# CONVERGENCE OF DIFFUSIVE BGK APPROXIMATIONS FOR NONLINEAR STRONGLY PARABOLIC SYSTEMS

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Abstract. We study a class of BGK approximations of parabolic systems in one space dimension. We prove stability and existence of global solutions for this model. Moreover, under certain conditions, we prove a rigorous result of convergence toward the formal limit, by using compensated compactness techniques.

## 1. Introduction

The aim of this paper is the study of a BGK approximation for the following parabolic system in one space dimension

$$\begin{cases} u_t + A(u)_x = B(u)_{xx}, & (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $u \in \mathbb{R}^N$, $A : \mathbb{R}^N \to \mathbb{R}^N$ is a regular function, $B(u)$ verifies

$$\nabla B(u) + (\nabla B(u))^* > 0 \tag{1.2}$$

for any $u$ in a suitable compact domain $\mathcal{U} \subset \mathbb{R}^N$ and $u_0 \in L^\infty(\mathbb{R})$. A kinetic BGK approximation for the system (1.1) is a semilinear system of balance laws of the form

$$\begin{cases} f_t^\varepsilon + \left(a(\xi) + \dfrac{b(\xi)}{\sqrt{\varepsilon}}\right) f_x^\varepsilon = \dfrac{1}{\varepsilon}\,(M_{f^\varepsilon} - f^\varepsilon), & (x,t;\xi) \in \mathbb{R} \times \mathbb{R}^+ \times \Xi \\ f^\varepsilon(x,0;\xi) = f_0(x;\xi), & (x;\xi) \in \mathbb{R} \times \Xi. \end{cases} \tag{1.3}$$

In the model (1.3), $f^\varepsilon \in \mathbb{R}^N$; $(\Xi, d\xi)$ is a probability space with $\Xi$ compact (eventually finite); $a$, $b \in L^\infty(\Xi)$ and $M_{f^\varepsilon}$ is a function defined as follows

$$M_{f^\varepsilon} = M(u^\varepsilon; \xi, \varepsilon),$$

where $M : \mathcal{U} \subset \mathbb{R}^N \times \Xi \times \mathbb{R}^+ \to \mathbb{R}^N$ is a $C^2$ function with respect to $u$, measurable with respect to $\xi$ and

$$u^\varepsilon(x,t) = \int_\Xi f^\varepsilon(x,t;\xi)\,d\xi. \tag{1.4}$$

In order to recover the equilibrium system (1.1) from the approximation (1.3) as $\varepsilon \downarrow 0$, the functions $a$, $b$ and $M$ must verify the following compatibility conditions:

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$(\mathbf{M_1})$ $\displaystyle\int_\Xi M(w;\xi,\varepsilon)d\xi = w$ for any $w\in\mathcal{U}$, for any $\varepsilon\in(0,1]$;

$(\mathbf{M_2})$ $\displaystyle\int_\Xi \left(a(\xi)+\frac{b(\xi)}{\sqrt{\varepsilon}}\right) M(w;\xi,\varepsilon)d\xi = A(w)$ for any $w\in\mathcal{U}$, for any $\varepsilon\in(0,1]$;

$(\mathbf{M_3})$ $\displaystyle\int_\Xi b(\xi)^2 M(w;\xi,\varepsilon)d\xi = B(w)$ for any $w\in\mathcal{U}$, for any $\varepsilon\in(0,1]$;

$(\mathbf{M_4})$ $M(w;\xi,\varepsilon)\to M(w;\xi,0)$ as $\varepsilon\downarrow 0$, uniformly for any $w\in\mathcal{U}$, a.e. in $\xi$.

If the function $M$ satisfies the above conditions, we call it local Maxwellian Function **MF** on $\mathcal{U}$. With the above conditions, it is easy to verify that, if $u^\varepsilon$ converges to $u$ in a sufficiently strong topology, then $u$ verifies the system (1.1). Indeed, setting

$$v^\varepsilon = \int_\Xi \left(a(\xi)+\frac{b(\xi)}{\sqrt{\varepsilon}}\right) f^\varepsilon(\xi)d\xi,$$

we get

$$\begin{aligned} &u_t^\varepsilon + v_x^\varepsilon = 0\\ &v_t^\varepsilon + \int_\Xi \left(a(\xi)+\frac{b(\xi)}{\sqrt{\varepsilon}}\right)^2 f^\varepsilon(\xi)_x d\xi = \frac{1}{\varepsilon}\left(A(u^\varepsilon)-v^\varepsilon\right), \end{aligned}$$

which gives (1.1) in the limit $\varepsilon\downarrow 0$, because

$$v^\varepsilon = A(u^\varepsilon) - \int_\Xi b(\xi)^2 f^\varepsilon(\xi)_x d\xi + O(\sqrt{\varepsilon}).$$

We shall prove this convergence result below, by imposing further conditions on the Maxwellian functions, which will replace the monotoniticy condition considered in [20, 5] for the scalar cases.

The model (1.3) can be obtained via the parabolic scaling $(\frac{x}{\sqrt{\varepsilon}},\frac{t}{\varepsilon})$ of the independent variables for a given semilinear hyperbolic system. This type of scaling was introduced first by Kurtz [15] and McKean [19] in the study of diffusive limits for the Carleman equation. Afterwards, it was used in the analysis of hyperbolic-parabolic relaxation limits for weak solutions of hyperbolic systems of balance laws with strongly diffusive source terms by means of compensated compactness techniques. For a partial list of results in this framework, we refer to the paper by Marcati and Rubino [18] and to the references therein.

In the kinetic contest, let us recall the paper by De Masi, Esposito and Lebowitz [10] for the discussion of the diffusion limit of the Boltzmann equation toward the incompressible Navier-Stokes system. Moreover, we recall the recent paper of Lions and Toscani [17] for the proof of the convergence of the diffusion limit for general two velocities models and [14] for various numerical investigations. A recent survey of the results concerning the Bolzmann equation and its various hydrodynamic limits is given in [12].

In the framework of $H^s$ solutions the study of diffusive limits has been performed in [16]. In this paper, the authors consider an appropriate *stability condition*, similar to the one utilized by Yong in [25, 26] for the hyperbolic-hyperbolic singular limits, to take advantage of the diffusion of the source term in the energy estimates. In our BGK framework, the conditions proposed in the following on the Maxwellian functions have the same rôle of this stability condition.

The interest in the study of a general kinetic BGK model of the form (1.3) comes from some good properties of this kind of approximation, in particular:

- it gives rise to a natural and easy way to design and implement new numerical schemes;
- it allows to study in the same framework either the hyperbolic and the parabolic case, and in particular it allows to use hyperbolic numerical schemes to approximate parabolic systems.

