ON THE STABILITY OF THE STANDARD RIEMANN SEMIGROUP

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Abstract. We consider the dependence of the entropic solution of a hyperbolic system of conservation laws

$$\begin{cases} u_t + f(u)_x = 0 \\ u(0, \cdot) = u_0 \end{cases}$$

on the flux function f. We prove that the solution in Lipschitz continuous w.r.t. the C^0 norm of the derivative of the perturbation of f. We apply this result to prove the convergence of the solution of the relativistic Euler equation to the classical limit.

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1. Introduction

Under suitable assumptions on the function $f\colon \Omega \mapsto \mathbb{R}^n$ (with $\Omega \subseteq \mathbb{R}^n$), the system

$$u_t + [f(u)]_x = 0 \tag{1.1}$$

generates a *Standard Riemann Semigroup* (SRS) $S\colon [0, +\infty[\times \mathcal{D} \mapsto \mathcal{D}$, see [6]. Aim of this paper is to investigate the dependence of S upon the flow function f.

Several papers in the current literature are concerned with the existence of an SRS, see for example [8] and the references in [6]. On the contrary, in the present paper the existence of an SRS is assumed as a starting point and the focus is on the correspondence $f \mapsto S$. In fact, the results in this paper imply that the SRS S is a Lipschitzean function of the flow f, with respect to the C^0 norm of Df. An immediate consequence is the following. Assume that f depends on the parameters $(p_1, \ldots, p_m)$ that may vary in a compact subset of $\mathbb{R}^m$. Given a continuous functional J defined on the solution u at time t to the Cauchy problem for (1.1), the present result ensures the continuity of the map $(p_1, \ldots, p_m) \mapsto J(u(t))$ hence, by Weierstrass Theorem, the optimization problem admits a solution.

For the sake of completeness, we only recall here that the existence of the SRS for the $n \times n$ system (1.1) was first proved in [8]. The main assumptions there are that Df is strictly hyperbolic with every characteristic field either linearly degenerate or genuinely nonlinear and that the initial data has sufficiently small total variation. More recently, the existence of the SRS was extended also to the non genuinely nonlinear setting in the 2×2 case, see [1].

Below we shall restrict our attention to *standard* solutions to Riemann problems and, hence, to general Cauchy problems for Conservation Laws. Here, by standard solutions we refer to those introduced by Lax [12] and then generalized by Liu [13]. Various extensions of the present work to other types of solvers are straightforward.

The present paper is organized as follows. In the next section we state the main results. The following Sections 4 and 5 are devoted to two applications: the classical – relativistic limit

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of Euler equations and scalar conservation laws with L^∞ initial data. The proofs are given in Section 3.

2. Notation and Main Results

Consider the following hyperbolic system of conservation laws in one space dimension

$$u_t + [f(u)]_x = 0 \tag{2.1}$$

where $f\colon \Omega \mapsto \mathbb{R}^n$ is in $\mathrm{Hyp}(\Omega)$, i.e. f is a sufficiently smooth function that generate a SRS $S^f\colon [0,+\infty[\, \times \mathcal{D}^f \mapsto \mathcal{D}^f$. Recall that by SRS generated by f (see [4]) we mean a map $S^f\colon [0,+\infty[\, \times \mathcal{D}^f \mapsto \mathcal{D}^f$ with the following properties:

(i) S^f is a semigroup: $S^f_0 = \mathrm{Id}$ and $S^f_t \circ S^f_s = S^f_{t+s}$;
(ii) S^f is Lipschitz continuous: there exists a positive L_f such that for all positive t, s and for all $u, w \in \mathcal{D}^f$, $\|S^f_t u - S^f_s w\|_{L^1} \leq L_f \cdot (|t-s| + \|u-w\|_{L^1})$;
(iii) if u is piecewise constant, then for t small, $S^f_t u$ coincides with the glueing of standard solutions to Riemann problems.

For all $u \in \mathcal{D}^f$, it is well known (see [4]) that the map $t \mapsto S^f_t u$ is a weak entropic solution to (2.1).

Given $f \in \mathrm{Hyp}(\Omega)$, let $\mathcal{R}(\mathcal{D}^f)$ be the set of all piecewise constant functions in $\mathcal{D}^f$ having a single jump at the origin. In other words, $\mathcal{R}(\mathcal{D}^f)$ is the set of initial data to the Riemann problems

$$\begin{cases} u_t + f(u)_x = 0 \\ u(0,x) = \begin{cases} u^- & \text{if } x < 0 \\ u^+ & \text{if } x > 0 \end{cases} \end{cases} \tag{2.2}$$

Below, by *solution* to (2.2) we always refer to the standard Lax (see [12]) self–similar entropic solutions.

Let $f, g \in \mathrm{Hyp}(\Omega)$ with

$$\mathcal{D}^g \subseteq \mathcal{D}^f \tag{2.3}$$

and define the *"distance"* between f and g as (cfr. [3])

$$\hat{d}(f,g) = \sup_{u \in \mathcal{R}(\mathcal{D}^g)} \frac{1}{|u^+ - u^-|} \cdot \left\| S^f_1 u - S^g_1 u \right\|_{L^1} \tag{2.4}$$

The distance $\hat{d}(f,g)$ is well defined due to (2.3). The main result of the present paper is the following theorem.

