# Entropy satisfying flux vector splittings and kinetic BGK models

F. Bouchut

Département de Mathématiques et Applications
Ecole Normale Supérieure et CNRS, UMR 8553
45, rue d'Ulm
75230 Paris cedex 05, France
e-mail: Francois.Bouchut@ens.fr

**Abstract**

We establish forward and backward relations between entropy satisfying BGK models such as those introduced previously by the author and the first order flux vector splitting numerical methods for systems of conservation laws. Classically, to a kinetic BGK model that is compatible with some family of entropies we can associate an entropy flux vector splitting. We prove that the converse is true: any entropy flux vector splitting can be interpreted by a kinetic model, and we obtain an explicit characterization of entropy satisfying flux vector splitting schemes. We deduce a new proof of discrete entropy inequalities under a sharp CFL condition that generalizes the monotonicity criterion in the scalar case. In particular, this gives a stability condition for numerical kinetic methods with noncompact velocity support. A new interpretation of general kinetic schemes is also provided via approximate Riemann solvers. We deduce the construction of finite velocity relaxation systems for gas dynamics.

**Key-words**: conservation laws – kinetic BGK model – flux vector splitting – discrete entropy inequality – approximate Riemann solver

**Mathematics Subject Classification**: 65M12, 35L65, 76N99

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# Contents



# 1 Introduction and BGK framework

Relaxation models such as in [11] have been used by many authors ([8], [9], [38], [25], [33], [1], [13]) to build numerical methods for hyperbolic systems of conservation laws. Kinetic BGK equations have been used especially in gas dynamics equations ([40], [24], [7], [31], [35], [36], [28], [50], [32], [27], [18], [3]), they give the so called kinetic or Boltzmann schemes. This approach has the advantage to provide a rigorous way to justify the good stability properties of the scheme: positivity of density and internal energy, and entropy consistency. This is obtained quite directly in the case of a compactly supported maxwellian equilibrium such as those introduced in [35], [36]. However, it is more delicate for noncompact distributions, and this has been studied in [23], [45], [30]. The problem of finding positive or/and entropy satisfying schemes for gas dynamics has been advised in [37], [29], [18], [4], [14], [17], [39], [12], [19], [20]. General entropy satisfying semi-discrete schemes have been studied in [34] and [44]. Concerning entropy schemes for scalar equations, we refer to [21].

Classically, kinetic methods lead, when discretized at first order, to particular flux vector splitting methods. An entropy condition was introduced in [10] for such flux vector splitting methods. A general framework for entropy compatible BGK models was introduced in [5] (see also [41] and [1]), and it was shown that flux vector splitting methods obtained via kinetic methods satisfy the entropy condition of [10]. The main result of this paper is that a flux vector splitting method is entropy satisfying if and only if it satisfies a refined *entropy dissipativity property* which is slightly stronger than that of [10], but equivalent to it locally. Moreover, such scheme can be obtained via a kinetic BGK model, with three velocities. Thus in particular, any kinetic scheme is equivalent, in the sense that it has the same first order numerical flux, to a three velocity relaxation model, and this gives a stability condition for numerical kinetic methods with noncompact velocity support. We obtain an explicit proof of discrete entropy inequalities, under a sharp CFL condition that generalizes the monotonicity criterion in the scalar case. Indeed the entropy dissipation can be expressed as a linear combination with nonnegative coefficients of *elementary dissipation terms* which have a generic form. We obtain also an interpretation of kinetic schemes and flux vector splitting methods as Godunov-type schemes defined by approximate Riemann solvers, in the sense of [24]. We apply the construction of three velocity relaxation models to gas dynamics.

The paper is organized as follows. After recalling the BGK framework introduced in [5], we explain the transport-projection method in Section 2, and its relation with flux vector splitting methods in Section 3. We establish the characterization of entropy satisfying flux vector splitting schemes and the converse relation from flux vector splitting methods to kinetic models in Section 4, and Section 5 is devoted to the application of our construction to gas dynamics. Finally, in Section 6 we give an extension to some parabolic problems.

In all the paper, all the nonlinearities are assumed implicitly to be sufficiently smooth, except the kinetic entropies $H_\eta$. The domains (such as $\mathcal{U}$ and $\mathcal{U}_{stab}$) are always supposed to have piecewise smooth boundary, and trivial topology, so that closed differential forms are always exact.

## 1.1 Continuous BGK models

Let us here recall the structure of time continuous BGK models as introduced in [5] (see also [41]). A BGK model is an equation

$$\partial_t f + a(\xi)\cdot\nabla_x f = \frac{M_f - f}{\varepsilon}, \tag{1.1}$$

where $t>0$, $x\in\mathbb{R}^N$,

$$\xi\in\Xi, \quad \text{a measure space with measure } d\xi, \tag{1.2}$$

$f(t,x,\xi)\in\mathbb{R}^p$ is the unknown,

$$a:\Xi\to\mathbb{R}^N \tag{1.3}$$

is the velocity,

$$M_f(t,x,\xi) = M(u(t,x),\xi), \qquad u(t,x)=\int f(t,x,\xi)\,d\xi - k, \tag{1.4}$$

and the equilibrium $M:\mathcal{U}\times\Xi\to\mathbb{R}^p$ satisfies the moment equations

$$\int M(u,\xi)\,d\xi = u+k, \qquad u\in\mathcal{U}, \tag{1.5}$$

$$\int a_j(\xi)M(u,\xi)\,d\xi = F_j(u)+k'_j, \qquad u\in\mathcal{U},\ j=1,\dots,N. \tag{1.6}$$

The BGK equation (1.1) is consistent, in the limit $\varepsilon \to 0$, with the system of conservation laws

$$\partial_t u + \sum_{j=1}^{N} \frac{\partial}{\partial x_j} F_j(u) = 0, \tag{1.7}$$

with $u(t,x) \in \mathcal{U}$ a convex subset of $\mathbb{R}^p$. In (1.1), we ask that

$$f(t,x,\xi) \in D_\xi, \tag{1.8}$$

where $D_\xi$ are convex subsets of $\mathbb{R}^p$ such that

$$a.e.\ \xi \quad \forall u \in \mathcal{U} \quad M(u,\xi) \in D_\xi. \tag{1.9}$$

Condition (1.8) is natural because it is easy to see, by integration of (1.1) along characteristics with $M_f$ as right-hand side, that under assumption (1.9), this property is preserved by the BGK equation.

We are also given a non empty family $\mathcal{E}$ of convex entropies for (1.7). We recall that an entropy for (1.7) is a scalar function $\eta(u)$ such that there exist scalar functions $\vartheta_j(u)$, called entropy fluxes, such that

$$\vartheta_j' = \eta' F_j', \tag{1.10}$$

where prime denotes differentiation with respect to $u$. It can be characterized by the property that the $(F_j')^t \eta''$ are symmetric. We look for weak solutions to (1.7) that satisfy the entropy inequalities

$$\partial_t \eta(u) + \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \vartheta_j(u) \leq 0, \qquad \eta \in \mathcal{E}. \tag{1.11}$$

For compatibility of the BGK model with (1.11), we ask that there exist kinetic entropies $H_\eta(f,\xi)$, $\eta \in \mathcal{E}$, satisfying

$$a.e.\ \xi \qquad H_\eta(.,\xi) : D_\xi \to \mathbb{R} \text{ is convex}, \tag{1.12}$$

$$\int H_\eta(M(u,\xi),\xi)\, d\xi = \eta(u) + c_\eta, \qquad u \in \mathcal{U}, \tag{1.13}$$

$$\text{for any } f : \Xi \to \mathbb{R}^p \text{ such that } a.e.\ \xi\ f(\xi) \in D_\xi \text{ and } u_f \equiv \int f(\xi) d\xi - k \in \mathcal{U},$$
$$\int H_\eta(M(u_f,\xi),\xi)\, d\xi \leq \int H_\eta(f(\xi),\xi)\, d\xi. \tag{1.14}$$