This approximation was proposed in [20] for multi-D hyperbolic conservation laws in the scalar case and in [2] in the case of systems. A rigorous result of convergence toward the scalar equilibrium was proved in [20]. Successively, Denis Serre proved in [21] the convergence of that approximation to equilibrium systems satisfying appropriate conditions. Under more general hypotheses, Bouchut in [4] gave a complete classification of BGK models for systems of conservation laws endowed with a family of kinetic entropies, namely entropies for the semilinear approximating system, which reduce to given entropies of the limit system, as the relaxation parameter tends to zero. This class of kinetic-type approximations has been intensively investigated in [1, 2] to build numerical schemes for multidimensional scalar conservation laws.

In the contest of diffusion limits, this general class of kinetic approximations for (possibly degenerate) parabolic equations in multi-D has been considered in [5]. In this paper, it was proved the convergence of the approximate solutions toward the weak solutions of the equilibrium equation and, in the case of degenerate diffusion, toward the entropy solutions in the sense of Carrillo [6]. Starting from this result, diffusive numerical schemes for parabolic systems have been considered in [3].

In the present paper, we prove both the $L^\infty$ stability of the BGK model (1.3), which yields to the existence of global solutions of this system for $\varepsilon > 0$ fixed, and the strong convergence toward weak solutions of the equilibrium system (1.1). As in the purely hyperbolic case, this last result is obtained with the aid of compensated compactness [23, 8]. However, as usual in the hyperbolic-parabolic relaxation limit (see, for instance, [18]), we do not need to control the entropy production for an infinite number of entropy-entropy flux pair, as in [21], but we get the result using the monotonicity of the operator defined by the diffusion $B$.

The paper is organized as follows. In Section 2 we give sufficient conditions to guarantee the $L^\infty$ stability of system (1.3) and the existence of global solutions for $\varepsilon > 0$ fixed. Section 3 concerns with the proof of the basic $L^2$ estimate of the relaxation term $M_{f^\varepsilon} - f^\varepsilon$, which controls the relaxation process. In Section 4 we prove the strong convergence in $L^p_{loc}$, $p < +\infty$, of $u^\varepsilon$, given by (1.4), toward the solution of (1.1), by using the uniform estimates for the BGK model of the previous sections. In the last section we describe some concrete parabolic systems for which the BGK approximation we propose converges.

Finally, let us notice that our results could be easily extended to the (slightly) more general case when there exists a constant, symmetric, positive definite matrix $P$ such that the diffusion $B(u)$ verifies

$$P\nabla B(u) + (\nabla B(u))^* P > P,$$

for any $u$ under consideration. For the theory of general parabolic systems, we refer, for instance, to the books by Eidel'man [11] and by Taylor [24]. For some recent results on parabolic systems with weaker parabolicity conditions, see [9].

## 2. $L^\infty$ bounds and global existence

In this section, following the ideas developed by Serre [21] in the hyperbolic case, we give sufficient conditions to obtain uniform estimates for the system (1.3) and the existence of global-in-time solutions of this system, when the relaxation time $\varepsilon$ is fixed. Throughout this section, we will refer to Maxwellians independent from $\varepsilon$. In this contest, conditions $(\mathbf{M_1})$–$(\mathbf{M_4})$ can be reformulated as follows

$(\mathbf{M_1'})$ $\displaystyle\int_\Xi M(w;\xi)d\xi = w$ for any $w \in \mathcal{U}$;

$(\mathbf{M_2'})$ $\displaystyle\int_\Xi a(\xi)M(w;\xi)d\xi = A(w)$ for any $w \in \mathcal{U}$;

$(\mathbf{M_3'})$ $\displaystyle\int_\Xi b(\xi)^2 M(w;\xi)d\xi = B(w)$ for any $w \in \mathcal{U}$;

$(\mathbf{M_4'})$ $\displaystyle\int_\Xi b(\xi)M(w;\xi)d\xi = 0$ for any $w \in \mathcal{U}$.

To our purpose, we shall assume

$(\mathbf{A_1})$ there exists a convex, compact, invariant set $\mathcal{U}$ for the system (1.1). Moreover, we require the tangent space $T_w\partial\mathcal{U}$ is stable with respect to $\nabla A(w)$ and $\nabla B(w)$;

$(\mathbf{A_2})$ $M$ is a **MF** on $\mathcal{U}$ such that $\rho(\nabla M(w;\xi)) > 0$ for any $w \in \mathcal{U}$, a.e. in $\xi$ and such that the commutators $[\nabla M(w;\xi), \nabla A(w)]$ and $[\nabla M(w;\xi), \nabla B(w)]$ are zero for any $w \in \mathcal{U}$, a.e. in $\xi$.

We recall that with $\rho(A)$ we denote the spectrum (the set of eigenvalues) of a given matrix $A$ and therefore the condition $\rho(A) > 0$ means that all eigenvalues of $A$ are real and positive.

*Remark* 2.1. Let us suppose the system (1.1) has a linear diffusion, namely $B(u) = Bu$, where $B$ is a constant, positive $N \times N$ matrix. In this case, the invariant region theory [7, 22] gives sufficient conditions in order to obtain hypothesis $(\mathbf{A_1})$. In particular, following this theory, we assume there exists a basis of common eigenvectors for the two matrices $\nabla A(w)$ and $B$ whose level sets define the invariant region $\mathcal{U}$. Hence, $T_w\partial\mathcal{U}$ is stable either with respect to $\nabla A(w)$ and with respect to $B$. Moreover, in $(\mathbf{A_2})$, we actually require that only one of the commutators $[\nabla M(w;\xi), \nabla A(w)]$ and $[\nabla M(w;\xi), B]$ vanishes.

*Remark* 2.2. The condition $[\nabla M(w;\xi), \nabla A(w)]$ yields that any eigenvector of the matrix $\nabla A(w)$ is also an eigenvector for the matrix $\nabla M(w;\xi)$, with a different eigenvalue, provided $\nabla A(w)$ has $N$ real, distinct eigenvalues, namely, if the system

$$w_t + A(w)_x = 0$$

is strictly hyperbolic. Indeed, let $v_1(w)$ be an eigenvector of $\nabla A(w)$, namely $\nabla A(w)v_1(w) = \lambda(w)v_1(w)$, and let $v_2(w;\xi) = \nabla M(w;\xi)v_1(w)$. Now,

$$\begin{aligned}\nabla A(w)v_2(w;\xi) &= \nabla A(w)\nabla M(w;\xi)v_1(w) = \nabla M(w;\xi)\nabla A(w)v_1(w)\\ &= \lambda(w)\nabla M(w;\xi)v_1(w) = \lambda(w)v_2(w;\xi).\end{aligned}$$

Hence, $v_2(w;\xi) = \overline{\lambda}(w;\xi)v_1(w)$, that is, $v_1(w)$ is an eigenvector of $\nabla M(w;\xi)$.

It is now possible to prove the $L^\infty$ stability for our BGK model, which comes from the next lemma.