Theorem 2.1. *Let $f \in \mathrm{Hyp}(\Omega)$. Then, for all $g \in \mathrm{Hyp}(\Omega)$ with $\mathcal{D}^g \subseteq \mathcal{D}^f$ and for all $u \in \mathcal{D}^g$*

$$\left\| S^f_t u - S^g_t u \right\|_{L^1} \leq L_f \cdot \hat{d}(f,g) \cdot \int_0^t \mathrm{Tot.Var.}\,(S^g_t u)\; dt\,. \tag{2.5}$$

Recall that L_f is the Lipschitz constant of the semigroup S^f, see (ii) above. The proof of Theorem 2.1 is deferred to Section 3.

Remark that $\hat{d}$ generalizes the analogous quantity $\hat{d}_{\mathrm{lin}}$ defined in [3] with reference to the linear case. Let $\mathbb{M}^{n\times n}_d$ denote the set of $n \times n$ diagonalizable matrices with real eigenvalues. Note that $\mathbb{M}^{n\times n}_d \subseteq \mathrm{Hyp}(\mathbb{R}^n)$. Fix a $v \in \mathbb{R}^n$, $v \neq 0$. Denote by $A^t \star v$ the solution evaluated at time t of the linear system

$$\begin{cases} u_t + A u_x = 0 \\ u(0,x) = v \cdot \chi_{[0,+\infty)}(x) \end{cases}$$

(here, χ_I is the characteristic function of the interval I). Theorem 2.3 in [3] shows that

$$\hat{d}_{\text{lin}}(A,B) \doteq \sup_{v\colon |v|=1} \left\| A^1 \star v - B^1 \star v \right\|_{L^1}, \tag{2.6}$$

is a distance on $\mathbb{M}_d^{n\times n}$ such that for all $A, B \in \mathbb{M}^{n\times n}$

$$\|B - A\| \leq \hat{d}_{\text{lin}}(A,B) \ , \tag{2.7}$$

$\|\cdot\|$ being the usual operator norm. Moreover, $\left(\mathbb{M}_d^{n\times n}, \hat{d}_{\text{lin}}\right)$ is a complete metric space. Clearly, if f and g are linear, then $\hat{d}_{\text{lin}}(f,g) = \hat{d}(f,g)$.

Furthermore, $\hat{d}$ is related to $\hat{d}_{\text{lin}}$ computed on the derivatives of the flow functions, as shown by the following Proposition.

Proposition 2.2. *Let $f, g \in \text{Hyp}(\Omega)$ with $\mathcal{D}^g \subseteq \mathcal{D}^f$. Then*

$$\hat{d}(f,g) \geq \sup_{u\in\Omega} \hat{d}_{\text{lin}}(Df(u), Dg(u)) \ . \tag{2.8}$$

Thus, $\hat{d}(f,g)$ seems stronger than the C^0 distance between Df and Dg, in the sense of (2.7). Nonetheless, Corollary 2.5 below shows that once the flow functions and the domains $\mathcal{D}^f$, $\mathcal{D}^g$ are fixed, i.e. the total variation of the solutions $S_t^f u$, $S_t^g u$ are uniformly bounded, we can estimate the r.h.s. in (2.5) by means of $\|Df - Dg\|_{C^0}$.

Theorem 2.1 shows that the key point in the stability of the SRS w.r.t. the flow function lies in the dependence of only the solution to Riemann problems upon the flow function. From the more abstract point of view of quasidifferential equations in metric spaces (see [5, 15]) this is equivalent to relate the distance between semigroups to the distance between the vector fields generated by the semigroups.

As in [5] (see also [15]), in a metric space (E,d) define an equivalence relation on all the Lipschitz curves $\gamma\colon [0,1] \mapsto E$ exiting a fixed point u as

$$\gamma \sim \gamma' \quad \text{iff} \quad \lim_{\theta\mapsto 0+} \frac{d(\gamma(\theta), \gamma'(\theta))}{\theta} = 0. \tag{2.9}$$

The quotient space T_u so obtained is naturally equipped with the metric

$$\hat{d}(v_1, v_2) \doteq \limsup_{\theta\to 0+} \frac{d(\gamma_1(\theta), \gamma_2(\theta))}{\theta}, \tag{2.10}$$

where γ_i is a representative of the equivalence class v_i. By (2.9), $\hat{d}$ does not depend on the particular representatives chosen. A map $v\colon E \mapsto \bigcup_{u\in E} T_u$ is a vector field, provided $v(u) \in T_u$ for all u.