An important result of [5] is that, under technical assumptions, we can characterize the existence of kinetic entropies $(H_\eta)_{\eta \in \mathcal{E}}$ satisfying (1.12)-(1.14) by the *reduced stability conditions*

$$a.e.\ \xi \in \Xi \quad M(.,\xi) \in \mathcal{M}_+^{\mathcal{E}}, \tag{1.15}$$

where the vector space of maxwellians and the convex cone of nondecreasing maxwellians are defined by

$$\mathcal{M}^{\mathcal{E}} = \Big\{ M : \mathcal{U} \to \mathbb{R}^p \ ;\ \forall \eta \in \mathcal{E} \quad (M')^t \eta'' \text{ is symmetric in } \mathcal{U} \Big\}, \tag{1.16}$$

$$\mathcal{M}_+^{\mathcal{E}} = \Big\{ M \in \mathcal{M}^{\mathcal{E}} \ ; \ \forall \eta \in \mathcal{E} \quad (M')^t \eta'' \geq 0 \text{ in } \mathcal{U} \Big\} . \tag{1.17}$$

Then the functions $H_\eta$ can be obtained via the relations

$$H_\eta(M(u,\xi),\xi) = G_\eta(u,\xi), \qquad G'_\eta(u,\xi) = \eta'(u)M'(u,\xi), \tag{1.18}$$

and by

$$H'_\eta(M(u,\xi),\xi) \ni \eta'(u) \tag{1.19}$$

in the sense of subdifferentials (i.e. $H_\eta(f,\xi) \geq H_\eta(M(u,\xi),\xi) + \eta'(u)(f - M(u,\xi))$ for any $f \in D_\xi$). Indeed, relation (1.13) can be derived from (1.18) by differentiation with respect to $u$ and by using (1.5), while (1.14) is obtained directly from (1.19) by integration with respect to $\xi$.

*Remark 1.1* If $\mathcal{E}$ contains at least one strictly convex entroy, a function $M \in \mathcal{M}^{\mathcal{E}}$ lies in $\mathcal{M}_+^{\mathcal{E}}$ if and only if $M'$ has nonnegative eigenvalues (Proposition 2.2 in [5]).

## 1.2 Entropy variable

In the case of a single entropy, the sets $\mathcal{M}^{\mathcal{E}}$ and $\mathcal{M}_+^{\mathcal{E}}$ can be described precisely by using the so called entropy variable.

Assume that $\eta$ is a strictly convex function, and take $\mathcal{E} = \{\eta\}$. Denote by $v = \eta'(u)$ the entropy variable, and $\mathcal{M}^\eta = \mathcal{M}^{\mathcal{E}}$. Then

$$\mathcal{M}^\eta = \{ M = \partial_v \psi, \quad \psi \text{ scalar function} \} , \tag{1.20}$$

$$\mathcal{M}_+^\eta = \{ M = \partial_v \psi, \quad \psi \text{ locally convex with respect to } v \} . \tag{1.21}$$

This is easily seen because $(\eta'')^{-1}(\eta'' M')(\eta'')^{-1} = M'(\eta'')^{-1} = \partial_v M$, thus this is symmetric if and only if $M = \partial_v \psi$ for some $\psi$. Then $\partial_v M = \partial^2_{vv}\psi$, and the characterization of $\mathcal{M}_+^\eta$ follows.

Moreover, if $M = \partial_v \psi \in \mathcal{M}^\eta$, we have

$$\psi' = \eta'' M, \tag{1.22}$$

and the entropy flux $G$ associated with $M$ (defined by $G' = \eta' M'$) is given by

$$G = \eta' M - \psi. \tag{1.23}$$

For example, if we take $\psi = \eta'(u) \cdot u - \eta(u)$ ($= \eta^*(v)$ with $\eta^*$ the dual convex function of $\eta$), we get $M = u$, and $G = \eta$.

The two following properties can be useful also. The first is that $M$ is colinear to a given constant vector $K$ if and only if $\psi$ is a function of $v \cdot K$ only. The second is that $G = a \cdot M$ for some given constant $a$ if and only if $\psi(v = a + \delta v)$ is homogeneous of degree one in $\delta v$.

Finally, with the above results, if we take $\mathcal{E} = \{\eta\}$ with $\eta$ a strictly convex entropy for (1.7), then a BGK model associated to (1.7) is always given through a scalar function $\psi(u,\xi)$, $u \in \mathcal{U}$, $\xi \in \Xi$, by

$$M(u,\xi) = \partial_v \psi(u,\xi), \tag{1.24}$$

with $v=\eta'(u)$, $\psi$ must be locally convex with respect to $v$, and the moment relations (1.5)-(1.6) write

$$\int \psi(u,\xi)\,d\xi = \eta'(u)\cdot(u+k)-\eta(u)+cst, \tag{1.25}$$

$$\int a_j(\xi)\psi(u,\xi)\,d\xi = \psi_j(u)+\eta'(u)\cdot k_j'+cst,\quad j=1,\dots,N, \tag{1.26}$$

where the functions $\psi_j(u)$ are defined through

$$\partial_v\psi_j(u)=F_j(u). \tag{1.27}$$

The models of [7] were presented in this setting.

*Remark 1.2* The property for $M$ to be maxwellian is characterized by the existence of some flux $G$ such that $G'=\eta' M'$. By defining $\psi$ by (1.23), we get that it is also equivalent to the existence of some $\psi$ such that (1.22) holds. This characterization is also valid for non strictly convex $\eta$.

# 2 The transport-projection method

The aim of this section is to provide, in the context of a time-discrete equation, a more precise analysis of entropy dissipation and stability than was possible in [5] in the time-continuous case.

The transport-projection method, introduced in [8], [9], [40], [24] (see [48], [49] for an analysis), consists in replacing $\varepsilon$ by a timestep $\Delta t>0$, and the BGK equation (1.1) by a time-discrete equation

$$\partial_t f + a(\xi)\cdot\nabla_x f = \sum_{n=1}^{\infty}\delta(t-t_n)(f^n-f^{n-}), \tag{2.1}$$

with $t_n=n\Delta t$, $f^{n-}(x,\xi)=\lim_{t\to t_n,t<t_n} f(t,x,\xi)$, and

$$f^n = M_{f^{n-}}. \tag{2.2}$$

Note that $\sum_{n=1}^{\infty}\delta(t-t_n)\sim 1/\Delta t$. Obviously it is equivalent to perform a transport step

$$\begin{gathered}\partial_t f + a(\xi)\cdot\nabla_x f = 0 \quad \text{in } ]t_n,t_{n+1}[\times\mathbb{R}^N\times\Xi,\\ f(t_n,.)=f^n,\end{gathered} \tag{2.3}$$

which gives $f(t,x,\xi)=f^n(x-(t-t_n)a(\xi),\xi)$ for $t_n\le t<t_{n+1}$, and the projection step (2.2), or equivalently

$$u^{n-}(x)=\int f^{n-}(x,\xi)\,d\xi - k, \tag{2.4}$$

$$f^n(x,\xi)=M(u^n(x),\xi), \tag{2.5}$$

and

$$u^n = u^{n-}. \tag{2.6}$$

Extra projection operators can also be involved in a more complex relation $u^n = Pu^{n-}$, see Section 3.

From the above formulas we obtain an operator which to a function $u^n(x)$ associates a new function $u^{n+1}(x)$, giving an approximate solution to (1.7), $u^n \simeq u(t_n)$. We have to notice that at this level, we have to assume as in the continuous case that $u^n$ from (2.6) remains in $\mathcal{U}$, it is is not necessarily true in general. For further reference, we also define the function $u$ for all times by

$$u(t,x) = \int f(t,x,\xi)\, d\xi - k. \tag{2.7}$$

As in the continuous case, we can obtain consistency with entropy inequalities, because for any $\eta \in \mathcal{E}$, we have

$$\partial_t[H_\eta(f,\xi)] + a(\xi)\cdot\nabla_x[H_\eta(f,\xi)] = \sum_{n=1}^{\infty} \delta(t-t_n)\Big(H_\eta(f^n,\xi) - H_\eta(f^{n-},\xi)\Big), \tag{2.8}$$

and after integration with respect to $\xi$,

$$\partial_t \int H_\eta(f,\xi)\, d\xi + \mathrm{div}_x \int a(\xi) H_\eta(f,\xi)\, d\xi = \sum_{n=1}^{\infty} \delta(t-t_n) S_\eta^n(x), \tag{2.9}$$

where the entropy production takes the form

$$\begin{aligned} S_\eta^n(x) &= \int H_\eta(f^n,\xi)\, d\xi - \int H_\eta(f^{n-},\xi)\, d\xi \\ &= \int \Big[H_\eta(M(u^n,\xi),\xi) - H_\eta(f^{n-},\xi) + \eta'(u^n)(f^{n-} - M(u^n,\xi))\Big]\, d\xi, \end{aligned} \tag{2.10}$$

and the integrand is nonpositive since $H_\eta$ is convex and because of (1.19). The integration of (2.9) on a time interval leads finally to time discrete entropy inequalities.

