A Wave-Front Tracking Algorithm for $N \times N$ Non Genuinely Nonlinear Conservation Laws*

FABIO ANCONA †   ANDREA MARSON ‡

Abstract

We introduce a wave-front tracking algorithm for $N \times N$ hyperbolic systems of conservation laws

$$u_t + F(u)_x = 0,$$

that admits characteristic fields which are neither genuinely nonlinear nor linearly degenerate in the sense of Lax. Instead we assume that, for any non genuinely nonlinear i-th characteristic family, the derivative of the i-th eigenvalue $\lambda_i(u)$ of $DF(u)$ in the direction of the i-th right eigenvector $r_i(u)$, vanishes on a single $(N-1)$-dimensional hypersurface in the u-space, transversal to the field $r_i(u)$. Systems that fulfill this type of assumptions are of particular interest in studying elastodynamic or rigid heat conductors at low temperature.

The first proof of the existence of weak solutions for non genuinely nonlinear systems was given by T.P. Liu [Li3], based on a Glimm scheme. Our construction here provides an alternative method to establish the global existence of weak solutions for such systems. Moreover, relying on the stability analysis developed in [AM3, AM5], we show that these solutions are entropy admissible in the sense of Lax.

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†Dipartimento di Matematica and C.I.R.A.M., P.zza Porta S. Donato, n. 5, 40123 - Bologna, Italy; E-mail: `ancona@ciram3.ing.unibo.it`.

‡Dipartimento di Matematica, Via Valotti, n. 9, 25133 - Brescia, Italy; E-mail: `marson@ing.unibs.it`.

1 Introduction

Consider the Cauchy problem for an $N \times N$ strictly hyperbolic system of conservation laws

$$u_t + F(u)_x = 0 = 0 \tag{1.1}$$

$$u(0, x) = \overline{u}(x). \tag{1.2}$$

Here, $t \geq 0$ and $x \in \mathbb{R}$ are, respectively, the time and the space variable, while the vector $u = u(t,x) \in \mathbb{R}^N$ represents the conserved quantities. The flux function F is assumed to be a smooth vector field defined on a neighborhood of the origin $\Omega \subseteq \mathbb{R}^N$ and taking values in $\mathbb{R}^N$. Denote with $\lambda_1(u) < \cdots < \lambda_N(u)$ the eigenvalues of the Jacobian matrix $DF(u)$, and with $r_1(u), \ldots, r_N(u)$ a corresponding basis of eigenvectors. We assume that system (1.1) admits **non genuinely non linear (NGNL)** characteristic fields, i.e. characteristic fields that are neither genuinely nonlinear (GNL) nor linearly degenerate (LD) in the sense of Lax [La, Sm]. Instead, we require that, for each NGNL characteristic field r_k, the directional derivative of λ_k in the direction of r_k

$$D_{r_k}\lambda_k(u) \doteq \lim_{h\to 0} \frac{\lambda_k\big(u + hr_k(u)\big) - \lambda_k(u)}{h}$$

vanishes on a smooth hypersurface transversal to the field r_k. More precisely, we make the following assumption (see Figure 1):

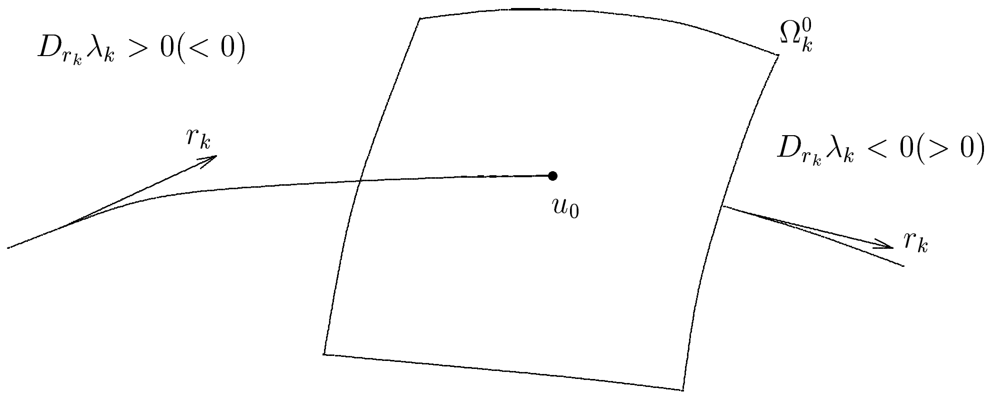

Figure 1: ASSUMPTION **(A)**

(A) If the k-th characteristic family is NGNL, then there exists an $(N-1)$-dimensional smooth manifold Ω_k^0, $k = 1, \ldots, N$, such that

$$\Omega_k^0 = \big\{u \in \Omega : \; D_{r_k}\lambda_k(u) = 0\big\}, \tag{1.3}$$

$$D^2_{r_k}\lambda_k(u) \doteq D_{r_k}\big(D_{r_k}\lambda_k\big)(u) \neq 0, \qquad \forall\, u \in \Omega_k^0. \tag{1.4}$$

Systems of conservation laws with characteristic fields that fulfill the assumption **(A)** physically arise in several contexts, for instance in studying elastodynamic (e.g. see [Dp2]) or rigid heat conductors at low temperature [RMS1, RMS2]. An example is given by the generalized Cattaneo model proposed by T. Ruggeri and co-workers [RMS1, RMS2] to describe the heat propagation in high-purity crystals (He, NaF, Bi):

$$\big[\rho\, e\big]_t + q_x = 0, \qquad \big[\alpha\, q\big]_t + \nu_x = -\frac{\nu'}{\kappa} q. \tag{1.5}$$

Here ρ denotes the constant mass density, $q = q(t,x)$ the one-dimensional heat flux, $e = e(\theta)$ the internal energy depending on the absolute temperature $\theta = \theta(t,x)$, while $\alpha = \alpha(\theta)$ and $\nu = \nu(\theta)$ are constitutive scalar functions (depending on the material), ν' is the derivative of ν with respect to θ, and $\kappa = \kappa(\theta)$ is the heat conductivity. Equation $(1.5)_1$ represents the energy balance, while $(1.5)_2$ is a generalization of the Maxwell-Cattaneo equation. In the range of temperature in which one can observe the so called "second sound" effect, the heat conductivity becomes very high and thus the system (1.5) can be assumed to be homogeneous. A direct computation of the corresponding characteristic speeds shows that such a system shares the property **(A)**.

It is well-known that, because of the nonlinear dependence of the characteristic speeds $\lambda_k(u)$ on the state variable u, systems of conservation laws, in general, do not admit smooth solutions globally defined in time. Therefore, weak solutions in the sense of distributions are considered. We recall that a continuous map $t \mapsto u(t,\cdot)$, with values in $\mathbb{L}^1_{loc}(\mathbb{R};\mathbb{R}^N)$, is a weak solution to the Cauchy problem (1.1-1.2) if the initial condition (1.2) is verified and, for any smooth function ϕ with compact support contained in $(0,+\infty)\times\mathbb{R}$, there holds

$$\int_0^\infty \int_{-\infty}^\infty \Big[u(t,\,x)\phi_t(t,\,x) + F\big(u(t,\,x)\big)\phi_x(t,\,x)\Big]\, dxdt = 0\,. \tag{1.6}$$

Since, in general, weak solutions to (1.1-1.2) are not unique, an entropy criterion for admissibility is usually added to rule out non-physical discontinuities. In [Li1, Li2] T.P. Liu proposed an admissibility criterion valid for general systems of conservation laws with NGNL characteristic fields. In the case of systems satisfying the assumption **(A)**, this criterion is equivalent to the classical stability condition introduced by Lax [La, Sm] (cfr. Def. 3.1).

For general $N\times N$ systems of the type considered in [Li2] the existence of global weak solutions to (1.1-1.2), with small total variation, was first obtained in [Li3], using the Glimm scheme. An alternative method for constructing solutions of the Cauchy problem, as limit of a sequence of piecewise constant approximate solutions defined by a front tracking algorithm, is developed in [AM2, AM3] for systems of two equations in which both characteristic families satisfy the assumption **(A)**.

The main goal of the present paper is to extend the result in [AM2, AM3] providing a front tracking algorithm valid for $N\times N$ systems with NGNL characteristic fields that satisfy the assumption **(A)**. We remark that the construction of weak solutions of (1.1) obtained as limit of a sequence of front tracking approximations, together with the $\mathbb{L}^1$-stability analysis developed in [AM3, AM5], is fundamental to establish the well-posedness theory for the Cauchy problem (1.1-1.2). Indeed, the $\mathbb{L}^1$-stability estimates obtained in [AM3, AM5] imply that any sequence of front tracking approximate solutions of (1.1-1.2) converges to a unique limit $u = u(t,x)$, and that the resulting map $(\overline{u}, t) \mapsto S_t\,\overline{u} \doteq u(t,\cdot)$ defines a uniformly Lipschitz continuous semigroup, whose trajectories are weak solutions to (1.1). Relying on these results, and using the same technique in [B3], we show here that a Lipschitz continuous semigroup of solutions to (1.1), compatible with the standard solutions of the Riemann problem, is necessarily unique (up to the domain) and that the trajectories of such a semigroup are entropy weak solutions of (1.1) in the sense of Lax. One can then obtain the well-posedness theory, with similar arguments as in [B-G, B-L], showing that any entropy weak solution to (1.1-1.2) must coincide with the trajectory of the semigroup, provided that the total variation of the solution does not grow too wildly.

The basic ideas involved in the construction of piecewise constant approximate solutions, based on wave-front tracking, were introduced in the papers of Dafermos [Da] for scalar equations and Di Perna [Dp1] for 2×2 systems, then extended in [B1, R, BJ] to general $N \times N$ systems with GNL or LD characteristic fields. The construction starts at time $t = 0$ by taking a piecewise constant approximation $u(0, x)$ of the initial data $\overline{u}(x)$. At each point of discontinuity of $\overline{u}$ the resulting Riemann problems are then solved within the class of piecewise constant functions by using an approximate Riemann solver that replace centered rarefaction waves with rarefaction fans containing several small jumps traveling with a speed close to the characteristic speed. Next, one tracks the outgoing fronts until the first time two waves interact. The corresponding Riemann problems can be solved applying again the approximate Riemann solver, etc... The main source of technical difficulty in this construction stems from the fact that the number of lines of discontinuity may approach infinity in finite time. To overcome such a difficulty, the algorithms in [B1, BJ] adopt two different procedures for approximate solving a Riemann problem within the class of piecewise constant functions: an *Accurate Riemann Solver* which introduces several new wave-fronts, and a *Simplified Riemann Solver*, which involves a minimum number of outgoing wave-fronts and collect all the remaining new waves into a single *non-physical* front, traveling with a speed strictly larger than all characteristic speeds. This second procedure is used whenever the product of the strengths of the incoming waves becomes smaller than a threshold parameter ρ, so to generate a new non-physical front with very small amplitude.

Unfortunately, in the case of NGNL systems, this algorithm fails to prevent the number of wave-fronts from approaching infinity within a finite time. This is the consequence of the different structure for such systems of the elementary waves contained in the exact solution of the Riemann Problem. In fact, in contrast with the standard GNL systems, the solution of a Riemann problem for NGNL systems may contain *composed waves* made of several contact discontinuities separated by centered rarefaction waves (see [Li2]). In particular, for systems satisfying the assumption **(A)**, the general self-similar solution of a Riemann problem consists of rarefaction waves, compressive shocks and composed waves made of a single one-sided contact-discontinuity adjacent to a rarefaction wave [AM1]. The presence of such composed waves in the solution of the Simplified Riemann Solver can possibly determine a blow-up of the number of wave-fronts.

We thus need to modify the algorithm by introducing two different types of Simplified Riemann Solvers. If ε is the small parameter that controls the maximum strength of the rarefaction fronts generated by the Accurate Riemann Solver, and if the exact solution of the Riemann problem contains rarefaction waves of strength greater than 3ε, then we basically define the approximate solution of the Riemann problem with the same procedure as for the Simplified Riemann Solvers in [B1, BJ]. In the case where the strength of an outgoing rarefaction wave is smaller than the threshold parameter 3ε, we lump together all new fronts of the outgoing (possibly composed) wave that contains the rarefaction into a single wave-front. In both cases we still collect the remaining new waves into a single non-physical front as in [B1, BJ]. This procedure allows to keep finite the total number of wave-fronts for any time $t > 0$.

A key part of the proof of the global existence of approximate solutions generated by wave-front tracking consists in deriving a-priori bounds on the total variation. As customary, these a-priori estimates are obtained using a functional Q that measures the *potential interaction* of waves in the solution. In particular, for NGNL systems, T.P. Liu [Li3] introduced a functional Q in which the amount of potential interaction between waves of the same family is proportional to the product of the strengths of the waves and of the negative part of their angle. This choice is motivated by the fact that no interaction between two waves is expected when their angle is nonnegative.

Instead here, following the same approach adopted in [AM3], we define the amount of potential interaction between *any* couple of wave-fronts in the approximate solution as proportional to the product of their strengths, no matter if they belong to the same family or not. More precisely, given an approximate solution $u = u(t,x)$ generated by wave-front tracking, with several wave-fronts of size $\sigma_\alpha,\ \alpha = 1,\dots,m$, say located at $x_\alpha = x_\alpha(t)$, we define the interaction potential $Q(t) \doteq Q(u(t,\cdot))$ by setting

$$Q(t) \doteq \sum_{\substack{i_\alpha = i_\beta \neq n+1 \\ \sigma_\alpha\sigma_\beta > 0}} \big|\sigma_\alpha\sigma_\beta\big| + \sum_{\substack{\alpha \\ i_\alpha \neq N+1}} \big|\sigma_\alpha^r \sigma_\alpha\big| + c\Bigg[\sum_{\substack{i_\alpha = i_\beta \neq N+1 \\ \sigma_\alpha\sigma_\beta < 0}} \big|\sigma_\alpha\sigma_\beta\big| + \sum_{\substack{i_\alpha > i_\beta \\ x_\alpha(t) < x_\beta(t)}} \big|\sigma_\alpha\sigma_\beta\big|\Bigg], \tag{1.7}$$

where we regard the non-physical fronts as belonging to a fictitious linearly degenerate $N+1$-th characteristic family, we let σ_α^r denote the (possibly zero) rarefaction component of the wave σ_α, and c is a suitable constant > 1. Notice that, since we are assuming by this definition that the amount of interaction-potential between two wave-fronts of the same family is non zero no matter which angle they make, one would expect that the global interaction-potential may increase after an interaction that produces a composed wave containing a piecewise constant rarefaction fan. However, interactions of this type always involve waves of different families or waves of the same family having an opposite sign. Therefore, observing that the total strength of, say i_α-rarefactions produced by such an interaction, is dominated by the total strength of i_β-waves, $i_\beta \neq i_\alpha$ (or of i_α-waves of opposite sign w.r.t. the outgoing waves) involved in the interaction, one can always control the increase of Q due to the new rarefaction waves by assigning a sufficiently large weight C to the amount of interaction-potential of the incoming waves.