Let $S\colon E \times [0, +\infty[\mapsto E$ be a Lipschitzean semigroup, i.e. S satisfies

$$\begin{array}{c} S_0 = \mathbb{I}_E \\ S_s \circ S_t = S_{s+t} \end{array} \qquad \text{and} \qquad \begin{array}{c} \exists\, L > 0 \text{ such that} \\ \forall\, t, s > 0 \text{ and } \forall\, u, w \in \mathcal{D} \\ d(S_t u, S_s w) \leq L \cdot (|t-s| + d(u,w)) \ . \end{array}$$

Then, S naturally defines a vector field v_S on E by

$$v_S(u) \quad \text{is the equivalence class of the orbit} \quad \theta \mapsto S_\theta u \ . \tag{2.11}$$

Theorem 2.1 has a natural abstract counterpart, namely

Proposition 2.3. *Let S, S' be two Lipschitz semigroups on E generating the vector fields v and v', respectively. Denote with L the Lipschitz constant of, say, S. Then*

$$d\left(S_t u, S'_t u\right) \leq L \cdot \int_0^t \hat{d}\left(v(S'_t u), v'(S'_t u)\right)\, dt \ . \tag{2.12}$$

The above proposition is an immediate corollary of the following widely used (see [1, 4, 5, 7, 8] and the references in [6]) error estimate:

Lemma 2.4. *Given a Lipschitz semigroup $S\colon E \times [0,+\infty[\mapsto E$ with Lipschitz constant L, for every Lipschitz continuous map $w\colon [0,T] \mapsto E$ one has*

$$d\left(w(T), S_T w(0)\right) \leq L \cdot \int_0^T \liminf_{h\mapsto 0+} \frac{d\left(w(t+h), S_h w(t)\right)}{h}\, dt \,. \tag{2.13}$$

The proof of Theorem 2.1 consists of the following steps.

(1) Find an explicit definition of the vector field v_S generated by the SRS S.
(2) Compute the r.h.s. in (2.12).
(3) Apply Lemma 2.4.

Note that this procedure requires the mere existence of the SRS. In several cases (see for instance [1, 2, 7]) the existence of the SRS is achieved through the construction of a sequence S^n of uniformly Lipschitzean approximate semigroups defined on piecewise constant functions. In all these cases, the vector field v_n generated by S^n on the set of piecewise constant functions simply consists in the gluing of solutions to Riemann problems. Thus, under the further assumption that such an approximating sequence S^n exists, the step (1) above could be avoided.

The quantity $\hat{d}$ in (2.4) is thus the natural tool to estimate the dependence of the SRS upon the flow function (note also that no constant is involved in (2.5)). However, in view of possible applications of Theorem 2.1, we provide an estimate of the r.h.s. in (2.5) in terms of handier quantities.

Corollary 2.5. *Let $f \in \mathrm{Hyp}(\Omega)$ and assume that*

$$\mathcal{D}^f \subseteq \left\{u \in L^1(\mathbb{R}, K)\colon \mathrm{Tot.Var.}(u) \leq M\right\}$$

for suitable positive M and compact $K \subseteq \mathbb{R}^n$. Then, there exists a positive constant C such that for all $g \in \mathrm{Hyp}(\Omega)$ with $\mathcal{D}^g \subseteq \mathcal{D}^f$ and for all $u \in \mathcal{D}^g$

$$\left\|S_t^f u - S_t^g u\right\|_{L^1} \leq C \cdot \left\|Df - Dg\right\|_{C^0(\Omega)} \cdot t \,. \tag{2.14}$$

The above is the counterpart of the well known estimate for solutions of bounded variation in the scalar case given in [9]. Note that, differently from the linear case, the "distance" $\hat{d}(f,g)$ is equivalent to $\|Df - Dg\|_{C^0}$ because the total variation of both solutions $S^f u$ and $S^g u$ is fixed by the domain $\mathcal{D}^f$: thus the case of example of Remark 3.3 in [3] is not valid here.

For scalar equations and assuming that the flow functions f and g are strictly convex, we are able to extend the estimate in [9] to L^∞ initial data.

Theorem 2.6. *Assume that the scalar flow functions f and g are uniformly strictly convex in a compact interval K: i.e. $f''(u), g''(u) \geq \kappa > 0$ for all $u \in K$. Let u (resp. w) denote the solution to*

$$\left\{\begin{array}{l} u_t + [f(u)]_x = 0 \\ u(0,x) = u_o(x), \end{array}\right. \qquad \textit{resp.} \qquad \left\{\begin{array}{l} w_t + [g(w)]_x = 0 \\ w(0,x) = u_o(x) \end{array}\right.$$

with the same initial data $u_o \in L^\infty(\mathbb{R}, K)$. Denote by $\hat{\lambda}$ an upper bound for the characteristic speeds, i.e. $\hat{\lambda} \geq \max_{u\in K}\{|f'(u)|, |g'(u)|\}$. Then

$$\int_a^b \left|u(t,x) - w(t,x)\right| dx \leq 2 \cdot \mathrm{diam}(K) \cdot t \cdot \frac{b - a + 4\hat{\lambda}t}{\kappa t} \cdot \max_{u\in K}\left|f'(u) - g'(u)\right| \,. \tag{2.15}$$

3. Proof of the main results

This section is devoted to the proofs of Theorem 2.1, Proposition 2.2 and Corollary 2.5. The first step consists in the explicit computation of the vector field $v(u)$ generated by a SRS in the sense of (2.11). This procedure follows [5]. Remark that, given an $f \in \mathrm{Hyp}(\Omega)$, such construction is accomplished for all $u \in \mathcal{D}^f$.