## 2.1 Accuracy

Let us prove that (2.3)-(2.2) gives a consistent method for solving (1.7). We recall that for (1.1), the Chapman-Enskog expansion gives

$$\partial_t u + \sum_{j=1}^{N} \frac{\partial}{\partial x_j} F_j(u) = \varepsilon \sum_{1\le i,j\le N} \frac{\partial}{\partial x_j}\Big[D_{ji}(u)\frac{\partial u}{\partial x_i}\Big] \tag{2.11}$$

up to terms in $\varepsilon^2$, with

$$D_{ji}(u) = Q'_{ji}(u) - F'_j(u)F'_i(u), \qquad Q_{ij}(u) = \int a_i(\xi)a_j(\xi)M(u,\xi)\, d\xi, \tag{2.12}$$

and that (2.11) is entropy compatible as soon as (1.15) holds. For the time discrete method, we have the same behavior, with $\varepsilon$ replaced by $\Delta t/2$.

**Proposition 2.1** *As $\Delta t \to 0$, the time discrete method (2.3)-(2.2) is consistent with (1.7), and the equivalent equation is*

$$\partial_t u + \sum_{j=1}^{N} \frac{\partial}{\partial x_j} F_j(u) = \frac{\Delta t}{2} \sum_{1\le i,j\le N} \frac{\partial}{\partial x_j} \Big[ D_{ji}(u) \frac{\partial u}{\partial x_i} \Big]. \tag{2.13}$$

**Proof.** From the transport step (2.3), we get obviously

$$f^{n+1-} = f^n - \Delta t\, a(\xi) \cdot \nabla_x f^n + \frac{\Delta t^2}{2} (a(\xi) \cdot \nabla_x)^2 f^n + O(\Delta t^3). \tag{2.14}$$

Integrating this with respect to $\xi$ and using (2.5) we get

$$\begin{aligned} &u^{n+1}(x) \\ &= u^n(x) - \Delta t \sum_j \frac{\partial}{\partial x_j} \Bigg[ \int a_j(\xi) M(u^n(x), \xi) d\xi \\ &\qquad\qquad - \frac{\Delta t}{2} \sum_i \frac{\partial}{\partial x_i} \int a_j(\xi) a_i(\xi) M(u^n(x), \xi) d\xi \Bigg] + O(\Delta t^3) \\ &= u^n(x) - \Delta t \sum_j \frac{\partial}{\partial x_j} \Bigg[ F_j(u^n(x)) - \frac{\Delta t}{2} \sum_i Q'_{ij}(u^n(x)) \frac{\partial u^n}{\partial x_i}(x) \Bigg] + O(\Delta t^3). \end{aligned} \tag{2.15}$$

Since for the exact solution $u_{ex}$ to (1.7) we have obviously

$$\begin{aligned} &u_{ex}(t_{n+1}, x) \\ &= u^n(x) - \Delta t \sum_{j=1}^{N} \frac{\partial}{\partial x_j} F_j(u^n(x)) + \frac{\Delta t^2}{2} \sum_{i,j} \frac{\partial}{\partial x_j} \left[ F'_j(u^n) F'_i(u^n) \frac{\partial u^n}{\partial x_i} \right] + O(\Delta t^3), \end{aligned} \tag{2.16}$$

we get

$$u^{n+1}(x) = u_{ex}(t_{n+1}, x) + \frac{\Delta t^2}{2} \sum_{i,j} \frac{\partial}{\partial x_j} \left[ D_{ji}(u^n) \frac{\partial u^n}{\partial x_i} \right] + O(\Delta t^3), \tag{2.17}$$

which yields (2.13). □

We have to notice that as in the time continuous case, the scheme cannot be second-order in time unless $D_{ji} \equiv 0$, which implies that the system (1.7) satisfies the very exceptional property that all components of the $F_j$ are entropies of (1.7). More relations need to be satisfied for higher order methods, see [26].

## 2.2 Entropy dissipation analysis

As we have seen in (2.10), the entropy production $S^n_\eta(x)$ is nonpositive as soon as $H_\eta$ is convex with respect to $f$ and satisfies (1.19). According to [5], the reduced stability condition (1.15) implies the existence of a convex $H_\eta$, defined through (1.18) and (1.19). However, this result needs some technical assumptions, such as for example the invertibility of $M$, and moreover the reduced condition (1.15) may not be satisfied for all states $u \in \mathcal{U}$, but only in a subdomain $u \in \mathcal{U}_{stab}$. This is the case for the models of Section 5. Therefore, we are going to introduce here an alternate way to write entropy dissipation, which does not involve the function $H_\eta$, and for which we have a precise information about for which states (1.15) should be checked. This analysis involves the notion of $\eta$-*dissipative* functions, which is more precise than the notion of maxwellian functions.

The idea is to use a very particular property of the transport projection algorithm, which is that $f(t, x, \xi) \in D_\xi$ is preserved in the transport step (2.3) for any family of sets $D_\xi$, which do not need to be convex (it is necessary for the continuous model (1.1)). Therefore, the algorithm (2.3), (2.4), (2.5) can be reformulated via a function $w(t, x, \xi)$,

$$\begin{gathered}\partial_t w + a(\xi)\cdot\nabla_x w = 0 \quad \text{in } ]t_n, t_{n+1}[\times\mathbb{R}^N \times \Xi,\\ w(t_n, x, \xi) = u^n(x),\end{gathered} \tag{2.18}$$

$$u^{n-}(x) = \int M(w^{n-}(x,\xi),\xi)\,d\xi - k, \tag{2.19}$$

because we have of course

$$f(t, x, \xi) = M(w(t, x, \xi), \xi). \tag{2.20}$$

Then, one can define $G_\eta(u, \xi)$ by

$$G'_\eta(u,\xi) = \eta'(u)M'(u,\xi), \tag{2.21}$$

and we obtain the entropy equation

$$\partial_t[G_\eta(w,\xi)]+a(\xi)\cdot\nabla_x[G_\eta(w,\xi)] = \sum_{n=1}^{\infty}\delta(t-t_n)[G_\eta(w^n,\xi)-G_\eta(w^{n-},\xi)]. \tag{2.22}$$

After integration with respect to $\xi$ we get

$$\partial_t \int G_\eta(w,\xi)\,d\xi + \operatorname{div}_x \int a(\xi)G_\eta(w,\xi)\,d\xi = \sum_{n=1}^{\infty}\delta(t-t_n)S^n_\eta(x), \tag{2.23}$$

$$\begin{aligned}S^n_\eta(x) &= \int \Big[G_\eta(w^n(x,\xi),\xi) - G_\eta(w^{n-}(x,\xi),\xi)\Big]\,d\xi\\ &= \int \Big[G_\eta(u^n(x),\xi) - G_\eta(w^{n-}(x,\xi),\xi)\\ &\qquad\qquad +\eta'(u^n(x))\Big(M(w^{n-}(x,\xi),\xi) - M(u^n(x),\xi)\Big)\Big]\,d\xi.\end{aligned} \tag{2.24}$$

Now we have an alternate formula replacing (2.10) which does not involve $H_\eta$, and we can deduce an entropy consistency result as follows.