The paper is organized as follows. In Section 2 we introduce the basic notations and review the general definition of the exact solution of a Riemann problem for non genuinely nonlinear systems. Section 3 contains the statements of the main results. In particular, Theorem 1 states the existence of front tracking approximate solutions globally defined in time, while Theorem 2 deals with the convergence of a suitable sequence of front tracking approximations to a function $u = u(t,x)$ that provides a weak solution to the Cauchy problem (1.1-1.2). The Lax admissibility condition for the weak solution follows from Theorem 3 which also states the uniqueness of a Lipschitz continuous semigroup of solutions to (1.1), compatible with the Lax admissible solutions of the Riemann problem. Section 3 contains the proofs of Theorem 2-3, while the proof of Theorem 1 is given in Section 5. The wave-front tracking algorithm is described in Section 4.

2 Preliminaries

Since the system at (1.1) is supposed to be strictly hyperbolic and because we will consider only solutions with small total variation, we shall assume throughout the paper that there exists a positive constant $\overline{\lambda}$ such that

$$\big|\lambda_k(u)\big| < \overline{\lambda}\,, \qquad \forall\, u \in \Omega\,, \quad k = 1,\ldots,N\,. \tag{2.1}$$

For any fixed $u^0 \in \Omega$, and $k = 1,\ldots,N$, let

$$\sigma \mapsto R_k(u^0)[\sigma]\,, \qquad \sigma \mapsto S_k(u^0)[\sigma]\,, \tag{2.2}$$

denote, respectively, the k-rarefaction and the k-shock curve through $u_0 \in \Omega$. Without loss of generality, by possibly performing a linear change of coordinates, we may assume that the k-th component u_k of the vector $u \in R_k(u^0)$ ($u \in S_k(u^0)$) is strictly monotone along $R_k(u^0)$ ($S_k(u^0)$). Thus, as in [Li3], we choose the parameter σ in (2.2) so that

$$\big(R_k(u^0)[\sigma]\big)_k = \big(S_k(u^0)[\sigma]\big)_k = u_k^0 + \sigma. \tag{2.3}$$

This, in particular, means that the strength of a k-wave (u^L, u^R) will be defined as

$$\big|(u^L, u^R)\big| \doteq \big|u_k^L - u_k^R\big|, \tag{2.4}$$

where u_k^L, u_k^R denote, respectively, the k-th component of the left state u^L and of the right state u^R. Recalling that, using a suitable parameterization, the shock curve $S_k(u^0)[\cdot]$ and the rarefaction curve $R_k(u^0)[\cdot]$ have a second order contact at their initial state u^0, we may normalize the k-th right eigenvector $r_k(u)$ of the Jacobian matrix $A(u) = DF(u)$ so that

$$\frac{d}{d\sigma} S_k(u)[\sigma]\Big|_{\sigma=0} = r_k(u)\,. \tag{2.5}$$

In this way, with the parameterization in (2.3), the two curves $R_k(u^0)[\cdot]$, $S_k(u^0)[\cdot]$ have a second order tangency at $\sigma = 0$. The speed of a shock wave connecting the left state u^L to the right state $u^R = S_k(u^L)[\sigma]$ is given by the k-th eigenvalue $\lambda_k(u^L, u^R)$ of the averaged matrix

$$A(u^L, u^R) \doteq \int_0^1 A\big(u^R + \xi(u^R - u^S)\big)d\xi$$

and will be equivalently denoted $\lambda_k^s(u^L)[\sigma]$.

The general entropy admissible solution of a Riemann Problem $\left(u^L,\, u^R\right)$ for a system with NGNL characteristic fields satisfying the assumption **(A)** consists of rarefaction waves, compressive shocks and *composed waves* made of one-sided contact-discontinuities adjacent to rarefaction waves (see [Li1, Li2, AM1]). In particular, if the k-th characteristic family is NGNL, because of (1.3)-(1.4) a composed wave of the k-th characteristic family, connecting a left state u^L with a right state u^R, may appear only in the case where u^L and u^R lie on opposite sides with respect to the manifold Ω_k^0. Such a wave consists of

CASE 1: a left contact-discontinuity on the right of a rarefaction wave, both of the k-th characteristic family, if

$$D_{r_k}^2\lambda_k\left(u^0\right) < 0\,, \qquad\qquad u^0 \in \Omega_k^0\,, \tag{2.6}$$

i.e. if, for each $u \in \Omega$, the characteristic speed λ_k attains its maximum value along the rarefaction curve $R_k(u)$ at the intersection point $R_k(u) \cap \Omega_k^0$;

CASE 2: a right contact-discontinuity on the left of a rarefaction wave, both of the k-th characteristic family, if

$$D_{r_k}^2\lambda_k\left(u^0\right) > 0\,, \qquad\qquad u^0 \in \Omega_k^0\,, \tag{2.7}$$

i.e. if, for each $u \in \Omega$, the characteristic speed λ_k attains its maximum value along the rarefaction curve $R_k(u)$ at the intersection point $R_k(u) \cap \Omega_k^0$;

In order to express the general solution of the Riemann problem, to take advantage of the symmetry of the system we define for the NGNL characteristic families two different types of elementary curves, according to which of the above two cases occurs (see [AM1]).

If (2.6) holds: we define the *elementary curve through u^0 of right states of the k-th characteristic family*, that consists of all right states connected with the left state u^0 by an elementary wave (rarefaction, shock or composed wave) of the k-th characteristic family.

If (2.7) holds: we define the *elementary curve through u^0 of left states of the k-th characteristic family*, that consists of all left states connected with the right state u^0 by an elementary wave (rarefaction, shock or composed wave) of the k-th characteristic family.

Of course, in the case the k-th characteristic family is GNL or LD, the elementary curve through u^0 is defined as usual as the curve of all right states connected with the left state u^0 by a k-th shock or a k-th rarefaction wave.

The elementary curve through u^0 of the k-th characteristic family, denoted by $\Psi_k\left(u^0\right)$, is in general a concatenation of a rarefaction curve, a shock curve and, in the case the k-th family is NGNL, a *mixed curves* $M_k\left(u\right)$ that represents all the states u^1 that are connected to u by a composed wave of the k-th characteristic family and thus consists of all the states u^1 for which there exists a (unique) point $\widetilde{u} \in R_k(u)$ such that

$$u^1 \in S_k(\widetilde{u}), \qquad \lambda_k(\widetilde{u}, u^1) = \lambda_k(\widetilde{u}). \tag{2.8}$$

Clearly, the mixed curve $M_k(u)$ describes all the right states u^1 which are connected to the left state u with a rarefaction wave $(u, \widetilde{u})$ followed by a left-sided contact discontinuity $(\widetilde{u}, u^1)$, in the case (2.6) holds, while $M_k(u)$ describes all the left states u^1 which are connected to the right state u with a right-sided contact discontinuity $(u^1, \widetilde{u})$ followed by a rarefaction wave $(\widetilde{u}, u)$, in the case (2.7) holds.

We collect next some basic properties of the mixed curve that are useful for the general definition of elementary curve and to establish the local wave interaction estimates. For a more detailed discussion see [AM1]. We shall consider only the case of a NGNL characteristic family for which (2.6) holds, the other case being entirely similar. Here and throughout the paper the Landau symbol $\mathcal{O}(1)$ shall always denote a quantity uniformly bounded by a constant depending only on the system (1.1).

Recalling that, because of (1.4), any rarefaction curve R_k is transversal to the manifold Ω_k^0 at (1.3), one can implicitly define the map $\delta_k : \Omega \to \mathbb{R}$ by setting

$$R_k(u)\big[\delta_k(u)\big] \in \Omega_k^0\,. \tag{2.9}$$

The quantity, $\delta_k(u)$ represents the signed distance of Ω_k^0 from u measured along the rarefaction curve $R_k(u)$. Next, let $\pi_k : \Omega \to \Omega_k^0$ be the projection on Ω_k^0 along a k-characteristic curve, i.e.

$$\pi_k(u) = R_k(u)\big[\delta_k(u)\big]\,. \tag{2.10}$$

The following properties hold.

1. There exists a smooth map $u \mapsto \nu_k(u)$ such that

$$\lambda_k^s(u)\big[\nu_k(u)\big] = \lambda_k(u) \qquad \forall\, u \in \Omega\,. \tag{2.11}$$

   The quantity $\nu_k(u)$ represents the size of a left-sided contact discontinuity connecting the left state u with the right state $S_k(u)[\nu_k(u)]$. The map ν_k has the following expansion

$$\nu_k(u) = 3\delta_k(u) + \mathcal{O}(1)\big|\delta_k(u)\big|^2 \qquad \forall\, u \in \Omega\,. \tag{2.12}$$

2. For any fixed $u^0 \in \Omega$ with $D_{r_k}\lambda_k(u^0) > 0$, there exists a smooth map

$$\sigma \mapsto \zeta_k(u^0)[\sigma] \qquad \sigma \in \big[\delta_k(u^0),\, \nu_k(u^0)\big]\,, \tag{2.13}$$

   such that

$$\zeta_k(u^0)\big[\delta_k(u^0)\big] = \delta_k(u^0), \qquad \zeta_k(u^0)\big[\nu_k(u^0)\big] = 0\,, \tag{2.14}$$

$$\lambda_k\Big(R_k(u^0)\big[\zeta_k(u^0)[\sigma]\big]\Big) = \lambda_k^s\Big(R_k(u^0)\big[\zeta_k(u^0)[\sigma]\big]\Big)\big[\sigma - \zeta_k(u^0)[\sigma]\big]\,. \tag{2.15}$$

The relation (2.15) says that the left state u^0 and the right state

$$S_k\Big(R_k(u^0)\,[\zeta_k(u^0)[\sigma]]\Big)\,[\sigma-\zeta_k(u^0)[\sigma]]$$

are connected by a composed wave of size σ consisting of a rarefaction wave of size $\zeta_k(u^0)[\sigma]$ followed by a left-sided contact discontinuity of size $\sigma-\zeta_k(u^0)[\sigma]$. The map $\zeta_k(u^0)$ has the following expansion

$$\zeta_k(u^0)[\sigma]=\frac{\big(3\delta_k^r(u^0)-\sigma\big)}{2}+\mathcal{O}(1)\big|\sigma-\delta_k^r(u^0)\big|^2. \tag{2.16}$$

3. For any fixed $u^0\in\Omega$, the mixed curve $M_k(u^0)$ is given by

$$M_k(u^0)[\sigma]=S_k\Big(R_k(u^0)\,[\zeta_k(u^0)[\sigma]]\Big)\,[\sigma-\zeta_k(u^0)[\sigma]]\,. \tag{2.17}$$

Moreover, $M_k(u^0)$ has a second order tangency at $M_k(u^0)[\delta_k(u^0)]$ with $R_k(u^0)$, and has a first order tangency at $M_k(u^0)[\nu_k(u^0)]$ with $S_k(u^0)$.

For any NGNL k-th characteristic family, consider the sets

$$\Omega_k^+\doteq\big\{u\in\Omega\ :\ D_{r_k}\lambda_k(u)>0\big\}\,,\qquad\qquad \Omega_k^-\doteq\big\{u\in\Omega\ :\ D_{r_k}\lambda_k(u)<0\big\}\,. \tag{2.18}$$

Then, in case (2.6) holds, define the elementary curve $\sigma\to\Psi_k(u)[\sigma]$ as follows

$$\Psi_k(u)[\sigma]=\begin{cases} S_k(u)[\sigma] \qquad \forall\,\sigma & \text{if}\quad u\in\Omega_k^0,\\[4pt] \begin{cases} S_k(u)[\sigma] & \text{if}\quad \sigma<0\\ R_k(u)[\sigma] & \text{if}\quad 0\le\sigma\le\delta_k(u)\\ M_k(u)[\sigma] & \text{if}\quad \delta_k(u)<\sigma\le\nu_k(u)\\ S_k(u)[\sigma] & \text{if}\quad \sigma>\nu_k(u)\end{cases} & \text{if}\quad u\in\Omega_k^+,\\[4pt] \begin{cases} S_k(u)[\sigma] & \text{if}\quad \sigma>0\\ R_k(u)[\sigma] & \text{if}\quad \delta_k(u)\le\sigma\le0\\ M_k(u)[\sigma] & \text{if}\quad \nu_k(u)\le\sigma<\delta_k(u)\\ S_k(u)[\sigma] & \text{if}\quad \sigma<\nu_k(u)\end{cases} & \text{if}\quad u\in\Omega_k^-.\end{cases} \tag{2.19}$$

Entirely similar definitions are given for $\Psi_k(u)[\cdot]$ in the case (2.7) holds instead of (2.6). Notice that, whenever a left state u^L and a right state u^R are connected by an entropy admissible discontinuity of the k-th NGNL family, one of the following two cases occurs:

E1. The two states u^L, u^R lie on the same side w.r.t. the manifold Ω_k^0 and

if (2.6) holds, letting σ be the size of the corresponding jump, i.e. $u^R=\Psi_k(u^L)[\sigma]$, one has

$$\mathrm{sgn}(\sigma)\neq\mathrm{sgn}\big(D_{r_k}\lambda_k(u^L)\big)\,;$$

if (2.7) holds, letting σ be the size of the corresponding jump, i.e. $u^L = \Psi_k(u^R)[\sigma]$, one has

$$\mathrm{sgn}(\sigma) = \mathrm{sgn}\big(D_{r_k}\lambda_k(u^R)\big)\,.$$

E2. The two states u^L, u^R lie on opposite sides w.r.t. the manifold Ω_k^0 and

if (2.6) holds, letting σ be the size of the corresponding jump, i.e. $u^R = \Psi_k(u^L)[\sigma]$, one has

$$\mathrm{sgn}(\sigma) = \mathrm{sgn}\big(D_{r_k}\lambda_k(u^L)\big), \qquad |\sigma| \geq \big|\nu_k(u^L)\big|\,; \tag{2.20}$$

if (2.7) holds, letting σ be the size of the corresponding jump, i.e. $u^L = \Psi_k(u^R)[\sigma]$, one has

$$\mathrm{sgn}(\sigma) \neq \mathrm{sgn}\big(D_{r_k}\lambda_k(u^R)\big), \qquad |\sigma| \geq \big|\nu_k(u^R)\big|\,. \tag{2.21}$$

In the case the k-th characteristic family is LD or GNL with

$$D_{r_k}\lambda_k(u) > 0 \qquad \forall\, u \in \Omega, \tag{2.22}$$

the corresponding elementary curve is defined as usual by setting

$$\Psi_k(u)[\sigma] = \begin{cases} R_k(u)[\sigma] & \text{if} \quad \sigma \geq 0, \\ S_k(u)[\sigma] & \text{if} \quad \sigma < 0. \end{cases} \tag{2.23}$$

Thus, for any u^L, u^R in a sufficiently small neighborhood of the origin $\Omega' \subset \Omega$, by the Implicit Function Theorem we can uniquely determine intermediate states $u^L = \omega_0$, $\omega_1, \ldots, \omega_N = u^R$ and wave sizes $\sigma_1, \ldots, \sigma_N$, such that there holds

$$\begin{aligned} \omega_k &= \Psi_k(\omega_{k-1})[\sigma_k] \qquad && \text{if} \qquad (2.6) \quad \text{occurs}, \\ \omega_{k-1} &= \Psi_k(\omega_k)[\sigma_k] \qquad && \text{if} \qquad (2.7) \quad \text{occurs}. \end{aligned} \tag{2.24}$$

Then, the solution to the Riemann Problem with initial data (u^L, u^R) can be constructed by piecing together the solutions to the Riemann problems with initial data

$$\overline{u}_k(x) = u_k(0, x) = \begin{cases} \omega_{k-1} & \text{if } x < 0, \\ \omega_k & \text{if } x < 0, \end{cases} \qquad k = 1, \ldots, N\,.$$

If (2.24) holds, we say that ω_{k-1} and ω_k are connected by a k-wave of size σ_k. Throughout the paper, with a slight abuse of notations, we shall simply call σ a wave of size σ.

