A DIFFERENCE SCHEME FOR CONSERVATION LAWS WITH A DISCONTINUOUS FLUX - THE NONCONVEX CASE

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Abstract. In a previous work by the author, convergence was established for a simple difference scheme approximating a scalar conservation law where the flux was concave, and had a discontinuous spatially varying coefficient. The main result of this paper is an extension of that convergence theorem to the situation where the flux may have any finite number of critical points. Additionally, spatially varying source terms are allowed. The spatially varying numerical flux is also shown to satisfy maximum and minimum principles, and to be Total Variation Decreasing (TVD) in time.

Key words. conservation laws, difference approximations, discontinuous coefficients, nonconvex, source terms

AMS subject classifications. 35L65, 65M06, 65M12, 35R05

1. Introduction. This paper discusses a finite difference algorithm for the scalar Cauchy problem

$$u_t + (k(x)f(u) - a(x))_x = 0, \quad u(x,0) = u_0(x), \tag{1.1}$$

where $(x,t) \in \mathbf{R} \times \mathbf{R}^+$. It continues the investigation initiated in [19], where it was assumed that f was concave and the source term a was not present. The flux $k(x)f(u) - a(x)$ in 1.1 has a possibly discontinuous spatial dependence through the positive coefficient $k(x)$ and the source term $a(x)$, which are allowed to have jump discontinuities. A simple physical model corresponding to 1.1 is traffic flow on a highway [11], [20]. Spatial variation of the coefficient k would correspond to changes in road conditions affecting the maximum speed, and jumps in the source term a can be used to model highway entrances and exits, as in [22]. One-dimensional scalar conservation laws with discontinuous coefficients also arise in dimensional splitting methods for solving multidimensional Hamilton-Jacobi equations [7].

The assumptions concerning the data are that $u_0 \in L^1 \bigcap L^\infty \bigcap BV$ and $a, k \in L^1_{loc} \bigcap L^\infty \bigcap BV$. Here BV denotes the space of locally integrable functions w having bounded total variation, denoted $TV(w)$. Even when $a(x)$, $k(x)$ and the initial data $u_0(x)$ are smooth, solutions to 1.1 develop discontinuities after finite time, and so weak solutions are sought, which satisfy

$$\int_{\mathbf{R}\times\mathbf{R}^+} (\phi_t u + \phi_x(kf(u) - a))dxdt + \int_{\mathbf{R}} \phi(x,0)u_0(x)dx = 0 \tag{1.2}$$

for every smooth test function ϕ with compact support in $\mathbf{R} \times \mathbf{R}^+$.

When $a = 0$, the scalar conservation law 1.1 can be written as a 2x2 system,

$$u_t + (kf(u))_x = 0, \quad k_t = 0. \tag{1.3}$$

In this form, it provides a simple model of a nonstrictly hyperbolic system. The eigenvalues of the system coincide wherever f' vanishes, a situation described as resonance. Resonant systems, and in particular simple models of the form 1.3, have been addressed by a number of authors [8], [9], [10], [21] in recent years.

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Like [19], this paper approaches 1.1 as a scalar conservation law. The previous paper established convergence for a simple monotone scalar difference scheme, with the flux assumed to be strictly concave and no source term. It is well known that when k and a are constant, monotone schemes converge to the physically relevant entropy solution to the Cauchy problem, and no concavity or convexity condition on the flux is required. (See references [1], [6], [16], or the textbook [11] for the relevant theory of monotone schemes in the setting where k and a are constant.) Thus it is natural to attempt to remove the concavity condition imposed in [19]. This provides the motivation for the present paper, which extends the convergence theorem of that paper to a large class of nonconvex flux functions, and allows for the presence of spatially varying source terms. As in the previous paper, compactness is established here by bounding the total variation of a singular function, referred to as Ψ. However, the proof of the variation bound in [19] relied on the fact that the flux had a single extremum, and so a different approach is required. In this paper the variation bound follows from a finite number of discrete entropy inequalities, one for each maximal interval where the flux f is monotone. In order to guarantee that the singular mapping Ψ is invertible, the flux is assumed to be nondegenerate in the sense that it has only finitely many critical points. For example, the degenerate, but possibly important, case where f is constant on some interval, is ruled out.

The construction of the numerical scheme, which is described next, is similar to the one in [19], with the following exceptions: the case of indefinite k is not addressed here, and source terms are included. The spatial domain $\mathbf{R}$ is discretized into uniform cells $I_j = [x_j - \Delta x/2, x_j + \Delta x/2)$ centered at the points $x_j = j\Delta x$ for $j \in \mathbf{Z}$. Similarly, the time domain $\mathbf{R}^+$ is divided into time strips $I^n = [t^n, t^{n+1})$, where $t^n = n\Delta t$ for $n \in \mathbf{Z}^+$. Let $\chi_{j,n}$ be the characteristic function for the rectangle $I_j \times I^n$. For each mesh size $\Delta = \Delta x$, the finite difference scheme generates a piecewise constant approximation

$$u^\Delta(x,t) = \sum_{n\geq 0} \sum_{j=-\infty}^{+\infty} \chi_{j,n}(x,t) U_j^n.$$

Cell averages are used to discretize the initial data:

$$u^\Delta(x,0) = \sum_{j=-\infty}^{+\infty} \chi_j(x) U_j^0, \qquad U_j^0 = \frac{1}{\Delta x} \int_{I_j} u_0(x)dx,$$

where $\chi_j(x)$ is the characteristic function for the interval I_j. The approximations U_j^n are generated via

$$U_j^{n+1} = U_j^n - \lambda((k_{j+\frac12} h_{j+\frac12} - a_{j+\frac12}) - (k_{j-\frac12} h_{j-\frac12} - a_{j-\frac12})). \tag{1.4}$$

Here, $\lambda = \Delta t/\Delta x$, and $h_{j+\frac12} = h(U_{j+1}^n, U_j^n)$ is the Engquist-Osher (EO) numerical flux [3], defined by

$$h(v,u) = \frac12 (f(u) + f(v)) - \frac12 \int_u^v |f'(w)| dw. \tag{1.5}$$

The EO flux is a two point monotone (nondecreasing with respect to the second argument, nonincreasing with respect to the first) numerical flux, $h(v,u)$, consistent with f, i.e., $h(u,u) = f(u)$. Employing the notational conventions $c_+ = \max(c,0)$, $c_- = \min(c,0)$, the partial derivatives of the EO flux satisfy

$$f'_-(v) = h_v(v,u) \le 0 \le h_u(v,u) = f'_+(u),$$

from which it is evident that $\|f'\|_\infty$ is a Lipschitz constant, with respect to both arguments, for the EO flux. In [19], convergence was established for versions of the scheme based on both the Godunov and EO numerical fluxes. For the more general nonconvex flux considered here, the author has only been able to prove convergence for the EO version. The EO flux and its associated numerical entropy flux each have a close functional relationship with both the singular function Ψ, and the entropy flux associated with the convex entropy $|u-c|$. The proof of Theorem 3.2 depends on these relationships.

In the scheme 1.4, the coefficients a and k are approximated at each cell boundary, resulting in discretized versions of k and a,

$$k^\Delta(x) = \sum_{j=-\infty}^{+\infty} \chi_{j+\frac{1}{2}}(x) k_{j+\frac{1}{2}} \qquad k_{j+\frac{1}{2}} = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} k(x)dx,$$

$$a^\Delta(x) = \sum_{j=-\infty}^{+\infty} \chi_{j+\frac{1}{2}}(x) a_{j+\frac{1}{2}} \qquad a_{j+\frac{1}{2}} = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} a(x)dx.$$

Here $\chi_{j+\frac{1}{2}}(x)$ is the characteristic function for the interval $I_{j+\frac{1}{2}} = [x_j, x_{j+1})$. The discretized versions of a and k, a^Δ and k^Δ, are staggered with respect to that of u, resulting in a reduction in complexity, compared with the approach where the three discretizations are aligned.

For $a = 0$, the theory for the conservation law 1.1 is already well established for the case of concave f. In that setting, the authors of reference [9] proved existence and uniqueness, and studied the asymptotic behavior of the solution. A front tracking scheme was used to construct the solution, and a singular function was employed as an analytical tool to establish compactness. The concept of achieving compactness for scalar conservation laws via a singular mapping, which appears in both [19] and the present paper, is borrowed from [9]. Reference [8] establishes continuous dependence on the initial data u_0 and the coefficient k. The nonconvex case appears to have received less attention in the research literature. One recent contribution to the theory for the nonconvex case is reference [10]. The authors of that article constructed a front-tracking scheme, and again used a singular function, to establish existence and uniqueness for the nonconvex flux $f(u) = sin(u)$ for $u \in [-\pi, \pi]$. That problem arose as an auxiliary equation to a resonant system, which was the primary focus of their paper. Presumably, the approach in that work could be applied to other equations having a nonconvex flux.

For constant k, the equation 1.1 having spatially varying source terms,

$$u_t + (f(u) - a)_x = 0, \qquad u(x,0) = u_0(x), \tag{1.6}$$

was studied in [4]. Existence and uniqueness were established for the case of convex f, and a assumed piecewise smooth. This was accomplished by finding an explicit solution to the Riemann problem, where a jump in both a and u are allowed. The correct entropy conditions for the solution across such a jump were also found, from which it was shown that the semigroup of piecewise continuous solutions is L^1-contractive. In addition, a Godunov scheme was constructed, using the solution to the Riemann problem, and L^∞ stability was proven for the scheme. In the terminology of the present work, the scheme constructed in [4] has discretizations of u and a that are aligned.