**Definition 2.2** *i) Let $\eta(u)$ be a convex function, and consider a vector function $W(u)$ such that*

$$W'(u)^t\eta''(u) \quad \textit{is symmetric.} \tag{2.25}$$

*We define $G_\eta[W](u)$ by*

$$G_\eta[W]'(u) = \eta'(u)W'(u), \tag{2.26}$$

*and the elementary entropy dissipation*

$$D_\eta[W](u,v) = G_\eta[W](u) - G_\eta[W](v) + \eta'(u)(W(v) - W(u)). \tag{2.27}$$

*ii) We shall say that $W$ is $\eta$-dissipative in $\mathcal{U}_{stab}$ if $W$ satisfies i) and*

$$D_\eta[W](u,v) \le 0 \quad \textit{for all } u, v \in \mathcal{U}_{stab}. \tag{2.28}$$

**Proposition 2.3** *Let $\mathcal{U}_{stab}$ be a (not necessarily convex) subset of $\mathcal{U}$ such that for a.e. $\xi \in \Xi$ and all $\eta \in \mathcal{E}$,*

$$M(.,\xi) \quad \textit{is } \eta\textit{-dissipative in } \mathcal{U}_{stab}. \tag{2.29}$$

*If $u^n(x) \in \mathcal{U}_{stab}$, $x \in \mathbb{R}^N$ is such that $u^{n+1}(x) \in \mathcal{U}_{stab}$, $x \in \mathbb{R}^N$, then $S_\eta^{n+1}$ from the right-hand side of (2.23) is nonpositive for any $\eta \in \mathcal{E}$ (and we do not need to check the existence of $H_\eta$ satisfying (1.12)-(1.14)).*

**Proof.** It is straightforward, because by (2.24),

$$S_\eta^{n+1}(x) = \int D_\eta[M(.,\xi)](u^{n+1}(x), w^{n+1-}(x,\xi))\, d\xi, \tag{2.30}$$

and the integrand is nonpositive since $w^{n+1-}(x,\xi) = u^n(x - \Delta t a(\xi))$. □

As was shown in [5], (2.29) is very close to the reduced condition (1.15). In order to make the paper complete, let us however recall more precisely why the $\eta$-dissipativity of $W$ is almost equivalent to $W \in \mathcal{M}_+^\eta$ (and is therefore also related to the fact that $W'$ has nonnegative eigenvalues according to Remark 1.1).

**Lemma 2.4** *If $W$ is $\eta$-dissipative in $\mathcal{U}_{stab}$, then $W \in \mathcal{M}_+^\eta$ in $\mathcal{U}_{stab}$.*

**Proof.** If $W$ satisfies (2.28), then for $u$ in the interior of $\mathcal{U}_{stab}$, $D_\eta[W](u,.)$ admits a maximum at $u$. Differentiating with respect to $v$, we get

$$\partial_v \left(D_\eta[W](u,v)\right) = \left(\eta'(u) - \eta'(v)\right) W'(v). \tag{2.31}$$

We deduce that the second derivative of $D_\eta[W](u,.)$ at $u$ is $-W'(u)^t\eta''(u)$, and it must be symmetric nonpositive. This gives the result since $\mathcal{U}_{stab}$ has dense interior. □

Conversely, we have the following lemma.

**Lemma 2.5** *Assume that $\eta$ is a strictly convex function, and that $W$ is as in Definition 2.2 (i). If $\mathcal{U}_{stab}$ is such that*

$$\eta'(\mathcal{U}_{stab}) \quad \textit{is convex} \tag{2.32}$$

*and*

$$(W')^t\eta'' \ge 0 \quad \textit{in } \mathcal{U}_{stab}, \tag{2.33}$$

*then $W$ is $\eta$-dissipative in $\mathcal{U}_{stab}$.*

**Proof.** Since $\eta$ is strictly convex, $\eta'$ is a diffeomorphism. Thus for given $u, v \in \mathcal{U}_{stab}$, because of (2.32), we can define $v(t) \in \mathcal{U}_{stab}$, $0 \le t \le 1$ by

$$\eta'(v(t)) = (1-t)\eta'(u) + t\eta'(v), \tag{2.34}$$

and $\varphi(t) = D_\eta[W](u, v(t))$. Then, with (2.33),

$$\begin{aligned} \varphi'(t) &= -\Big(\eta'(v(t)) - \eta'(u)\Big) W'(v(t))v'(t) \\ &= -t\Big(\eta''(v(t))v'(t)\Big) W'(v(t))v'(t) \\ &\le 0, \end{aligned} \tag{2.35}$$

and therefore $\varphi(1) \le \varphi(0)$, which yields (2.28). $\square$

*Remark 2.1* Let $\eta$ be a convex function. Another way of stating that a vector function $W$ is $\eta$-dissipative in $\mathcal{U}_{stab}$ is to ask that there exists some scalar function $G$ such that

$$G(u) - G(v) + \eta'(u)(W(v) - W(u)) \le 0 \quad \text{for all } u, v \in \mathcal{U}_{stab}. \tag{2.36}$$

This is easily obtained because if (2.36) is satisfied, then the left-hand side of (2.36), seen as a function of $v$, admits a maximum at $u$. Writing that its derivative must vanish there, we obtain $G'(u) = \eta'(u)W'(u)$, which implies that (2.25) holds and that $G_\eta[W] = G + cst$.

*Remark 2.2* If $W$ is $\eta$-dissipative in $\mathcal{U}_{stab}$, then we can solve formally the problem of finding the kinetic entropy $H_\eta(f)$ such that $H_\eta$ is convex, and

$$H_\eta(W(u)) = G_\eta[W](u), \tag{2.37}$$

$$H'_\eta(W(u)) \ni \eta'(u). \tag{2.38}$$

One can easily check that the formula

$$H_\eta(f) = \sup_{u \in \mathcal{U}_{stab}} \Big( G_\eta[W](u) + \eta'(u)\big(f - W(u)\big)\Big) \le \infty \tag{2.39}$$

gives a convex solution to (2.37)-(2.38).

*Remark 2.3* In practice, in order to verify the $\eta$-dissipativity assumption (2.29) of Proposition 2.3, several strategies can be used. The most obvious is to use Lemma 2.5. But the convexity assumption (2.32) can be rather restrictive. Another important case is if we are able to find directly a convex function $H$ such that $\eta'(u) = H'(W(u))$. Then $G_\eta[W](u) = H(W(u))$, and $D_\eta[W](u,v) = H(W(u)) - H(W(v)) + H'(W(u))(W(v) - W(u)) \le 0$ by convexity of $H$. If $H$ is not given, one can also obtain it under the condition $W \in \mathcal{M}_+^\eta$ and if $W$ is a diffeomorphism and $W(\mathcal{U}_{stab})$ is convex, as was proved in [5]. Finally if these three methods fail, one can try to check (2.28) directly, by an explicit computation.

*Remark 2.4* According the the above analysis, the reduced condition (1.15) is not fully sufficient, the $\eta$-dissipativity (2.29) is more precise, but is equivalent to it for a strictly convex entropy $\eta$ and if the states are sufficiently close, because then we can apply Lemma 2.5. An example where they are not globally equivalent is provided in Subsection 5.2, Remark 5.9. However, (1.15) is easier to check, this is the reason why we call it *reduced* stability condition.

*Remark 2.5* The function $D_\eta[W]$ also appears in the weak formulation of boundary conditions, see [39].

# 3 Space discretization and flux vector splitting

From now on, we only deal with the one-dimensional case $N = 1$. We consider a uniform spatial grid $(x_{i+1/2})_{i\in\mathbb{Z}}$ of size $\Delta x > 0$,

$$x_{i+1/2} - x_{i-1/2} = \Delta x. \tag{3.1}$$

In order to obtain a fully discrete method for solving (1.7), the usual Godunov approach is to consider a function $u^n(x)$ that is piecewise constant,

$$u^n(x) = u_i^n, \qquad x_{i-1/2} < x < x_{i+1/2}. \tag{3.2}$$

Then, in order to recover a piecewise constant function at step $n+1$, one has to perform a piecewise constant projection

$$u^{n+1} = P^0 u^{n+1-}, \tag{3.3}$$

or more explicitly

$$u_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} u^{n+1-}(x)\,dx, \tag{3.4}$$

where $u^{n+1-}$ is obtained by an exact or approximate resolution of (1.7) in $]t_n, t_{n+1}[$ with initial data $u^n$.