Fix $f \in \mathrm{Hyp}(\Omega)$ and $u \in \mathcal{D}^f$. Du stands for the total variation of the weak derivative of u. Let $\hat\lambda$ be a constant strictly greater than all the characteristic speeds induced by f on $\mathcal{D}^f$. Let $\xi \in \mathbb{R}$ be given. We denote by ω the self similar solution of the Riemann problem

$$\begin{cases} \omega_t + f(\omega)_x = 0 \\ \omega(0,x) = \begin{cases} u(\xi-) & \text{if } x < 0 \\ u(\xi+) & \text{if } x > 0 \end{cases} \end{cases}$$

and let $U^\sharp_\xi$ be the function

$$U^\sharp_\xi(\theta,x) \doteq \begin{cases} \omega(\theta, x-\xi) & \text{if } |x-\xi| \le \hat\lambda\theta \\ u(x) & \text{if } |x-\xi| > \hat\lambda\theta \end{cases} \tag{3.1}$$

Moreover, define $U^\flat_\xi$ as the solution to the linear hyperbolic problem with constant coefficients

$$\begin{cases} \omega_\theta + Df(\xi)\omega_x = 0 \\ \omega(0,x) = u(x). \end{cases} \tag{3.2}$$

Given $\epsilon > 0$ by an ϵ–covering of the real line we mean a family

$$\mathcal{F} \doteq \{I_1, \ldots, I_N, I'_1, \ldots, I'_M\} \tag{3.3}$$

of open intervals which cover $\mathbb{R}$ such that:

1) the intervals I_α are mutually disjoint and no point $x \in \mathbb{R}$ lies inside more than two distinct intervals I'_β;
2) for every α there exists $\xi_\alpha \in I_\alpha$ such that $Du(I_\alpha \setminus \{\xi_\alpha\}) < \epsilon/N$;
3) for every β, $Du(I'_\beta) < \epsilon$.

Now, let ϵ_n be a sequence strictly decreasing to 0. If $\mathcal{F}_n = \{I_1, \ldots, I_{N(n)}, I'_1, \ldots, I'_{M(n)}\}$ is an ϵ_n–covering of $\mathbb{R}$, then also $\mathcal{F}_{n,\theta} = \{I_{1,\theta}, \ldots, I_{N(n),\theta}, I'_{1,\theta}, \ldots, I'_{M(n),\theta}\}$ is an ϵ_n–covering, where $I_{j,\theta} \doteq I_j + \lambda(-\theta,\theta)$ and $\theta \in [0,\theta_n]$, for θ_n sufficiently small and such that the sequence θ_n strictly decreases to 0. Define now

$$u^\theta_n(x) \doteq \begin{cases} U^\sharp_{\xi_\alpha}(\theta,x) & \text{if } x \in I_{\alpha,\theta} \\ U^\flat_{\xi_\beta}(\theta,x) & \text{if } x \in I'_{\beta,\theta}, x \notin \bigcup_\alpha I_{\alpha,\theta}, x \notin \bigcup_{\beta'<\beta} I'_{\beta',\theta} \end{cases}$$

We finally obtain the vector field v_{S^f} letting $v_{S^f}(u)$ be the equivalence class (w.r.t. (2.9)) of the curve $\theta \mapsto u^\theta$, where

$$u^\theta \doteq s u_n^{\theta_n} + (1-s) u_{n-1}^{\theta_{n-1}} \quad \text{if} \quad \theta = s\theta_n + (1-s)\theta_{n-1},\ s \in [0,1]\ . \tag{3.4}$$

In [5] it is shown that the trajectories of SRS S^f generated by the hyperbolic system of conservation laws $u_t + f(u)_x = 0$ are the solution of the quasilinear equation $\dot u = v_{S^f}(u)$, where the vector field $v_{S^f}(u)$ is generated by the curve (3.4).

Let f, g be as in Theorem 2.1, and denote with u^θ, w^θ the two curves generating the vector fields v_{S^f} and v_{S^g} induced by the SRSs S^f and S^g. In view of (2.12), we now pass to compute the distance (2.10) between $v_{S^f}(u)$ and $v_{S^g}(u)$. A simple computation shows that by Proposition 2.2

$$
\begin{aligned}
\frac{d(u^{\theta_n}, w^{\theta_n})}{\theta_n} &= \frac{1}{\theta_n} \cdot \left\| w^{\theta_n} - u^{\theta_n} \right\|_{L^1} \\
&\leq \sum_\alpha Du(\xi_\alpha) \cdot \hat{d}_{\mathrm{lin}}(f,g) + \sum_\beta \mathrm{Tot.Var.}(u, I'_\beta) \cdot \hat{d}\left(Df(\xi_\beta), Dg(\xi_\beta)\right) \\
&\leq \mathrm{Tot.Var.}(u) \cdot \hat{d}(f,g) \ .
\end{aligned} \tag{3.5}
$$

As a consequence, if s is as in (3.4)

$$
\begin{aligned}
\frac{d(u^\theta, w^\theta)}{\theta} &= \frac{\left\| s \cdot (u_n^{\theta_n} - w_n^{\theta_n}) + (1-s) \cdot (u_{n-1}^{\theta_{n-1}} - w_{n-1}^{\theta_{n-1}}) \right\|_{L^1}}{\theta} \\
&\leq \frac{s \cdot \theta_n + (1-s) \cdot \theta_{n-1}}{\theta} \cdot \mathrm{Tot.Var.}(u) \cdot \hat{d}(f,g) \\
&= \mathrm{Tot.Var.}(u) \cdot \hat{d}(f,g)
\end{aligned} \tag{3.6}
$$

hence

$$
\hat{d}\left(v_{S^f}(u), v_{S^g}(u)\right) = \limsup_{n \to +\infty} \frac{d(u^\theta, w^\theta)}{\theta} \leq \mathrm{Tot.Var.}(u) \cdot \hat{d}(f,g) \ . \tag{3.7}
$$

By Proposition 2.3, applying (3.7) to $S_t^g u$, the proof of Theorem 2.1 is completed.

