead of $E_-(G)$, and the necessity to revise the form of the friction term near a well balanced state.

4.3 The Splash Down

This model corresponds to the last phase of the reentry of a space module, when it reaches the surface of the sea. It is a velocity pressure model in 2 space dimensions, where the interface computation corresponds to solve (1) (2) with $u = \frac{m}{q}$ and

$$h(q,m) \ = \ q \ g \ sin\varphi \ + \ k \ u \quad , \quad c(q) \ = \ c_0 \ q^{\frac{\mu}{2}} \quad ,$$

with μ a constant depending on the medium ($5 \leq \mu \leq 10$ in practice), c_0 the soundpeed at rest. The quantity q is linked to the pressure p by

$$q \ = \ (1 + \beta_0 p)^{\frac{1}{\mu}}$$

with β_0 also depending to the medium ($10^{-6} \leq \beta_0 \leq 10^{-12}$ in practice). The parameter φ is the angle between the horizontal line and the normal to the interface. This model is analysed in [3]. It is interesting to notice that on H_0, we have

$$u \ \approx \ - \ \frac{g}{k} \ sin\varphi \ \approx \ - \ 10^4 \ sin\varphi \ ,$$

and the solver will reduce to cases 1 and 2 above when the mesh is built in such a way that (for example) $|sin\varphi| \geq \frac{1}{2}$, in order to get rid of a vertical interface. In [3], the mesh is made of regular hexagones, with no vertical side. If a vertical interface is present, we have to extend the investigation to the four cases, which makes the solver far slower, and to modify the friction term since $\psi'(q)$ will be not integrable near the interface, which has strictly no physical meaning.

This example underlines the fact that the mesh itself may develop some unstabilities with some source term corresponding to geometrical data, or at least may demand a more complete (thus slower) solver to handle with.

4.4 The Relaxation Approximation for Scalar Laws

The idea is to consider a scalar conservation law of the form

$$q_t \; + \; f(q)_x \; = \; 0, \tag{23}$$

with $f \in C^1(\mathbb{R})$ a given convex function, and then to approach $f(q)$ by a new parameter m, in the following system

$$q_t \; + \; m_x \; = \; 0 \;\; , \tag{24}$$

$$m_t \; + \; c^2 \, q_x \; = \; - \; \mu \, (m - f(q)) \; , \tag{25}$$

where $c > 0, \; \mu > 0$ are some parameters to be fitted (see [4,7]). When μ goes to infinity, we get $m = f(q)$ and get (23) again. This system corresponds to the formulation (1) (2) and we can set $m = Aq + B$ to obtain

$$\psi'(q) \, q_x \; = \; 1 \qquad \textit{with } \psi'(q) \; = \; \frac{c^2 - A^2}{\mu(f(q) - Aq - B)} \; , \tag{26}$$

where $\psi'(q) \geq 0$, for $c > |A|$, and $f(q) > \; Aq + B$, which is always a possible choice since f is convex. This is consistant with the entropy condition which reduces here to $q_x \geq 0$ on a regular wave and a shock wave, whose velocity must be $\pm \; c$, corresponds to $c = |A|$ (i.e extremal lines). That means that the set of the possible values of a shock wave corresponds to a tangent line of $H_0 = \{(q, m) \mid m = f(q) \; \}$, which is the graph of f. The points in the phase plane are all left sided since $f(q) \leq m, \; c \geq |A|$, and a regular solution to (23) corresponds to a creeping wave of (24), (25). For a given concave function f, $q_x \leq 0$ is expected on a regular wave, and $c > |A|$ and $f(q) < Aq + B$ will be chosen, and the points are all right sided.

5 Applications

We propose or suggest several applications of the previous solver for the numerical developments. This concerns the generation of interface or boundary solvers, and some other possibilities as to put a small constant to a homogenous problem in order to get a simpler and faster solver.

5.1 The Interface solver

The solver may apply in a finite volume method, in place of any interface solver, such that the well known Riemann solver or some of its approximations.