**Lemma 2.3.** *Let us assume the hypotheses* $(\mathbf{A_1})$ *and* $(\mathbf{A_2})$ *hold. Then the domains* $\mathcal{U}_\xi = M(\mathcal{U};\xi)$ *are convex and* $M(\cdot;\xi) \to \mathcal{U}_\xi$ *are global diffeomorphisms.*

The proof follows the one performed in [21] and it is omitted. We emphasize that the sets $\mathcal{U}_\xi = M(\mathcal{U};\xi)$ defined above are also compact, as continuous images of compact sets.

Let us define the domain $\mathcal{F}$ as follows

$$\mathcal{F} = \{\xi \mapsto f(\xi) \text{ measurable such that } f(\xi) \in \mathcal{U}_\xi, \text{ a.e. in } \xi\}\,. \tag{2.1}$$

With the aid of Lemma 2.3, it is easy to prove that $\mathcal{F}$ defines an invariant set for (1.3).

**Proposition 2.4.** *Assume the hypotheses* $(\mathbf{A_1})$ *and* $(\mathbf{A_2})$ *hold. Then*

$$\int_\Xi f(\xi) d\xi \in \mathcal{U}$$

*for any* $f \in \mathcal{F}$ *and* $\mathcal{F}$ *is invariant for the system (1.3).*

*Proof.* The first part of the proposition follows from the relation

$$\int_\Xi \mathcal{U}_\xi d\xi = \mathcal{U}.$$

The inclusion

$$\mathcal{U} \subset \int_\Xi \mathcal{U}_\xi d\xi = \int_\Xi M(\mathcal{U};\xi) d\xi$$

is a direct consequence of the condition $(\mathbf{M_1})$ on the Maxwellian $M$, while the opposite one can be proved observing that the set

$$\int_\Xi \mathcal{U}_\xi d\xi$$

is compact and convex. In order to prove that $\mathcal{F}$ is invariant for (1.3), we note that $\mathcal{F}$ is invariant for the homogeneous system

$$f_t^\varepsilon + \left(a(\xi) + \frac{b(\xi)}{\sqrt{\varepsilon}}\right) f_x^\varepsilon = 0,$$

because $\mathcal{U}_\xi$ are convex sets (if $\Xi$ is finite, $\mathcal{F} = \prod_{\xi\in\Xi} \mathcal{U}_\xi$ is the cartesian product of convex sets). Now, let us consider $f \in \mathcal{F}$. For any $\xi_0$ such that $f(\xi_0) \in \partial\mathcal{U}_{\xi_0}$, we know, from the convexity of $\mathcal{U}_{\xi_0}$, that the vector

$$M_f(\xi_0) - f(\xi_0) = M(u;\xi_0) - f(\xi_0)$$

points inside $\mathcal{U}_{\xi_0}$, because

$$u = \int_\Xi f(\xi) d\xi \in \mathcal{U}$$

implies $M(u;\xi_0) \in \mathcal{U}_{\xi_0}$, which concludes the proof. □

The Proposition 2.4 gives the $L^\infty$ stability for the BGK model (1.3), because the set $\mathcal{F}$ defined in (2.1) is compact. In particular, the previous results yield an a priori estimate in $L^\infty$ for the solution of (1.3), provided the initial data are chosen in $\mathcal{F}$. Since this system is semilinear, this $L^\infty$ estimate implies that the local-in-time solution (1.3) is defined for any $t \geq 0$. Hence, by standard arguments, the following theorem holds.

**Theorem 2.5.** *Let us assume the hypotheses* $(\mathbf{A_1})$ *and* $(\mathbf{A_2})$ *hold and let* $f_0 : \mathbb{R} \to \mathcal{F}$. *Then the Cauchy problem (1.3) admits an unique global-in-time solution* $f^\varepsilon(x,t;\xi)$ *such that* $f^\varepsilon : \mathbb{R} \times \mathbb{R}^+ \to \mathcal{F}$.

*Remark* 2.6. Let $u_0 \in L^\infty(\mathbb{R})$ such that $u_0(x) \in \mathcal{U}$ a.e. Then the *well-prepared* initial datum $f_0(x;\xi) = M(u_0(x);\xi)$ belongs to the invariant set $\mathcal{F}$. Hence, if $u_0(x) \in \mathcal{U}$, Theorem 2.5 implies the existence of an unique global-in-time solution $f^\varepsilon$ of (1.3) with $f_0(x;\xi) = M(u_0(x);\xi)$ as initial datum. Moreover,

$$u^\varepsilon = \int_\Xi f^\varepsilon(\xi)d\xi \in \mathcal{U},$$

which gives the following estimate

$$\|u^\varepsilon\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)} \leq C, \tag{2.2}$$

uniformly with respect to $\varepsilon$.

## 3. Energy-type estimate

This section concerns with the basic energy estimate which allows to control the difference $M_{f^\varepsilon} - f^\varepsilon$ in $L^2(\mathbb{R} \times [0,T] \times \Xi)$. As usual in this kind of problems, this is the key estimate to justify rigorously the relaxation limit by using compensated compactness. Due to the discussion of the previous section, it is natural at this point to consider the following framework

$(\mathbf{H_1})$ the function $M$ are **MF** such that, for any $\varepsilon \in (0,1]$, a.e. in $\xi$, $M(\cdot;\xi,\varepsilon) : \mathcal{U} \to \mathcal{U}_{\xi,\varepsilon}$ are global diffeomorphisms and $\mathcal{U}$, $\mathcal{U}_{\xi,\varepsilon}$ are compact, convex sets. Moreover, $f^\varepsilon(x,t;\xi)$ is a global-in-time solution of (1.3), uniformly bounded, such that $f^\varepsilon : \mathbb{R} \times \mathbb{R}^+ \to \mathcal{F}^\varepsilon$.

The above assumption allows in particular to get the results of Proposition 2.4 and therefore an $L^\infty$ bound for $u^\varepsilon$ of the form (2.2) is obtained, because, as we pointed out in the previous section,

$$u^\varepsilon = \int_\Xi f^\varepsilon(\xi)d\xi \in \int_\Xi \mathcal{U}_{\xi,\varepsilon} d\xi = \mathcal{U}.$$

We recall that our condition is very similar to the one assumed in [4] to give a complete description of kinetic entropies in the strictly hyperbolic case, namely, when no diffusion $B(u)_{xx}$ is present in (1.1).

Concerning the Maxwellians, let us make the following further assumption

$(\mathbf{H_2})$ there exist a function $F : \mathcal{U} \to \mathbb{R}^N$ and a positive constant $\alpha$, independent from $\varepsilon$, such that the matrix $(\nabla M)^* \nabla F$ is symmetric and it verifies $(\nabla M)^* \nabla F \geq \alpha > 0$, for any $u \in \mathcal{U}$, for any $\varepsilon \in (0,1]$, a.e. in $\xi$.