Remark 2.1 Given a wave (u^L, u^R) of size σ, belonging to the k-th NGNL family, we will often use a superscript ^{r} and a superscript ^{s} to denote, respectively, the sizes of its rarefaction and shock components, i.e.

$$\begin{aligned} (u^L, u^R)^r &\doteq \sigma^r \doteq \zeta_k\big(u^L\big)[\sigma], \qquad && \text{if} \quad (2.6) \quad \text{holds}, \\ (u^L, u^R)^r &\doteq \sigma^r \doteq \zeta_k\big(u^R\big)[\sigma], \qquad && \text{if} \quad (2.7) \quad \text{holds}, \\ (u^L, u^R)^s &\doteq \sigma^s \doteq \sigma - \sigma^r\,, \end{aligned} \tag{2.25}$$

where $\sigma \to \zeta_k\big(u\big)[\sigma]$ denotes the map defined at (2.13-2.15).

We collect next the basic estimates on the change of wave-size after an interaction whose proofs are quite standard and can be derived with the same arguments in [B2, Section 7], using the implicit function theorem, the geometrical properties of rarefaction, shock and mixed curves, and the expansions (2.12), (2.16). Let us give first the following

Definition 2.1 Given two incoming waves of size σ', σ'', we define the *quantity of interaction* $\mathcal{I}(\sigma', \sigma'')$ between σ' and σ'' as

$$\mathcal{I}(\sigma', \sigma'') \doteq \begin{cases} \Big[\big|(\sigma'+\sigma'')^r - \sigma'^{\,r}\big| + \big|\sigma'^{\,s}\big|\Big]\big|\sigma''\big| & \text{if } \sigma',\, \sigma'' \text{ belong to the } k\text{-th NGNL} \\ & \text{characteristic family, } \ \sigma'\sigma''>0 \\ & \text{and 2.6 holds,} \\ \Big[\big|(\sigma'+\sigma'')^r - \sigma''^{\,r}\big| + \big|\sigma''^{\,s}\big|\Big]\big|\sigma'\big| & \text{if } \sigma',\, \sigma'' \text{ belong to the } k\text{-th NGNL} \\ & \text{characteristic family, } \ \sigma'\sigma''>0 \\ & \text{and 2.7 holds,} \\ \big|\sigma'\sigma''\big| & \text{in all the other cases,} \end{cases} \tag{2.26}$$

where, in the case σ', σ'' belong to the same k-th NGNL characteristic family, we interpret $\sigma'+\sigma''$ as the size of a k-wave having the same left (right) state as σ' (σ'') if (2.6) ((2.7)) holds.

Lemma 2.1 *For any compact set $K \subset \Omega$, there exists constants $C_0, C_1, \mu_1 > 0$, such that, whenever $u^L, u^R \in K$, $|\sigma'_i|, |\sigma''_i||\sigma'_j| \leq \mu_1$, the following statements hold.*

(i) Let σ'_i, σ'_j be the sizes of two incoming waves belonging to the distinct characteristic families $i > j$. Assume that σ'_i is located on the left of σ'_j, and that u^L, u^R are, respectively, the left state of σ'_i and the right state of σ'_j. Then, the sizes of the outgoing waves σ_k, $k = 1, \ldots, N$, defined at (2.24), satisfy

$$\big|\sigma_i - \sigma'_i\big| + \big|\sigma_j - \sigma'_j\big| + \sum_{k\neq i,j} \big|\sigma_k\big| \leq C_1\ \mathcal{I}(\sigma', \sigma'')\,. \tag{2.27}$$

If σ_i, σ_j are composed waves and σ_i^r, σ_j^r are the sizes of their rarefaction components, then there holds

$$C_0 \min\Big\{\big|\sigma'_j\big|,\, \big|\sigma'_i\big|\Big\} \leq \big|\sigma_i^r - \sigma_i'^{\,r}\big|,\, \big|\sigma_j^r - \sigma_j'^{\,r}\big| \leq C_1 \min\Big\{\big|\sigma'_j\big|,\, \big|\sigma'_i\big|\Big\}\,. \tag{2.28}$$

Moreover, if we consider the auxiliary right state $\widetilde{u}$, connected on the left to u^L by a j-wave of size σ'_j followed by an i-wave of size σ'_i, then one has

$$\big|u^R - \widetilde{u}\big| \leq C_1\ \mathcal{I}(\sigma', \sigma'')\,. \tag{2.29}$$

(ii) *Let σ_i', σ_i'' be the sizes of two incoming waves, both belonging to the i-th characteristic family. As before, assume that σ_i' is located on the left of σ_i'', and that u^L, u^R are, respectively, the left state of σ_i' and the right state of σ_i''. Then, the sizes of the outgoing waves σ_k, $k = 1, \ldots, N$, defined at (2.24), satisfy*

$$\left|\sigma_i - \sigma_i' - \sigma_i''\right| + \sum_{k \neq i} \left|\sigma_k\right| \leq C_1 \ \mathcal{I}(\sigma', \sigma'') . \tag{2.30}$$

If σ_i is a composed wave, σ_i^r is the size of its rarefaction component, and $\sigma' + \sigma''$ represents the size of an i-wave having left state u^L, then one has

$$\left|\sigma_i^r - (\sigma_i' + \sigma_i'')^r\right| \leq C_1 \ \mathcal{I}(\sigma', \sigma'') . \tag{2.31}$$

Furthermore, whenever $\sigma_i' \cdot \sigma_i'' < 0$, there holds

$$C_0 \min\left\{\left|\sigma_i'\right|, \left|\sigma_i''\right|\right\} \leq \left|\sigma_i^r - \sigma_i'^{\,r} + \sigma_i''^{\,r}\right| \leq C_1 \min\left\{\left|\sigma_i'\right|, \left|\sigma_i''\right|\right\} . \tag{2.32}$$

Moreover, if we consider the auxiliary right state $\widetilde{u}$, connected on the left to u^L by an i-wave of size $\sigma_i' + \sigma_i''$, then one has

$$\left|u^R - \widetilde{u}\right| \leq C_1 \ \mathcal{I}(\sigma', \sigma'') . \tag{2.33}$$

(iii) *Let $\left(u^L, u^R\right)$, $\left(v^L, v^R\right)$, $\ v^L, v^R \in K$, be two i-waves of size σ, and assume that $|\sigma|$, $\left|u^L - v^L\right| \leq \mu_1$. Then, one has*

$$\left|v^R - u^R\right| - \left|v^L - u^L\right| \leq C_1 |\sigma| \left|v^L - u^L\right| . \tag{2.34}$$

3 Statements of Main Results

In this section we first present a result that states the existence of front tracking approximate solutions to (1.1)-(1.2) defined for all time $t > 0$, whose proof is postponed to Sections 4-5. Next, we establish the convergence of a suitable sequence of approximate solutions to a limit $u = u(t,x)$ that provides a weak solution of (1.1)-(1.2). Finally, relying on the $\mathbb{L}^1$-stability estimates obtained in [AM3, AM5], we show that weak solutions of the Cauchy problem constructed as limit of front tracking approximations actually satisfy the Lax entropy conditions and generate a unique Lipschitz continuous semigroup of solutions to (1.1)-(1.2), compatible with the Lax entropy admissible solutions of the Riemann problem.

Theorem 1 *Assume that system (1.1) is strictly hyperbolic and that each NGNL characteristic family satisfy the assumption* **(A)**. *Then, there exists $\delta_0 > 0$ with the following property. For every initial condition $\overline{u} \in \mathbb{L}^1$ satisfying*

$$\mathrm{Tot.Var.}(\overline{u}) < \delta_0 \,, \tag{3.1}$$

and for every $\varepsilon > 0$, there exists a uniformly (in ε) Lipschitz continuous map

$$t \mapsto u_\varepsilon(t, \cdot), \qquad t \in [0, \infty),$$

with values in $\mathbb{L}^1_{loc}(\mathbb{R}; \mathbb{R}^N)$, that provides an ε-approximate solution of (1.1)-(1.2) in the following sense:

1.

$$\left\| u_\varepsilon(0, \cdot) - \overline{u} \right\|_{\mathbb{L}^1} < \varepsilon \,. \tag{3.2}$$

2. *There exists $\delta > 0$ such that*

$$\text{Tot.Var.}\big(u_\varepsilon(t, \cdot)\big) \leq \delta, \qquad \forall\, t > 0\,, \quad \varepsilon > 0\,. \tag{3.3}$$

3. *As a function of two variables, $u_\varepsilon = u_\varepsilon(t, x)$ is piecewise constant with discontinuities occurring along finitely many straight lines in the t-x plane. Only finitely many wave-fronts interactions occur, each involving exactly two incoming fronts. Jumps can be of three types: shocks (or one-sided contact discontinuities), rarefactions and non-physical waves, denoted, respectively, as $\mathcal{S}, \mathcal{R}, \mathcal{NP}$. The set of all jumps is denoted $\mathcal{J} = \mathcal{S} \cup \mathcal{R} \cup \mathcal{NP}$.*

4. *Along each shock (or contact discontinuity) $x = x_\alpha(t)$, $\alpha \in \mathcal{S}$, the values $u^L \doteq u_\varepsilon(t, x_\alpha-)$ and $u^R \doteq u_\varepsilon(t, x_\alpha+)$ satisfy one of the following conditions.*

   (a) *There exists some wave size σ_α and some index $k_\alpha \in \{1, \dots, N\}$ corresponding to a NGNL characteristic family for which (2.6) holds, such that*

$$\left| u^R - S_{k_\alpha}(u^L)[\sigma_\alpha] \right| = \mathcal{O}(1) \cdot \varepsilon |\sigma_\alpha| \,. \tag{3.4}$$

   If the states u^L, $S_{k_\alpha}(u^L)[\sigma_\alpha]$ lie on the same side w.r.t. the manifold $\Omega^0_{k_\alpha}$, the Lax entropy admissibility conditions **E1** *for $\big(u^L, S_{k_\alpha}(u^L)[\sigma_\alpha]\big)$ also holds. Otherwise, there holds*

$$\text{sgn}(\sigma_\alpha) = \text{sgn}\big(D_{r_{k_\alpha}} \lambda_{k_\alpha}(u^L)\big)\,, \qquad |\sigma_\alpha| \geq \left| \nu_{k_\alpha}(u^L) \right| - \mathcal{O}(1) \cdot \varepsilon \,. \tag{3.5}$$

   (b) *There exists some wave size σ_α and some index $k_\alpha \in \{1, \dots, N\}$ corresponding to a NGNL characteristic family for which (2.7) holds, such that*

$$\left| u^L - S_{k_\alpha}(u^R)[\sigma_\alpha] \right| = \mathcal{O}(1) \cdot \varepsilon |\sigma_\alpha| \,. \tag{3.6}$$

   If the states $S_{k_\alpha}(u^R)[\sigma_\alpha]$, u^R lie on the same side w.r.t. the manifold $\Omega^0_{k_\alpha}$, the Lax entropy admissibility conditions **E2** *for $\big(S_{k_\alpha}(u^R)[\sigma_\alpha], u^R\big)$ also holds. Otherwise, there holds*

$$\text{sgn}(\sigma_\alpha) \neq \text{sgn}\big(D_{r_{k_\alpha}} \lambda_{k_\alpha}(u^R)\big), \qquad |\sigma_\alpha| \geq \left| \nu_{k_\alpha}(u^R) \right| - \mathcal{O}(1) \cdot \varepsilon \,. \tag{3.7}$$

(c) *There exists some wave size* σ_α *and some index* $k_\alpha \in \{1, \ldots, N\}$ *corresponding to a GNL or LD characteristic family, such that*

$$\left|u^R - S_{k_\alpha}(u^L)[\sigma_\alpha]\right| = \mathcal{O}(1) \cdot \varepsilon |\sigma_\alpha| \,. \tag{3.8}$$

If the k_α*-th family is GNL, then the Lax entropy admissibility condition* $\sigma_\alpha < 0$ *also holds.*

Moreover, in any of the above cases, the speed $\dot{x}_\alpha$ *of the shock front satisfies*

$$\left|\dot{x}_\alpha - \lambda^s_{k_\alpha}(u^L,\, u^R)\right| = \mathcal{O}(1) \cdot \varepsilon \,. \tag{3.9}$$

5. *Along each rarefaction front* $x = x_\alpha(t)$, $\alpha \in \mathcal{R}$, *the values* $u^L \doteq u_\varepsilon(t, x_\alpha -)$ *and* $u^R \doteq u_\varepsilon(t, x_\alpha +)$ *are related by*

$$u^R = R_{k_\alpha}(u^L)[\sigma_\alpha] \,,$$

for some $k_\alpha \in \{1, \ldots, N\}$ *corresponding to a GNL or NGNL characteristic family satisfying (2.6), and some wave size* σ_α *such that*

$$\sigma_\alpha = \mathcal{O}(1) \cdot \varepsilon, \qquad \operatorname{sgn}(\sigma_\alpha) = \operatorname{sgn}\big(D_{r_{k_\alpha}} \lambda_{k_\alpha}(u^L)\big) \,, \tag{3.10}$$

or by

$$u^L = R_{k_\alpha}(u^R)[\sigma_\alpha] \,,$$

for some $k_\alpha \in \{1, \ldots, N\}$ *corresponding to a NGNL characteristic family satisfying (2.7), and some wave size* σ_α *such that*

$$\sigma_\alpha = \mathcal{O}(1) \cdot \varepsilon, \qquad \operatorname{sgn}(\sigma_\alpha) \neq \operatorname{sgn}\big(D_{r_{k_\alpha}} \lambda_{k_\alpha}(u^L)\big) \,. \tag{3.11}$$

Moreover, the speed $\dot{x}_\alpha$ *of the rarefaction front satisfies*

$$\left|\dot{x}_\alpha - \lambda_{k_\alpha}(u^L)\right| = \mathcal{O}(1) \cdot \varepsilon \,. \tag{3.12}$$

6. *All non-physical fronts* $x = x_\alpha(t)$, $\alpha \in \mathcal{NP}$ *have the same speed*

$$\dot{x}_\alpha \equiv \widehat{\lambda} \,, \tag{3.13}$$

where $\widehat{\lambda}$ *is a fixed constant strictly greater than all characteristic speeds* $\lambda_k(u)$, $u \in \Omega$, $k = 1, \ldots, N$, *i.e.*

$$\widehat{\lambda} > \lambda_i(u), \qquad \forall\, u \in \Omega, \quad i = 1, \ldots, N \,. \tag{3.14}$$

The total strength of all non-physical fronts in $u(t, \cdot)$ *remains uniformly small, namely one has*

$$\sum_{\alpha \in \mathcal{NP}} \left|u_\varepsilon(t, x_\alpha +) - u_\varepsilon(t, x_\alpha -)\right| = \mathcal{O}(1) \cdot \varepsilon \qquad \forall\, t \geq 0 \,. \tag{3.15}$$

The proofs of the next two theorems rely only on the qualitative properties (3.2)-(3.15) of the approximate solutions stated in Theorem 1, and not on the particular front-tracking algorithm used to construct them.