In our case, $u^{n+1-}$ is obtained by the transport-projection method of Section 2, and we get the so called kinetic scheme

$$\begin{gathered}\partial_t f + a(\xi)\cdot\nabla_x f = 0 \quad \text{in } ]t_n, t_{n+1}[\times\mathbb{R}^N\times\Xi,\\ f(t_n, x, \xi) = f^n(x,\xi) = M(u^n(x),\xi),\end{gathered} \tag{3.5}$$

$$u^{n-}(x) = \int f^{n-}(x,\xi)\,d\xi - k, \qquad u^n = P^0 u^{n-}. \tag{3.6}$$

We can extend the definition of $u$ to all times by (2.7). As in Subsection 2.2, we have $f(t,x,\xi) = M(w(t,x,\xi),\xi)$, and we obtain

$$\begin{gathered}\partial_t w + a(\xi)\cdot\nabla_x w = 0 \quad \text{in } ]t_n, t_{n+1}[\times\mathbb{R}^N\times\Xi,\\ w(t_n, x, \xi) = u^n(x),\end{gathered} \tag{3.7}$$

$$u^{n-}(x) = \int M(w^{n-}(x,\xi),\xi)\,d\xi - k, \qquad u^n = P^0 u^{n-}. \tag{3.8}$$

Next, in order to obtain entropy consistency, we define $G_\eta(u,\xi)$ by (2.21), and we write (2.22) and (2.23), with

$$S_\eta^n(x) = \int \Big[G_\eta(w^n(x,\xi),\xi) - G_\eta(w^{n-}(x,\xi),\xi)\Big]\,d\xi. \tag{3.9}$$

We then integrate (2.23) over $]t_n, t_{n+1}]\times]x_{i-1/2}, x_{i+1/2}[$, and get

$$\eta(u_i^{n+1}) - \eta(u_i^n) + \frac{\Delta t}{\Delta x}(\vartheta^n_{\eta,i+1/2} - \vartheta^n_{\eta,i-1/2}) = S^{n+1}_{\eta i}, \tag{3.10}$$

with

$$\vartheta^n_{\eta,i+1/2} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\int_\xi a(\xi)G_\eta(w(t,x_{i+1/2},\xi),\xi)\,d\xi dt, \tag{3.11}$$

and

$$S^{n+1}_{\eta i} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} S^{n+1}_\eta(x)\,dx. \tag{3.12}$$

In particular, if we take for $\eta$ the components of $u$, we get

$$u_i^{n+1} - u_i^n + \frac{\Delta t}{\Delta x}(F^n_{i+1/2} - F^n_{i-1/2}) = 0 \tag{3.13}$$

because $u^n - u^{n-}$ has vanishing average in the cells, with

$$F^n_{i+1/2} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\int_\xi a(\xi)M(w(t,x_{i+1/2},\xi),\xi)\,d\xi dt. \tag{3.14}$$

In order to explicit the numerical fluxes, we use the CFL condition

$$\Delta t \operatorname*{ess\,sup}_{\xi\in \mathrm{supp}M'(\mathcal{U}_{stab},.)} |a(\xi)| \le \Delta x, \tag{3.15}$$

where $\xi \in \mathrm{supp}M'(\mathcal{U}_{stab},.)$ means that there exists some $u\in\mathcal{U}_{stab}$ such that $M'(u,\xi)\neq 0$. Obviously, for $\xi\notin \mathrm{supp}M'(\mathcal{U}_{stab},.)$, $M(u,\xi) = C(\xi)$, $u\in\mathcal{U}_{stab}$. We observe that for $t_n < t < t_{n+1}$,

$$w(t,x_{i+1/2},\xi) = u^n(x_{i+1/2} - (t-t_n)a(\xi)) \tag{3.16}$$

remains in $\mathcal{U}_{stab}$ if $u^n(x)\in\mathcal{U}_{stab}$. Then, since for $\xi\in \mathrm{supp}M'(\mathcal{U}_{stab},.)$ we have $\Delta t|a(\xi)|\le\Delta x$ by (3.15), we obtain

$$w(t,x_{i+1/2},\xi) = \begin{cases} u_i^n & \text{if } a(\xi)>0,\\ u_{i+1}^n & \text{if } a(\xi)<0,\end{cases} \qquad \xi\in \mathrm{supp}M'(\mathcal{U}_{stab},.). \tag{3.17}$$

Thus

$$M\Big(w(t,x_{i+1/2},\xi),\xi\Big) = \begin{cases} M(u_i^n,\xi) & \text{if } a(\xi)>0,\\ M(u_{i+1}^n,\xi) & \text{if } a(\xi)<0,\end{cases} \tag{3.18}$$

and this remains true also for $\xi\notin \mathrm{supp}M'(\mathcal{U}_{stab},.)$. Therefore, by splitting the integral in two domains according to the sign of $a(\xi)$ in (3.14), we get

$$F^n_{i+1/2} = F^+(u_i^n) + F^-(u_{i+1}^n), \tag{3.19}$$

with

$$F^\pm(u) = \int_{\pm a(\xi)>0} a(\xi)M(u,\xi)\,d\xi. \tag{3.20}$$

Similarly,

$$\vartheta^n_{\eta,i+1/2} = \vartheta^+_\eta(u^n_i) + \vartheta^-_\eta(u^n_{i+1}), \tag{3.21}$$

with

$$\vartheta^\pm_\eta(u) = \int_{\pm a(\xi)>0} a(\xi) G_\eta(u,\xi)\, d\xi. \tag{3.22}$$

We remark that with (2.21),

$$(\vartheta^\pm_\eta)' = \eta'(F^\pm)', \tag{3.23}$$

therefore $F^+, F^- \in \mathcal{M}^{\mathcal{E}}$, and that by (1.6),

$$F^+ + F^- = F + k'. \tag{3.24}$$

Also, from (3.20) and by linearity, if $M(.,\xi)$ is $\eta$-dissipative in $\mathcal{U}_{stab}$, then $F^+, -F^-$ are $\eta$-dissipative in $\mathcal{U}_{stab}$.

As in Subsection 2.2, we obtain entropy consistency.

**Theorem 3.1** *Let $\mathcal{U}_{stab}$ be a (not necessarily convex) subset of $\mathcal{U}$ such that for a.e. $\xi \in \Xi$ and all $\eta \in \mathcal{E}$,*

$$M(.,\xi) \quad \textit{is } \eta\textit{-dissipative in } \mathcal{U}_{stab}, \tag{3.25}$$

*and assume that the CFL condition (3.15) holds. Define the fully discrete kinetic scheme by (3.13), (3.19), and (3.20). If $u^n_i \in \mathcal{U}_{stab}$, $i \in \mathbb{Z}$, is such that $u^{n+1}_i \in \mathcal{U}_{stab}$, $i \in \mathbb{Z}$, then $S^{n+1}_{\eta i}$ in the right-hand side of (3.10) is nonpositive for any $\eta \in \mathcal{E}$, and the scheme is entropy satisfying.*

**Proof.** From (3.12) and (3.9), and the fact that $u^{n+1-}(x)$ has average $u^{n+1}_i$ over $]x_{i-1/2}, x_{i+1/2}[$, we get

$$\begin{aligned} S^{n+1}_{\eta i} &= \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \int \Big[G_\eta(u^{n+1}_i,\xi) - G_\eta(w^{n+1-}(x,\xi),\xi)\Big]\, d\xi dx \\ &= \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \int \Big[G_\eta(u^{n+1}_i,\xi) - G_\eta(w^{n+1-}(x,\xi),\xi) \\ &\qquad + \eta'(u^{n+1}_i)\Big(M(w^{n+1-}(x,\xi),\xi) - M(u^{n+1}_i,\xi)\Big)\Big]\, d\xi dx \\ &= \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \int D_\eta[M(.,\xi)](u^{n+1}_i, w^{n+1-}(x,\xi))\, d\xi dx, \end{aligned} \tag{3.26}$$

which is obviously nonpositive by (3.25), since $w^{n+1-}(x,\xi) = u^n(x - \Delta t a(\xi))$. □

*Remark 3.1* The assumption that $u^{n+1}_i \in \mathcal{U}_{stab}$ is used only in (3.26). In the CFL condition (3.15), the property $u^n_i \in \mathcal{U}_{stab}$ is sufficient.

*Remark 3.2* Another proof is by using the decomposition

$$
\begin{aligned}
S_{\eta i}^{n+1} = \eta(u_i^{n+1}) &- \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \eta(u^{n+1-}(x))\,dx \\
&+ \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \int D_\eta[M(.,\xi)](u^{n+1-}(x), w^{n+1-}(x,\xi))\,d\xi dx.
\end{aligned} \tag{3.27}
$$

The first term is nonpositive according to Jensen's inequality, but for the second we need to assume that $u^{n+1-}(x) \in \mathcal{U}_{stab}$, $x \in \mathbb{R}$ instead of $u_i^{n+1} \in \mathcal{U}_{stab}$, which is somehow more natural.