Scalar conservation laws with spatial and temporal discontinuities in the flux have also been investigated from the point of view of Hamilton-Jacobi equations. Reference [15] applies control theory techniques to establish uniqueness and existence for the Hamilton-Jacobi equation $u_t + H(K(x,t), u_x) = 0$ where H is convex as a function of u_x, and K is allowed to be discontinuous along a finite number of curves. Via the correspondence between scalar conservation laws and Hamilton-Jacobi equations, this establishes the corresponding facts for the conservation law $u_t + H(K(x,t), u)_x$. Reference [14] provides an example of a nonconvex Hamiltonian having a discontinuous parameter, derived from a radar shape-from-shading problem. In this application, the discontinuous parameter is the radar intensity function, corresponding to the density of radar signal returns. Convergence is established for a monotone difference scheme for the Hamilton-Jacobi equation of that paper, under the assumption that the limits of certain upper and lower approximating sequences coincide.

The organization of the remainder of the paper is as follows. Section 2 discusses the monotonicity property of the scheme and a priori bounds on the numerical approximations. Section 3 provides a proof of the main result of the paper, convergence of a subsequence to a weak solution of 1.1. Section 4 presents results concerning maximum principles and variation stability for the flux $kf(u) - a$. Section 5 discusses areas for further investigation.

2. Monotonicity and L^∞ stability. As in [19], monotonicity is an important property of the algorithm. For a fixed mesh size Δ and λ, let Γ^Δ denote the time advance operator for the scheme, $\Gamma^\Delta(u^\Delta(\cdot, t^n)) = u^\Delta(\cdot, t^{n+1})$. Let $\mathcal{S}^\Delta$ denote a set of step functions of the form $s^\Delta = \sum_j \chi_j(x) S_j$, i.e., constant on the spatial mesh associated with the mesh size Δ. A finite difference scheme such as 1.4 is monotone on $\mathcal{S}^\Delta$ if

$$s_1^\Delta \le s_2^\Delta \Rightarrow \Gamma^\Delta(s_1^\Delta) \le \Gamma^\Delta(s_2^\Delta)$$

for any pair of step functions $s_1^\Delta, s_2^\Delta \in \mathcal{S}^\Delta$.

PROPOSITION 2.1. *Assume that* $\underline{a} \le a(x) \le \overline{a}$, $0 < \underline{k} \le k(x) \le \overline{k}$ *and that there are real numbers* $\overline{w} \le \widetilde{w} \le \widetilde{v} \le \overline{v}$ *such that*

1. $\widetilde{w} \le \inf_x u_0(x) \le u_0(x) \le \sup_x u_0(x) \le \widetilde{v}$ *for all* $x \in \mathbf{R}$.

2. $f \in C^1([\overline{w}, \overline{v}])$, *and* f *is monotone on both of the intervals* $I_w = [\overline{w}, \widetilde{w}]$ *and* $I_v = [\widetilde{v}, \overline{v}]$.

3. There are constants η *and* ξ *such that the equations*

$$kf(v) - a = \eta, \qquad kf(w) - a = \xi \tag{2.1}$$

have solutions $v(k,a,x) \in [\widetilde{v}, \overline{v}]$, $w(k,a,x) \in [\overline{w}, \widetilde{w}]$ *for all* $k \in [\underline{k}, \overline{k}]$ *and all* $a \in [\underline{a}, \overline{a}]$.

4. The CFL condition $\lambda \|k\|_\infty |f'(\rho)| \le 1$ *holds for all* $\rho \in [\overline{w}, \overline{v}]$.

Then,

A. The computed approximations satisfy $u^\Delta(x, t^n) \in [\overline{w}, \overline{v}]$ *for all* $n \ge 0$ *and all* $\Delta > 0$.

B. $u^\Delta(\cdot, t^n) \in L^1(\mathbf{R}) \bigcap BV(\mathbf{R})$ *for all* $n \ge 0$ *and all* $\Delta > 0$.

C. For all $n > 0$ and all $\Delta > 0$,

$$\begin{aligned} \|u^\Delta(\cdot,t^{n+1}) - u^\Delta(\cdot,t^n)\|_1 &\le \|u^\Delta(\cdot,t^n) - u^\Delta(\cdot,t^{n-1})\|_1 \\ &\le \|u^\Delta(\cdot,t^1) - u^\Delta(\cdot,t^0)\|_1 \\ &\le \{2\lambda\|k\|_\infty\|f'\|_\infty TV(u_0) + \lambda\|h\|_\infty TV(k) + \lambda TV(a)\}\Delta x. \end{aligned} \tag{2.2}$$

Proof. Suppose that f is nondecreasing on $[\widetilde{v},\overline{v}]$; the case where f is nonincreasing is similar. Let $V_j = f^{-1}((\eta + a_{j+\frac{1}{2}})/k_{j+\frac{1}{2}})$, where f^{-1} denotes the branch of f^{-1} such that $V_j \in [\widetilde{v},\overline{v}]$. Since f is nondecreasing on $[\widetilde{v},\overline{v}]$, $h(V_{j+1},V_j) = f(V_j)$, due to the upwind property of the EO flux. Thus

$$k_{j+\frac{1}{2}} h(V_{j+1},V_j) - a_{j+\frac{1}{2}} = \eta,$$

which does not depend on j. Let $v^\Delta(x) = \sum_j \chi_j(x)V_j$. It follows that v^Δ is a fixed point of the time advance mapping, i.e. $\Gamma^\Delta(v^\Delta) = v^\Delta$. Similarly, it is possible to construct a fixed point $w^\Delta = \sum_j \chi_j(x)W_j$ such that $W_j \in [\overline{w},\widetilde{w}]$. By construction,

$$\underline{w} \le w^\Delta \le u^\Delta(\cdot,0) \le v^\Delta \le \overline{v}.$$

Let $\mathcal{S}^\Delta$ be the set of step functions of the form $s^\Delta(x) = \sum_j \chi_j(x)S_j$, where $\overline{w} \le S_j \le \overline{v}$ for all j. The CFL condition guarantees monotonicity of the scheme on $\mathcal{S}^\Delta$ (see Proposition 2.1 of [19]), so

$$\underline{w} \le w^\Delta = \Gamma^\Delta(w^\Delta) \le \Gamma^\Delta(u^\Delta(\cdot,0)) \le \Gamma^\Delta(v^\Delta) = v^\Delta \le \overline{v}.$$

Thus $u^\Delta(\cdot,t^1) = \Gamma^\Delta(u^\Delta(\cdot,0))$ also lies in $[\overline{w},\overline{v}]$ and $w^\Delta \le u^\Delta(\cdot,t^1) \le v^\Delta$. (It may happen that $\widetilde{w} \le u^\Delta(\cdot,t^n) \le \widetilde{v}$ fails to hold after the first step; this does not affect the validity of the induction step.) Continuing this inductively shows that $u^\Delta(\cdot,t^n)$ lies in $[\overline{w},\overline{v}]$ for all $n \ge 0$, completing the proof of A.

To prove B., let

$$\|f\|_\infty = \max_{[\overline{w},\overline{v}]} |f(\rho)|, \qquad \|f'\|_\infty = \max_{[\overline{w},\overline{v}]} |f'(\rho)|,$$

$$\|h\|_\infty = \max\{|h(\rho_1,\rho_2)| : \rho_1,\rho_2 \in [\overline{w},\overline{v}]\}.$$

Then,

$$\begin{aligned} \|u^\Delta(\cdot,t^1) - u^\Delta(\cdot,t^0)\|_1 &\le 2\lambda\|f'\|_\infty \overline{k} TV(u_0)\Delta x \\ &\quad + \lambda\|h\|_\infty TV(k)\Delta x + \lambda TV(a)\Delta x, \end{aligned} \tag{2.3}$$

which shows that $u^\Delta(\cdot,t^1) \in L^1$. Also,

$$TV(u^\Delta(\cdot,t^1)) \le 2\|u^\Delta(\cdot,t^1) - u^\Delta(\cdot,t^0)\|_1/\Delta x + TV(u^\Delta(\cdot,t^0)),$$

which shows that $TV(u^\Delta(\cdot,t^1)) < \infty$. Continuing this process inductively yields the following pair of inequalities:

$$\begin{aligned} \|u^\Delta(\cdot,t^{n+1}) - u^\Delta(\cdot,t^n)\|_1 &\le \lambda(2\|f'\|_\infty \overline{k} TV(u(\cdot,t^n)) + \|h\|_\infty TV(k) + TV(a))\Delta x, \\ TV(u^\Delta(\cdot,t^{n+1})) &\le 2\|u^\Delta(\cdot,t^{n+1}) - u^\Delta(\cdot,t^n)\|_1/\Delta x + TV(u^\Delta(\cdot,t^n)), \end{aligned} \tag{2.4}$$

which completes the proof of B.

Finally, C. follows from the Crandall-Tartar lemma [1], [2], applied inductively to two successive time steps, $u^{\Delta}(\cdot, t^n)$ and $u^{\Delta}(\cdot, t^{n+1})$, of the approximate solutions. A slight extension of the usual version of the Crandall-Tartar lemma is required here, since the scheme 1.4 does not generally preserve the integral. In fact,

$$\int_{\mathbf{R}} u^{\Delta}(x, t^{n+1}) dx = \int_{\mathbf{R}} u^{\Delta}(x, t^n) dx - (k(x)f(0) - a(x))\Big|_{-\infty}^{+\infty}.$$

It is easily checked that the the Crandall-Tartar lemma remains valid if the integral is preserved modulo a constant (with respect to n), as is the case here.