In the next part of this section we give a proof of Corollary 2.5. We only need to prove that there exists a constant C such that $\hat{d}(f,g) \leq C \cdot \|Df - Dg\|_{C^0}$: this means that for all Riemann problems (2.2) we have

$$
\left\| S_1^f u - S_1^g u \right\|_{L^1} \leq C \cdot \left\| Df - Dg \right\|_{C^0} \cdot \left| u^+ - u^- \right|. \tag{3.8}
$$

We recall that $S_t^g u$ is a self similar solution, obtained by piecing together centered rarefaction waves and jump discontinuities. By the L^1_{loc} dependence of S^f, formula (3.8) is proved if we can verify it for Riemann problems generating a single wave in the solution $S_t^g u$. We then have to consider two cases.

If $S_t^g u$ is a centered rarefaction wave, then by the Lipschitz continuity of the solution for $t > 0$, the functions $S_{t+h}^g u$ and $S_h^f \circ S_t^g u$ solves in the broad sense the quasilinear versions of (2.1):

$$
\left[S_h^f \circ S_t^g u \right]_h + Df \left[S_h^f \circ S_t^g u \right]_x = 0 \quad \text{and} \quad \left[S_{t+h}^g u \right]_h + Dg \left[S_{t+h}^g u \right]_x = 0. \tag{3.9}
$$

Applying Lemma 2.4, we obtain

$$
\left\| S_1^f u - S_1^g u \right\|_{L^1} \leq L_f \cdot \int_0^1 \liminf_{h \to 0} \frac{\left\| S_h^f \circ S_t^g u - S_{t+h}^g u \right\|_{L^1}}{h} dt \tag{3.10}
$$

$$
\begin{aligned}
&= L_f \cdot \int_0^1 \left\| Df \left[S_t^g u \right]_x - Dg \left[S_t^g u \right]_x \right\|_{L^1} \\
&\leq C \cdot \left\| Df - Dg \right\|_{C^0} \cdot |u^+ - u^-|,
\end{aligned} \tag{3.11}
$$

since in this case the L^1 limits as $h \to 0$ of $(S_h^f \circ S_t^g u)/h$ and $(S_{t+h}^g u)/h$ exist by (3.9) and are equal to $(Df - Dg)[S_t^g u]_x$.

Now we consider the case in which the jump u^-, u^+ is solved by a shock travelling with speed σ, where σ is given by the Rankine–Hugoniot condition

$$
g(u^+) - g(u^-) = \sigma \cdot \left(u^+ - u^- \right). \tag{3.12}
$$

To prove (3.8), we approximate the solution $S_t^g u$ with the Lipschitz continuous function $\tilde{u}(t)$ defined as

$$\tilde{u}(t,x) = u^- \cdot \chi_{(-\infty,0]}\big(x-\sigma t\big) + \big(u^+ - u^-\big) \cdot \min\left(\frac{x-\sigma t}{\delta}, 1\right) \cdot \chi_{(0,+\infty)}\big(x-\sigma t\big).$$

Roughly speaking, $\tilde{u}$ is obtained from $S_t^g u$ by substituting the jump u^-, u^+ at $x-\sigma t$ with a linear function. Using the Lipschitzeanity of S^g we can write

$$\begin{aligned}\left\|S_h^f \circ S_t^g u - S_{t+h}^g u\right\|_{L^1} &\le \left\|S_h^f \circ S_t^g u - S_h^f \tilde{u}(t)\right\|_{L^1} + \left\|S_h^f \tilde{u}(t) - \tilde{u}(t+h)\right\|_{L^1} \\ &\quad + \left\|\tilde{u}(t+h) - S_{t+h}^g u\right\|_{L^1} \\ &\le \frac{\delta}{2} \cdot (1+L^f) \cdot \left|u^+ - u^-\right| + \left\|\tilde{u}(t+h) - S_h^f \tilde{u}(t)\right\|_{L^1}. \end{aligned} \tag{3.13}$$

The last term above can be evaluated again by means of Lemma 2.4:

$$\begin{aligned}\left\|\tilde{u}(t+h) - S_h^f \tilde{u}(t)\right\|_{L^1} &\le L^f \cdot \int_t^{t+h} \liminf_{\xi\to 0+} \frac{\left\|\tilde{u}(\eta+\xi) - S_\xi^f \tilde{u}(\eta)\right\|_{L^1}}{\xi}\, d\eta \\ &= h \cdot L^f \cdot \int_0^\delta \left|\left(\dot{\sigma}_\alpha \mathbb{I} - Dg\left(u_\alpha^- + (u_\alpha^+ - u_\alpha^-)\frac{y}{\delta}\right)\right)\left(\frac{u_\alpha^- + (u_\alpha^+ - u_\alpha^-)}{y\delta}\right)\right| dy \\ &\le h \cdot L^f \cdot \sup_{x\in K}\left\|Df(x) - Dg(x)\right\| \cdot \left|u_\alpha^+ - u_\alpha^-\right|, \end{aligned} \tag{3.14}$$

where we use $\dot{x}_\alpha \in [\sigma-\epsilon, \sigma+\epsilon]$ and the relation

$$\sigma \cdot (u_\alpha^+ - u_\alpha^-) = f(u_\alpha^+) - f(u_\alpha^-) = \int_0^1 Df\left((1-s)u_\alpha^- + s u_\alpha^+\right) ds \cdot (u_\alpha^+ - u_\alpha^-).$$