**Proposition 3.1.** *Assume hypotheses* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold. Then there exist a positive constant* $\overline{c}$ *independent from* $\varepsilon$ *and a positive, strictly convex function* $e(f;\xi,\varepsilon)$ *such that*

$$\int_\Xi e(M_f;\xi)d\xi \leq \int_\Xi e(f;\xi)d\xi - \overline{c}\int_\Xi (M_f - f)^2 d\xi, \tag{3.1}$$

*for any* $f \in \mathcal{F}^\varepsilon$.

*Proof.* Due to $(\mathbf{H_2})$, we can define a function $G(u;\xi,\varepsilon)$ such that

$$\nabla G(u;\xi,\varepsilon) = F(u)\nabla M(u;\xi,\varepsilon) \tag{3.2}$$

for any $u \in \mathcal{U}$, for any $\varepsilon \in (0,1]$, a.e. in $\xi$. Since in our case the Maxwellian functions $M$ are invertible (condition $(\mathbf{H_1})$), we can define the functions $e(f;\xi,\varepsilon)$ as follows

$$e(f;\xi,\varepsilon) = G(M^{-1}(f);\xi,\varepsilon). \tag{3.3}$$

We can assume, without loss of generality, $e > 0$, by adding eventually to this function a sufficiently large positive constant. Therefore, (3.2) yields

$$\nabla e = F(M^{-1}(f)),$$

that is

$$\nabla e(M_f;\xi,\varepsilon) = F(u). \tag{3.4}$$

Thus

$$\nabla^2 e = (\nabla F(M^{-1}(f)))(\nabla M)^{-1} \geq \alpha > 0$$

in view of $(\mathbf{H_2})$. In particular, $e(f;\xi,\varepsilon)$ is strictly convex in $f$,for any $\varepsilon \in (0,1]$, a.e. in $\xi$, that is

$$e(f;\xi,\varepsilon) - e(M_f;\xi,\varepsilon) - \nabla e(M_f;\xi,\varepsilon)(f - M_f) \geq \overline{c}(M_f - f)^2,$$

where $\overline{c}$ is independent from $\varepsilon$. Hence, in view of (3.4), we have

$$e(f;\xi,\varepsilon) - e(M_f;\xi,\varepsilon) - F(u)(f - M_f) \geq \overline{c}(M_f - f)^2 \tag{3.5}$$

which yields (3.1), if we choose

$$u = \int_\Xi f(\xi)d\xi$$

and we integrate with respect to $\xi$. □

The previous proposition allows us to define a strictly convex, strictly dissipative function for the system (1.3), if the condition $(\mathbf{H_2})$, together with $(\mathbf{H_1})$, is fulfilled. However, in practical cases, it is quite simple to obtain such a function for (1.3) (see equation (3.3) above), because the condition $(\mathbf{H_2})$ is easily verified. In the next example, we show how it is possible to construct such a function when the convective part of the equilibrium system defines a strictly hyperbolic system endowed with a strictly convex entropy. For the case of a purely parabolic system, see Example 5.2 in Section 5 below.

**Example 3.2.** Assume the condition $(\mathbf{H_1})$ and, as in Remark 2.2, assume the system (1.1) has a convection $A(w)$ such that the reduced system

$$w_t + A(w)_x = 0 \tag{3.6}$$

is strictly hyperbolic. Moreover, assume there exists a strictly convex entropy $\eta(u)$ for that system, for any $u \in \mathcal{U}$. Then, if $[\nabla M(w;\xi,\varepsilon), \nabla A(w)] = 0$ (see $(\mathbf{A_2})$ in Section 2), we can consider the function

$$e(f;\xi,\varepsilon) = q(M(f;\xi,\varepsilon)^{-1};\xi,\varepsilon),$$

where $q(w;\xi,\varepsilon)$ is the flux of the entropy $\eta(w)$ for the hyperbolic system

$$w_t + M(w;\xi,\varepsilon)_x = 0. \tag{3.7}$$

We emphasize that such a function $q(\cdot;\xi,\varepsilon)$ exists because the systems (3.6) and (3.7) have the same entropies (with different fluxes), provided the commutator $[\nabla M(w;\xi,\varepsilon),\nabla A(w)]$ vanishes. Therefore

$$\nabla e(M_f;\xi,\varepsilon)=\nabla\eta \tag{3.8}$$

and

$$\nabla^2 e(M_f;\xi,\varepsilon)\nabla M_f=\nabla^2\eta.$$

Hence, $e$ is strictly convex and therefore it verifies

$$e(f;\xi,\varepsilon)\geq e(M_f;\xi,\varepsilon)+\nabla e(M_f;\xi,\varepsilon)(f-M_f)+C(f-M_f)^2, \tag{3.9}$$

for a constant $C>0$ independent from $\varepsilon$. Finally, integrating (3.9) in $\xi$ and using (3.8), we recover (3.1) in view of the compatibility conditions the Maxwellian $M_f$ must satisfy. Naturally, the condition $(\mathbf{H_2})$ is satisfied with the simple choice $F(u)=\nabla\eta(u)$.

Now we pass to the proof of the uniform $L^2$ estimate for the difference $M_{f^\varepsilon}-f^\varepsilon$.

**Proposition 3.3.** *Assume the conditions* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold and assume the initial datum* $f_0(x;\xi)$ *verify, uniformly with respect to* $\varepsilon$, *the following finite-energy condition*

$$\int_{-\infty}^{+\infty}\int_\Xi e(f_0(x;\xi);\xi,\varepsilon)d\xi dx<+\infty, \tag{3.10}$$

*where* $e$ *is the function constructed in Proposition 3.1. Then*

$$\|M_{f^\varepsilon}-f^\varepsilon\|_{L^2(\mathbb{R}\times[0,T]\times\Xi)}=O(\sqrt{\varepsilon}), \tag{3.11}$$

*for any fixed* $T>0$.