□

The immediate reason for establishing B. in Proposition 2.1 is that $u^{\Delta}(\cdot, t^n) \in L^1$ is required for Crandall-Tartar lemma. The variation estimate was needed to prove that $u^{\Delta}(\cdot, t^n) \in L^1$. These bounds will also be required in the next section. It is clear that Proposition 2.1 only establishes $u^{\Delta}(\cdot, t^n) \in L^1 \bigcap BV$ for each fixed $\Delta > 0$, and does not provide any useful information when $\Delta \to 0$. In fact, Example 5. provides an example where $TV(u^{\Delta}(\cdot, t)) \to \infty$ as $\Delta \to 0$.

The main ingredient in establishing bounds for the approximate solutions u^{Δ} in Proposition 2.1 is the existence of fixed point solutions, above and below the initial data. This idea was previously used in [4] for the equation 1.6 with a convex flux, with similar results. (See example 4. below.)

Example 1. In the case where a and k are constant, the usual maximum and minimum principles for monotone schemes are recovered, i.e.,

$$u^{\Delta}(x, t^n) \in [\inf_x u_0(x), \sup_x u_0(x)]$$

for all x and all $n \geq 0$. Within the framework of Proposition 2.1,

$$\eta = kf(\sup_x u_0(x)) - a, \quad v(k, a, x) = \sup_x u_0(x), \quad \widetilde{v} = \overline{v} = \sup_x u_0(x),$$

$$\xi = kf(\inf_x u_0(x)) - a, \quad w(k, a, x) = \inf_x u_0(x), \quad \widetilde{w} = \overline{w} = \inf_x u_0(x).$$

Example 2. Take the case with no source terms, $a = 0$, and let $f(\widetilde{v}) \to 0$, $f(\widetilde{w}) \to 0$. In that case, the only restriction on the range of k is the CFL condition, and the maximum and minimum principles $u^{\Delta}(x, t^n) \in [\widetilde{w}, \widetilde{v}]$ are again recovered. From the point of view of Proposition 2.1, the equations 2.1 are satisfied trivially by $\eta = \xi = 0$, $v(k, a, x) = \widetilde{v}$, $w(k, a, x) = \widetilde{w}$. This condition has been assumed in a number of works, eg., [8], [9], [10], [19].

Example 3. Take the source-free equation $u_t + (kf(u))_x = 0$, with f strictly convex, symmetric about $u = 0$, and $f(0) = 0$, and assume that the initial data takes both signs. Let f_l^{-1} and f_r^{-1} denote the negative and positive branches of f^{-1}. With the notation of Propostion 2.1

$$\eta = \overline{k} f(\sup_x u_0(x)), \quad v(k, x) = f_r^{-1}\Big(\frac{\overline{k}}{k(x)} f(\widetilde{v})\Big), \quad \widetilde{v} = \sup_x u_0(x),$$

$$\xi = \overline{k} f(\inf_x u_0(x)), \quad w(k, x) = f_l^{-1}\Big(\frac{\overline{k}}{k(x)} f(\widetilde{w})\Big), \quad \widetilde{w} = \inf_x u_0(x).$$

Taking the minimum of w and the maximum of v gives

$$\overline{v} = f_r^{-1}\Big(\frac{\overline{k}}{\underline{k}} f(\widetilde{v})\Big), \quad \overline{w} = f_l^{-1}\Big(\frac{\overline{k}}{\underline{k}} f(\widetilde{w})\Big),$$

from which the following bounds are produced

$$f_l^{-1}\Big(\frac{\overline{k}}{\underline{k}} f(\inf_x u_0(x))\Big) \leq u^\Delta(x,t^n) \leq f_r^{-1}\Big(\frac{\overline{k}}{\underline{k}} f(\sup_x u_0(x))\Big)$$

for all x and $n \geq 0$. If the initial data only takes one sign, then $u = 0$ can be used for one of the bounds on u^Δ.

Example 4. With the same assumptions about f and u_0 as in the previous example, take the equation $u_t + f(u)_x = a_x$. Let $\widetilde{w} = \inf_x u_0(x) < 0$, $\widetilde{v} = \sup_x u_0(x) > 0$, and $\overline{w} = f_l^{-1}(f(\inf_x u_0(x)) + \overline{a} - \underline{a})$, $\overline{v} = f_r^{-1}(f(\sup_x u_0(x)) + \overline{a} - \underline{a})$, $\eta = f(\tilde{v}) - \underline{a}$, $\xi = f(\tilde{w}) - \underline{a}$,

$$v(a,x) = f_r^{-1}(f(\sup_x u_0(x)) + a(x) - \underline{a}), \quad w(a,x) = f_l^{-1}(f(\inf_x u_0(x)) + a(x) - \underline{a}),$$

resulting in the bounds

$$f_l^{-1}(f(\inf_x u_0(x)) + \overline{a} - \underline{a}) \leq u^\Delta(x,t^n) \leq f_r^{-1}(f(\sup_x u_0(x)) + \overline{a} - \underline{a})$$

for all x and $n \geq 0$. Applying f to this inequality gives

$$f(u^\Delta(x,t^n)) \leq f(\sup_x u_0(x)) + \overline{a} - \underline{a}, \quad f(u^\Delta(x,t^n)) \leq f(\inf_x u_0(x)) + \overline{a} - \underline{a}.$$

Combining these two inequalities yields

$$f(u^\Delta(x,t^n)) \leq \sup_x f(u_0(x)) + \overline{a} - \underline{a},$$

which is closely related to (but weaker than) the bound derived in [4] for the solution to the Cauchy problem constructed in that paper.

As a practical matter, Proposition 2.1 provides a recipe for determining a priori an allowable time step Δt for a given mesh refinement $\Delta = \Delta x$: Use 1., 2., and 3. to find $\overline{w}$ and $\overline{v}$ (assuming that they exist), then compute $\lambda = \Delta t/\Delta x$ such that the CFL condition 4. holds.

3. Convergence. This section establishes the main result of the paper, convergence of a subsequence to a weak solution of the conservation law.

THEOREM 3.1. *Let $a, k \in L^1_{loc} \bigcap BV \bigcap L^\infty$, $\underline{a} \leq a(x) \leq \overline{a}$ and $0 < \underline{k} \leq k(x) \leq \overline{k}$. Assume that the conditions stated in Proposition 2.1 are satisfied. Let $f \in C^1([\overline{w}, \overline{v}])$ with finitely many critical points. Let $u_0 \in BV \bigcap L^1 \bigcap L^\infty$, $\widetilde{w} \leq u_0(x) \leq \widetilde{v}$. Let the mesh size $\Delta \to 0$ with λ fixed and satisfying the CFL condition stated in Proposition 2.1, resulting in the sequence of approximations u^Δ. Then, there is a weak solution u of 1.2 and a subsequence u^{Δ_i} such that $u^{\Delta_i} \to u$ a.e. and in $L^1_{loc}(\mathbf{R} \times [0,\infty))$. The limit solution u satisfies the bounds $\overline{w} \leq u(x,t) \leq \overline{v}$ a.e.*

The strategy for establishing Theorem 3.1 is basically the same, with some modifications where necessary, as was used to prove Theorem 3.2 of [19]. The proof of that theorem was patterned after standard compactness arguments as found in [16] and [17], with the exception that the compactness argument in [19] was applied to the transformed quantity z^Δ, rather than the conserved variable u^Δ. Once compactness was established for z^Δ, the variable of ultimate interest, u, was recovered by inverting the singular mapping Ψ. The same approach is used in what follows.

By way of establishing notation, let Δ_+ and Δ_- denote forward and backward difference operators. For example, $\Delta_+ h_{j-\frac{1}{2}} = \Delta_- h_{j+\frac{1}{2}} = h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}$. Let $\sigma(w) = w/|w|$, for $w \neq 0$, and $\sigma(w) = 0$ if $w = 0$. Let $\Omega_0 = [\overline{w}, \overline{v}]$, and $|\Omega_0| = \overline{v} - \overline{w}$, where $\overline{w}$ and $\overline{v}$ are the bounds referred to in Proposition 2.1 . Finally, let $\chi_r(w;c)$ be the characteristic function for the interval $[c, +\infty)$, and let $\chi_l(w;c)$ be the characteristic function for $(-\infty, c]$.

One other preliminary matter is the convergence of the various infinite sums over the domain $j \in \mathbf{Z}$ which will appear in what follows. Absolute convergence of these sums in each case results ultimately from the bounds $TV(a^\Delta) < \infty$, $TV(k^\Delta) < \infty$, and $\sum_j |U_j^n|\Delta x < \infty$, $\sum_j |U_{j+1}^n - U_j^n| < \infty$. The last two of these inequalities are 3. of Proposition 2.1

As discussed in [19], a singular mapping has been used by a number of authors, (eg., [9], [12], [13], [18]) to establish compactness for resonant systems. For the case of f concave with a single critical point located at u^*, the singular function of [9], $\Psi(u,k) = (k/f(u^*)) \int_{u^*}^u |f'(w)| dw$, due originally (in a somewhat different form) to Temple, was used in [19] to prove compactness. In this paper, the following version of Ψ will be used:

$$\Psi(u,k,a) = k \int_0^u |f'(w)| dw - a.$$

The normalizing factor f^* has been dropped and the source term a has been added. It is clear that since $k > 0$ and $|f'| > 0$ a.e., $\Psi(u,k,a)$ is strictly increasing with respect to u. Zero is chosen as the lower limit of integration in the formula for Ψ purely for convenience; any real number within $[\overline{w}, \overline{v}]$ would work. With $u_0 \in L^1$, $[\overline{w}, \overline{v}]$ must contain the origin, possibly as an endpoint.