In fact, since $u_{\alpha,\delta}$ is Lipschitz continuous, we can use Lebesgue's dominated convergence theorem as in the previous case. Letting δ tend to 0, we obtain finally

$$\frac{1}{h}\left\|S_h^f \circ S_t^g u - S_{t+h}^g\right\|_{L^1} \le L_f \cdot \sup_{x\in K}\|Df(x) - Dg(x)\| \cdot \left|u_\alpha^+ - u_\alpha^-\right|. \tag{3.15}$$

This concludes the proof of Corollary 2.5: in fact an application of Lemma 2.4 gives immediately (3.8).

To end this section, we prove Proposition 2.2. Fix $u_o \in \Omega$ and $v \in \mathbb{R}^n$ with $|v| = 1$. Assume that for all positive and sufficiently small h the function $u_h = u_o + h \cdot \chi_{[0,+\infty)} \cdot v$ is in $\mathcal{D}^g$. Let

$$\tilde{f}_h(u) = \frac{1}{h} \cdot \big(f(u) - f(u_o)\big), \qquad \tilde{g}_h(u) = \frac{1}{h} \cdot \big(g(u) - g(u_o)\big)$$

and note that using a simple rescaling, we can write

$$\left\|S_1^{\tilde{f}_h}\left(\frac{1}{h}(u_h - u_o)\right) - S_1^{\tilde{g}_h}\left(\frac{1}{h}(u_h - u_o)\right)\right\|_{L^1} = \frac{1}{h} \cdot \left\|S_1^f u_h - S_1^g u_h\right\|_{L^1}$$

(recall that $(1/h) \cdot (u_h - u_o) = \chi_{[0,+\infty)} \cdot v$ is independent from h). Hence, passing to the limit $h \to 0$

$$\left\|Df(u_o)^1 \star v - Dg(u_o)^1 \star v\right\|_{L^1} = \lim_{h\to 0} \frac{1}{h} \cdot \left\|S_1^f u_h - S_1^g u_h\right\|_{L^1} \tag{3.16}$$

$$\hat{d}_{\mathrm{lin}}\left(Df(u_o), Dg(u_o)\right) = \sup_v \lim_{h\to 0} \frac{1}{h} \cdot \left\|S_1^f u_h - S_1^g u_h\right\|_{L^1}$$

and the proof is completed.

4. The Classical Limit of the Relativistic Euler Equations

In this section we apply Corollary 2.5 to the classical limit of the relativistic Euler equations, generalizing what was obtained in [14].

The relativistic p–system, (see [16, 17]) is

$$\left\{\begin{array}{l} \left[\rho + \left(\rho + \frac{1}{c^2}p\right)\frac{(v/c)^2}{1-(v/c)^2}\right]_t + \left[\left(\rho + \frac{1}{c^2}p\right)\frac{v}{1-(v/c)^2}\right]_x = 0 \\ \left[\left(\rho + \frac{1}{c^2}p\right)\frac{v}{1-(v/c)^2}\right]_t + \left[\left(\rho + \frac{1}{c^2}p\right)\frac{v^2}{1-(v/c)^2} + p\right]_x = 0 \end{array}\right. \tag{4.1}$$

Above, ρ is the mass–energy density of the fluid, v the classical coordinate velocity, p the pressure and c the light speed. We show below that as $c \to +\infty$, the problem (4.1) approaches to its classical counterpart

$$\left\{\begin{array}{r} [\rho]_t + [\rho v]_x = 0 \\ [\rho v]_t + \left[\rho v^2 + p\right] = 0 \end{array}\right. \tag{4.2}$$

in the sense that the SRS S^c generated by (4.1) converges to the SRS S generated by (4.2) on a domain containing all physically reasonable data. In particular, the total variation of the data need not be small.

In (4.1) a standard choice (see [16] and the references therein) for the pressure law is

$$p = \sigma^2 \cdot \rho\ , \tag{4.3}$$

σ being the sound speed.

Fix a positive lower bound for the density $\rho_{\min}$ and for the light speed c_o. Without any loss in generality, we may assume $\sigma < c_o$.

Let $\mathcal{V}_M = \{(\rho, \rho v) \in \mathrm{BV}\left(\mathbb{R}, (\rho_{\min}, +\infty) \times \mathbb{R}\right) : \mathrm{Tot.Var.}(\rho) + \mathrm{Tot.Var.}(\rho v) \leq M\}$. In [10] it is proved that for any $M > 0$, (4.1) generates a SRS $S^{c,M}$ defined on a domain $\mathcal{D}^{c,M}$ containing $\mathcal{V}_M$ and consisting of functions of total variation bounded by, say, M^3 (provided M is sufficiently large). Similarly, (4.2) generates a SRS S^M on a domain $\mathcal{D}^M$ containing $\mathcal{V}_{M^3}$ and contained, say, in $\mathcal{V}_{M^9}$.