*Remark 3.3* Instead of assumption (3.25), a weaker form of stability condition for (3.27) to be nonpositive is

$$
\int D_\eta[M(.,\xi)]\Big(u^{n+1-}(x), u^n(x-\Delta t a(\xi))\Big)\,d\xi \le 0, \quad x_{i-1/2} < x < x_{i+1/2}. \tag{3.28}
$$

*Remark 3.4* We have

$$
F^+(u) - F^-(u) = \int |a(\xi)| M(u,\xi)\,d\xi, \tag{3.29}
$$

and under assumptions (3.15) and (3.25),

$$
\mathrm{Id} - \frac{\Delta t}{\Delta x}(F^+ - F^-) \quad \text{is } \eta\text{-dissipative in } \mathcal{U}_{stab}. \tag{3.30}
$$

It is shown in Section 4 that this CFL condition is sufficient for the scheme to be entropy satisfying. This condition is weaker than (3.15) since it does not need $a(\xi)$ to be bounded in the support of $M$, only the integrals in $\xi$ of $M$ are involved.

*Remark 3.5* There is another interpretation of the scheme (3.13), (3.19), and (3.20), which is to write an upwind scheme for the kinetic equation (3.5),

$$
\begin{aligned}
&f_i^{n+1-}(\xi) - f_i^n(\xi) + \frac{\Delta t}{\Delta x} a(\xi)\Big(f_i^n(\xi) - f_{i-1}^n(\xi)\Big) = 0 \qquad \text{if } a(\xi) \ge 0, \\
&f_i^{n+1-}(\xi) - f_i^n(\xi) + \frac{\Delta t}{\Delta x} a(\xi)\Big(f_{i+1}^n(\xi) - f_i^n(\xi)\Big) = 0 \qquad \text{if } a(\xi) \le 0,
\end{aligned} \tag{3.31}
$$

with

$$
f_i^n(\xi) = M(u_i^n, \xi), \tag{3.32}
$$

and

$$
u_i^{n+1} = \int f_i^{n+1-}(\xi)\,d\xi - k. \tag{3.33}
$$

Indeed, under the CFL condition (3.15), this is exactly satisfied by the space averages

$$
f_i(t,\xi) = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} f(t,x,\xi)\,dx, \tag{3.34}
$$

with as usual $f_i^n(\xi) = f_i(t_n, \xi)$ and $f_i^{n+1-}(\xi) = f_i(t_{n+1}-, \xi)$. It is not possible to perform the above analysis for this formulation, because here $f_i^{n+1-}(\xi)$ cannot be written easily as $M(w_i^{n+1-}(\xi), \xi)$ for some $w_i^{n+1-}(\xi)$. We have to use discrete analogues of (2.8)-(2.10), and we really need the function $H_\eta$. However, this formulation has the advantage to extend to multidimensional settings, by replacing (3.31) by a suitable multidimensional scheme for the advection problem (2.3).

# 4 Kinetic interpretation of flux vector splitting methods

In this section, we establish the characterization of entropy satisfying flux vector splitting methods, and we prove that any such method for solving the system of conservation laws (1.7) in dimension $N = 1$ can be interpreted as a kinetic method as described in Section 2, with three velocities. Moreover we are able to write explicitly the entropy dissipation as a linear combination of elementary entropy dissipation terms as defined in (2.27). We also obtain an interpretation of the scheme as a Godunov-type scheme associated to an approximate Riemann solver.

Classically, a flux vector splitting method for the one-dimensional system

$$\partial_t u + \partial_x F(u) = 0, \tag{4.1}$$

with $u(t, x) \in \mathcal{U}$ a convex subset of $\mathbb{R}^p$, is a scheme

$$u_i^{n+1} - u_i^n + \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n) = 0, \tag{4.2}$$

where $\Delta t > 0$ is the timestep, $\Delta x > 0$ the size of the uniform spatial grid $(x_{i+1/2})_{i\in\mathbb{Z}}$, and

$$F_{i+1/2}^n = F^+(u_i^n) + F^-(u_{i+1}^n). \tag{4.3}$$

Consistency with (4.1) writes as

$$F^+ + F^- = F, \tag{4.4}$$

eventually up to an additive constant.

Let $\mathcal{E}$ be a non empty family of convex entropies for (4.1). An entropy compatibility condition with $\mathcal{E}$ was proposed in [10],

$$F^+,\ -F^- \in \mathcal{M}_+^{\mathcal{E}}, \tag{4.5}$$

where $\mathcal{M}_+^{\mathcal{E}}$ is defined in (1.17). With this assumption and under a CFL condition, it was proved in [10] that if the nonlinearities $F^+$, $F^-$ lead to well-posed Riemann problems satisfying entropy conditions, the scheme (4.2)-(4.3) satisfies discrete entropy inequalities. Here we are going to obtain this result, but without any assumption on these Riemann problems. In order to do so, we introduce a refined entropy compatibility condition,

$$F^+,\ -F^- \quad \text{are } \eta\text{-dissipative in } \mathcal{U}_{stab}, \quad \eta \in \mathcal{E}, \tag{4.6}$$

in the sense of Definition 2.2, which is more precise than (4.5) according to the analysis of Subsection 2.2. Notice that (4.5) and (4.6) are closely related to the classical condition that $(F^+)'$ and $-(F^-)'$ have nonnegative eigenvalues, see Remark 1.1. In the same spirit, we introduce the CFL-1 condition

$$\mathrm{Id} - \frac{\Delta t}{\Delta x}(F^+ - F^-) \quad \text{is } \eta\text{-dissipative in } \mathcal{U}_{stab}, \quad \eta \in \mathcal{E}, \tag{4.7}$$

which is a refined way to state that the eigenvalues of $(F^+ - F^-)'$ are less than $\Delta x/\Delta t$. According to Section 3, the fully discrete scheme derived from any kinetic transport-projection method takes the form (4.2)-(4.3), and the assumptions in Theorem 3.1 imply that (4.4)-(4.7) are satisfied.

These conditions alone are indeed sufficient for the scheme (4.2)-(4.3) to be entropy satisfying.

**Theorem 4.1** *Let $\mathcal{U}_{stab}$ be a (not necessarily convex) subset of $\mathcal{U}$, and $F^+$, $F^-$ verify (4.4), such that the stability condition (4.6) holds, and assume the CFL-1 condition (4.7). Then the flux vector splitting scheme (4.2)-(4.3) verifies the discrete entropy inequalities*

$$\eta(u_i^{n+1}) - \eta(u_i^n) + \frac{\Delta t}{\Delta x}(\vartheta^n_{\eta,i+1/2} - \vartheta^n_{\eta,i-1/2}) \equiv S^{n+1}_{\eta i} \leq 0, \quad \eta \in \mathcal{E}, \tag{4.8}$$

*for any sequence $u_i^n \in \mathcal{U}_{stab}$, $i \in \mathbb{Z}$, such that $u_i^{n+1} \in \mathcal{U}_{stab}$, $i \in \mathbb{Z}$, where*

$$\vartheta^n_{\eta,i+1/2} = \vartheta^+_\eta(u_i^n) + \vartheta^-_\eta(u^n_{i+1}), \tag{4.9}$$

$$(\vartheta^\pm_\eta)' = \eta'(F^\pm)'. \tag{4.10}$$

**Proof.** Let us define a kinetic BGK model by

$$\Xi = \{-1, 0, 1\} \tag{4.11}$$

with the counting measure,

$$a(\xi) = \xi c \tag{4.12}$$

for some parameter $c > 0$, and

$$M(u, 1) = F^+(u)/c, \qquad M(u, -1) = -F^-(u)/c, \tag{4.13}$$

$$M(u, 0) = u - \Big(F^+(u) - F^-(u)\Big)/c. \tag{4.14}$$

Then with (4.4) we get the moment relations

$$\int M(u,\xi)\, d\xi = u, \qquad \int a(\xi) M(u,\xi)\, d\xi = F(u), \tag{4.15}$$

where of course the integral in $\xi$ means only the sum over $\xi \in \Xi$. We observe also that by construction

$$\int_{\pm a(\xi) > 0} a(\xi) M(u,\xi)\, d\xi = F^\pm(u). \tag{4.16}$$

In order to verify the CFL condition (3.15), we need that $c \leq \Delta x/\Delta t$. Then, (4.6), (4.7) give that (3.25) is verified for any $\xi \in \Xi$ and $\eta \in \mathcal{E}$, under the condition $c \geq \Delta x/\Delta t$. Therefore, we obtain the result by taking $c = \Delta x/\Delta t$ and by applying Theorem 3.1. □

*Remark 4.1* The scheme (4.2)-(4.3) can also be written

$$u_i^{n+1} = \frac{\Delta t}{\Delta x} F^+(u_{i-1}^n) + u_i^n - \frac{\Delta t}{\Delta x}\Big(F^+(u_i^n) - F^-(u_i^n)\Big) - \frac{\Delta t}{\Delta x} F^-(u_{i+1}^n), \quad (4.17)$$

and conditions (4.6)-(4.7) exactly say that this expression is monotone (in a generalized sense) with respect to each argument, which is a well-known sufficient condition for entropy consistency in the scalar case. This indicates that the CFL condition (4.7) should be optimal.