A first step is to fit up the data in the cell in such a way that the equilibria are preserved. As a matter of fact, a constant state (q_0, m_0) is generally never a solution of (1) (2) when a source term is present. For a given state $M_0 = (q_0, m_0)$ in the cell $]x_0 - \frac{\Delta x}{2},\ x_0 + \frac{\Delta x}{2}[$ we propose a distribution of the values corresponding to a stationary state. That can be represented on a line of equation $m = m_0$ (that is with $A = 0$) in the phase plane by an inverse profile function such that

$$\psi'(q) \;=\; \frac{u(q,m_0)^2 \;-\; c(q)^2}{h(q,m_0)} \quad , \quad \psi(q) \;-\psi(q_0) \;=\; x - x_0 \;. \tag{27}$$

We can integrate (27) or, since Δx is a small parameter, compute an approximation of this stationary profile. To the first order we have

$$\psi'(q_0)\;(q - q_0) \;=\; x - x_0 \;,$$

which leads to

$$q \;= q_0 \;+\; \frac{x - x_0}{\psi'(q_0)} \;, \tag{28}$$

when $\psi'(q_0) \neq 0$. Another approximation to the second order may be derived, for instance when $\psi'(q_0) = 0$. The trace at $x = x_0 - \frac{\Delta x}{2}$ will be used as a state D in the solver, and the trace at $x = x_0 + \frac{\Delta x}{2}$ as a state G. That way, an interface has a state G on the left hand and a state D on the right hand.

The resolution involves the computation of the intersection point between a curve $T(M_0)$ with a straight line, that to solve an equation of the form

$$m_0 \;+\; (q - q_0)\;(u(q, Aq + B) \;\pm\; c(q)) \;=\; Aq + B \;,$$

and we recommend to apply a few Newton iterations to

$$\frac{m_0 - B}{q} \;+\; (q - q_0)\frac{u(q, Aq + B) \;\pm\; c(q)}{q} \;-\; A \;=\; 0 \tag{29}$$

in order to avoid the solution $q = 0$ in some cases. In practice, for the two important cases, the expression in simpler that the one appearing in (29).

The solver itself reduces to select the state S as detailed in section 3, when the slopes meeting in S are of different signs. If they are of the same sign, we select either G or D as for the decentered scheme.

We may notice that the computation of a stationary solution may lead to a transgression of the usual entropy condition, since we can have $u_x < 0$ in a stable way.

5.2 The Boundary solver

We point out at one extremity of the domain of computation $]a, b[$, say b for example. The mirror situation will be seen at $x = a$. The value in the last cell is represented by a state denoted $M_0 = (q_0, m_0)$ and we denote by G the trace at $x = b$ when this state is shaped as a stationary state.

5.2.1 An active boundary solver

We denote by $D = (q_b, m_b)$ the value imposed for the boundary condition. Then we use the solver described above, with G and D playsing the same roles, and build that way an intermediary value S. If the line joining G to S in the phase plane has a negative slope, then the corresponding values will enter into the domain of computation.

5.2.2 A reflexive boundary condition

This corresponds to set $m = 0$ in the general case. If G is a left sided point in the phase plane, one can follow $E_-(G)$ until we reach the q-axis at a point denoted by $F = (q_F, m_F)$, which gives

$$q_F \ = \ q_g \ + \ \frac{m_g}{c(q_g) - u_g} \qquad ; \qquad m_F \ = \ 0 \ .$$

If G is right sided, then F is to be found at the intersection of $T_-(G)$ with the q-axis, for $q_F \neq 0$. This corresponds to solve the equation

$$m_g \ = \ (q - q_g) \ (u(q, 0) - c(q)) \ ,$$

which reduces to

$$m_g \ = \ c(q) \ (q - q_g) \ \ ,$$

in the usual cases and is easy solved thru a few Newton iterations as for (29). For a critical (that is for $u_g = c_g$) left sided state G, a solution as in Section 3.4 will be considered.