Thus, for all sufficiently large c and M, there exist domains $\mathcal{D}^{c,M}$, $\mathcal{D}^M$ such that

$$\mathcal{V}_M \subset \mathcal{D}^{c,M} \subset \mathcal{V}_{M^3} \subset \mathcal{D}^M \subset \mathcal{V}_{M^9} \tag{4.4}$$

and, moreover, the problems (4.1) and (4.2) generate the SRSs

$$S^{c,M} \colon \mathcal{D}^{c,M} \times (0, +\infty) \mapsto \mathcal{D}^{c,M} \quad \text{and} \quad S^M \colon \mathcal{D}^M \times (0, +\infty) \mapsto \mathcal{D}^M\ .$$

We are now ready to state and prove the following application of Corollary 2.5.

Theorem 4.1. *Fix a positive $\rho_{\min}$ and sufficiently large M, c_o. Let $\mathcal{D}^{c,M}$ and $\mathcal{D}^M$ satisfy to (4.4) for $c > c_o$. Then there exists a constant C such that for all $c \geq c_0$ and for all $u \in \mathcal{D}^{M,c}$*

$$\left\| S_t^{c,M} u - S_t^M u \right\|_{L^1} \leq C \cdot \frac{1}{c^2} \cdot t. \tag{4.5}$$

In particular, by (4.4), the bound (4.5) holds for all initial data u with $\mathrm{Tot.Var.}(u) < M$.

Proof. Note that (4.1) and (4.2) in conservation form become, respectively,

$$\left\{\begin{array}{c} \rho_t + q_x = 0 \\ q_t + \left(\phi_c(\rho, q) \cdot \frac{q^2}{\rho} + p\right)_x = 0 \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{c} \rho_t + q_x = 0 \\ q_t + \left(\frac{q^2}{\rho} + p\right)_x = 0 \end{array}\right. \tag{4.6}$$

where

$$\phi_c(\rho,q) = 1 + \frac{1}{c^2}\cdot\left(1-\frac{v^2(\rho,q)}{c^2}\right)\cdot\frac{p}{\rho+\dfrac{v^2(\rho,q)}{c^2}\cdot\dfrac{p}{c^2}}\,.$$

Call f_c and, respectively, f the fluxes in the two systems (4.6). Then, the estimate $\|Df_c - Df\|_{C^0} \le C\cdot(1/c^2)$ follows from straightforward computations and completes the proof. □

We remark that the particular pressure law (4.3) is necessary only to ensure the existence of the SRS $S^{c,M}$ in the large. The above procedure remains true under much milder assumptions on the equation of state. Note moreover that the rate of convergence $\mathcal{O}(1/c^2)$ is exactly the one expected by the convergence of relativistic to classical mechanics.

It is of interest to mention that (4.5) proves also the uniform Lipschitz continuity of all semigroups $S^{c,M}$ for all sufficiently large c.

Remark 4.2. As is well known, linearizing (4.2) around $\rho=\rho_o$, $v=0$ at constant entropy leads to the wave equation. Corollary 2.5 ensures that the solutions to (4.2) converge to the linearized equation in L^1 over finite time intervals.

5. Stability of a scalar equation w.r.t. flux

In this section we prove Theorem 2.6. We consider two scalar equation,

$$u_t + f(u)_x = 0, \tag{5.1}$$

$$v_t + g(u)_x = 0, \tag{5.2}$$

with the same initial condition: $u(0,x) = v(0,x) = u_0$. We assume that f, g are strictly convex C^2 functions, precisely there exists a constant κ such that

$$\min_{u\in K}\{|f''(u)|,|g''(u)|\} \ge \kappa\,, \tag{5.3}$$

where K is a compact interval of $\mathbb{R}$ such that $u_0(x)\in K$ for all $x\in\mathbb{R}$. We recall that by maximum principle also the entropic solutions of (5.1), (5.2) will satisfies the same bound.

We recall that, by [11], given a point (t,x) we can consider the set of characteristics $\xi(t,x)$ passing through (t,x). If we denote with $\xi^-(t,x)$ and $\xi^+(t,x)$ the minimal and maximal backward characteristics, then either $\xi^-(t,x)=\xi^+(t,x)$ and the solution u is continuous in (t,x), or we have an admissible shock and the jump is exactly given by the (constant) values of u on the characteristics ξ^-, ξ^+. By condition (5.3) we have that if at time t two characteristics $\xi^+(t)$ and $\xi^-(t)$ meet, then we have

$$\frac{d}{dt}(\xi^+-\xi^-) \le -\kappa(u^+-u^-)\,. \tag{5.4}$$

Suppose that $\xi^-(0) < \xi^+(0)$, and consider now an initial datum $\tilde u_0$ defined as

$$\widetilde{u}_0(x) = \begin{cases} u_0(x) & x\le \xi^-(0)\\ u_0^- & \xi^-(0) < x \le \Xi\\ u_0^+ & \Xi < x \le \xi^+(0)\\ u_0(x) & x > \xi^+(0)\end{cases} \tag{5.5}$$

where $\Xi\in\mathbb{R}$ is chosen such that

$$\Xi \doteq \frac{1}{u_0^- - u_0^+}\int_{\xi^-}^{\xi^+} u_0(x)dx + \frac{u_0^-\xi^- - u_0^+\xi^+}{u_0^- - u_0^+}\,. \tag{5.6}$$