Theorem 2 *In the same setting as Theorem 1, assume that the initial condition $\overline{u} \in \mathbb{L}^1$ satisfies (3.1). Let $\{\varepsilon_\nu\}_{\nu\in\mathbb{N}}$ be a sequence of positive real number converging to zero, and let $u_\nu \doteq u_{\varepsilon_\nu}(t,x)$ be an ε_ν-approximate solution of (1.1)-(1.2) constructed as in Theorem 1. Then, there exists a subsequence of $\{u_\nu\}_{\nu\in\mathbb{N}}$ converging in $\mathbb{L}^1_{loc}([0,+\infty)\times\mathbb{R};\ \mathbb{R}^N)$ to a weak solution $u = u(t,x)$ of the Cauchy problem (1.1)-(1.2).*

Proof The proof is standard and relies on Helly's Compactness Theorem (see [B2] and references therein). Since the functions $t \mapsto u_\nu(t,\cdot)$ are uniformly Lipschitz continuous and thanks to (3.3), by Helly's Theorem there exists a subsequence of $\{u_\nu\}_{\nu\in\mathbb{N}}$ which converges in $\mathbb{L}^1_{loc}((0,+\infty)\times\mathbb{R})$ to a function $u = u(t,x)$. Because of (3.2), the initial condition (1.2) clearly holds. Hence, by (1.6), in order to prove that u is a weak solution to (1.1)-(1.2) it suffices to show that, for any smooth function ϕ with compact support in $(0,+\infty)\times\mathbb{R}$, there holds

$$\lim_{\nu\to+\infty}\int_0^\infty\int_{-\infty}^\infty \Big[u_\nu(t,\,x)\phi_t(t,\,x) + F\big(u_\nu(t,\,x)\big)\phi_x(t,\,x)\Big]\ dxdt = 0\,. \tag{3.16}$$

Choose $T>0$ so that ϕ is supported in $(0,T)\times\mathbb{R}$. By the divergence theorem we have

$$\begin{aligned}\int_0^{+\infty}\int_{\mathbb{R}} \Big[u_\nu(t,\,x)\phi_t(t,\,x) + f\big(u_\nu(t,\,x)\big)\phi_x(t,\,x)\Big]\ dx\ dt = \\ = \int_0^T \sum_\alpha \bigg(\dot{x}_\alpha\Big[u_\nu(t,\,x_\alpha+) - u_\nu(t,\,x_\alpha-)\Big] + \\ - \Big[F\big(u_\nu(t,\,x_\alpha+)\big) - F\big(u_\nu(t,\,x_\alpha-)\big)\Big]\bigg)\phi(t,\,x_\alpha)\ dt\,,\end{aligned}$$

where $x_\alpha = x_\alpha(t)$ denotes a line of discontinuity of u_ν in the strip $[0,T]\times\mathbb{R}$. Denote with $\mathcal{R}(t)$, $\mathcal{S}(t)$, $\mathcal{N}P(t)$, respectively, the sets of all rarefactions, shocks and non-physical fronts of $u_\nu(t,\cdot)$. Relying on (3.3)-(3.12), (3.15), and proceeding as in [B2], we obtain

$$\begin{aligned}\int_0^T \sum_\alpha \bigg(\dot{x}_\alpha\Big[u_\nu(t,\,x_\alpha+) - u_\nu(t,\,x_\alpha-)\Big] - \Big[F\big(u_\nu(t,\,x_\alpha+)\big) - F\big(u_\nu(t,\,x_\alpha-)\big)\Big]\bigg)\phi(t,\,x_\alpha)\ dt \\ = \mathcal{O}(1)\cdot\max_{t,\,x}|\phi|\int_0^T\bigg(\sum_{\alpha\in\mathcal{R}(t)}|\sigma_\alpha|\big(\varepsilon_\nu + |\sigma_\alpha|\big) + \sum_{\alpha\in\mathcal{S}(t)}\varepsilon_\nu|\sigma_\alpha| + \sum_{\alpha\in\mathcal{N}P(t)}|\sigma_\alpha|\bigg)\ dt \\ = \mathcal{O}(1)\cdot\varepsilon_\nu(\delta+1)\,T\max_{t,\,x}|\phi|\,.\end{aligned}$$

By letting $\nu\to+\infty$, the limit (3.16) follows, thus proving that u is a weak solution to (1.1)-(1.2).

□

Before stating the last theorem proved in the paper, we recall here the Lax stability condition [La] as reformulated in [B2] and the definition of Standard Riemann Semigroup [B2, B3] generated by system (1.1).

Definition 3.1 Let $u = u(t,x)$ be a function with values in Ω. A point (τ,ξ) is of approximate jump for u if there exist states u^L, $u^R \in \Omega$, and a speed $\lambda \in \mathbb{R}$ such that, calling

$$U(t,\,x) \doteq \begin{cases} u^L & \textit{if} \quad x < \lambda t\,, \\ u^R & \textit{if} \quad x > \lambda t\,, \end{cases} \tag{3.17}$$

there holds

$$\lim_{s\to 0+} \frac{1}{s^2} \int_{\tau-s}^{\tau+s} \int_{\xi-s}^{\xi+s} \Big| u(t,\,x) - U(t-\tau,\,x-\xi) \Big| \; dx\; dt = 0\,. \tag{3.18}$$

A weak solution $u = u(t,x)$ of (1.1) is *Lax entropy admissible* if, at every point (τ,ξ) of approximate jump, the states u^L, $u^R \in \Omega$ and the speed $\lambda \in \mathbb{R}$ determined by (3.17-3.18) satisfy the Lax entropy inequality

$$\lambda_k\big(u^L\big) \geq \lambda \geq \lambda_k\big(u^R\big)\,, \tag{3.19}$$

for some $k \in \{1,\ldots,N\}$.

Definition 3.2 Let $\mathcal{D} \subset \mathbb{L}^1(\mathbb{R};\,\mathbb{R}^N)$ be a closed domain. A map $S : \mathcal{D} \times [0,+\infty) \to \mathcal{D}$ is a *Standard Riemann Semigroup* generated by the system of conservation laws (1.1) if the following conditions hold.

1. For every $\overline{u} \in \mathcal{D}$, t, $s \geq 0$, one has
$$S_0\overline{u} = \overline{u}\,, \qquad\qquad S_t S_s \overline{u} = S_{t+s}\overline{u}\,.$$
2. There exists a Lipschitz constant L such that, for all $\overline{u}$, $\overline{v} \in \mathcal{D}$, $\;t$, $s \geq 0$, one has
$$\big\| S_t\overline{u} - S_s\overline{v} \big\|_{\mathbb{L}^1} \leq L\Big(\big|\overline{u} - \overline{v}\big|_{\mathbb{L}^1} + |t-s| \Big)\,.$$
3. Given any piecewise constant initial data $\overline{u} \in \mathcal{D}$, there exists $\delta > 0$ such that, for all $t \in [0,\,\delta]$, the trajectory $u(t,\cdot) = S_t\overline{u}$ coincides with the solution of the Cauchy problem (1.1)-(1.2) obtained by piecing together the Lax entropy admissible solutions of the Riemann problem determined by the jumps of $\overline{u}$.

Theorem 3 *In the same setting as Theorem 1, there exists a closed domain $\mathcal{D} \subset \mathbb{L}^1(\mathbb{R};\,\mathbb{R}^N)$ containing all functions with sufficiently small total variation such that, for any initial data $\overline{u} \in \mathcal{D}$, any sequence of ε-approximate front tracking solutions of the Cauchy problem (1.1)-(1.2) converges to a unique limit $u = u(t,\,x)$, as $\varepsilon \to 0$. The map*

$$(\overline{u},\,t) \mapsto S_t\overline{u} \doteq u(t,\cdot) \tag{3.20}$$

defines a Standard Riemann Semigroup, whose trajectories are entropy weak solutions of (1.1) in the sense of Lax. Moreover, if another Standard Riemann Semigroup

$$\widetilde{S} : \widetilde{\mathcal{D}} \times [0, +\infty) \to \widetilde{\mathcal{D}}$$

exists on a domain $\widetilde{\mathcal{D}} \supseteq \mathcal{D}$, then

$$\widetilde{S}_t \overline{u} = S_t \overline{u} \qquad \forall\, \overline{u} \in \mathcal{D}\,, \quad t \geq 0\,. \tag{3.21}$$

Proof Consider a domain of the form

$$\mathcal{D} = cl\Big\{u \in \mathbb{L}^1\left(\mathbb{R};\, \mathbb{R}^N\right); \quad u \text{ is piecewise constant}, \quad V(u) + c_1 \cdot Q(u) < \delta_1\Big\},$$

where V, Q are functionals measuring, respectively, the total strength of waves in u and the wave interaction potential (for a precise definition see (5.1), (5.2)), while cl denotes closure in $\mathbb{L}^1$, and c_1, $\delta_1 > 0$ are suitable constants. Then, a proof of the uniqueness of the limit of front tracking approximate solutions to (1.1-1.2), with $\overline{u} \in \mathcal{D}$, can be obtained with the same arguments in [BLY, Theorem 2], relying on the $\mathbb{L}^1$-stability estimates established in [AM3, AM5]. In turn, the semigroup property is an immediate consequence of uniqueness, and the trajectories of the semigroup are weak solutions of (1.1) because of Theorem 2. Regarding the uniqueness of the Standard Riemann Semigroup generated by system (1.1), proceeding as in [B2, Theorem 9.1] we provide a bound on the distance between any ε-approximate front tracking solution $u_\varepsilon = u_\varepsilon(t, x)$, satisfying the additional property $u_\varepsilon(t,\cdot) \in \mathcal{D}$ for all $t \geq 0$, and the corresponding semigroup trajectory $t \mapsto \widetilde{S}_t \overline{u}$, by means of the error estimate

$$\left\|u_\varepsilon(t) - \widetilde{S}_t u_\varepsilon(0)\right\| \leq L \cdot \int_0^t \left\{ \liminf_{h \to 0+} \frac{\left\|u_\varepsilon(\tau + h) - \widetilde{S}_h u_\varepsilon(\tau)\right\|}{h} \right\} d\tau, \tag{3.22}$$

valid for every Lipschitz continuous map $u_\varepsilon : [0, t] \mapsto \mathcal{D}$. To estimate the right-hand side of (3.22), consider any time $\tau \in [0,\, t]$ where no wave-front interaction occurs. Then, using (3.4)-(3.12), (3.14)-(3.15), with the same arguments in [B2, Theorem 9.1] we derive

$$\begin{aligned}
\limsup_{h \to 0+} & \frac{\left\|u_\varepsilon(\tau + h, \cdot) - \widetilde{S}_h u_\varepsilon(\tau)\right\|_{\mathbb{L}^1}}{h} = \\
&= \sum_{\alpha \in \mathcal{S}} \mathcal{O}(1) \cdot \varepsilon |\sigma_\alpha| + \sum_{\alpha \in \mathcal{R}} \mathcal{O}(1) \cdot |\sigma_\alpha| (\varepsilon + |\sigma_\alpha|) + \sum_{\alpha \in \mathcal{NP}} \mathcal{O}(1) \cdot |\sigma_\alpha| \\
&\leq \mathcal{O}(1) \cdot \varepsilon\,.
\end{aligned} \tag{3.23}$$

Thus, if we construct a sequence $\left\{u_\nu\right\}_{\nu \in \mathbb{N}}$ of ε_ν-approximate solution of (1.1)-(1.2) that converge to the semigroup trajectory $u(t,\cdot) = S_t \overline{u}$, relying on (3.22)-(3.23) and (3.2), we obtain

$$\begin{aligned}
\limsup_{\nu \to \infty} \left\|u_\nu(t) - \widetilde{S}_t \overline{u}\,\right\|_{\mathbb{L}^1} &\leq \limsup_{\nu \to \infty} \Big(\left\|u_\nu(t) - \widetilde{S}_t u_\nu(0)\,\right\|_{\mathbb{L}^1} + \left\|\widetilde{S}_t u_\nu(0) - \widetilde{S}_t \overline{u}\,\right\|_{\mathbb{L}^1} \Big) \\
&\leq \mathcal{O}(1) \cdot \limsup_{\nu \to \infty} \Big(t \cdot \varepsilon_\nu + \left\|u_\nu(0) - \overline{u}\right\|_{\mathbb{L}^1} \Big) = 0\,,
\end{aligned}$$

proving that $u(t,\cdot) = \widetilde{S}_t \overline{u}$ for all $t \geq 0$.