Since f has only finitely many critical points, there are finitely many points $\overline{w} = u_0^* < u_1^* < \cdots < u_m^* < u_{m+1}^* = \overline{v}$ such that f is strictly monotone in $(u_\nu^*, u_{\nu+1}^*)$, for $0 \le \nu \le m$, and each of $u_1^*, \cdots, u_m^*$ is an extremum of f. Obviously, $\{u_1^*, \ldots, u_m^*\}$ is possibly only a subset of the critical points of f; for the purpose of this analysis, those which do not correspond to extrema are simply ignored. Without loss of generality, assume that f is strictly increasing in $(\overline{w}, u_1^*)$, so that

$$u \in (u_i^*, u_{i+1}^*) \Rightarrow \sigma(f'(u)) = (-1)^i.$$

The case where f is strictly decreasing in $(\overline{w}, u_1^*)$ is completely analogous.

The following Lipschitz continuity relationships in u and k follow directly from the definition of Ψ and the assumptions about the flux f:

$$|\Psi(u,k_1,a) - \Psi(u,k_2,a)| \le |\Omega_0| \|f'\|_\infty |k_1 - k_2|, \tag{3.1}$$

$$|\Psi(u_1,k,a) - \Psi(u_2,k,a)| \le \|k\|_\infty \|f'\|_\infty |u_1 - u_2| \tag{3.2}$$

$$|\Psi(u,k,a_1) - \Psi(u,k,a_2)| \le |a_1 - a_2|. \tag{3.3}$$

Let $z^\Delta(x,t) = \Psi(u^\Delta(x,t), k^\Delta(x), a^\Delta(x))$. This is the sequence of functions for which compactness will be proven.

THEOREM 3.2. *Let $z^n(x) = z^\Delta(x,t^n)$. With the assumptions stated in Theorem 3.1, $TV(z^n)$ is bounded uniformly, for all $n \ge 0$, and all $\Delta > 0$.*

The proof proceeds by a series of lemmas. The first four lemmas lead up to the pair of inequalities stated in Lemma 3.6, which can be viewed as a pair of discrete entropy inequalities, parameterized by c. The singular mapping is then decomposed into components with support limited to maximal segments where f is monotone. The next two lemmas (3.7 and 3.8) then use the entropy inequalities to show that each of those components of the singular function has bounded variation. Only a finite set of entropy inequalities are used; specifically, those resulting from values of the parameter c located at critical points corresponding to extrema of f.

LEMMA 3.3. *Let $c \in \mathbf{R}$. For the convex entropy pair $V(u) = |u - c|$, $F(u) = \sigma(u - c)(f(u) - f(c))$, and W_j computed using the EO numerical flux via*

$$W_j = U_j - \lambda k_{j+\frac{1}{2}} \Delta_+ h_{j+\frac{1}{2}}, \tag{3.4}$$

the following discrete entropy inequality holds

$$V(W_j) \leq V(U_j) - \lambda k_{j+\frac{1}{2}} \Delta_+ H_{j-\frac{1}{2}}, \tag{3.5}$$

where the numerical entropy flux is defined by

$$H_{j+\frac{1}{2}} = \frac{1}{2}(F(U_j) + F(U_{j+1})) - \frac{1}{2}\int_{U_j}^{U_{j+1}} \sigma(f'(w))\sigma(w - c)f'(w)dw. \tag{3.6}$$

Proof. The proof is an application of the numerical entropy flux constructed in the proof of Proposition 4.1 of [1]. Specifically, the numerical entropy flux

$$k_{j+\frac{1}{2}} H(U_{j+1}, U_j) = k_{j+\frac{1}{2}} h(U_{j+1} \vee c, U_j \vee c) - k_{j+\frac{1}{2}} h(U_{j+1} \wedge c, U_j \wedge c) \tag{3.7}$$

is applied to the EO flux $k_{j+\frac{1}{2}} h$ consistent with the flux $k_{j+\frac{1}{2}} f$. Here $a \vee b = \max(a, b)$, $a \wedge b = \min(a, b)$. Dropping the multiplier $k_{j+\frac{1}{2}}$, formula 3.7 gives

$$\begin{aligned} H(U_{j+1}, U_j) = {} & \frac{1}{2}(f(U_j \vee c) - f(U_j \wedge c)) + \frac{1}{2}(f(U_{j+1} \vee c) - f(U_{j+1} \wedge c)) \\ & - \frac{1}{2}\int_{U_j \vee c}^{U_{j+1} \vee c} |f'(w)|dw + \frac{1}{2}\int_{U_j \wedge c}^{U_{j+1} \wedge c} |f'(w)|dw \end{aligned} \tag{3.8}$$

The formula 3.6 for the EO entropy flux then follows from the identities

$$f(U \vee c) - f(U \wedge c) = \sigma(U - c)(f(U) - f(c)) = F(U),$$

and

$$-\int_{U \vee c}^{V \vee c} |f'(w)|dw + \int_{U \wedge c}^{V \wedge c} |f'(w)|dw = -\int_{U}^{V} \sigma(f'(w))\sigma(w - c)f'(w)dw,$$

which are readily verified by considering the three cases corresponding to whether c is less than, greater than, or between, U and V.

□

LEMMA 3.4. *For the EO flux and its associated numerical entropy flux 3.6 consistent with entropy flux $F(u) = \sigma(u - c)(f(u) - f(c))$, the following identities are satisfied:*

$$\frac{1}{2}(\Delta_+ H_{j-\frac{1}{2}} + \Delta_+ h_{j-\frac{1}{2}}) = \int_{U_j^n}^{U_{j+1}^n} \chi_r(w; c) f'_-(w)dw + \int_{U_{j-1}^n}^{U_j^n} \chi_r(w; c) f'_+(w)dw, \tag{3.9}$$

$$\frac{1}{2}(\Delta_+ H_{j-\frac{1}{2}} - \Delta_+ h_{j-\frac{1}{2}}) = -\int_{U_j^n}^{U_{j+1}^n} \chi_l(w;c) f'_-(w)dw - \int_{U_{j-1}^n}^{U_j^n} \chi_l(w;c) f'_+(w)dw. \tag{3.10}$$

Proof. The definition of the EO flux 1.5 and its associated discrete entropy flux 3.6 yield the identities

$$\Delta_+ H_{j-\frac{1}{2}} = \int_{U_j^n}^{U_{j+1}^n} \sigma(w-c) f'_-(w)dw + \int_{U_{j-1}^n}^{U_j^n} \sigma(w-c) f'_+(w)dw, \tag{3.11}$$

$$\Delta_+ h_{j-\frac{1}{2}} = \int_{U_j^n}^{U_{j+1}^n} f'_-(w)dw + \int_{U_{j-1}^n}^{U_j^n} f'_+(w)dw. \tag{3.12}$$

Adding 3.12 to 3.11, then dividing by two, and using the identities

$$\frac{1}{2}(\sigma(w-c)+1) = \chi_r(w;c), \quad \frac{1}{2}(\sigma(w-c)-1) = -\chi_l(w;c) \tag{3.13}$$

gives 3.9. Similarly, 3.10 results by subtracting 3.12 from 3.11, then dividing by two. ∎

LEMMA 3.5. *With the assumptions stated in Theorem 3.1, the following relationships hold:*

$$U_j^{n+1} = U_j^n - \lambda k_{j+\frac{1}{2}} \Delta_+ h_{j-\frac{1}{2}} + C_j^n \Delta_+ k_{j-\frac{1}{2}} + \lambda \Delta_+ a_{j-\frac{1}{2}}, \quad |C_j^n| \le \lambda \|h\|_\infty, \tag{3.14}$$

$$V(U_j^{n+1}) \le V(U_j^n) - \lambda k_{j+\frac{1}{2}} \Delta_+ H_{j-\frac{1}{2}} + \lambda \|h\|_\infty |\Delta_+ k_{j-\frac{1}{2}}| + \lambda |\Delta_+ a_{j-\frac{1}{2}}|. \tag{3.15}$$

Proof. The equation in 3.14 is just 1.4 with $C_j^n = -\lambda h_{j-\frac{1}{2}}$. The inequality follows from

$$|C_j^n| \le \lambda |h_{j-\frac{1}{2}}| \le \lambda \|h\|_\infty \tag{3.16}$$

For 3.15, with W_j defined by 3.4, the discrete entropy inequality 3.5 gives

$$\begin{aligned} V(U_j^{n+1}) &\le V(U_j^n) - \lambda k_{j+\frac{1}{2}} \Delta_+ H_{j-\frac{1}{2}} + (V(U_j^{n+1}) - V(W_j)) \\ &= V(U_j^n) - \lambda k_{j+\frac{1}{2}} \Delta_+ H_{j-\frac{1}{2}} + |U_j^{n+1} - c| - |W_j - c|. \end{aligned}$$

Inequality 3.15 then follows from

$$|U_j^{n+1} - c| - |W_j - c| \le |U_j^{n+1} - W_j| \le \lambda \|h\|_\infty |\Delta_+ k_{j-\frac{1}{2}}| + \lambda |\Delta_+ a_{j-\frac{1}{2}}|.$$

∎

LEMMA 3.6. *With the assumptions stated in Theorem 3.1, the following inequalities are valid:*

$$\begin{aligned} & k_{j+\frac{1}{2}} \int_{U_j^n}^{U_{j+1}^n} \chi_r(w;c) f'_-(w)dw + k_{j-\frac{1}{2}} \int_{U_{j-1}^n}^{U_j^n} \chi_r(w;c) f'_+(w)dw \\ \le \; & \frac{1}{\lambda}(U_j^n - U_j^{n+1})_+ + (\|h\|_\infty + \|f'\|_\infty |\Omega_0|)|\Delta_+ k_{j-\frac{1}{2}}| + |\Delta_+ a_{j-\frac{1}{2}}|, \end{aligned} \tag{3.17}$$

$$-k_{j+\frac{1}{2}}\int_{U_j^n}^{U_{j+1}^n}\chi_l(w;c)f'_-(w)dw - k_{j-\frac{1}{2}}\int_{U_{j-1}^n}^{U_j^n}\chi_l(w;c)f'_+(w)dw$$

$$\leq \frac{-1}{\lambda}(U_j^n - U_j^{n+1})_- + (\|h\|_\infty + \|f'\|_\infty|\Omega_0|)|\Delta_+ k_{j-\frac{1}{2}}| + |\Delta_+ a_{j-\frac{1}{2}}|. \tag{3.18}$$