*Proof.* Let us set

$$E^\varepsilon(f)=\int_\Xi e(f;\xi,\varepsilon)d\xi;\qquad F^\varepsilon(f)=\int_\Xi\left(a(\xi)+\frac{b(\xi)}{\sqrt{\varepsilon}}\right)e(f;\xi,\varepsilon)d\xi,$$

for any $f=f(\xi)$, for any $\varepsilon\in(0,1]$, a.e. in $\xi\in\Xi$. Thus,

$$\begin{aligned}E^\varepsilon(f^\varepsilon(x,t;\xi))_t+F^\varepsilon(f^\varepsilon(x,t;\xi))_x&=\frac{1}{\varepsilon}\int_\Xi\nabla e(f^\varepsilon;\xi,\varepsilon)\cdot(M_{f^\varepsilon}-f^\varepsilon)\,d\xi\\&\leq\frac{1}{\varepsilon}\int_\Xi\left(e(M_{f^\varepsilon};\xi,\varepsilon)-e(f^\varepsilon;\xi,\varepsilon)\right)d\xi\\&\leq-\frac{\overline{c}}{\varepsilon}\int_\Xi|M_{f^\varepsilon}-f^\varepsilon|^2\,d\xi,\end{aligned} \tag{3.12}$$

thanks to the convexity of $e$ and thanks to (3.1). Finally, integrating (3.12) in $dx$ and in $dt$, in view of (3.10), we get (3.11). □

We conclude this section by exploiting the convergence given by (3.11) in term of the mean variables

$$u^\varepsilon=\int_\Xi f^\varepsilon(\xi)d\xi;\qquad v^\varepsilon=\int_\Xi\left(a(\xi)+\frac{b(\xi)}{\sqrt{\varepsilon}}\right)f^\varepsilon(\xi)d\xi.$$

We rewrite the BGK model in the following way

$$\begin{cases}u_t^\varepsilon+v_x^\varepsilon=0\\ v_t^\varepsilon+z_x^\varepsilon+\dfrac{1}{\sqrt{\varepsilon}}w_x^\varepsilon+\dfrac{1}{\varepsilon}\widetilde{B}_x^\varepsilon=\dfrac{1}{\varepsilon}\left(A(u^\varepsilon)-v^\varepsilon\right),\end{cases} \tag{3.13}$$

where

$$z^\varepsilon = \int_\Xi a^2(\xi) f^\varepsilon(\xi) d\xi; \quad w^\varepsilon = \int_\Xi a(\xi) b(\xi) f^\varepsilon(\xi) d\xi; \quad \widetilde{B}^\varepsilon = \int_\Xi b^2(\xi) f^\varepsilon(\xi) d\xi.$$

Due to the assumption $a, b \in L^\infty(\Xi)$, the uniform bound on $f^\varepsilon$ gives in a straightforward way the following result.

**Proposition 3.4.** *Under the condition* $(\mathbf{H_1})$*, we have*

$$\|z^\varepsilon\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)};\ \|w^\varepsilon\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)};\ \|\widetilde{B}^\varepsilon\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)};\ \sqrt{\varepsilon}\|v^\varepsilon\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)} \le C, \qquad (3.14)$$

*uniformly for* $\varepsilon < 1$.

**Proposition 3.5.** *Assume the conditions* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold and assume the initial datum* $f_0(x;\xi)$ *verify the finite-energy condition (3.10). Then*

$$\|A(u^\varepsilon) - v^\varepsilon\|_{L^2(\mathbb{R}\times[0,T])} = O(1); \qquad \|\widetilde{B}^\varepsilon - B(u^\varepsilon)\|_{L^2(\mathbb{R}\times[0,T])} = O(\sqrt{\varepsilon}), \qquad (3.15)$$

*uniformly with respect to* $\varepsilon$ *and for any fixed* $T > 0$.

*Proof.* The $L^\infty$ bounds for $a$ and $b$ and the Jensen inequality yield the estimate

$$\begin{aligned}\|A(u^\varepsilon) - v^\varepsilon\|^2_{L^2(\mathbb{R}\times[0,T])} \le O(1) \Bigg[ &\int_{-\infty}^{+\infty} \int_0^T \left| \int_\Xi a(\xi) \left(M_{f^\varepsilon} - f^\varepsilon\right) d\xi \right|^2 dt dx \\ &+ \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} \int_0^T \left| \int_\Xi b(\xi) \left(M_{f^\varepsilon} - f^\varepsilon\right) d\xi \right|^2 dt dx \Bigg] \\ &\le O(1) \left(1 + \frac{1}{\varepsilon}\right) \|M_{f^\varepsilon} - f^\varepsilon\|^2_{L^2(\mathbb{R}\times[0,T]\times\Xi)} = O(1).\end{aligned}$$

In a similar way, one can show

$$\|\widetilde{B}^\varepsilon - B(u^\varepsilon)\|^2_{L^2(\mathbb{R}\times[0,T])} \le \|b^4\|_{L^\infty(\Xi)} \|M_{f^\varepsilon} - f^\varepsilon\|^2_{L^2(\mathbb{R}\times[0,T]\times\Xi)} = O(\sqrt{\varepsilon}),$$

which concludes the proof. □

## 4. Justification of the limit

In this section we analyze the limit, as $\varepsilon \downarrow 0$, of (1.3). Our aim is to prove a rigorous result of convergence of the (weak) solutions of this model toward the solution (in the sense of distributions) of the parabolic system (1.1). In particular, we shall prove the strong convergence in $L^p_{loc}(\mathbb{R}\times\mathbb{R}^+)$ of the mean variable

$$u^\varepsilon(x,t) = \int_\Xi f^\varepsilon(x,t;\xi) d\xi, \qquad (4.1)$$

by using the uniform estimates of the previous section (Proposition 3.5) and the compensated compactness techniques.

Our main theorem states as follows.

**Theorem 4.1.** *Let hypotheses* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold and assume* $f_0(x;\xi)$ *verifies the finite-energy condition (3.10). Then*

$$u^\varepsilon \to u \quad \textit{strongly in } L^p_{loc}(\mathbb{R}\times\mathbb{R}^+) \textit{ for any } p < +\infty, \qquad (4.2)$$

*where* $u(x,t)$ *verifies*

$$u_t + A(u)_x = B(u)_{xx},$$

*in the sense of distributions.*

*Proof.* We consider again the approximation system (3.13)

$$\begin{cases} u_t^\varepsilon + v_x^\varepsilon = 0 \\ v_t^\varepsilon + z_x^\varepsilon + \dfrac{1}{\sqrt{\varepsilon}} w_x^\varepsilon + \dfrac{1}{\varepsilon}\widetilde{B}_x^\varepsilon = \dfrac{1}{\varepsilon}\left(A(u^\varepsilon) - v^\varepsilon\right), \end{cases}$$

with the same notations of Section 2. From (3.14) and (3.15), it follows that

$$v^\varepsilon - A(u^\varepsilon) + B(u^\varepsilon)_x = -\sqrt{\varepsilon} w_x^\varepsilon - \varepsilon\left(v_t^\varepsilon + z_x^\varepsilon\right) + B(u^\varepsilon)_x - \widetilde{B}_x^\varepsilon$$

tends to zero in $H^{-1}_{loc}$, namely, in the sense of distributions. Thus it remains to prove the strong convergence (4.2). The uniform estimates (3.14) and (3.15) yield also the following differential constraints

$$\begin{pmatrix} u_t^\varepsilon + v_x^\varepsilon \\ B(u^\varepsilon)_x \end{pmatrix} \in \text{comp } H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+). \tag{4.3}$$