Finally, to conclude the proof of the Theorem, it remains to show that the semigroup trajectories satisfy the Lax entropy admissibility condition. To this purpose first observe that, calling $U^\sharp_{(u;\tau,\xi)}$ the solution of the Riemann problem with data

$$u^L = \lim_{x\to\xi-} u(\tau,\, x)\,, \qquad\qquad u^R = \lim_{x\to\xi+} u(\tau,\, x)\,,$$

and relying on the same type of error estimate in (3.22), one can obtain as in [B2, Theorem 9.2] the inequality

$$\limsup_{h\to 0+} \frac{1}{h}\int_{\xi-h\widehat{\lambda}}^{\xi+h\widehat{\lambda}} \left| u(\tau+h,\, x) - U^\sharp_{(u;\tau,\xi)}(h,\, x-\xi) \right| \, dx = 0\,. \tag{3.24}$$

Next, consider any point $(\tau,\, \xi)$ of approximate jump for $u(t,x) = S_t\overline{u}(x)$, and let $U = U(x,t)$ be the map defined according with (3.17)-(3.18). Then, observing that both U and $U^\sharp$ are self-similar, and using (3.18), (3.24), one deduces as in [B2] that $U^\sharp_{(u;\tau,\xi)} = U$. Since, by construction, $U^\sharp_{(u;\tau,\xi)}$ satisfies condition (3.19), it follows that the same is true for U, thus proving that u is entropy admissible in the sense of Lax.

□

4 Construction of the Approximate Solutions

In order to construct a piecewise constant approximate solution of (1.1) defined for all positive times, we shall introduce three different ways of solving a Riemann Problem generated by wave-fronts interactions. We will use accurate solvers whenever the quantity of interactions is sufficiently large, while we will use two types of simplified solvers when the quantity of interaction is "small". In particular, to construct the simplified solvers we shall define the following two kinds of wave-fronts that satisfy, respectively, conditions (3.4)-(3.9) and (3.14)-(3.15).

- Sometimes we will force a composed wave (rarefaction plus contact discontinuity) to behave like a single front. More precisely, let ε be the parameter controlling the maximum strength of rarefaction fronts defined by the accurate Riemann Solver, and let $\big(u^L,\, u^R\big)$ be a composed k-wave of size σ, with rarefaction and shock components σ^r, σ^s defined as in Remark 2.1. In our approximate solution, provided σ^r is smaller than 3ε, we treat $\big(u^L,\, u^R\big)$ as a single wave-front traveling with speed

$$\lambda_k^c\big(u^L,\, u^R\big) \doteq \begin{cases} \dfrac{\sigma^r\lambda_k\big(u^L\big) + \sigma^s\lambda_k^s\big(R_k(u^L)[\sigma^r]\big)[\sigma^s]}{\sigma} & \text{if} \quad (2.6) \text{ holds,} \\[2ex] \dfrac{\sigma^r\lambda_k\big(u^R\big) + \sigma^s\lambda_k^s\big(R_k(u^R)[\sigma^r]\big)[\sigma^s]}{\sigma} & \text{if} \quad (2.7) \text{ holds.} \end{cases} \tag{4.1}$$

- Following [BJ, B1], in the simplified solution we collect all the "small" waves into a so-called non-physical wave-front. All non-physical waves travel with a fixed speed $\widehat{\lambda}$ satisfying (3.14). For notational convenience, we regard a non-physical front as belonging to a fictitious linearly degenerate $(N+1)$-th characteristic family.

Fix a positive parameter ε controlling the maximum strength of a rarefaction front defined by the Accurate Riemann Solver. Assume that at a positive time $\bar{t}$ an interaction occurs at $\bar{x}$ involving two wave-fronts of sizes σ', σ'' and families k', k'', respectively. Here σ' is the left incoming wave. Denote u^L, u^M, u^R the left, middle and right states before the interaction.

4.1 Approximation of a k-th wave

In the previous setting, assume that the Riemann problem (u^L, u^R) is exactly solved by a k-wave of size σ. We describe now two different procedures adopted to construct a piecewise constant approximate solution of (u^L, u^R).

Shock Wave. Assume that (u^L, u^R) is (exactly) solved by an entropy admissible shock wave. Then we set

$$\mathcal{R}_k^s\big(u^L, u^R\big)(t, x) \doteq \begin{cases} u^L & \text{if} \quad x < \bar{x} + \lambda_k^s\big(u^L, u^R\big)(t - \bar{t}), \\ u^R & \text{if} \quad x > \bar{x} + \lambda_k^s\big(u^L, u^R\big)(t - \bar{t}). \end{cases} \tag{4.2}$$

Rarefaction Wave. Assume that (u^L, u^R) is (exactly) solved by a rarefaction wave. We can solve the Riemann problem (u^L, u^R) in two ways. In the first case we treat (u^L, u^R) as a single front. Call

$$x_k^r(t) \doteq \begin{cases} \bar{x} + \lambda_k\big(u^L\big)(t - \bar{t}) & \text{if the } k\text{-th family is GNL or is NGNL and (2.6) holds,} \\ \bar{x} + \lambda_k\big(u^R\big)(t - \bar{t}) & \text{if the } k\text{-th family is NGNL and (2.7) holds.} \end{cases} \tag{4.3}$$

Then we set

$$\mathcal{R}_k^{ru}\big(u^L, u^R\big)(t, x) \doteq \begin{cases} u^L & \text{if} \quad x < x_k^r(t), \\ u^R & \text{if} \quad x > x_k^r(t)). \end{cases} \tag{4.4}$$

In the second case, we split (u^L, u^R) into a rarefaction fan. Let

$$p_k \doteq \big\lfloor |\sigma|/\varepsilon \big\rfloor,$$

where $\lfloor s \rfloor$ denote the smallest integer number greater than s. Then define

$$\omega_{k,\ell} \doteq \begin{cases} R_k\big(u^L\big)\big[\ell\sigma_k/p_k\big] & \text{if the } k\text{-th family is GNL or is NGNL and (2.6) holds,} \\ R_k\big(u^R\big)\big[(p_k - \ell)\sigma_k/p_k\big] & \text{if the } k\text{-th family is NGNL and (2.7) holds,} \end{cases} \qquad \ell = 0, \dots p_k \tag{4.5}$$

$$x^r_{k,\ell}(t) \doteq \begin{cases} \bar{x} + \lambda_k\big(\omega_{k,\ell-1}\big)(t-\bar{t}) & \text{if the } k\text{-th family is GNL or} \\ & \text{is NGNL and (2.6) holds,} \\ \bar{x} + \lambda_k\big(\omega_{k,\ell}\big)(t-\bar{t}) & \text{if the } k\text{-th family is NGNL} \\ & \text{and (2.7) holds,} \end{cases} \qquad \ell = 1,\dots,p_k. \tag{4.6}$$

Notice that $\omega_{k,0} = u^L$ and $\omega_{k,p_k} = u^R$. Then we define the approximate solution as

$$\mathcal{R}^{rf}_k\big(u^L,\, u^R\big)(t,\,x) = \begin{cases} u^L & \text{if} \quad x < x^r_{k,1}(t), \\ \omega_{k,\ell} & \text{if} \quad x^r_{k,\ell}(t) < x < x^r_{k,\ell+1}(t), \quad \ell = 1,\dots,p_k-1, \\ u^R & \text{if} \quad x > x^r_{k,p_k}(t). \end{cases} \tag{4.7}$$

Composed Wave. Assume that $\big(u^L,\, u^R\big)$ is (exactly) solved by an entropy admissible composed wave. Again, we can solve the Riemann problem $\big(u^L,\, u^R\big)$ in two ways. In the first case we look at $\big(u^L,\, u^R\big)$ as a single wave-front traveling with speed $\lambda^c_k\big(u^L,\, u^R\big)$ defined at (4.1). Hence we call

$$x^c_k(t) \doteq \bar{x} + \lambda^c_k\big(u^L,\, u^R\big)(t-\bar{t})\,, \tag{4.8}$$

and set

$$\mathcal{R}^{cu}_k\big(u^L,\, u^R\big)(t,\,x) \doteq \begin{cases} u^L & \text{if} \quad x < x^c_k(t)\,, \\ u^R & \text{if} \quad x > x^c_k(t)\,. \end{cases} \tag{4.9}$$

In the second case, we split the rarefaction component of $\big(u^L,\, u^R\big)$ into a rarefaction fan. First assume that (2.6) holds: the approximate solution $\mathcal{R}^{cf}_k\big(u^L,\, u^R\big)$ consists of a rarefaction fan followed by a left contact discontinuity. Let ω^r_k be such that

$$\omega^r_k = R_k\big(u^L\big)\big[\zeta_k(u^L)[\sigma]\big]\,.$$

Observe that

$$u^R = S_k\big(\omega^r_k\big)\big[\nu_k(\omega^r_k)\big]\,.$$

Set

$$q_k \doteq \left\lfloor \frac{\big|\zeta_k\big(u^L\big)[\sigma]\big|}{\varepsilon} \right\rfloor\,.$$

Then, consider the rarefaction fan consisting of the states

$$\omega_{k,\ell} \doteq R_k\big(u^L\big)\left[\ell\,\frac{\zeta_k\big(u^L\big)[\sigma_k]}{q_k}\right]\,,$$

where $\ell = 0,\dots q_k$. Notice that $\omega_{k,q_k} = \omega^r_k$. Set

$$x_{k,\ell}(t) \doteq \bar{x} + \lambda_k\big(\omega_{k,\ell-1}\big)(t-\bar{t}), \qquad \ell = 1,\dots,q_k+1.$$

We have

$$\mathcal{R}_k^{cf}\big(u^L,\, u^R\big)(t,\, x) \doteq \begin{cases} u^L & \text{if} \quad x < x_{k,1}(t), \\ \omega_{k,\ell} & \text{if} \quad x_{k,\ell}(t) < x < x_{k,\ell+1}(t), \qquad \ell = 1, \dots q_k, \\ u^R & \text{if} \quad x > x_{k,q_k+1}(t). \end{cases} \tag{4.10}$$

Whenever (2.7) holds, $\mathcal{R}_k^{cf}\big(u^L,\, u^R\big)$ consists of a right contact discontinuity followed by a rarefaction fan. The construction of the approximate solution is symmetric of the case in which (2.6) holds, with the rarefaction component split as at (4.7)-(4.6).

4.2 Riemann Solvers

We describe now how construct a piecewise constant approximate solution for a general Riemann problem $\big(u^L,\, u^R\big)$ that is not necessarily exactly solved by an elementary wave of a single characteristic family.

Let $\omega_1, \dots, \omega_N$ and $\sigma_1, \dots, \sigma_N$ be as in (2.24). Fix constants $\widehat{\lambda}_1, \dots, \widehat{\lambda}_{N-1}$ such that

$$\lambda_i(u) < \widehat{\lambda}_i < \lambda_{i+1}(u), \qquad \forall\, u \in \Omega, \quad i = 1, \dots, N-1\,. \tag{4.11}$$

Moreover, call

$$\widehat{x}_i(t) \doteq \bar{x} + \widehat{\lambda}_i(t - \bar{t})\,. \tag{4.12}$$

Accurate Solvers. Here we define the approximate solution $u_k = u_k(t,\, x)$ of the Riemann Problem $(\omega_{k-1},\, \omega_k)$, $k = 1, \dots, N$. The general solution of the Riemann Problem $\big(u^L,\, u^R\big)$ is then obtained by piecing together all of the u_k's. In this case we always use the accurate solvers (4.2), (4.7), (4.10) describe in Section 4.1. Hence we set

$$u_k \doteq \begin{cases} \mathcal{R}_k^{s}\big(\omega_{k-1},\, \omega_k\big) & \text{if } \big(\omega_{k-1},\, \omega_k\big) \text{ is an entropy admissible shock wave,} \\ \mathcal{R}_k^{rf}\big(\omega_{k-1},\, \omega_k\big) & \text{if } \big(\omega_{k-1},\, \omega_k\big) \text{ is a rarefaction wave,} \\ \mathcal{R}_k^{cf}\big(\omega_{k-1},\, \omega_k\big) & \text{if } \big(\omega_{k-1},\, \omega_k\big) \text{ is an entropy admissible composed wave.} \end{cases} \tag{4.13}$$

Simplified Solvers. Define

$$x^{\mathcal{N}P}(t) \doteq \bar{x} + \widehat{\lambda}(t - \bar{t}). \tag{4.14}$$

We consider two cases

1. Assume that σ', σ'' belong to different characteristic families, i.e. $k' > k''$. Let $\widetilde{u}^M$, $\widetilde{u}^R$ be such that

$$\begin{aligned} &\widetilde{u}^M = \Psi_{k''}\big(u^L\big)[\sigma''] && \text{if the } k''\text{-th family is GNL, LD or} \\ & && \text{is NGNL and (2.6) holds for } k = k'', \\ &u^L = \Psi_{k''}\big(\widetilde{u}^M\big)[\sigma''] && \text{if the } k''\text{-th family is NGNL and (2.7) holds for } k = k''. \end{aligned} \tag{4.15}$$

$$\begin{aligned}
&\widetilde{u}^R = \Psi_{k''}\big(\widetilde{u}^M\big)[\sigma'] && \text{if the } k''\text{-th family is GNL, LD or}\\
& && \text{is NGNL and (2.6) holds for } k=k'',\\
&\widetilde{u}^M = \Psi_{k''}\big(\widetilde{u}^R\big)[\sigma'] && \text{if the } k''\text{-th family is NGNL and (2.7) holds for } k=k'.
\end{aligned} \tag{4.16}$$