Proof. Rearranging 3.15 and dividing by λ results in

$$\begin{aligned} k_{j+\frac{1}{2}}\Delta_+ H_{j-\frac{1}{2}} &\leq \frac{1}{\lambda}(V(U_j^n) - V(U_j^{n+1})) + \|h\|_\infty|\Delta_+ k_{j-\frac{1}{2}}| + |\Delta_+ a_{j-\frac{1}{2}}| \\ &= \frac{1}{\lambda}(|U_j^n - c| - |U_j^{n+1} - c|) + \|h\|_\infty|\Delta_+ k_{j-\frac{1}{2}}| + |\Delta_+ a_{j-\frac{1}{2}}| \\ &\leq \frac{1}{\lambda}|U_j^n - U_j^{n+1}| + \|h\|_\infty|\Delta_+ k_{j-\frac{1}{2}}| + |\Delta_+ a_{j-\frac{1}{2}}|. \end{aligned} \tag{3.19}$$

Adding 3.19 to the following rearrangement of 3.14,

$$k_{j+\frac{1}{2}}\Delta_+ h_{j-\frac{1}{2}} = \frac{1}{\lambda}(U_j^n - U_j^{n+1}) + \frac{1}{\lambda}C_j^n\Delta_+ k_{j-\frac{1}{2}} + \Delta_+ a_{j-\frac{1}{2}}, \tag{3.20}$$

dividing by two, then applying 3.9 yields

$$k_{j+\frac{1}{2}}\int_{U_j^n}^{U_{j+1}^n}\chi_r(w;c)f'_-(w)dw + k_{j+\frac{1}{2}}\int_{U_{j-1}^n}^{U_j^n}\chi_r(w;c)f'_+(w)dw$$

$$\leq \frac{1}{\lambda}(U_j^n - U_j^{n+1})_+ + \|h\|_\infty|\Delta_+ k_{j-\frac{1}{2}}| + |\Delta_+ a_{j-\frac{1}{2}}|. \tag{3.21}$$

Similarly, subtracting 3.20 from 3.19, dividing by two, then applying 3.10 gives

$$-k_{j+\frac{1}{2}}\int_{U_j^n}^{U_{j+1}^n}\chi_l(w;c)f'_-(w)dw - k_{j+\frac{1}{2}}\int_{U_{j-1}^n}^{U_j^n}\chi_l(w;c)f'_+(w)dw$$

$$\leq \frac{-1}{\lambda}(U_j^n - U_j^{n+1})_- + \|h\|_\infty|\Delta_+ k_{j-\frac{1}{2}}| + |\Delta_+ a_{j-\frac{1}{2}}|. \tag{3.22}$$

The second term on the left in 3.17 results from the corresponding term in 3.21:

$$\begin{aligned} k_{j-\frac{1}{2}}\int_{U_{j-1}^n}^{U_j^n}\chi_r(w;c)f'_+(w)dw &= (k_{j+\frac{1}{2}} - \Delta_+ k_{j-\frac{1}{2}})\int_{U_{j-1}^n}^{U_j^n}\chi_r(w;c)f'_+(w)dw \\ &\leq k_{j+\frac{1}{2}}\int_{U_{j-1}^n}^{U_j^n}\chi_r(w;c)f'_+(w)dw + |\Delta_+ k_{j-\frac{1}{2}}|\left|\int_{U_{j-1}^n}^{U_j^n}|f'(w)|dw\right|. \end{aligned}$$

Then, since U_j^n remains within the invariant domain $\Omega_0 = [\overline{w}, \overline{v}]$, 3.17 follows from

$$\left|\int_{U_{j-1}^n}^{U_j^n}|f'(w)|dw\right| \leq \|f'\|_\infty|U_{j+1}^n - U_j^n| \leq \|f'\|_\infty|\Omega_0|.$$

The inequality 3.18 follows in a similar way from 3.22. $\square$

Let $\Phi(u) = (\Psi(u,k,a) + a)/k = \int_0^u |f'(w)|dw$. For $\nu = 0, 1, \ldots, m$, let $\chi^\nu(u)$ be the characteristic function of the interval $[u_\nu^*, u_{\nu+1}^*)$. Decompose Φ as follows:

$$\Phi(u) = \sum_{\nu=0}^{m}\phi^\nu(u), \qquad \phi^\nu(u) = \int_0^u \chi^\nu(w)|f'(w)|dw.$$

LEMMA 3.7. *Let E^i_+ (O^i_+) denote the even (odd) integers ν with $0 \le i \le \nu \le m$, and let E^i_- (O^i_-) denote the even (odd) integers ν with $0 \le \nu < i < m$. For each $i = 1, \ldots, m$, the following pair of inequalities holds:*

$$k_{j+\frac{1}{2}} \sum_{\nu \in O^i_+} (\phi^\nu(U^n_j) - \phi^\nu(U^n_{j+1})) + k_{j-\frac{1}{2}} \sum_{\nu \in E^i_+} (\phi^\nu(U^n_j) - \phi^\nu(U^n_{j-1}))$$

$$\le \frac{1}{\lambda}(U^n_j - U^{n+1}_j)_+ + (\|h\|_\infty + \|f'\|_\infty |\Omega_0|)|\Delta_+ k_{j-\frac{1}{2}}| + |\Delta_+ a_{j-\frac{1}{2}}|, \tag{3.23}$$

$$k_{j+\frac{1}{2}} \sum_{\nu \in O^i_-} (\phi^\nu(U^n_{j+1}) - \phi^\nu(U^n_j)) + k_{j-\frac{1}{2}} \sum_{\nu \in E^i_-} (\phi^\nu(U^n_{j-1}) - \phi^\nu(U^n_j))$$

$$\le \frac{-1}{\lambda}(U^n_j - U^{n+1}_j)_- + (\|h\|_\infty + \|f'\|_\infty |\Omega_0|)|\Delta_+ k_{j-\frac{1}{2}}| + |\Delta_+ a_{j-\frac{1}{2}}|. \tag{3.24}$$

Proof. For each $i = 1, \ldots, m$, substitute $c = u^*_i$ into 3.17 and 3.18. With the assumption that f is increasing in $(\overline{w}, u^*_1)$, it is clear that $f'_+ = 0$ in the odd-numbered intervals, and $f'_- = 0$ in the even-numbered intervals. Thus,

$$\int_{U^n_j}^{U^n_{j+1}} \chi_r(w; u^*_i) f'_-(w)dw = \sum_{\nu \in O^i_+} \int_{U^n_j}^{U^n_{j+1}} \chi^\nu(w) f'_-(w)dw = -\sum_{\nu \in O^i_+} \Delta_+ \phi^\nu(U^n_j), \tag{3.25}$$

$$\int_{U^n_{j-1}}^{U^n_j} \chi_r(w; u^*_i) f'_+(w)dw = \sum_{\nu \in E^i_+} \int_{U^n_{j-1}}^{U^n_j} \chi^\nu(w) f'_+(w)dw = \sum_{\nu \in E^i_+} \Delta_- \phi^\nu(U^n_j), \tag{3.26}$$

$$\int_{U^n_j}^{U^n_{j+1}} \chi_l(w; u^*_i) f'_-(w)dw = \sum_{\nu \in O^i_-} \int_{U^n_j}^{U^n_{j+1}} \chi^\nu(w) f'_-(w)dw = -\sum_{\nu \in O^i_-} \Delta_+ \phi^\nu(U^n_j), \tag{3.27}$$

$$\int_{U^n_{j-1}}^{U^n_j} \chi_l(w; u^*_i) f'_+(w)dw = \sum_{\nu \in E^i_-} \int_{U^n_{j-1}}^{U^n_j} \chi^\nu(w) f'_+(w)dw = \sum_{\nu \in E^i_-} \Delta_- \phi^\nu(U^n_j). \tag{3.28}$$

The ith version of 3.17 yields the corresponding version of 3.23, using the definition of the ϕ^ν, along with the pair of identities 3.25, 3.26. The ith version of 3.24 follows in the same way from 3.18, 3.27, and 3.28. ∎

LEMMA 3.8. *For each $i = 0, \ldots, m$, there is a constant $E^i \ge 0$ such that*

$$\sum_j k_{j+\frac{1}{2}} |\phi^i(U^n_{j+1}) - \phi^i(U^n_j)| \le E^i, \tag{3.29}$$

where E^i is independent of Δ and n.