Let us consider the sequence $(u^\varepsilon, v^\varepsilon, B(u^\varepsilon)) \in \mathbb{R}^{3N}$. From (3.15) and from the bound on $u^\varepsilon$, we know that $v^\varepsilon = v^\varepsilon - A(u^\varepsilon) + A(u^\varepsilon)$ is uniformly bounded in $L^2_{loc}(\mathbb{R}\times\mathbb{R}^+)$. Hence, the sequence $(u^\varepsilon, v^\varepsilon, B(u^\varepsilon))$ converges to $(u, v, \overline{B})$, weakly in $L^2_{loc}(\mathbb{R}\times\mathbb{R}^+)$. Moreover, let us consider the quadratic function $Q : \mathbb{R}^{3N} \to \mathbb{R}$ defined by $Q(\lambda,\mu,\nu) = \lambda\cdot\nu$, for any $(\lambda,\mu,\nu) \in \mathbb{R}^{3N}$. From the compensated compactness techniques [23, 8], it follows $Q(u^\varepsilon, v^\varepsilon, B(u^\varepsilon)) \stackrel{*}{\rightharpoonup} Q(u, v, \overline{B})$ in $L^\infty(\mathbb{R}\times\mathbb{R}^+)$, if $Q(\lambda,\mu,\nu) = 0$, for any $(\lambda,\mu,\nu)\in\Lambda$, where the characteristic manifold $\Lambda$ is defined by

$$\Lambda = \left\{(\lambda,\mu,\nu)\in\mathbb{R}^{3N} : \exists\gamma\in\mathbb{R}^2 - \{0\} \text{ with } L(\lambda,\mu,\nu,\gamma) = 0\right\}$$

and, due to (4.3), the matrix $L(\lambda,\mu,\nu,\gamma)$ is given by

$$L(\lambda,\mu,\nu,\gamma) = \begin{pmatrix} \gamma_1\lambda + \gamma_2\mu \\ \gamma_2\nu \end{pmatrix}.$$

Let $(\lambda,\mu,\nu)\in\Lambda$ and let assume $\nu\neq 0$. Thus we must have $\gamma_2 = 0$ and $\lambda\gamma_1 = 0$, which implies $\lambda = 0$, since $\gamma = (\gamma_1,\gamma_2)\neq 0$. Hence $Q|_\Lambda \equiv 0$ and the compensated compactness yield the following convergence result

$$u^\varepsilon\cdot B(u^\varepsilon) \stackrel{*}{\rightharpoonup} u\cdot\overline{B}, \quad \text{in } L^\infty(\mathbb{R}\times\mathbb{R}^+). \tag{4.4}$$

The matrix $B$ is continuous and, due to the condition (1.2), it verifies

$$\langle B(u) - B(v), u - v\rangle \geq \alpha\left\|u - v\right\|^2.$$

Therefore, $B$ defines a maximal monotone operator in $L^p_{loc}(\mathbb{R}\times\mathbb{R}^+)$. Thus, thanks to (4.4), the Minty arguments imply $\overline{B} = B(u)$. Let $\nu_{(x,t)}(y)$ be the (vectorial) Young measure associated with the sequence $u^\varepsilon(x,t)$. We know that $\operatorname{supp}\nu_{(x,t)}(y)\subset\mathcal{U}$ and moreover

$$0 = \int_{\mathcal{U}} (y - u(x,t))\cdot(B(y) - B(u(x,t)))d\nu_{(x,t)}(y) \geq \alpha\int_{\mathcal{U}} \|y - u(x,t)\|^2 d\nu_{(x,t)}(y).$$

Finally, $\operatorname{supp}\nu_{(x,t)}(y)\subset\{y = u(x,t)\}$, namely $\nu(y) = \delta_{u(x,t)}$, which implies (4.2). $\square$

## 5. Examples

In this section we present some examples of BGK models for parabolic systems of the form (1.1). As we pointed out in the previous sections, in order to obtain the strong convergence of our BGK model, we have to verify conditions $(\mathbf{H_1})$ and $(\mathbf{H_2})$. In the first two examples, instead of assuming $(\mathbf{H_1})$, we rewrite explicitly the conditions $(\mathbf{A_1})$ and $(\mathbf{A_2})$ of Section 2, which guarantee the corresponding BGK models verify $(\mathbf{H_1})$. We are allowed to consider those sufficient conditions because, for these examples, we will utilize only Maxwellians independent from $\varepsilon$. We recall once again that if we know *a priori* that condition $(\mathbf{H_1})$ is fulfilled, we have to check only condition $(\mathbf{H_2})$ on the Maxwellians to conclude our BGK approximation converges strongly to the parabolic equilibrium system.

**Example 5.1.** Given a general system of the form (1.1), we consider a BGK approximation for that system with a finite number of velocities, namely

$$f^\varepsilon_{it} + \left(a_i + \frac{b_i}{\sqrt{\varepsilon}}\right) f^\varepsilon_{ix} = \frac{1}{\varepsilon}(M_{f^\varepsilon_i} - f^\varepsilon_i), \qquad i = 1, \dots, N.$$

Then, the compatibility condition for the Maxwellian functions read [5]

$$\sum_{i=1}^N M_i(u) = u; \qquad \sum_{i=1}^N a_i M_i(u) = A(u);$$
$$\sum_{i=1}^N b_i^2 M_i(u) = B(u); \qquad \sum_{i=1}^N b_i M_i(u) = 0.$$

For this model, we choose $N = 4$ and we fix the coefficient to be

$$a_1 = -\lambda; \qquad a_2 = \lambda; \qquad a_3 = a_4 = 0;$$
$$b_1 = b_2 = 0; \qquad b_3 = -\theta; \qquad b_4 = \theta.$$

Hence, the Maxwellian functions are given by

$$\begin{aligned} M_1(u) &= \frac{1}{2}\left(u - \frac{A(u)}{\lambda} - \frac{B(u)}{\theta^2}\right); \\ M_2(u) &= \frac{1}{2}\left(u + \frac{A(u)}{\lambda} - \frac{B(u)}{\theta^2}\right); \\ M_3(u) = M_4(u) &= \frac{1}{2}\frac{B(u)}{\theta^2}. \end{aligned}$$

Let us assume that the equilibrium system (1.1) admits a convex, compact, invariant region $\mathcal{U}$ (with $T_u\partial\mathcal{U}$ stable with respect to $\nabla A(u)$ and $\nabla B(u)$ for $u \in \mathcal{U}$), so that condition $(\mathbf{A_1})$ is satisfied. Moreover, if we require $[\nabla A(u), \nabla B(u)] = 0$, then

$$[\nabla A(u), \nabla M_i(u)] = [\nabla B(u), \nabla M_i(u)] = 0, \qquad i = 1, \dots, 4,$$

because any matrix $\nabla M_i(u)$, $i = 1, \dots, 4$, is a linear combination of the matrices $I$, $\nabla A(u)$ and $\nabla B(u)$. Moreover, if $\rho(\nabla B(u)) > 0$, then $\rho(\nabla M_i(u)) > 0$, $i = 1, \dots, 4$, provided

$$\frac{\|\nabla A(u)\|}{\lambda} + \frac{\|\nabla B(u)\|}{\theta^2} < 1$$

and therefore condition $(\mathbf{A_2})$ is fulfilled (concerning the condition above on the parameter $\lambda$ and $\theta$, see also the discussion of [3]). We already know (Example 3.2) that if $\nabla A(u)$ has real and distinct eigenvalues and if we assume the system

$$u_t + A(u)_x = 0$$

is endowed with a strictly convex entropy, condition $(\mathbf{H_2})$ is easily satisfied and therefore we obtain, for this particular BGK model, the convergence result of Theorem 4.1, provided the initial data $f_{0i}(x) = M_i(u_0(x))$ verifies the finite-energy condition (3.10).