Then, the approximate solution of the Riemann problem $\big(u^L,\, u^R\big)$ will consist of three waves, one, $\big(u^L,\, \widetilde{u}^M\big)$, of the k''-th characteristic family, another, $\big(\widetilde{u}^M,\, \widetilde{u}^R\big)$, of the k'-th characteristic family, and the third, $\big(\widetilde{u}^R,\, u^R\big)$, will be a non-physical front, traveling with speed $\widehat{\lambda}$. Each physical wave $\big(u^L,\, \widetilde{u}^M\big)$, $\big(\widetilde{u}^M,\, \widetilde{u}^R\big)$ will be solved according to the following scheme. We describe the solver $\mathcal{R}_{k''}\big(u^L,\, \widetilde{u}^M\big)$ of $\big(u^L,\, \widetilde{u}^M\big)$, the one of $\big(\widetilde{u}^M,\, \widetilde{u}^R\big)$ being entirely similar. Roughly speaking, whenever the rarefaction component $\big(u^L,\, \widetilde{u}^M\big)^r$ of $\big(u^L,\, \widetilde{u}^M\big)$ is nonzero, we compare its strength $\big|\big(u^L,\, \widetilde{u}^M\big)^r\big|$ with 3ε in order to decide to treat $\big(u^L,\, \widetilde{u}^M\big)$ as a single front or not. More precisely we set

$$\mathcal{R}_{k''}\big(u^L,\, \widetilde{u}^M\big) \doteq \begin{cases}
\mathcal{R}_k^{s}\big(u^L,\, \widetilde{u}^M\big) & \text{if } \big(u^L,\, \widetilde{u}^M\big) \text{ is an entropy admissible}\\
& \text{shock wave},\\
\mathcal{R}_k^{ru}\big(u^L,\, \widetilde{u}^M\big) & \text{if } \big(u^L,\, \widetilde{u}^M\big) \text{ is a rarefaction wave}\\
& \text{and } \big|\big(u^L,\, \widetilde{u}^M\big)\big| \le 3\varepsilon\,,\\
\mathcal{R}_k^{rf}\big(u^L,\, \widetilde{u}^M\big) & \text{if } \big(u^L,\, \widetilde{u}^M\big) \text{ is a rarefaction wave}\\
& \text{and } \big|\big(u^L,\, \widetilde{u}^M\big)\big| > 3\varepsilon\,,\\
\mathcal{R}_k^{cu}\big(u^L,\, \widetilde{u}^M\big) & \text{if } \big(u^L,\, \widetilde{u}^M\big) \text{ is an entropy admissible}\\
& \text{composed wave and } \big|\big(u^L,\, \widetilde{u}^M\big)^r\big| \le 3\varepsilon\,,\\
\mathcal{R}_k^{cf}\big(u^L,\, \widetilde{u}^M\big) & \text{if } \big(u^L,\, \widetilde{u}^M\big) \text{ is an entropy admissible}\\
& \text{composed wave and } \big|\big(u^L,\, \widetilde{u}^M\big)^r\big| > 3\varepsilon\,.
\end{cases} \tag{4.17}$$

Hence, recalling (4.11), (4.12), (4.14), the approximate solution of the Riemann problem $\big(u^L,\, u^R\big)$ will be

$$u(t,\, x) \doteq \begin{cases}
\mathcal{R}_{k''}\big(u^L,\, \widetilde{u}^M\big)(t,\, x) & \text{if} \quad x < \widehat{x}_{k''}(t)\,,\\
\mathcal{R}_{k'}\big(\widetilde{u}^M,\, \widetilde{u}^R\big)(t,\, x) & \text{if} \quad \widehat{x}_{k''}(t) < x < x^{\mathcal{N}P}(t)\,,\\
u^R & \text{if} \quad x > x^{\mathcal{N}P}(t)\,.
\end{cases} \tag{4.18}$$

Observe that we collect all of the waves not belonging to the families k', k'' into the non-physical front $\big(\widetilde{u}^R,\, u^R\big)$, traveling with speed $\widehat{\lambda}$.

2. Assume that σ', σ'' belong to the same characteristic families, i.e. $k' = k'' \doteq k$. Let $\widetilde{u}^R$ be such that

$$\begin{aligned}
&\widetilde{u}^R = \Psi_k\big(u^L\big)[\sigma'+\sigma''] && \text{if the } k\text{-th family is GNL, LD or}\\
& && \text{is NGNL and (2.6) holds},\\
&u^L = \Psi_k\big(\widetilde{u}^R\big)[\sigma'+\sigma''] && \text{if the } k\text{-th family is NGNL and (2.7) holds.}
\end{aligned} \tag{4.19}$$

Now we define $\mathcal{R}_k\big(u^L,\, \widetilde{u}^R\big)$ similarly to (4.17), looking at the rarefaction component of the wave $\big(u^L,\, \widetilde{u}^R\big)$. Hence the solution of the Riemann problem $\big(u^L,\, u^R\big)$ is defined in the following way.

$$u(t,\, x) \doteq \begin{cases} \mathcal{R}_k\big(u^L,\, \widetilde{u}^R\big)(t,\, x) & \text{if} \quad x < x^{\mathcal{NP}}(t)\,, \\ u^R & \text{if} \quad x > x^{\mathcal{NP}}(t)\,. \end{cases} \tag{4.20}$$

Observe that we collect all of the waves not belonging to the k-th family into the non-physical front $\big(\widetilde{u}^R,\, u^R\big)$, traveling with speed $\widehat{\lambda}$.

Crude Solvers. Assume that $\big(u^M,\, u^R\big)$ is a physical wave-front, i.e. a rarefaction front, a shock front, or a composed wave traveling as a single front. Let $\big(u^L,\, u^M\big)$ be a non-physical wave-front, traveling with speed $\widehat{\lambda} > \lambda_i(u)$, $\forall\, u$, $i = 1, \dots, N$ and with strength $\left|\sigma'\right|$ defined by

$$|\sigma'| \doteq \left|u^L - u^M\right|. \tag{4.21}$$

Let $\widetilde{u}^R$ be defined as in (4.19). Set

$$\mathcal{R}_k\big(u^L,\, \widetilde{u}^R\big) \doteq \begin{cases} \mathcal{R}_k^s\big(u^L,\, \widetilde{u}^R\big) & \text{if } \big(u^L,\, \widetilde{u}^M\big) \text{ is an entropy admissible shock wave,} \\ \mathcal{R}_k^{ru}\big(u^L,\, \widetilde{u}^R\big) & \text{if } \big(u^L,\, \widetilde{u}^M\big) \text{ is a rarefaction wave,} \\ \mathcal{R}_k^{cu}\big(u^L,\, \widetilde{u}^R\big) & \text{if } \big(u^L,\, \widetilde{u}^M\big) \text{ is an entropy admissible composed wave.} \end{cases} \tag{4.22}$$

Then the approximate solution of the Riemann problem $\big(u^L,\, u^R\big)$ is

$$u(t,\, x) \doteq \begin{cases} \mathcal{R}_k\big(u^L,\, \widetilde{u}^R\big) & \text{if} \quad x < x^{\mathcal{NP}}(t)\,, \\ u^R & \text{if} \quad x > x^{\mathcal{NP}}(t)\,. \end{cases} \tag{4.23}$$

Observe that

$$\left|u^R - \widetilde{u}^R\right| \le C_1\left|\sigma'\sigma''\right|, \tag{4.24}$$

holds, where $C_1 > 0$ is the constant at Lemma 2.1.

4.3 The Approximate Solutions

For any fixed $\varepsilon > 0$, now we construct an approximate solution $u_\varepsilon = u_\varepsilon(t,\, x)$ of (1.1)-(1.2). For a given initial data $\bar{u}$, let $u_\varepsilon(0, \cdot) = \overline{u}_\varepsilon$ be a piecewise constant $\mathbb{L}^1$ function satisfying (3.2) and such that

$$\text{Tot.Var.}(\overline{u}_\varepsilon) \le \text{Tot.Var.}(\overline{u})\,.$$

At time $t = 0$ we use Accurate Solvers to construct approximate solutions to the Riemann Problems at the points of discontinuity of $\overline{u}_\varepsilon$. By performing a slight perturbation of the speeds of the waves, we can assume that at any positive time any interaction involves only two wave-fronts. Now suppose that at time $t > 0$ a collision occurs involving two wave-fronts of sizes σ', σ'' respectively, σ' on the left of σ''. The Riemann Problem generated by this interaction is solved as follows. Let $\mathcal{I}(\sigma', \sigma'')$ be defined as in Definition 2.1. Let $\rho \ll \varepsilon$ be a small positive parameter to be specified later. We will use

- an Accurate Solver if none of the waves is a non-physical wave-front and $\mathcal{I}(\sigma', \sigma'') > \rho$;
- a Simplified Solver if none of the waves is a non-physical wave-front and $\mathcal{I}(\sigma', \sigma'') \leq \rho$;
- a Crude Solver if one of the wave is a non-physical wave-front.

Observe that, due to the choice of the speed $\widehat{\lambda}$ at (3.14), a non-physical front can interact only with physical ones.

Remark 4.1 By construction, a rarefaction front of a NGNL k-th characteristic family

$$\left(u^L,\, u^R\right), \qquad u^R = R_k\left(u^L,\, \widetilde{\sigma}\right),$$

always travels with speed $\widetilde{\lambda}$, that satisfies

$$\begin{aligned} \widetilde{\lambda} = \lambda_k\left(u^L\right) &\leq \lambda_k\left[R_k\left(u^L,\, \sigma\right)\right], \qquad \text{if} \quad (2.6) \text{ holds}, \\ \widetilde{\lambda} = \lambda_k\left(u^R\right) &\geq \lambda_k\left[R_k\left(u^L,\, \sigma\right)\right], \qquad \text{if} \quad (2.7) \text{ holds}, \end{aligned} \tag{4.25}$$

for any σ such that $|\sigma| \leq |\widetilde{\sigma}|$, $\operatorname{sgn} \sigma = \operatorname{sgn} \widetilde{\sigma}$. In particular this implies that two wave-fronts σ', σ'' which belong to the same characteristic family and such that $\sigma'^{\,r}\sigma''^{\,r} > 0$, never interact. Indeed, by construction the speed $\lambda_k^c\left(u^L,\, u^R\right)$ of a wave-front $\left(u^L,\, u^R\right)$ satisfies

$$\begin{aligned} \lambda_k^c\left(u^L,\, u^R\right) &> \lambda_k\left(u^L\right), \qquad \text{if} \quad (2.6) \text{ holds}, \\ \lambda_k^c\left(u^L,\, u^R\right) &< \lambda_k\left(u^R\right), \qquad \text{if} \quad (2.7) \text{ holds}. \end{aligned}$$

Remark 4.2 Assume that an interaction involves two wave-fronts σ', σ'' belonging to the same NGNL k-th characteristic family and $\sigma'\sigma'' > 0$. Then, due to the choice of the speed of a rarefaction front (4.25), and to the definitions (2.11), (2.17), if

- one of them is a rarefaction front,
- $\sigma'^{\,r} = \sigma''^{\,r} = 0$,

then there is only one outgoing wave-front of the k-family and it is a shock front. Moreover, in case one of them, say σ', has a nonzero rarefaction component $\sigma'^{\,r}$, i.e. it is a composed wave behaving like a single front, then using a Simplified Solver the outgoing k-th wave has a rarefaction component whose strength does not exceed $|\sigma'^{\,r}|$.

Remark 4.3 Observe that the strength of any rarefaction front belonging to a GNL characteristic family does not exceed ε. Indeed, using an Accurate Solver, by construction the outgoing rarefaction fronts have strength which is at most ε. The use of a Simplified Solver cannot origin rarefaction fronts whose strength is larger than ε. Infact, a rarefaction front of a GNL characteristic family has always positive size, while a shock front has negative size. Moreover, there is no composed wave belonging to a GNL family. If an interaction involves a wave front σ' of a GNL k-th characteristic family and we use a Simplified Solver, two cases can occur:

1. the interacting wave fronts belong to different characteristic families: in this case, due to (4.17)-(4.18), the size of the outgoing wave of the k-th characteristic family equals σ'. Hence, if a shock front enter the interaction point, a shock front exit, and if a rarefaction front of strength less than ε enter, a rarefaction front of strength less than ε exit the interaction point.

2. the interacting wave fronts both belong to the k-th family: in this case at least one of the interacting waves is a shock wave. Hence the sizes of the interacting waves, say σ' and σ'', have opposite signs. Due to (4.20), it follows that if a rarefaction front outcomes from the interaction point, its size σ satisfies
$$0 \leq \sigma = \sigma' + \sigma'' \leq \varepsilon\,.$$

Moreover, if we use a Crude Solver, due to (4.22)-(4.23) again an outgoing rarefaction wave has the same size of the physical wave involved in the interaction. Hence its strength can not exceed ε.

5 Proof of Theorem 1

After observing that (3.2) holds by construction, the technique involved in the proof of the other properties of front tracking solutions claimed at Theorem 1 is standard. We first introduce two functionals V and Q measuring the total variation and the interaction potential, respectively.

Let $\varepsilon > 0$ be fixed. At each time $t > 0$ at which no interaction occur, call $x_1(t) < \cdots < x_m(t)$ the position of the jumps in $u_\varepsilon(t,\cdot)$, and $\sigma_1, \dots, \sigma_m$, $i_1, \dots, i_m$ their respective sizes and characteristic families. We define the *total strength* $V = V(t)$ of waves in $u_\nu(t,\cdot)$ as
$$V(t) \doteq \sum_\alpha |\sigma_\alpha|, \tag{5.1}$$
and the *interaction potential* $Q = Q(t)$ as
$$Q(t) \doteq \sum_{\substack{i_\alpha = i_\beta \neq n+1\\ \sigma_\alpha\sigma_\beta > 0}} \left|\sigma_\alpha\sigma_\beta\right| + \sum_{\substack{\alpha\\ i_\alpha \neq N+1}} \left|\sigma^r_\alpha \sigma_\alpha\right| + c\Bigg[\sum_{\substack{i_\alpha = i_\beta \neq N+1\\ \sigma_\alpha\sigma_\beta < 0}} \left|\sigma_\alpha\sigma_\beta\right| + \sum_{\substack{i_\alpha > i_\beta\\ x_\alpha(t) < x_\beta(t)}} \left|\sigma_\alpha\sigma_\beta\right|\Bigg], \tag{5.2}$$
where $c > 1$ is a suitable constant to be specified later.

Remark 5.1 The term

$$\sum_{\substack{\alpha \\ i_\alpha \neq N+1}} \left|\sigma^r_\alpha \sigma_\alpha\right|$$

in the above definition of the interaction potential takes into account of the possible splitting, after an interaction, into two or more wave-fronts of an incoming wave σ belonging to a NGNL characteristic family, with

$$\sigma^r \sigma^s \neq 0 \qquad \text{or} \qquad |\sigma^r| > \varepsilon\,.$$

Indeed, in such a case the sizes of the new outgoing wave-fronts cannot be controlled by the size of the other interacting front, by means of (2.28) or (2.32) at Lemma 2.1.

5.1 Bound on the Total Variation

First we need a technical Lemma, whose proof is standard and relies on Lemma 2.1.

Lemma 5.1 *For any compact set $K \subset \Omega$ there exist constants C_2, $\mu_2 > 0$ such that the following holds. Let $u_\varepsilon = u_\varepsilon(t, x)$ be a piecewise constant approximate solution of (1.1), constructed as above and defined on the strip $[0, T[\times\mathbb{R}$. Assume that* $\text{Tot.Var.}\big(u_\varepsilon(0, \cdot)\big) < \mu_2$, $\lim\limits_{x\to-\infty} u(0, x) \in K$. *Then*

$$\text{Tot.Var.}\big(u_\varepsilon(t, \cdot)\big) \leq C_2 \text{Tot.Var.}\big(u_\varepsilon(0, \cdot)\big), \qquad \forall\, t > 0\,. \tag{5.3}$$

Moreover, if at time $\tau > 0$ two wave-fronts of sizes σ', σ'' interact, then

$$\Delta Q(\tau) \leq -\frac{1}{2}\, \mathcal{I}(\sigma', \sigma'')\,, \tag{5.4}$$

$$\Delta V(\tau) + 2C_1 \Delta Q(\tau) \leq 0\,, \tag{5.5}$$

with $\mathcal{I}(\sigma', \sigma'')$ defined at (2.26).