**Proposition 4.2** *Let $\mathcal{U}_{stab}$ be a (not necessarily convex) subset of $\mathcal{U}$, and $F^+$, $F^-$ verify (4.4), such that the stability condition (4.6) holds, and assume the CFL conditions*

$$c \leq \Delta x/\Delta t, \quad (4.18)$$

$$\mathrm{Id} - (F^+ - F^-)/c \quad \textit{is } \eta\textit{-dissipative in } \mathcal{U}_{stab}, \quad \eta \in \mathcal{E}, \quad (4.19)$$

*for some $c > 0$, which obviously imply (4.7). Then the flux vector splitting scheme (4.2)-(4.3) can be interpreted as the transport-projection method with piecewise constant data and piecewise constant projection associated to the entropy satisfying relaxation system*

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x z = (F(u) - v)/\varepsilon, \\ \partial_t z + c^2 \partial_x v = \Big(c(F^+(u) - F^-(u)) - z\Big)/\varepsilon. \end{cases} \quad (4.20)$$

**Proof.** Obviously, in the proof of Theorem 4.1, one can use the interpretation (4.11)-(4.14) with $c$ different from $\Delta x/\Delta t$, but satisfying (4.18)-(4.19). This allows for example to treat non uniform grids, then (4.18) is replaced by $c \leq \Delta x_i/\Delta t$ for all $i$.

Then, the continuous BGK model associated to (4.11)-(4.14) can be written

$$\begin{cases} \partial_t f(t,x,1) + c\partial_x f(t,x,1) = (M(u(t,x),1) - f(t,x,1))/\varepsilon, \\ \partial_t f(t,x,-1) - c\partial_x f(t,x,-1) = (M(u(t,x),-1) - f(t,x,-1))/\varepsilon, \\ \partial_t f(t,x,0) = (M(u(t,x),0) - f(t,x,0))/\varepsilon. \end{cases} \quad (4.21)$$

We define as in (2.7)

$$u(t,x) = \int f(t,x,\xi)\, d\xi = f(t,x,1) + f(t,x,-1) + f(t,x,0), \quad (4.22)$$

$$v(t,x) = \int a(\xi) f(t,x,\xi)\, d\xi = cf(t,x,1) - cf(t,x,-1), \quad (4.23)$$

$$z(t,x) = \int a(\xi)^2 f(t,x,\xi)\, d\xi = c^2(f(t,x,1) + f(t,x,-1)), \quad (4.24)$$

and we easily obtain (4.20) by taking the moments of (4.21), since (4.15) holds and

$$\int a(\xi)^2 M(u,\xi)\, d\xi = c(F^+(u) - F^-(u)). \quad (4.25)$$

Indeed, (4.21) is the diagonal form of (4.20). □

The stability condition (4.6) is indeed necessary for the flux vector splitting scheme (4.2)-(4.3) to satisfy the discrete entropy inequalities (4.8). This can be seen by looking at the semidiscrete method obtained by letting $\Delta t \to 0$ in (4.2)-(4.3), as in [34], [44].

**Theorem 4.3** *The semidiscrete method*

$$\frac{du_i}{dt} + \frac{1}{\Delta x}\Big(F_{i+1/2} - F_{i-1/2}\Big) = 0, \qquad F_{i+1/2} = F^+(u_i) + F^-(u_{i+1}) \tag{4.26}$$

*satisfies entropy inequalities of the form*

$$\frac{d}{dt}\eta(u_i) + \frac{1}{\Delta x}\Big(\vartheta_{\eta,i+1/2} - \vartheta_{\eta,i-1/2}\Big) \leq 0, \quad \vartheta_{\eta,i+1/2} = \vartheta_\eta^+(u_i) + \vartheta_\eta^-(u_{i+1}) \tag{4.27}$$

*for some functions $\vartheta_\eta^+$, $\vartheta_\eta^-$, if and only if (4.6) holds.*

**Proof.** Expressing $d(\eta(u_i))/dt$ from (4.26) by multiplying by $\eta'(u_i)$, we get that (4.27) is equivalent to

$$\vartheta_{\eta,i+1/2} - \vartheta_{\eta,i-1/2} - \eta'(u_i)\Big(F_{i+1/2} - F_{i-1/2}\Big) \leq 0, \tag{4.28}$$

and this must be true for any $u_{i-1}$, $u_i$, $u_{i+1}$. By taking successively $u_{i+1} = u_i$ and $u_{i-1} = u_i$, we obtain

$$\vartheta_\eta^+(u_i) - \vartheta_\eta^+(u_{i-1}) - \eta'(u_i)\Big(F^+(u_i) - F^+(u_{i-1})\Big) \leq 0, \tag{4.29}$$

$$\vartheta_\eta^-(u_{i+1}) - \vartheta_\eta^-(u_i) - \eta'(u_i)\Big(F^-(u_{i+1}) - F^-(u_i)\Big) \leq 0, \tag{4.30}$$

and this is obviously sufficient since by summing up (4.29) and (4.30) we recover (4.28). Now, according to Remark 2.1, the properties (4.29) and (4.30) exactly mean that $F^+$, $-F^-$ are $\eta$-dissipative, and we have necessarily the relations $(\vartheta_\eta^\pm)' = \eta'(F^\pm)'$. □

*Remark 4.2* In order to determine an entropy flux vector splitting method, one has only to find a decomposition of $F$ as (4.4), (4.6), or (4.4)-(4.5) as a first approach. In the case of a single strictly convex entropy $\mathcal{E} = \{\eta\}$, we can define $\psi$ by

$$\partial_v \psi = F, \tag{4.31}$$

with $v = \eta'(u)$ the entropy variable, and according to Section 1.2, a decomposition (4.4)-(4.5) is equivalent to a decomposition

$$\psi = \psi^+ + \psi^-, \qquad \psi^+,\ -\psi^- \text{ locally convex with respect to } v, \tag{4.32}$$

via the relations

$$F^\pm = \partial_v \psi^\pm. \tag{4.33}$$

## 4.1 A direct proof of discrete entropy inequalities

In order to make Theorem 4.1 more clear, Let us finally give a quite explicit proof of it. We notice that with the BGK model (4.11)-(4.14), formula (3.26) can be written as

$$S_{\eta i}^{n+1} = \sum_{\xi \in \Xi} S_{\eta i}^{n+1}(\xi), \tag{4.34}$$

with

$$S_{\eta i}^{n+1}(\xi) = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} D_\eta[M(.,\xi)]\Big(u_i^{n+1}, u^n(x - \Delta t a(\xi))\Big)\, dx. \tag{4.35}$$

Since $c\Delta t = \Delta x$, the value of $u^n(x - \Delta t a(\xi))$ in (4.35) is respectively $u_{i+1}^n$, $u_i^n$, $u_{i-1}^n$ if $\xi = -1, 0, 1$, and the integral in $x$ disappears. We observe that

$$G_\eta(u, 1) = \vartheta_\eta^+(u)/c, \qquad G_\eta(u, -1) = -\vartheta_\eta^-(u)/c, \tag{4.36}$$

$$G_\eta(u, 0) = \eta(u) - \vartheta_\eta^+(u)/c + \vartheta_\eta^-(u)/c, \tag{4.37}$$

therefore

$$S_{\eta i}^{n+1}(1) = \vartheta_\eta^+(u_i^{n+1})/c - \vartheta_\eta^+(u_{i-1}^n)/c + \eta'(u_i^{n+1}) \left(F^+(u_{i-1}^n) - F^+(u_i^{n+1})\right)/c, \tag{4.38}$$

$$S_{\eta i}^{n+1}(-1) = -\vartheta_\eta^-(u_i^{n+1})/c + \vartheta_\eta^-(u_{i+1}^n)/c - \eta'(u_i^{n+1}) \left(F^-(u_{i+1}^n) - F^-(u_i^{n+1})\right)/c, \tag{4.39}$$

$$\begin{aligned} S_{\eta i}^{n+1}(0) = {} & \eta(u_i^{n+1}) - \vartheta_\eta^+(u_i^{n+1})/c + \vartheta_\eta^-(u_i^{n+1})/c \\ & - \Big[\eta(u_i^n) - \vartheta_\eta^+(u_i^n)/c + \vartheta_\eta^-(u_i^n)/c\Big] \\ & + \eta'(u_i^{n+1})\Big(u_i^n - F^+(u_i^n)/c + F^-(u_i^n)/c \\ & \quad - \Big[u_i^{n+1} - F^+(u_i^{n+1})/c + F^-(u_i^{n+1})/c\Big]\Big). \end{aligned} \tag{4.40}$$

With these formulas and by using (4.2)-(4.3), one can check directly that $S_{\eta i}^{n+1}$ from (4.8) is the sum of (4.38)-(4.40), as stated in (4.34). This is just an algebraic identity. Then obviously $S_{\eta i}^{n+1} \leq 0$, since each term $S_{\eta i}^{n+1}(\xi)$, $\xi = -1, 0, 1$ has the form (2.27) of an elementary entropy dissipation associated to $M(., \xi)$.