**Example 5.2.** Let us consider a parabolic system without convection part

$$u_t = B(u)_{xx}$$

verifying condition $(\mathbf{A_1})$ and let us consider again a finite-velocity BGK model for that system, namely

$$f^\varepsilon_{it} + \left(a_i + \frac{b_i}{\sqrt{\varepsilon}}\right) f^\varepsilon_{ix} = \frac{1}{\varepsilon}(M_{f^\varepsilon_i} - f^\varepsilon_i), \qquad i = 1, \dots, N. \tag{5.1}$$

In particular, for $N = 4$, the constants $a_i$, $b_i$ and the Maxwellians in this case take the form

$$\begin{aligned}
&a_1 = a_2 = a_3 = a_4 = 0;\\
&b_1 = b_2 = 0; \qquad\qquad b_3 = -\theta; \qquad b_4 = \theta;\\
&M_1(u) = M_2(u) = \frac{1}{2}\left(u - \frac{B(u)}{\theta^2}\right);\\
&M_3(u) = M_4(u) = \frac{1}{2}\frac{B(u)}{\theta^2}.
\end{aligned}$$

Therefore, condition $(\mathbf{A_2})$ is satisfied provided $\rho(\nabla B(u)) > 0$ and $\theta^2 > \|B(u)\|$, without any further restriction. Moreover, in order to verify $(\mathbf{H_2})$, we have to assume the existence of a function $F(u)$ such that the matrices $(\nabla M_i(u))^*\nabla F(u)$ are symmetric and they verify $(\nabla M_i(u))^*\nabla F(u) \geq \alpha > 0$, for a given constant $\alpha$ and for any $i = 1, \dots, 4$ (in order to fulfill this condition, it is possible to assume the existence of a scalar function $g(u)$ such that $(\nabla B(u))^*\nabla^2 g(u)$ is symmetric and it verifies $(\nabla B(u))^*\nabla^2 g(u) \geq \alpha > 0$). However, we want to show how it is possible to construct directly strictly convex functions $e_i(f)$, $i = 1, \dots, 4$, verifying the dissipativity condition (3.1) in the particular case when $\nabla B(u)$ is symmetric. The strategy is to define the functions $G_i(u)$, $i = 1, \dots, 4$, from (3.2) and hence we recover, as in (3.3), the expression for $e_i(f)$, $i = 1, \dots, 4$, because our the Maxwellians $M_i(u)$, $i = 1, \dots, 4$, are invertible. Since $\nabla B(u)$ is symmetric, $\operatorname{rot} B = 0$ and therefore there exists a function $\Psi = \Psi(u)$ such that $B(u) = \nabla\Psi(u)$. Then, with the notation

$$B(u) = \begin{pmatrix} b_1(u) \\ \vdots \\ b_n(u) \end{pmatrix},$$

the functions $G_i(u)$, $i = 1, \dots, 4$, are given by

$$\begin{aligned}
G_1(u) = G_2(u) &= \frac{1}{2}\left[\frac{1}{2}|u|^2 - \frac{1}{\theta^2}\left(\sum_{i=1}^n u_i b_i(u) - \Psi(u)\right)\right];\\
G_3(u) = G_4(u) &= \frac{1}{2}\left[\frac{1}{\theta^2}\left(\sum_{i=1}^n u_i b_i(u) - \Psi(u)\right)\right].
\end{aligned}$$

*In other words, if $\nabla B(u)$ is symmetric, the Maxwellians $M_i(u)$, $i=1,\dots,4$, are symmetric and therefore $(\mathbf{H_2})$ is fulfilled with the choice $F(u)=u$, thanks to (1.2) and provided*

$$\frac{\|\nabla B(u)\|}{\theta^2}<1, \qquad \textit{for any } u\in\mathcal{U}. \tag{5.2}$$

In the linear case, namely, if $B(u)=Bu$ with $B$ constant and symmetric, the *potential* $\Psi$ is given by

$$\Psi(u)=\frac{1}{2}\sum_{i,j=1}^{n} b_{ij}u_iu_j.$$

At this point, we want to determine an explicit formula for $e$ in the diagonal case, to show that this function is simply a classical quadratic energy function. To this aim, let us assume $B(u)=Bu$, where $B$ is a positive, constant, diagonal matrix, namely

$$B=\begin{pmatrix} b_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & b_n\end{pmatrix}.$$

The Maxwellians $M_i(u)$, $i=1,\dots,4$, become linear, that is

$$\begin{aligned} M_1(u)=M_2(u)&=\frac{1}{2}\left(u-\frac{Bu}{\theta^2}\right);\\ M_3(u)=M_4(u)&=\frac{1}{2}\frac{Bu}{\theta^2}\end{aligned}$$

and the condition (5.2) becomes

$$\frac{|b_i|}{\theta^2}<1, \qquad \text{for any } i=1,\dots,n. \tag{5.3}$$

Thus, the relation (3.2) can be rewritten as

$$\nabla G_i(u)=u\nabla M_i(u)=u\mathcal{M}_i=(m_1^iu_1,\dots,m_n^iu_n), \qquad i=1,\dots,4,$$

where the diagonal matrices

$$\mathcal{M}_i=\begin{pmatrix} m_1^i & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & m_n^i\end{pmatrix}, \qquad i=1,\dots,4,$$

are

$$\begin{aligned}\mathcal{M}_1=\mathcal{M}_2&=\frac{1}{2}\left(I-\frac{B}{\theta^2}\right);\\ \mathcal{M}_3=\mathcal{M}_4&=\frac{1}{2}\frac{B}{\theta^2}\end{aligned}$$

and $m_j^i\neq 0$ for any $i=1,\dots,4$, $j=1,\dots,n$, in view of (5.3) and the positivity of $B$. Hence, the functions $G_i(u)$, $i=1,\dots,4$, are given by

$$G_i(u)=\frac{1}{2}\sum_{j=1}^{n} m_j^iu_j^2, \qquad i=1,\dots,4.$$

At this point, in order to determine the functions $e_i(f)$, $i = 1, \dots, 4$, we invert the relations

$$f = M_i(u) = \mathcal{M}_i u, \qquad i = 1, \dots, 4,$$

and we obtain

$$u_j = \left(m_j^i\right)^{-1} f_j, \qquad i = 1, \dots, 4, \; j = 1, \dots, n.$$