Proof Let τ be as in the assumptions and call i', i'' the characteristic family of σ', σ'' respectively. Assume that σ' connect u^L to u^M, and σ'' u^M to u^R. Hence, from (2.27), (2.30), (4.24) we derive

$$\Delta V(\tau) = V(\tau+) - V(\tau-) \leq C_1\, \mathcal{I}(\sigma', \sigma'')\,. \tag{5.6}$$

Concerning Q, we will prove (5.4) only for $\mathcal{I}(\sigma', \sigma'') > \rho$, and assuming that both the i^prime-th and the i''-th characteristic families are NGNL, the other cases being similar.

After time τ the two fronts σ', σ'' are no longer approaching. On the other hand new wave-fronts, which appear after the interaction and belong to characteristic families different to both i', i'', may approach all the other waves. Call $\sigma^r_{\ell,j}$ the outgoing rarefaction fronts of the ℓ-th family, $j = 1, \ldots, m_\ell$, and σ^s_ℓ the outgoing shock front, whenever they exist. Here we consider two cases

Case 1: $i' \neq i''$ or $i' = i'' = i$ and $\sigma'\sigma'' < 0$.

Assume $i' \neq i''$, the other case being similar. By definition

$$\mathcal{I}(\sigma',\, \sigma'') = \left|\sigma'\sigma''\right|. \tag{5.7}$$

Due to (2.28) we have

$$\left|\left|\sigma'^{\,r}\right| - \sum_{j=1}^{m_{i'}} \left|\sigma^r_{i',j}\right|\right| \leq C_1 \min\left\{\left|\sigma'\right|,\, \left|\sigma''\right|\right\},$$
$$\left|\left|\sigma'^{\,s}\right| - \left|\sigma^s_{i'}\right|\right| \leq C_1 \min\left\{\left|\sigma'\right|,\, \left|\sigma''\right|\right\},$$

whenever a composed wave of the i'-th family outgoes. Entirely similar estimates hold for the i''-th characteristic family. We obtain

$$\begin{aligned}\Delta Q(\tau) &\leq -c\left|\sigma'\sigma''\right| - \left|\sigma'^{\,r}\sigma'\right| - \left|\sigma''^{\,r}\sigma''\right| + \\ &\quad + \frac{1}{2}\sum_{\ell=i',i''}\sum_{j,k=1}^{m_\ell} \left|\sigma^r_{\ell,j}\sigma^r_{\ell,k}\right| + \sum_{\ell=i',i''}\left(\left|\sigma^s_\ell\right|\sum_{j=1}^{m_\ell}\left|\sigma^r_{\ell,j}\right|\right) + cC_1\left|\sigma'\sigma''\right|V(\tau-) \\ &\leq -\big(c - 8C_1 - 4C_1^2 - cC_1V(\tau-)\big)\left|\sigma'\sigma''\right|.\end{aligned}$$

By choosing

$$V(\tau-) < \frac{1}{2C_1}, \qquad c > 1 + 16C_1 + 8C_1^2\,, \tag{5.8}$$

we get (5.4).

Case 2: $i' = i'' = i$ and $\sigma'\sigma'' > 0$.

For sake of simplicity, assume that (2.6) holds for $k = i$ and that $D_{r_i}\lambda_i(u^L) \geq 0$, the other cases being entirely similar. If

$$\sigma'^{\,r}\sigma'^{\,s} = \sigma''^{\,r}\sigma''^{\,s} = 0\,,$$

using Remark 4.2, we get (5.7). Hence, with arguments similar to the ones used above and choosing

$$V(\tau-) < \frac{1}{2cC_1}\,, \tag{5.9}$$

we get (5.4).

Otherwise, due to Remark 4.1 we have

$$\sigma'^{\,r}\sigma'^{\,s} \neq 0, \qquad \sigma' > 0 \qquad \sigma'' = \sigma''^{\,s} > 0\,.$$

If the only outgoing wave of the i-th family is a shock front, again (5.7) holds and we can use the above arguments to get (5.4). Otherwise, by definition 2.1

$$\mathcal{I}(\sigma',\, \sigma'') = \left[\left|\zeta(u^L)[\sigma' + \sigma''] - \sigma'^{\,r}\right| + \left|\sigma'^{\,s}\right|\right]\left|\sigma''\right|.$$

From (2.31) we derive

$$\left| \sum_{j=1}^{m_i} \left|\sigma^r_{i,j}\right| - \zeta(u^L)[\sigma' + \sigma''] \right| \le C_1 \, \mathcal{I}(\sigma', \sigma'') \,,$$

$$\left| \sigma^s_i - \sigma' - \sigma'' + \zeta(u^L)[\sigma' + \sigma''] \right| \le C_1 \, \mathcal{I}(\sigma', \sigma'') \,.$$

It follows that, using (2.30), we get

$$\begin{aligned}
\Delta Q(\tau) &\le -\left|\sigma'\sigma''\right| - \left|\sigma'^{\,r}\sigma'\right| + \frac{1}{2}\sum_{j,k=1}^{m_i} \left|\sigma^r_{i,j}\sigma^r_{i,k}\right| + \left|\sigma^s_i\right| \sum_{j=1}^{m_i} \left|\sigma^r_{i,j}\right| + cC_1 \, \mathcal{I}(\sigma', \sigma'')V(\tau-) \\
&\le -\left|\sigma'\sigma''\right| - \left|\sigma'^{\,r}\sigma'\right| + \left|\zeta(u^L)[\sigma' + \sigma'']\right|^2 + \\
&\quad + \left|\sigma' + \sigma'' - \zeta(u^L)[\sigma' + \sigma'']\right| \left|\zeta(u^L)[\sigma' + \sigma'']\right| + C_1\big(3 + c\big) \, \mathcal{I}(\sigma', \sigma'')V(\tau-) \\
&= -\big[1 - C_1(3 + c)V(\tau-)\big]\mathcal{I}(\sigma', \sigma'') \,.
\end{aligned}$$

If we choose

$$V(\tau-) < \frac{1}{2C_1(3 + c)} \,, \tag{5.10}$$

we obtain (5.4).

From (5.4), (5.6), (5.8)-(5.10), it follows that (5.5) holds, provided that

$$V(\tau-) < \frac{1}{2C_1(3 + c)} \,, \qquad\qquad c > 1 + 16C_1 + 8C_1^2 \,,$$

hold. Now let $C_0 > 1$ be such that

$$\frac{1}{C_0}\text{Tot.Var.}\big\{u_\varepsilon(t, \cdot)\big\} \le V(t) \le C_0\text{Tot.Var.}\big\{u_\varepsilon(t, \cdot)\big\} \,. \tag{5.11}$$

Let

$$\text{Tot.Var.}\big\{u_\varepsilon(0, \cdot)\big\} < \frac{1}{3C_0C_1(1 + 2cC_1)(3 + c)} \doteq \mu_2 \,, \tag{5.12}$$

and call τ the first interaction time. We have

$$V(\tau-) \le C_0\text{Tot.Var.}\big\{u_\varepsilon(0, \cdot)\big\} \le \frac{1}{3C_1(1 + 2cC_1)(3 + c)} < \frac{1}{2C_1(3 + c)} \,,$$

and hence, due to (5.5) and observing that $Q(t) \le cV^2(t) \le cV(t)$,

$$V(\tau+) \le V(\tau+) + 2C_1Q(\tau+) \le V(\tau-) + 2C_1Q(\tau-) \le \frac{1}{3C_1(3 + c)} < \frac{1}{2C_1(3 + c)} \,, \tag{5.13}$$

holds. By an induction argument, (5.13), and hence (5.5), holds for any $\tau > 0$. To complete the proof of the Lemma, we need to prove (5.3). This follows easily from (5.5) and (5.11)-(5.12). Indeed

$$\begin{aligned} \text{Tot.Var.}\{u_\varepsilon(t,\cdot)\} &\leq C_0\left[V(t) + 2C_1 Q(t)\right] \\ &\leq C_0\left[V(0) + 2C_1 Q(0)\right] \\ &\leq C_0(1 + 2cC_1)V(0) \\ &\leq C_0^2(1 + 2cC_1)\text{Tot.Var.}\{u_\varepsilon(0,\cdot)\}\,, \end{aligned}$$

which is (5.3) with $C_2 = C_0^2(1 + 2cC_1)$.

□

From Lemma 5.1, (3.3) follows easily, provided that in (3.1) we choose $\delta_0 = \mu_2$.

5.2 Bound on the Number of Wave Fronts

Observe that new wave-fronts can origin only at interaction points. Moreover, it suffices to prove that the total number of physical fronts is finite. Indeed, a new non-physical wave-front can origin only at the interaction between two physical fronts. Now, observe that, due to (5.4), the total number of interactions involving two wave-fronts σ', σ'' such that $\mathcal{I}(\sigma', \sigma'') > \rho$, is finite, and by construction, when an interaction involves a non-physical wave-front, no new physical front outgoes. Hence we can take into account only of interactions involving two physical wave-fronts σ', σ'' with $\mathcal{I}(\sigma', \sigma'') \leq \rho$. Moreover, due to definitions (4.17)-(4.20), whenever in the outgoing wave fronts the strength of the rarefaction components do not exceed 3ε, then the total number of physical wave-fronts do not increase. It follows that we can take into account only of interactions between two physical fronts σ', σ'' such that $\mathcal{I}(\sigma', \sigma'') \leq \rho$ and at least one of the strengths of the rarefaction components of the outgoing wave fronts is greater than 3ε. We claim that the total number of this kind of interactions is finite. This follows from the next lemma.

Lemma 5.2 *Assume that at time $t = \bar\tau$ an interaction occurs involving two physical wave-fronts (u^L, u^M), (u^M, u^R) of sizes σ', σ'' respectively. Assume that $\mathcal{I}(\sigma', \sigma'') < \rho$ and that, using a simplified solver for the Riemann problem (u^L, u^R), one of the outgoing wave-fronts has a rarefaction component whose strength is strictly greater than 3ε. Then there exists a constant $\tilde\tau < \bar\tau$ such that, with the definitions at (5.2), there holds*

$$Q(\bar\tau) - Q(\tilde\tau-) < -\frac{1}{2}C_0\varepsilon^2\,, \tag{5.14}$$

where C_0 is the constant at Lemma 2.1.

Proof As usual, call respectively k', k'' the characteristic families (u^L, u^M), (u^M, u^R) belong to. For sake of simplicity, assume that $k' > k''$, the other cases being similar. Hence, because of (4.15)-(4.18) the outgoing solution contains one wave $(u^L,, \widetilde{u}^M)$ of the k''-th characteristic family of size σ'', one wave $(\widetilde{u}^M, \widetilde{u}^R)$ of the k'-th characteristic family of size σ', and possibly a non-physical wave-front $(\widetilde{u}^R, u^R)$. Due to Remark 4.3 at least one of the interacting fronts belong to a NGNL characteristic family. For sake of simplicity assume it is σ' and that the rarefaction component $(\widetilde{u}^M, \widetilde{u}^R)^r$ of $(\widetilde{u}^M, \widetilde{u}^R)$ has strength $|(\widetilde{u}^M, \widetilde{u}^R)^r|$ satisfying

$$|(\widetilde{u}^M, \widetilde{u}^R)^r| > 3\varepsilon\,, \tag{5.15}$$

the other cases being similar. First observe that $|\sigma'| > 3\varepsilon$. Moreover, the rarefaction component $(u^L, u^M)^r$ of (u^L, u^M) is nonzero. Otherwise, due to the definition of the Simplified Solvers and to (2.28) we have

$$\min\{|\sigma'|, |\sigma''|\} > C\varepsilon, \qquad \exists\, C > 0\,,$$

and hence

$$\mathcal{I}(\sigma', \sigma'') \geq C^2\varepsilon^2,$$

in contrast with the assumptions and the choice of $\rho \ll \varepsilon$. Now we define inductively the set $W(\sigma') \subset \mathbb{R} \times \mathbb{R}^{\geq 0}$. Let $(\tau_0, \xi_0) \doteq (\bar{\tau}, \bar{\xi}) \in W(\sigma')$, where $(\bar{\tau}, \bar{\xi})$ is the interaction point of σ', σ''. Assuming $(\tau_n, \xi_n) \in W(\sigma')$, then $(\tau_{n+1}, \xi_{n+1}) \in W(\sigma')$, $\tau_{n+1} < \tau_n$, iff

1. (τ_{n+1}, ξ_{n+1}) is an interaction point at which we use a Simplified or a Crude Solver;
2. the interaction at (τ_{n+1}, ξ_{n+1}) involves a wave-front of the k'-th characteristic family whose size σ satisfies
$$\sigma\sigma' > 0\,; \tag{5.16}$$
3. there exists a segment in the (t, x)-plane joining (τ_{n+1}, ξ_{n+1}) with (τ_n, ξ_n). Such a segment is a wave-front of the k'-th characteristic family whose size σ satisfies (5.16);
4. there is only one wave-front of the k'-th family outgoing from (τ_{n+1}, ξ_{n+1}), σ_{n+1}, and it has a rarefaction component σ^r_{n+1} that satisfies
$$|\sigma^r_{n+1}| \geq \varepsilon\,. \tag{5.17}$$

Conditions 1-4 imply that at each point $(\tau_n, \xi_n) \in W(\sigma')$, when using a Simplified Solver, the rarefaction component of the strength of outgoing wave of the k'-th family does not exceed 3ε. Let

$$\tilde{\tau} \doteq \tau_{\widetilde{n}} \doteq \min\{\tau_n \;:\; \exists\, \xi_n \;:\; (\tau_n, \xi_n) \in W(\sigma')\}\,. \tag{5.18}$$

Let $\xi_{\widetilde{n}} \in \mathbb{R}$ be such that $(\tau_{\widetilde{n}}, \xi_{\widetilde{n}}) \in W(\sigma')$. For any $(\tau_n, \xi_n) \in W(\sigma')$, let σ_n be the size of the k'-th wave outgoing from (τ_n, ξ_n), so that $\sigma' = \sigma_1$ and σ_{n+1} is one of the interacting fronts at (τ_n, ξ_n), $n < \widetilde{n}$. Moreover call $\widehat{\sigma}_{n+1}$ the wave-front interacting with σ_{n+1} at (τ_n, ξ_n). By construction and due to the above remarks we have

$$\sigma_n\sigma' > 0\,, \qquad |\sigma_n| \geq \varepsilon,\,, \qquad \forall\ (\tau_n, \xi_n) \in W(\sigma')\,. \tag{5.19}$$