Proof. Starting from the inequality 3.24, with $i = 1$, gives

$$\begin{aligned} k_{j-\frac{1}{2}}(\phi^0(U^n_{j-1}) - \phi^0(U^n_j)) &\le \frac{-1}{\lambda}(U^n_j - U^{n+1}_j)_- \\ &\quad + (\|h\|_\infty + \|f'\|_\infty |\Omega_0|)|\Delta_+ k_{j-\frac{1}{2}}| + |\Delta_+ a_{j-\frac{1}{2}}|, \end{aligned} \tag{3.30}$$

since only the $\nu = 0$ term is nonzero in this case. Since the right side of 3.30 is nonnegative, it follows that

$$
\begin{aligned}
-\sum_j k_{j-\frac{1}{2}}(\phi^0(U_j^n) - \phi^0(U_{j-1}^n))_- \\
&\leq \sum_j \Big(\frac{-1}{\lambda}(U_j^n - U_j^{n+1})_- + (\|h\|_\infty + \|f'\|_\infty|\Omega_0|)|\Delta_+ k_{j-\frac{1}{2}}| + |\Delta_+ a_{j-\frac{1}{2}}|\Big) \\
&\leq \frac{1}{\lambda}\sum_j |U_j^n - U_j^{n+1}| + (\|h\|_\infty + \|f'\|_\infty|\Omega_0|)TV(k) + TV(a) \\
&\leq \frac{1}{\lambda}\sum_j |U_j^1 - U_j^0| + (\|h\|_\infty + \|f'\|_\infty|\Omega_0|)TV(k) + TV(a).
\end{aligned}
$$

(3.31)

Using summation by parts gives, for each ν,

$$
\begin{aligned}
\sum_j k_{j+\frac{1}{2}}|\phi^\nu(U_{j+1}^n) - \phi^\nu(U_j^n)| &= \sum_j k_{j+\frac{1}{2}}(\Delta_+\phi^\nu(U_j^n))_+ - \sum_j k_{j+\frac{1}{2}}(\Delta_+\phi^\nu(U_j^n))_- \\
&= 2\sum_j k_{j+\frac{1}{2}}(\Delta_+\phi^\nu(U_j^n))_+ - \sum_j k_{j+\frac{1}{2}}\Delta_+\phi^\nu(U_j^n) \\
&= 2\sum_j k_{j+\frac{1}{2}}(\Delta_+\phi^\nu(U_j^n))_+ + \sum_j \phi^\nu(U_j^n)\Delta_+ k_{j-\frac{1}{2}} - \sum_j \Delta_+(k_{j+\frac{1}{2}}\phi^\nu(U_j^n)) \\
&= 2\sum_j k_{j+\frac{1}{2}}(\Delta_+\phi^\nu(U_j^n))_+ + \sum_j \phi^\nu(U_j^n)\Delta_+ k_{j-\frac{1}{2}} \\
&\leq 2\sum_j k_{j+\frac{1}{2}}(\Delta_+\phi^\nu(U_j^n))_+ + |\Omega_0|\|f'\|_\infty TV(k).
\end{aligned}
$$

(3.32)

Similarly,

$$
\sum_j k_{j+\frac{1}{2}}|\phi^\nu(U_{j+1}^n) - \phi^\nu(U_j^n)| \leq -2\sum_j k_{j+\frac{1}{2}}(\Delta_+\phi^\nu(U_j^n))_- + |\Omega_0|\|f'\|_\infty TV(k).
$$

(3.33)

From 3.31 and 3.33, it follows that

$$
\sum_j k_{j+\frac{1}{2}}|\phi^0(U_{j+1}^n) - \phi^0(U_j^n)| \leq E^0,
$$

where

$$
E^0 = \frac{2}{\lambda}\sum_j |U_j^1 - U_j^0| + (2\|h\|_\infty + 3\|f'\|_\infty|\Omega_0|)TV(k) + 2TV(a).
$$

Now taking 3.24 with $i = 2$, since only contributions from $\nu = 0, 1$ are present, gives

$$
\begin{aligned}
k_{j+\frac{1}{2}}(\phi^1(U_{j+1}^n) - \phi^1(U_j^n)) + k_{j-\frac{1}{2}}(\phi^0(U_{j-1}^n) - \phi^0(U_j^n)) \\
&\leq \frac{-1}{\lambda}(U_j^n - U_j^{n+1})_- + (\|h\|_\infty + \|f'\|_\infty|\Omega_0|)|\Delta_+ k_{j-\frac{1}{2}}| + |\Delta_+ a_{j-\frac{1}{2}}|.
\end{aligned}
$$

It follows that

$$k_{j+\frac{1}{2}}(\phi^1(U_{j+1}^n) - \phi^1(U_j^n))_+ \leq k_{j-\frac{1}{2}}|\phi^0(U_j^n) - \phi^0(U_{j-1}^n)| + \frac{-1}{\lambda}(U_j^n - U_j^{n+1})_- + (\|h\|_\infty + \|f'\|_\infty|\Omega_0|)|\Delta_+ k_{j-\frac{1}{2}}| + |\Delta_+ a_{j-\frac{1}{2}}|. \tag{3.34}$$

Combining 3.32 and 3.34 yields

$$\begin{aligned}
\sum_j k_{j+\frac{1}{2}}|\phi^1(U_{j+1}^n) - \phi^1(U_j^n)| &\leq 2\sum_j k_{j+\frac{1}{2}}(\Delta_+\phi^1(U_j^n))_+ + |\Omega_0|\|f'\|_\infty TV(k) \\
&\leq 2\sum_j k_{j-\frac{1}{2}}|\Delta_+\phi^0(U_{j-1}^n)| - \frac{2}{\lambda}\sum_j (U_j^n - U_j^{n-1})_- \\
&+ 2\sum_j (\|h\|_\infty + \|f'\|_\infty|\Omega_0|)|\Delta_+ k_{j-\frac{1}{2}}| + |\Omega_0|\|f'\|_\infty TV(k) + 2TV(a) \\
&\leq 2E^0 + \frac{2}{\lambda}\sum_j |U_j^n - U_j^{n+1}| + (2\|h\|_\infty + 3\|f'\|_\infty|\Omega_0|)TV(k) + 2TV(a) \\
&\leq 3E^0 \equiv E^1.
\end{aligned}$$

Continuing this way inductively from left to right (increasing i) with 3.24 gives, for each $i = 0, \ldots, m-1$

$$\sum_j k_{j+\frac{1}{2}}|\phi^i(U_{j+1}^n) - \phi^i(U_j^n)| \leq 2(E^0 + \cdots + E^{i-1}) + E^0 \equiv E^i.$$

This leaves only ϕ^m to deal with. Starting from the inequality 3.23, with $i = m$, and proceeding as in the case where $i = 0$, gives

$$\sum_j k_{j+\frac{1}{2}}|\phi^m(U_{j+1}^n) - \phi^m(U_j^n)| \leq E^0 \equiv E^m.$$

$\square$

It is now possible to prove Theorem 3.2.

Proof. Taking into account the jumps in z^n at the cell boundaries (due to jumps in u^Δ) and the jumps at the cell centers (due to jumps in k^Δ and a^Δ), the total variation of $z^n(x)$ is

$$TV(z) = \sum_j |\Delta_+^u z_j^n| + \sum_j |\Delta_+^{k,a} z_{j-\frac{1}{2}}^n|, \tag{3.35}$$

where

$$\begin{aligned}
\Delta_+^u z_j^n &= \Psi(U_{j+1}^n, k_{j+\frac{1}{2}}, a_{j+\frac{1}{2}}) - \Psi(U_j^n, k_{j+\frac{1}{2}}, a_{j+\frac{1}{2}}), \\
\Delta_+^{k,a} z_{j-\frac{1}{2}}^n &= \Psi(U_j^n, k_{j+\frac{1}{2}}, a_{j+\frac{1}{2}}) - \Psi(U_j^n, k_{j-\frac{1}{2}}, a_{j-\frac{1}{2}}).
\end{aligned}$$

The second sum in 3.35, due to jumps in k^Δ and a^Δ, is bounded by $|\Omega_0|\|f'\|_\infty TV(k) + TV(a)$, as is seen from the Lipschitz continuity estimates 3.1 and 3.3. For the first sum,

$$\sum_j |\Delta_+^u z_j^n| = \sum_j k_{j+\frac{1}{2}}|\Phi(U_{j+1}^n) - \Phi(U_j^n)|$$

$$= \sum_j k_{j+\frac{1}{2}} \Big| \sum_{i=0}^{m} (\phi^i(U_{j+1}^n) - \phi^i(U_j^n)) \Big|$$
$$\le \sum_{i=0}^{m} \sum_j k_{j+\frac{1}{2}} |\phi^i(U_{j+1}^n) - \phi^i(U_j^n)| \le \sum_{i=0}^{m} E^i.$$

□

The proof of Theorem 3.1 follows. It is similar in outline to the proof of Theorem 3.2 of [19], but is necessarily somewhat different, due to the absence of concavity, and the presence of source terms.