Using the conservation of mass it is easy to conclude that the solution in unchanged at time t. In fact consider the triangle $\mathcal{T}$ whose vertices are $(0, \xi^-(0))$, $(0, \xi^+(0))$ and (t, x). Since the equation (5.1) can be written as

$$\operatorname{div}\begin{pmatrix} u \\ f(u) \end{pmatrix} = 0\,,$$

and u is constant along the lines $\xi(t)$, we have

$$\begin{aligned}\iint_{\mathcal{T}} (u_t + f(u)_x) dt dx &= -\int_{\xi^-}^{\xi^+} u_0(x) dx + \int_0^t (f(u^+) - f'(u^+)u^+) dt + \int_0^t (f'(u^-)u^- - f(u^-)) dt \\ &= -\int_{\xi^-}^{\xi^+} u_0(x) dx + (f'(u^-)u^- - f'(u^+)u^+ + f(u^+) - f(u^-))t = 0.\end{aligned}$$

Using the relation

$$x = \xi^- + f'(u^-)t = \Xi + \frac{f(u^+) - f(u^-)}{u^+ - u^-} t = \xi^+ + f'(u^+)t\,,$$

we obtain (5.6). Using (5.6) we can change the initial of (5.1) or (5.2) so that $u(t)$ or $v(t)$ are unchanged. Since with this procedure we collect all the interactions at time 0, the total variation of $\tilde{u}_0$ has the same value of Tot.Var.$(u(t))$.

To change the initial datum in such a way that both $u(t)$ and $v(t)$ are the same, consider now the test system

$$w_t + \left(\frac{\kappa}{2} w^2\right)_x = 0\,. \tag{5.7}$$

By (5.3) and (5.4) if two characteristics meet is $u(t)$ and $v(t)$, then they also meet in (5.7). Let us denote with $\tilde{u}_0$ the new initial condition, obtained by the above procedure using equation (5.7).

Consider now an interval $[a, b]$. By the definition of $\hat{\lambda}$ we have that the values of $u(t)$ and $v(t)$ in $[a, b]$ depends only on $\tilde{u}_0$ in $[a - \hat{\lambda}t, b + \hat{\lambda}t]$. Using the standard estimates we have

$$\int_a^b |u(t, x) - v(t, x)| dx \le t \cdot \max_{u \in K} |f'(u) - g'(u)| \cdot \text{Tot.Var.}\Big(\tilde{u}_0; [a - \widehat{\lambda}t, b + \widehat{\lambda}t]\Big). \tag{5.8}$$

To estimate the total variation, an easy computation gives

(5.9)

$$\text{Tot.Var.}\Big(\tilde{u}_0; [a - \widehat{\lambda}t, b + \widehat{\lambda}t]\Big) \le \text{Tot.Var.}\Big(w(t); [a - 2\widehat{\lambda}t, b + 2\widehat{\lambda}t]\Big) \le 2 \cdot \operatorname{diam}(K) \cdot \frac{b - a + 4\widehat{\lambda}t}{\kappa t}\,.$$

Combining (5.8) and (5.9) we get

$$\int_a^b |u(t, x) - v(t, x)| dx \le \max_{u \in K} \big|f'(u) - g'(u)\big| \cdot 2 \cdot \operatorname{diam}(K) \cdot \frac{b - a + 4\widehat{\lambda}t}{\kappa}. \tag{5.10}$$

Remark 5.1. When $t \to 0$ the integral does not converge to 0. This is clear since the initial datum is in L^∞, and then the semigroup is continuous but not Lipschitz continuous in time, since the amount of interaction at $t = 0$ is infinite. Consider for example the following two equations

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \qquad v_t + \left(-v + \frac{v^2}{2}\right)_x = 0\;,$$

with the periodic initial condition

$$u_{0,n}(x) = \begin{cases} 1 & \text{if } x \in \big[k2^{-n+1}, k2^{-n+1} + 2^{-n}\big], k \in \mathbb{Z} \\ -1 & \text{otherwise} \end{cases}$$

At time $t_n = 2^{-n}$ the two solution are

$$\begin{cases} u_n(t_n,x) = 2^n(x - k2^{-n+1}) & \text{if } x \in \left(k2^{-n+1} - 2^{-n}, k2^{-n+1} + 2^{-n}\right] \\ v_n(t_n,x) = 2^n(x - k2^{-n+1} - 2^{-n}) & \text{if } x \in \left(k2^{-n+1}, (k+1)2^{-n+1}\right] \end{cases}$$

so that

$$\int_0^1 \left| v_n(t_n,x) - u_n(t_n,x) \right| dx = 1 \ .$$

This depends on the fact that the modulus of continuity of the semigroup can be arbitrarily large.

Note moreover that since the solutions u, v are limits of wave front tracking approximations, the continuous dependence of the solution on the flux function f can be stated also if f non convex. However in general one cannot prove any uniform continuous dependence.

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