Set

$$\widehat{\widehat{\sigma}}_n \doteq \begin{cases} \widehat{\sigma}_n & \text{if } \widehat{\sigma}_n \text{ does not belong to the } k'\text{-th family} \\ & \text{or is a } k'\text{-wave such that } \widehat{\sigma}_n \sigma_n < 0\,, \\ 0 & \text{otherwise.} \end{cases} \tag{5.20}$$

Observe that whenever $\widehat{\widehat{\sigma}}_n = \widehat{\sigma}_n$, then $\mathcal{I}\big(\sigma_n,\, \widehat{\sigma}_n\big) = \big|\sigma_n \widehat{\sigma}_n\big|$. We claim that

$$\sum_{(\tau_n,\,\xi_n)\in W(\sigma')} \big|\widehat{\widehat{\sigma}}_{n+1}\big| \geq C_0 \varepsilon \tag{5.21}$$

holds. With the notations introduced above, observe that, due to the definition of $\tau_{\widetilde{n}}$ (5.18), we have

$$|\sigma^r_{\widetilde{n}+1}| < \varepsilon\,. \tag{5.22}$$

By Remark 4.2 the rarefaction component of a wave-front σ can increase only interacting with a wave-front of a different family or a wave-front of the same family but different sign. The rarefaction component of the σ_n's increases from ε (5.22) to 3ε (5.15). Hence due to (2.28), (5.21) holds.
Now observe that

$$|\sigma_{\widetilde{n}+1}| \geq \varepsilon\,. \tag{5.23}$$

Indeed, assume that $\widehat{\sigma}_{\widetilde{n}+1}$ belongs to the k'-th family, the other case being similar. Then, due to Remarks 4.1-4.2, to (5.22) and being $|\sigma^r_{\widetilde{n}}| \geq \varepsilon$, there holds $\sigma_{\widetilde{n}+1}\widehat{\sigma}_{\widetilde{n}+1} < 0$. Thus $|\sigma_{\widetilde{n}+1}| > |\widehat{\sigma}_{\widetilde{n}+1}|$, being $\sigma_{\widetilde{n}+1} + \widehat{\sigma}_{\widetilde{n}+1} = \sigma_{\widetilde{n}}$. Hence (5.23) holds.
It follows that we can deduce (5.14) from (5.4), (5.19), (5.21), (5.23).

□

5.3 Bound on Non-Physical Waves

We need to prove (3.15) only, (3.13), (3.14) being true by construction. Observe that, whenever a new non-physical wave-front σ origins, it satisfies

$$|\sigma| = \mathcal{O}(1)\cdot\rho\,. \tag{5.24}$$

Indeed, by construction, a new non-physical wave origins at an interaction involving two physical fronts σ', σ'' satisfying

$$\mathcal{I}(\sigma',\, \sigma'') \leq \rho\,, \tag{5.25}$$

with $\mathcal{I}(\sigma',\, \sigma'')$ defined at (2.26). Hence, due to the definition of the Simplified Solvers at Section 4.2), (5.24) follows from (5.4), (5.5). Now, set

$$\sigma_\alpha(t) = \big|u_\varepsilon(t, x_\alpha(t)+) - u_\varepsilon(t, x_\alpha(t)-)\big|\,, \qquad \alpha \in \mathcal{NP}\,, \tag{5.26}$$

$\mathcal{NP}$ being the set of all non-physical wave-fronts. If we prove that there exists a positive constant C_3 such that

$$\sigma_\alpha(t) \le C_3 \sigma_\alpha(s)\,, \qquad \forall\, 0 < s \le t\,, \tag{5.27}$$

then using the same arguments as in [BJ, B2], we can choose $\rho > 0$ so small in order that (3.15) holds. Assume that σ_α origins at time $t = t_\alpha$. Let

$$V_\alpha^*(t) \doteq \sum_{\sigma_\alpha \in \mathcal{A}(\alpha)} |\sigma_\alpha|\,, \tag{5.28}$$

where $\mathcal{A}(\alpha)$ is the set of all wave-fronts that are approaching $\sigma_\alpha(t)$, i.e., denoting $x_\beta(t)$ the position of σ_β at time t,

$$\mathcal{A}(\alpha) \doteq \Big\{ \sigma_\beta \;:\; x_\beta(t) > x_\alpha(t) \Big\}\,. \tag{5.29}$$

Now consider an interaction time $\tau > t_\alpha$. Two cases can occur.

1. The interaction does not involve σ_α. Then, due to (5.5) we have

$$\Delta\sigma_\alpha(\tau) = 0\,, \qquad \Delta V_\alpha^*(\tau) + 2C_1 \Delta Q(\tau) \le 0\,. \tag{5.30}$$

2. The interaction does involve σ_α. Call σ_β the other interacting wave-front. Then, due to the definition of V_α^* (5.28)-(5.29), to (5.4) and to the interaction estimates at Lemma 2.1, we deduce that there exists a positive constant c_2 such that

$$\Delta V_\alpha^*(\tau) = -|\sigma_\beta|\,, \qquad \Delta Q(\tau) < 0\,, \qquad |\Delta\sigma_\alpha(\tau)| \le c_2 \sigma_\alpha(\tau-)|\sigma_\beta|\,. \tag{5.31}$$

In any case, we can choose a positive constant c_3 in order to make the map

$$t \mapsto \sigma_\alpha(t) \exp\Big\{ c_3 \big[V_\alpha^*(t) + 2C_1 Q(t) \big] \Big\}\,,$$

decreasing. Hence, if $t > s$, we have

$$\sigma_\alpha(t) \le \sigma_\alpha(s) \exp\Big\{ c_3 \big[V_\alpha^*(s) - V_\alpha^*(t) + 2C_1 (Q(s) - Q(t)) \big] \Big\}\,,$$

from which (5.27) follows, due to (3.3).

5.4 Estimates on Shock and Rarefaction Fronts

First we need a preliminary Lemma.

Lemma 5.3 *There exists $\rho > 0$ such that, in connection with the construction of an approximate solution $u_\varepsilon = u_\varepsilon(t,x)$ at Section 4.3, the following holds. Let $\widehat{\lambda}$ be chosen as in (3.14). Let $x = x_{\widehat{\lambda}}(t)$ be any time-like polygonal line in the (t,x)-plane with slope strictly less than $\widehat{\lambda}$, i.e.*

$$|\dot{x}_{\widehat{\lambda}}(t)| < \widehat{\lambda}\,. \tag{5.32}$$

Call $x_1(t),\ldots,x_\nu(t)$ the non-physical wave-fronts in the approximate solution u_ε. Let $(t_\alpha,\overline{x}_\alpha)$, $t_\alpha \geq 0$, $\alpha = 1,\ldots,\nu$ be the intersecting point in the (t,x)-plane of $x = x_\alpha(t)$ with $x = x_{\widehat{\lambda}}(t)$, whenever it exists, i.e.

$$x_\alpha(t_\alpha) = x_{\widehat{\lambda}}(t_\alpha) = \overline{x}_\alpha, \qquad \alpha = 1,\ldots,\nu\,. \tag{5.33}$$

Then we have

$$\sum_{\alpha=1}^{\nu} \big|u_\varepsilon(t_\alpha,\overline{x}_\alpha+) - u_\varepsilon(t_\alpha,\overline{x}_\alpha-)\big| = \mathcal{O}(1)\cdot\varepsilon\,. \tag{5.34}$$

Proof Observe that (5.34) can be derived by (3.15), once we have shown that, if $\sigma_\alpha(t)$ is defined as in (5.26), then there exists a positive constant C_4 such that

$$\sigma_i(s) \leq C_4\sigma_i(t), \qquad \forall\; 0 < s \leq t, \quad \forall\; \alpha = 1,\ldots,\nu\,. \tag{5.35}$$

In order to prove (5.35), assume that at time $t = t_\alpha$, σ_α origins from an interaction point between two physical wave-fronts. Let $V_\alpha^*(t)$ be defined as at (5.28), (5.29). Consider an interaction time $\tau > t_\alpha$. Then (5.30) and (5.31) still hold. Hence we can choose a positive constant c_4 in order to make the map

$$t \mapsto \sigma_\alpha(t)\exp\Big\{-c_4\big[V_\alpha^*(t) + 2C_1Q(t)\big]\Big\}\,,$$

increasing. Hence, if $t > s$, we have

$$\sigma_\alpha(s) \leq \sigma_\alpha(t)\exp\Big\{c_4\big[V_\alpha^*(s) - V_\alpha^*(t) + 2C_1(Q(s) - Q(t))\big]\Big\}\,,$$

from which (5.35) follows, due to (3.3).

□

Now it is easy to prove (3.4), (3.6). Any shock front in the approximate solution u_ε either it is a real shock wave, fulfilling the Rankine-Hugoniot conditions, or it is a composed wave traveling as a unique wave-front. The rarefaction component can increase only due to interactions with non-physical wave-fronts, or using a Simplified Solver. In the first case, the strength of this component can be bound by $\mathcal{O}(1)\cdot\varepsilon$, due to Lemma 5.3. In the latter case, by construction the strength of the rarefaction components of the outgoing waves is bound by 3ε. Since rarefaction and shock curves have a second order contact at their starting point, it follows that (3.4) or (3.6) holds, depending on the characteristic family the shock front belongs to. By construction and with the above arguments, it can be proved that (3.5), (3.7), (3.8), (3.9) hold too.

(3.10), (3.11) follow with similar arguments. Indeed, a rarefaction front can origin

1. using an Accurate Solver or a Simplified Solver: in this case its strength is less than 3ε by construction;

2. using a Crude Solver, whenever one of the interacting fronts is a composed wave we have forced to behave like a single front: in this case its strength can be bound by the increasing of the rarefaction component in this front. Due to Lemma 5.3 and using the above arguments, it follows that its strength is $\mathcal{O}(1) \cdot \varepsilon$.

Hence at the origin the strength of a rarefaction front is $\mathcal{O}(1) \cdot \varepsilon$. Following [B2] and with arguments similar to the ones used for the non-physical fronts, it can be proved that an estimate similar to (5.27) holds for a rarefaction front, proving $(3.10)_1$, $(3.11)_1$. Moreover, $(3.10)_2$, $(3.10)_2$ and (3.12) hold by construction.
This ends the proof of Theorem 1.

References

[AM1] F. Ancona, A. Marson, A note on the Riemann problem for general $n \times n$ systems of conservation laws, *Journal Math. Anal. Appl.*, to appear.

[AM2] F. Ancona, A. Marson, Basic estimates for a front tracking algorithm for general 2×2 conservation laws, *Math. Models & Methods in Appl. Sc. - M3AS*, submitted.

[AM3] F. Ancona, A. Marson, Well-posedness for general 2×2 systems of conservation laws, preprint S.I.S.S.A.-I.S.A.S. 27/99/M, 1999, submitted.

[AM4] F. Ancona, A. Marson, Well-posedness for non genuinely nonlinear conservation laws, *Proc. 8^{th} Intern. Conf. on Hyperb. Probl. - Theory Numeric and Appl.*, Magdeburg, to appear.

[AM5] F. Ancona, A. Marson, $\mathbb{L}^1$-Stability of weak solutions for $N \times N$ non genuinely nonlinear conservation laws, preprint, 2000.

[BJ] P. Baiti, H.K. Jenssen, On the front-tracking algorithm, *J. Math. Anal. Appl.*, **217** (1998), no. 2, pp. 395-404.

[B1] A. Bressan, Global solutions to systems of conservation laws by wave-front tracking, *J. Math. Anal. Appl.*, **170** (1992), pp. 414-432.

[B2] A. Bressan, *Hyperbolic Systems of Conservation Laws - The one-dimensional Cauchy problem*, Oxford Univ. Press, to appear.

[B3] A. Bressan, The unique limit of the Glimm scheme, *Arch. Rational Mech. Anal.*, **130** (1995), pp. 205-230.

[B-G] A. Bressan, P. Goatin, Oleinik type estimates and uniqueness for $n \times n$ conservation laws, *J. Differential Equations*, to appear.

[B-L] A. Bressan, M. Lewicka, A uniqueness condition for hyperbolic systems of conservation laws, *Discr. Cont. Dynam. Syst.*, to appear.

[BLY] A. Bressan, T.P. Liu, T. Yang, L^1 stability estimates for $n \times n$ conservation laws, *Arch. Rational Mech. Anal.*, to appear.

[BM] A. Bressan, A. Marson, Error bounds for a deterministic version of the Glimm scheme, *Arch. Rational Mech. Anal.*, **142** (1998), pp. 155-176.

[Da] C. Dafermos, Polygonal approximations of solutions of the initial value problem for a conservation law, *J. Math. Anal. Appl.*, **38** (1972), pp. 33-41.

[Dp1] R. DiPerna, Global existence of solutions to nonlinear hyperbolic systems of conservation laws, *J. Differential Equations*, **20** (1976), pp. 187-212.

[Dp2] R. DiPerna, Uniqueness of solutions to hyperbolic conservation laws *Indiana Univ. Math. J.*, **28** (1979), pp. 137-188.

[Gl] J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.*, **18** (1965), pp. 697-715.

[La] P.D. Lax, Hyperbolic systems of conservation laws II, *Comm. on Pure and Applied Math.*, **10** (1957), pp. 537-566.

[Li1] T.P. Liu, The Riemann problem for general 2×2 conservation laws, *Trans. Amer. Math. Soc.*, **199** (1974), pp. 89-112.

[Li2] T.P. Liu, The Riemann problem for general systems of conservation laws, *J. Differential Equations*, **18** (1975), pp. 218-234.

[Li3] T.P. Liu, Admissible solutions of hyperbolic conservation laws, *Memoir Amer. Math. Soc.*, **30** (1981), no. 240.

[L-Y] T.P. Liu, T. Yang, Weak solutions of general systems of hyperbolic conservation laws, submitted.

[R] N.H. Risebro, A front-tracking alternative to the random choice method, *Proc. Amer. Math. Soc.*, **117** (1993), pp. 1125-1139.

[RMS1] T.Ruggeri, A.Muracchini, L.Seccia, Continuum approach to phonon gas and shape changes of second sound via shock waves theory, *Nuovo Cimento D* **16** (1994), pp. 15-44.

[RMS2] T.Ruggeri, A.Muracchini, L.Seccia, Second sound and characteristic temperature in solids, *Phisical Review B* **54** (1996), pp. 332-339.

[Sm] J. Smoller, *Shock Waves and Reaction-diffusion Equations*, Second Edition, Springer-Verlag, 1994.