Proof. The computed solutions u^Δ remain within $[\overline{w}, \overline{v}]$, thanks to the CFL condition and Proposition 2.1. Fix $B > 0$. The following estimates

$$\|z^\Delta(\cdot,t)\|_\infty \le \overline{k}\|f'\|_\infty|\Omega_0| + \|a\|_\infty, \quad \int_{-B}^{+B} |z^\Delta(x,t)|dx \le 2B(\overline{k}\|f'\|_\infty|\Omega_0| + \|a\|_\infty)$$

which hold uniformly for $t \ge 0$, $\Delta > 0$, prove that for each $t \ge 0$, $z^\Delta(\cdot,t) \in L^\infty$, and $z^\Delta(\cdot,t) \in L^1_{loc}(\mathbf{R})$. Theorem 3.2 provides a uniform bound on $TV(z^\Delta(\cdot,t))$. The following time-continuity estimate is a direct consequence of 2.2 of Proposition 2.1, along with the Lipschitz continuity relationship 3.2:

$$\|z^\Delta(\cdot,t+\tau) - z^\Delta(\cdot,t)\|_1 \le C(|\tau| + \Delta).$$

Here the constant C is independent of Δ and t. Using the same argument as in [19], there is a subsequence denoted z^{Δ_i}, which converges in $L^1_{loc}(\mathbf{R}\times\mathbf{R}^+)\bigcap L^\infty$ and pointwise a.e. to some function $z \in L^1_{loc}(\mathbf{R}\times\mathbf{R}^+)\bigcap L^\infty$. For fixed (x,t), let $u(x,t) = \Psi^{-1}(z(x,t),k(x),a(x))$. Due to the strict monotonicity of $\Psi(\cdot,k,a)$, the function u is well-defined a.e., $u \in [\overline{w},\overline{v}]$ a.e., and $u \in L^1_{loc}(\mathbf{R}\times\mathbf{R}^+)\bigcap L^\infty$. The proof is completed by dropping the subscript on Δ, and showing that $u^\Delta \to u$ a.e. and in $L^1_{loc}(\mathbf{R}\times\mathbf{R}^+)$. It is clear from its definition that Φ is strictly increasing and that Ψ and Φ are related via $\Psi^{-1}(z,k,a) = \Phi^{-1}((z+a)/k)$. Fix $T > 0$. Using the identities $u = \Phi^{-1}((z+a)/k)$, $u^\Delta = \Phi^{-1}((z^\Delta + a^\Delta)/k^\Delta)$ gives

$$\int_0^T \int_{-B}^{B} |u - u^\Delta| dxdt = \int_0^T \int_{-B}^{B} |\Phi^{-1}((z+a)/k) - \Phi^{-1}((z^\Delta + a^\Delta)/k^\Delta)| dxdt. \tag{3.36}$$

Since $a,k \in BV$, $(a^\Delta,k^\Delta) \to (a,k)$ pointwise a.e., and so $(z^\Delta + a^\Delta)/k^\Delta \to (z+a)/k$ a.e. It follows from the continuity of Φ^{-1} that $u^\Delta \to u$ a.e. The bounded convergence theorem then guarantees that the integral in 3.36 converges to zero as Δ approaches zero. Proving that the limit u is a weak solution of 1.1 requires a version of the Lax-Wendroff theorem. The statement and proof of Theorem 3.6 of [19] are applicable here with only minor modifications. □

Example 5. Consider the Cauchy problem for $u_t + (k(x)f(u))_x = 0$ with $f(u) = 1-u^2$, initial data $u_0(x) = 0$, and the coefficient $k(x) = \underline{k}+1/n = k_n$ for $x \in [n-1,n)$, $n \ge 1$; and $k(x) = 1 + \underline{k}$ for $x \le 1$. Here $\underline{k} > 0$. Then $k(x)$ and u_0 satisfy the regularity conditions of Theorem 3.1. Also, f has zeros at -1 and $+1$, so the solution remains in $[-1,1]$. For small time, the solution can be found by solving the Riemann problems that arise at the jumps in k. Each Riemann problem gives rise to a shock originating from the jump in k located at $x = n$, and traveling to the left with speed

$s_n = -k_n\sqrt{1-(k_{n+1}/k_n)}$. The value of u to the right of the shock and the left of $x = n$ is $u_n = \sqrt{1-(k_{n+1}/k_n)} = -s_n/k_n$. The solution at a small positive time t can be pictured as a sequence of rectangles with their bases touching the x-axis, and right sides touching the vertical lines $x = n$. The rectangle whose right edge is located at $x = n$ has height u_n and width $s_n t$. This solution satisfies the geometric entropy conditions in [19]. For this problem $\Psi(u,k) = k\sigma(u)u^2$. For x at the rectangle, $\Psi(u,k) = \Psi(u_n,k_n) = k_n u_n^2 = (k_n - k_{n+1})$. To the left and right of the rectangle, $\Psi(u,k) = 0$. There is a finite time $t_0 > 0$ such that no wave interactions occur until $t = t_0$, so for $0 < t < t_0$,

$$TV(u(\cdot,t)) = 2\sum_{n>0} u_n = 2\sum_{n>0}\frac{1}{n\sqrt{(1+1/n)(\underline{k}+1/n)}} = \infty,$$
$$\int_{\mathbf{R}} |u(x,t)|dx = t\sum_{n>0} u_n|s_n| = t\sum_{n>0} k_n(1-\frac{k_{n+1}}{k_n}) = t\sum_{n>0}\frac{1}{n(n+1)} < \infty,$$
$$TV(\Psi(u(\cdot,t),k(\cdot))) = 2\sum_{n>0}(k_n - k_{n+1}) = 2TV(k) < \infty.$$

This example shows the usefulness of the singular mapping. There is no hope of proving a uniform (in n and Δ) bound on the total variation of the numerical approximations for this Cauchy problem. It still leaves open the possibility of proving compactness via local boundedness of the variation. Before leaving this example, it is worth noting that $TV(\Psi(u(\cdot,0),k(\cdot))) = 0$, which shows that even though Ψ is known to have bounded total variation, the variation may actually increase.

It is possible to construct an analogous example for the equation $u_t+(u^2/2)_x = a_x$, using initial data $u_0(x) = 0$, and the source term $a(x) = 1/n$ for $x \in [n-1,n)$, $n \geq 1$; and $a(x) = 1$ for $x \leq 1$. The same type of calculation shows that for small but finite time, $TV(u(\cdot,t)) = \infty$, while $TV(\Psi(u(\cdot,t),a(\cdot))) < \infty$.

4. Stability properties of the flux approximations. This brief section discusses two stability properties of the flux approximations. Monotone schemes for the equation $u_t + f(u)_x = 0$ have associated maximum and minimum principles, and the Total Variation Decreasing (TVD) property:

$$\min_j U_j^n \leq U_j^{n+1} \leq \max_j U_j^n, \qquad \sum_j |\Delta_+ U_j^{n+1}| \leq \sum_j |\Delta_+ U_j^n|.$$

Clearly these properties do not hold for the approximations U_j^n produced by any numerical scheme consistent with the conservation law 1.1, since the actual solution to the Cauchy problem does not have these properties, unless a and k happen to be constants. In the variable (k,a) situation, those properties evidently do hold for the generated flux approximations $k_{j+\frac{1}{2}}h(U_{j+1}^n, U_j^n) - a_{j+\frac{1}{2}}$.

THEOREM 4.1. *With the assumptions of Theorem 3.1, and the more restrictive CFL condition $\lambda\|k\|_\infty\|f'\|_\infty \leq 1/2$, the flux approximations $k_{j+\frac{1}{2}}h(U_{j+1}^n, U_j^n) - a_{j+\frac{1}{2}}$ satisfy the maximum and minimum principles*

$$\min_j(k_{j+\frac{1}{2}}h(U_{j+1}^0, U_j^0) - a_{j+\frac{1}{2}}) \leq k_{j+\frac{1}{2}}h(U_{j+1}^n, U_j^n) - a_{j+\frac{1}{2}}$$
$$\leq \max_j(k_{j+\frac{1}{2}}h(U_{j+1}^0, U_j^0) - a_{j+\frac{1}{2}})$$

and the following flux-TVD property

$$\sum_j |\Delta_+(k_{j+\frac{1}{2}} h(U_{j+1}^{n+1}, U_j^{n+1}) - a_{j+\frac{1}{2}})| \le \sum_j |\Delta_+(k_{j+\frac{1}{2}} h(U_{j+1}^{n}, U_j^{n}) - a_{j+\frac{1}{2}})|.$$

If k is constant, the flux-TVD property holds with the CFL condition $\lambda k \|f'\|_\infty \le 1$.

Proof. Let $h^n_{j+\frac{1}{2}} = h(U^n_{j+1}, U^n_j)$, and $D_{j+\frac{1}{2}} = \Delta_+(k_{j+\frac{1}{2}} h^n_{j+\frac{1}{2}} - a_{j+\frac{1}{2}})$. Then

$$\begin{aligned} k_{j+\frac{1}{2}} h^{n+1}_{j+\frac{1}{2}} - a_{j+\frac{1}{2}} &= k_{j+\frac{1}{2}} h(U^n_{j+1} - \lambda D_{j+\frac{1}{2}}, U^n_j - \lambda D_{j-\frac{1}{2}}) - a_{j+\frac{1}{2}} \\ &= k_{j+\frac{1}{2}} h^n_{j+\frac{1}{2}} - a_{j+\frac{1}{2}} + k_{j+\frac{1}{2}} \int_0^1 \frac{d}{d\theta} h(U^n_{j+1} - \theta\lambda D_{j+\frac{1}{2}}, U^n_j - \theta\lambda D_{j-\frac{1}{2}}) d\theta. \end{aligned}$$

It follows that

$$\begin{aligned} k_{j+\frac{1}{2}} h^{n+1}_{j+\frac{1}{2}} - a_{j+\frac{1}{2}} &= k_{j+\frac{1}{2}} h^n_{j+\frac{1}{2}} - a_{j+\frac{1}{2}} + C^+_{j+\frac{1}{2}} D_{j+\frac{1}{2}} - C^-_{j-\frac{1}{2}} D_{j-\frac{1}{2}} \\ &= (1 - C^+_{j+\frac{1}{2}} - C^-_{j-\frac{1}{2}})(k_{j+\frac{1}{2}} h^n_{j+\frac{1}{2}} - a_{j+\frac{1}{2}}) \\ &\quad + C^+_{j+\frac{1}{2}}(k_{j+\frac{3}{2}} h^n_{j+\frac{3}{2}} - a_{j+\frac{3}{2}}) + C^-_{j-\frac{1}{2}}(k_{j-\frac{1}{2}} h^n_{j-\frac{1}{2}} - a_{j-\frac{1}{2}}), \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} C^+_{j+\frac{1}{2}} &= -\lambda k_{j+\frac{1}{2}} \int_0^1 h_v(U^n_{j+1} - \theta D_{j+\frac{1}{2}}, U^n_j - \theta D_{j-\frac{1}{2}}) d\theta \\ &= -\lambda k_{j+\frac{1}{2}} \int_0^1 f'_-(U^n_{j+1} - \theta D_{j+\frac{1}{2}}) d\theta \\ C^-_{j-\frac{1}{2}} &= \lambda k_{j+\frac{1}{2}} \int_0^1 h_u(U^n_{j+1} - \theta D_{j+\frac{1}{2}}, U^n_j - \theta D_{j-\frac{1}{2}}) d\theta \\ &= \lambda k_{j+\frac{1}{2}} \int_0^1 f'_+(U^n_j - \theta D_{j-\frac{1}{2}}) d\theta \end{aligned} \tag{4.2}$$