Therefore, the functions $e_i(f)$, $i = 1, \dots, 4$, are quadratic functions in $f$ with a wedge depending on the Maxwellians $M_i$, $i = 1, \dots, 4$, that is

$$e_i(f) = \frac{1}{2} \sum_{j=1}^{n} m_j^i \left(m_j^i\right)^{-2} f_j^2 = \sum_{j=1}^{n} \frac{1}{m_j^i} f_j^2, \qquad i = 1, \dots, 4.$$

It is clear that the above functions are strictly convex and, moreover, the relation (3.1) can be checked directly. Indeed, we compute

$$\begin{aligned}
\mathcal{A} &= \frac{1}{2} \sum_{j=1}^{n} \frac{1}{m_j^i} f_j^2 - \frac{1}{2} \sum_{j=1}^{n} m_j^i u_j^2 - \sum_{j=1}^{n} u_j (f_j - m_j^i u_j) - \overline{c} \sum_{j=1}^{n} (f_j - m_j^i u_j)^2 \\
&= \sum_{j=1}^{n} \left[ \left( \frac{1}{2m_j^i} - \overline{c} \right) f_j^2 + m_j^i \left( \frac{1}{2} - \overline{c} m_j^i \right) u_j^2 + \left( 2\overline{c} m_j^i - 1 \right) f_j u_j \right],
\end{aligned}$$

and thus

$$\begin{aligned}
\left( \frac{1}{2m_j^i} - \overline{c} \right) \mathcal{A} &= \sum_{j=1}^{n} \left( \frac{1}{2m_j^i} - \overline{c} \right)^2 \left[ f_j^2 + \left(m_j^i\right)^2 u_j^2 - 2m_j^i f_j u_j \right] \\
&= \sum_{j=1}^{n} \left( \frac{1}{2m_j^i} - \overline{c} \right)^2 \left[ f_j - m_j^i u_j \right]^2 \geq 0.
\end{aligned}$$

Hence, if

$$\overline{c} < \min \left\{ \frac{1}{2m_j^i} \,:\, i = 1, \dots, 4, \; j = 1, \dots, n \right\},$$

we have proved $\mathcal{A} \geq 0$, namely

$$\frac{1}{2} \sum_{j=1}^{n} \frac{1}{m_j^i} f_j^2 \geq \frac{1}{2} \sum_{j=1}^{n} m_j^i u_j^2 + \sum_{j=1}^{n} u_j (f_j - m_j^i u_j) + \overline{c} \sum_{j=1}^{n} (f_j - m_j^i u_j)^2, \tag{5.4}$$

which is exactly the relation (3.5) for our particular example. Finally, we get (3.1) by summing up (5.4) for any $i = 1, \dots, 4$ and using the properties of the Maxwellians $M_i$, namely, the explicit expressions for the constants $m_j^i$.

**Example 5.3.** Let us consider a parabolic system of the form

$$u_t + A(u)_x = \mu u_{xx}, \tag{5.5}$$

where $\mu > 0$ is a given constant. S. Jin and H.L. Liu proposed in [13] to approximate (5.5), in the scalar case, with the following system

$$\begin{cases} u_t^\varepsilon + v_x^\varepsilon = 0 \\ \varepsilon^2 v_t^\varepsilon + \mu u_x^\varepsilon = A(u^\varepsilon) - v^\varepsilon. \end{cases} \tag{5.6}$$

As it was pointed out in [3] in the scalar case, we want to prove that this approximation can be viewed in terms of a BGK model of the form (5.1) actually in the general system case. Indeed, we set $N = 2$,

$$a_1 = a_2 = 0; \qquad b_1 = \sqrt{\mu}; \qquad b_2 = -\sqrt{\mu}$$

and

$$M_1(u;\varepsilon) = \frac{u}{2} + \frac{\sqrt{\varepsilon}A(u)}{2\sqrt{\mu}}; \qquad M_2(u;\varepsilon) = \frac{u}{2} - \frac{\sqrt{\varepsilon}A(u)}{2\sqrt{\mu}}.$$

The functions $M_1$ and $M_2$ verify conditions $(\mathbf{M_1})$–$(\mathbf{M_4})$ (they are local Maxwellian functions) and the BGK model takes the form

$$\begin{cases} f^\varepsilon_{1t} + \dfrac{\sqrt{\mu}}{\sqrt{\varepsilon}} f^\varepsilon_{1x} = \dfrac{1}{\varepsilon}\left(\dfrac{u}{2} + \dfrac{\sqrt{\varepsilon}A(u)}{2\sqrt{\mu}} - f^\varepsilon_1\right) \\ f^\varepsilon_{2t} - \dfrac{\sqrt{\mu}}{\sqrt{\varepsilon}} f^\varepsilon_{2x} = \dfrac{1}{\varepsilon}\left(\dfrac{u}{2} - \dfrac{\sqrt{\varepsilon}A(u)}{2\sqrt{\mu}} - f^\varepsilon_2\right). \end{cases} \tag{5.7}$$

Hence, we recover the system (5.6) with the choice

$$u^\varepsilon = f^\varepsilon_1 + f^\varepsilon_2; \qquad v^\varepsilon = \frac{\sqrt{\mu}}{\sqrt{\varepsilon}}(f^\varepsilon_1 - f^\varepsilon_2).$$

Finally, we have the strong convergence of $u^\varepsilon$ toward the solution of (5.5), with initial data in $L^\infty$, provided (5.7) has global-in-time, bounded, solutions and $M_1$, $M_2$ are global diffeomorphisms, verifying $(\mathbf{H_2})$, from a compact, convex set $\mathcal{U}$ into compact, convex sets $\mathcal{U}_{1,\varepsilon}$, $\mathcal{U}_{2,\varepsilon}$. We recall that $M_1$ and $M_2$ verify the above conditions if there exists a compact, convex, invariant set $\mathcal{U}$ for (5.5) (see condition $(\mathbf{A_1})$ of Section 2) and if, for instance, $A(u)$ is symmetric, provided $\varepsilon$ is sufficiently small to verify

$$\frac{\sqrt{\varepsilon}}{\sqrt{\mu}}\|\nabla A(u)\| < 1, \qquad \text{for any } u \in \mathcal{U},$$

so that condition $(\mathbf{H_2})$ is clearly fulfilled for $\varepsilon$ small enough.

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