5.2.3 An absorbing boundary condition

The idea for an exit boundary is to let any wave travelling outwards the domain of computation and to forbid any wave to enter back into this domain. This means that the state near the boundary must be either a stationary one or have a positive velocity (corresponding with a slope $m'(q) \geq 0$). The absorbing boundary condition aims to kill the wave when crossing the boundary to be certain that nothing will come back into the domain of computation.

If G is left sided, one can follow any line starting from G, for $q > q_g$, with a slope A such that $0 \leq A \leq u_g + c_g$, that is lines between the stationary state $m = m_g$ and $E_+(G)$. If possible, the stationary case may be preferred which corresponds to take $F = (q_g + \epsilon, m_g)$, for $\epsilon > 0$.

If G is right sided the choice is the same but with $q < q_g$ and will corresponds to take $F = (q_g - \epsilon, m_g)$, for $\epsilon > 0$.

The absorption may be performed by modifying the source term $h(q, m)$ at the boundary, in such a way that a prescribed value (usually $u = 0$) will be hit in a short time. Recall that we need to have the corresponding function $\psi'(q)$ lying in L^1_{loc} as observed in Section 2 above.

5.2.4 A forced boundary condition

Such a condition corresponds to the reverse of the absorbing boundary condition, and wil enforced a given value to enter into the domain of computation. We have only to exchange the roles of G and F in the previous suggestion to handle the absorbing boundary condition, in such a way that the linkage between G and F is made by lines with a nonpositive slope in the phase plane. Such a forced boundaru condition may be used in a multidomain computation technique on the artificial boundaries since it enforced the already computed values to be taken in account.

5.3 Other applications

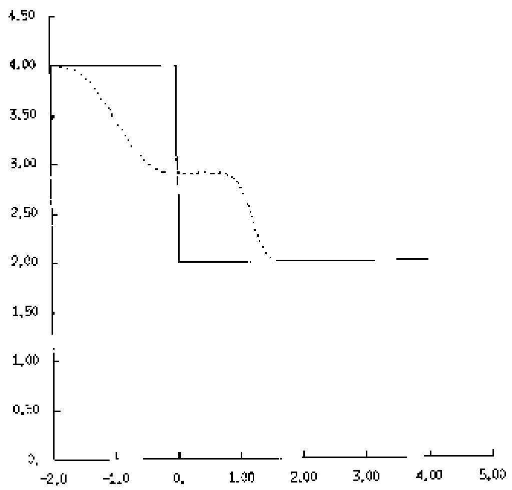
We propose some other applications of the solver for very usual and simple purposes.

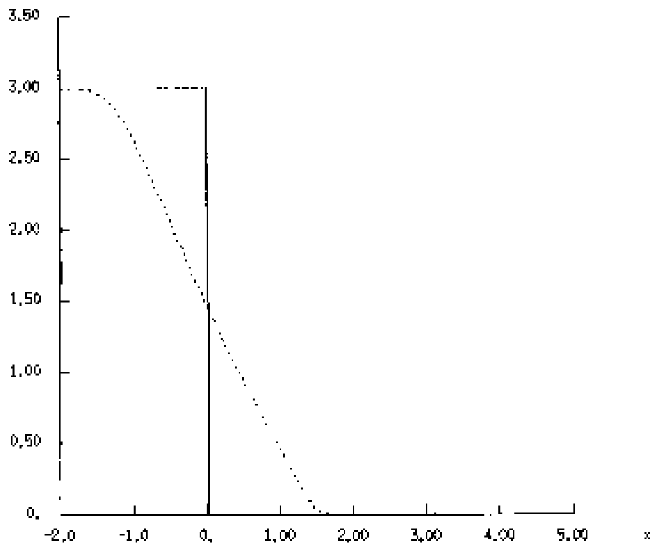
5.3.1 The splitting technique

In these methods used for problems involving several operators, only one operator is taken in account at each step and the other terms are temporarily ignored. This can break up the balance between the several terms and we suggest to use the solver proposed here at least when one is able to know the sigh of these remaining terms.