For $\theta \in [0,1]$, $U^n_{j+1} - \theta D_{j+\frac{1}{2}}$ lies between U^n_{j+1} and U^{n+1}_{j+1}, so

$$\lambda k_{j+\frac{1}{2}} |f'(U^n_{j+1} - \theta D_{j+\frac{1}{2}})| \le 1/2,$$

due to the CFL condition. Thus,

$$0 \le C^+_{j+\frac{1}{2}} \le 1/2, \quad 0 \le C^-_{j-\frac{1}{2}} \le 1/2,$$

and so, with the notation $co\{\cdot\}$ for the convex hull,

$$k_{j+\frac{1}{2}} h^{n+1}_{j+\frac{1}{2}} - a_{j+\frac{1}{2}} \in co\{k_{j+\frac{3}{2}} h^n_{j+\frac{3}{2}} - a_{j+\frac{3}{2}}, k_{j+\frac{1}{2}} h^n_{j+\frac{1}{2}} - a_{j+\frac{1}{2}}, k_{j-\frac{1}{2}} h^n_{j-\frac{1}{2}} - a_{j-\frac{1}{2}}\}.$$

The maximum and minimum principles then follow by induction.

The first line of 4.1 contains the following incremental form for a single time step of the flux:

$$\begin{aligned} k_{j+\frac{1}{2}} h^{n+1}_{j+\frac{1}{2}} - a_{j+\frac{1}{2}} = k_{j+\frac{1}{2}} h^n_{j+\frac{1}{2}} - a_{j+\frac{1}{2}} &+ C^+_{j+\frac{1}{2}} \Delta_+(k_{j+\frac{1}{2}} h^n_{j+\frac{1}{2}} - a_{j+\frac{1}{2}}) \\ &- C^-_{j-\frac{1}{2}} \Delta_+(k_{j-\frac{1}{2}} h^n_{j-\frac{1}{2}} - a_{j-\frac{1}{2}}). \end{aligned}$$

Since $C^+_{j+\frac{1}{2}} \geq 0$, $C^-_{j-\frac{1}{2}} \geq 0$, and $C^+_{j+\frac{1}{2}} + C^-_{j+\frac{1}{2}} \leq 1$, the flux-TVD property follows from Lemma 2.2 of [5]. When k is constant

$$\begin{aligned} C^+_{j+\frac{1}{2}} + C^-_{j+\frac{1}{2}} &= -\lambda k \int_0^1 f'_-(U^n_{j+1} - \theta D_{j+\frac{1}{2}}) d\theta + \lambda k \int_0^1 f'_+(U^n_{j+1} - \theta D_{j+\frac{1}{2}}) d\theta \\ &= \lambda k \int_0^1 |f'(U^n_{j+1} - \theta D_{j+\frac{1}{2}})| d\theta \leq 1, \end{aligned}$$

and this inequality holds with the less restrictive CFL condition $\lambda k \|f'\|_\infty \leq 1$.
$\square$

Letting $\Delta \to 0$ gives the following fact about solutions of the Cauchy problem 1.1.

COROLLARY 4.2. *With the assumptions in Theorem 3.1, the Cauchy problem 1.1 has a weak solution $u(x,t)$ satisfying a.e. the maximum and mininmum principles*

$$\inf_x (k(x) f(u_0(x)) - a(x)) \leq k(x) f(u(x,t)) - a(x) \leq \sup_x (k(x) f(u_0(x)) - a(x)).$$

For the conservation law 1.6 with a convex flux, reference [4] establishes a maximum principle similar to the one in Theorem 4.1, for the Godunov scheme of that paper, without the additional restriction on the CFL condition. It also proves the maximum principle of Corollary 4.2, for the solution to 1.6. The maximum principles appearing in [4] result from a detailed study of entropy satisfying solutions to the Riemann problem, while the approach here is via a slight generalization of standard techniques from the theory of monotone and TVD schemes.

It is not clear whether the flux-TVD portion of Theorem 4.1 can be used in place of the singular mapping to establish convergence, unless the flux $f(u)$ happens to be strictly monotone. In that case the singular function is equal to $kf(u) - a$. The flux-TVD property does have some utility in the context of this paper. The variation bound on z^Δ of Theorem 3.2 depends on L^1 contractiveness (via Lemma 3.8), which requires uniform time steps. The flux-TVD property of Theorem 4.1 remains valid with variable time steps, as long as the CFL condition is satisfied. In addition, the flux-TVD property implies a uniform time-continuity estimate for both u^Δ and z^Δ. The time-continuity estimate for u^Δ is sufficient to provide the required bound on $TV(z^\Delta)$, thus allowing an extension of the convergence result to variable time steps.

5. Conclusion. This paper has extended the convergence result of [19] to a large class of nonconvex flux functions, and allowed for spatially varying source terms. A number of important issues have not been addressed, and are potential areas for further investigation. For example, the related questions of uniqueness and entropy satisfaction have not been discussed at all. These are important issues for conservation laws, since the presence of discontinuities requires the satisfaction of entropy conditions to ensure that the correct, physically relevant solution is singled out. In [19], the author derived geometric entropy conditions for the limit of the scheme for the case of a concave flux. Carrying out that program for the present situation appears feasible, but is significantly more complicated.

The value of the algorithm investigated here is its simplicity. Given a working computer program implementing a simple upwind difference scheme for the conservation law $u_t + f(u)_x = 0$, only minor modifications are required to produce a program implementing the scheme 1.4 for the significantly harder Cauchy problem 1.1. The price paid for simplicity in this case is performance on some, but not all, problems where a discontinuity in u_0 is perfectly aligned with a discontinuity in one or both of a, k. The author tested the algorithm on the four numerical test cases provided in [4]. In all cases, the results were very similar to those described in that reference. However, for examples 2. and 3. of that paper, the scheme 1.4 produced persistent overshoots where the shock waves exit from the interval $[-1, 1]$, at the location of a perfectly aligned discontinuity in u_0 and a. The overshoot has width of only 1 or 2 mesh points (for a small enough mesh size Δ), but fixed (with respect to Δ) amplitude at steady state. These overshoots are actually part of the steady solution for the scheme 1.4, as was verified by checking that the flux approximations $h_{j+\frac{1}{2}} - a_{j+\frac{1}{2}}$ (corresponding to the quantity referred to as the diagnostic variable in [4]) converged to zero. The scheme described in [4] was able to resolve these interfaces without overshoot. The author has also observed small, spurious traveling waves generated by some Riemann problems for the scheme described in [19], which is a special case of the one described here. As expected, the waves approach zero as the mesh size shrinks, in agreement with the convergence theory developed here, but their presence is distracting. It is not clear why these spurious numerical artifacts are present in some cases, and not others. It is clear that there are situations where algorithms which use solutions to the full (jumps in u, k, a) Riemann problem are worth the extra complexity.

In an effort to devise an algorithm that has aligned discretizations, but does not require a solution for the full Riemann problem, the author has performed some experiments with the flux

$$\frac{1}{2}(F_j + F_{j+1}) - \frac{1}{2}\Big(k_{j+1}\int_{u^*}^{U_{j+1}} |f'(w)|dw - k_j \int_{u^*}^{U_j} |f'(w)|dw\Big).$$

Here $F_j = k_j f(U_j) - a_j$ and $u^* \in [\overline{w}, \overline{v}]$. This flux is monotone, and the resulting algorithm seems to have all of the analytical and computational qualities (good and bad) of the staggered mesh version. Although this scheme does not provide a remedy for the overshoots and spurious traveling waves described above, it is worth mentioning since there may be situations where staggering the meshes is inconvenient.

The nondegeneracy assumption (finitely many critical points) rules out any flux that is constant over an interval, a seemingly important case. In this situation, Ψ would fail to be monotone, and so it would not be possible to recover the conserved quantity u from the limit function z. The question here is whether there is a real problem with convergence, or just a limitation of the analytical tool.

This paper only addressed the EO version of the scheme. A convergence proof for the Godunov version would also be of interest. Since the proof of convergence for the EO version exploited the close relationship between the EO flux and the singular mapping Ψ, it may be that a different form of Ψ will be required.

The convergence proofs in both [19] and the present work depended on monotonicity. This property is well known to imply that the scheme is limited to first order accuracy. Some numerical experiments (with a concave flux and no source terms) indicate that second order schemes, constructed by applying flux corrections and minmod-type limiters to the scheme 1.4, give very satisfactory results. Due to the essential way that monotonicity was used, it seems unlikely that it will be possible to develop a convergence theory for such schemes using the approach of this paper and the previous one.

The conditions 1., 2., and 3. of Proposition 2.1, which were used to guarantee boundedness of the computed approximations, are by no means a complete treatment of this subject. For example, there may be less restrictive conditions which allow for solutions which grow beyond any fixed bounds as $t \to +\infty$.

One more potential avenue of investigation is the case where the parameters a and k are allowed to be time dependent as well as spatially dependent, with both spatial and temporal discontinuities, as in [14] and [15].

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