5.3.2 The homogeneous case

The solver proposed here becomes much simpler when a source term of constant sign is present. For a homogeneous problem of the form (1) (2), one may take $h = \varepsilon$ small instead of $h = 0$ and get a faster solver than the usual ones. The following pictures present a computation for the Shallow water model with $h = 10^{-8}$, using our solver with the usual Godunov scheme, for the Dambreak problem on a wet ground and on a dry ground. The height of water is presented.





The quality of the results is similar to the one obtained by using the exact Godunov scheme, which was expected. We can suggest to put either $h = \varepsilon$ or $h = -\varepsilon$ at each time step, with a centered random choice of the sign, to have homogeneous numerical model in average, but the ersults will be very similar to the previous ones. The case $h = -\varepsilon$ corresponds to a small friction term or a small upward slope and the case $h = \varepsilon$ to a small downward slope.

6 Conclusion

This study was motivated by some unstable results observed in the computational results when source terms are present. A first method preserving the balance between the flow and the source term has been proposed in [2] for the Shallow water model, using a constant piecewis bottom (that is flat on each cell) and involving the source term only in the solver where it appears as a Dirac mass. This solver was very efficient and robust, but strongly dependent on the complexity of the source term. See also [1,5,6] for others applications of this idea in the scalar case.

Another study, in [9,10] using a diphasic (air water) model for the Dambreak example on a flat bottom and with a friction term clearly shows that other solutions than the one obtained with the usual splitting methods may occur. It appears that the interface solvers based on a Riemann problem may be unadapted in some cases. As a mattrer of fact, among the hypotheses to settle the Shallow water model, one assume the atmospheric pressure to be a constant and the vertical component of the flow acceleration to be neglected. These two arguments are obviously ruled out at the first moment of a dambreak, even if they become trus later. This was an argument to look at the interface solver as a two point boundary value problem instead of an initial value problem. The solution may be continued for later times by developing a stationary state from the value named S in the solver.

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References

[1] R.Botchorishvili, B.Perthame, A.Vasseur, *Solvers which preserve equilibria for conservation laws with stiff sources,* Preprint Paris (2000).

[2] A.Chinnayya, A.Y.LeRoux, *A new general Riemann Solver for the Shallow Water Equations with Friction and Topography,* Available on the conservation law preprint server http://www.math.ntnu.no/conservation/ (1999-021).

[3] C.Ferchal, *Un nouveau modèle mathématique pour la simulation de l'impact en mer.* Rapport DEA, Université Bordeaux1 (2000).

[4] L.Gosse, *Localisation effects and measure source terms in numerical schemes for balance laws,* Preprint 1999-51 Univ. dell'Aquila, Italy.

[5] L.Gosse, A.Y.LeRoux, *Un schéma équilibre adapté aux lois de conservations scalaires non homogenes,* C.R.Acad.Sc. Paris Série 1, 323 (1996) pp 543-546.

[6] J.M.Greenberg, A.Y.LeRoux, *A well balanced scheme for the numerical processing of source terms in hyperbolic equation,* SIAM J. Numer. Anal., 33(1996)pp1-16.

[7] S.Jin, Z.Xin, *The relaxing schemes for systems of conservation laws in arbitrary space dimensions,* Comm. Pure Appl. Math. , 48 (1995)pp235-277.

[8] J.J.Stoker, *Water Waves,* Interscience Publisher, 1957.

[9] S.Vincent, *Modélisation d'écoulements incompressible de fluides non miscibles,* Thèse de Mécanique, Université Bordeaux1 (1999)

[10] S.Vincent, P.Bonneton, J.P.Caltagirone, *Numerical modelling of bore propagation and run-up on sloping beaches using a Mac-Cormack TVD scheme.* to appear in Journal of Hydraulic Research (2000).