*Remark 4.3* If we take $c \neq \Delta x/\Delta t$ satisfying (4.18)-(4.19) as in Proposition 4.2, the integral in (4.35) has to be split over subintervals, this is a bit more complicate. With the decomposition (3.27) instead of (3.26), the computation is even more tedious, but in the case of a half CFL condition $c\Delta t \leq \Delta x/2$, this can be done, see Proposition 4.4 and Remark 4.5.

## 4.2 Interpretation as an approximate Riemann solver

In this subsection, we give an interpretation of the flux vector splitting scheme (4.2)-(4.3) via an approximate Riemann solver. Then we give also the interpretation for general kinetic schemes.

According to Proposition 4.2, the scheme can be interpreted as the transport-projection method with piecewise constant data described in Section 3 associated to the BGK model (4.11)-(4.14). We can write obviously the explicit value of the kinetic function $f(t,x,\xi)$, $t_n < t < t_{n+1}$, solving (3.5),

$$f(t,x,1) = F^+(u_i^n)/c, \quad x_{i-1/2} < x - c(t-t_n) < x_{i+1/2}, \tag{4.41}$$

$$f(t,x,-1) = -F^-(u_i^n)/c, \quad x_{i-1/2} < x + c(t-t_n) < x_{i+1/2}, \tag{4.42}$$

$$f(t,x,0) = u_i^n - F^+(u_i^n)/c + F^-(u_i^n)/c, \quad x_{i-1/2} < x < x_{i+1/2}. \tag{4.43}$$

Therefore, for $x_{i-1/2} < x < x_{i+1/2}$, we obtain with the CFL condition (4.18)

$$f(t,x,1) = F^+(u_i^n)/c - \mathbb{1}_{x<x_{i-1/2}+c(t-t_n)} \Big[F^+(u_i^n) - F^+(u_{i-1}^n)\Big]/c, \tag{4.44}$$

$$f(t,x,-1) = -F^-(u_i^n)/c - \mathbb{1}_{x>x_{i+1/2}-c(t-t_n)} \Big[F^-(u_{i+1}^n) - F^-(u_i^n)\Big]/c, \tag{4.45}$$

and $f(t,x,0)$ is given by (4.43). Then we define

$$u(t,x) = \int f(t,x,\xi)\,d\xi = f(t,x,1) + f(t,x,-1) + f(t,x,0), \tag{4.46}$$

and we obtain for $t_n < t < t_{n+1}$ and $x_{i-1/2} < x < x_{i+1/2}$

$$\begin{aligned} u(t,x) = u_i^n &- \mathbb{1}_{x<x_{i-1/2}+c(t-t_n)} \Big[F^+(u_i^n) - F^+(u_{i-1}^n)\Big]/c \\ &- \mathbb{1}_{x>x_{i+1/2}-c(t-t_n)} \Big[F^-(u_{i+1}^n) - F^-(u_i^n)\Big]/c. \end{aligned} \tag{4.47}$$

The value of $u^{n+1-}(x)$ is obtained for $t - t_n = \Delta t$ in (4.47). Now, in order to simplify, let us assume the half CFL condition

$$c\Delta t \le \Delta x/2, \tag{4.48}$$

so that the two intervals in (4.47) do not overlap. We get

$$u(t,x) = \begin{cases} u_i^n - [F^+(u_i^n) - F^+(u_{i-1}^n)]/c & \text{if } x_{i-1/2} < x < x_{i-1/2} + c(t-t_n), \\ u_i^n & \text{if } x_{i-1/2} + c(t-t_n) < x < x_{i+1/2} - c(t-t_n), \\ u_i^n - [F^-(u_{i+1}^n) - F^-(u_i^n)]/c & \text{if } x_{i+1/2} - c(t-t_n) < x < x_{i+1/2}. \end{cases} \tag{4.49}$$

This formula is indeed an approximate Riemann solver formula.

Let us recall (see [24]) that an approximate Riemann solver for (4.1) is a function $R(x/t, u_l, u_r)$ satisfying the consistency relations

$$R(x/t, u, u) = u, \tag{4.50}$$

$$\frac{1}{\Delta t}\int_{-\Delta x/2}^{\Delta x/2} R(x/\Delta t, u_l, u_r)\,dx - \frac{\Delta x}{2\Delta t}(u_l+u_r) = -[F(u_r)-F(u_l)]. \tag{4.51}$$

It is called compatible with the family of convex entropies $\mathcal{E}$ if for any $\eta\in\mathcal{E}$,

$$\frac{1}{\Delta t}\int_{-\Delta x/2}^{\Delta x/2} \eta\Big(R(x/\Delta t, u_l, u_r)\Big)\,dx - \frac{\Delta x}{2\Delta t}(\eta(u_l)+\eta(u_r)) \leq -[\vartheta(u_r)-\vartheta(u_l)], \tag{4.52}$$

where $\vartheta$ is the entropy flux associated to $\eta$, $\vartheta'=\eta' F'$. A CFL condition 1/2 has to be assumed (in the sense that $R(x/t,u_l,u_r)=u_l$ if $x/t<-\Delta x/2\Delta t$, and $R(x/t,u_l,u_r)=u_r$ if $x/t>\Delta x/2\Delta t$), in order that (4.51) and (4.52) make sense. To the approximate Riemann solver we can associate a *Godunov-type scheme* by

$$\begin{aligned} u_i^{n+1} &= \frac{1}{\Delta x}\int_0^{\Delta x/2} R(x/\Delta t, u_{i-1}^n, u_i^n)\,dx + \frac{1}{\Delta x}\int_{-\Delta x/2}^{0} R(x/\Delta t, u_i^n, u_{i+1}^n)\,dx \\ &= u_i^n - \frac{\Delta t}{\Delta x}[F(u_i^n,u_{i+1}^n)-F(u_{i-1}^n,u_i^n)], \end{aligned} \tag{4.53}$$

with

$$\begin{aligned} F(u_l,u_r) &= F(u_r) + \frac{1}{\Delta t}\int_0^{\Delta x/2}\Big(R(x/\Delta t,u_l,u_r)-u_r\Big)\,dx \\ &= F(u_l) - \frac{1}{\Delta t}\int_{-\Delta x/2}^{0}\Big(R(x/\Delta t,u_l,u_r)-u_l\Big)\,dx. \end{aligned} \tag{4.54}$$

Because of Jensen's inequality, the scheme satisfies discrete entropy inequalities

$$\eta(u_i^{n+1})-\eta(u_i^n)+\frac{\Delta t}{\Delta x}[\vartheta_l(u_i^n,u_{i+1}^n)-\vartheta_r(u_{i-1}^n,u_i^n)]\leq 0, \tag{4.55}$$

where

$$\begin{aligned} \vartheta_r(u_l,u_r) &= \vartheta(u_r) + \frac{1}{\Delta t}\int_0^{\Delta x/2}\Big(\eta\Big(R(x/\Delta t,u_l,u_r)\Big)-\eta(u_r)\Big)\,dx, \\ \vartheta_l(u_l,u_r) &= \vartheta(u_l) - \frac{1}{\Delta t}\int_{-\Delta x/2}^{0}\Big(\eta\Big(R(x/\Delta t,u_l,u_r)\Big)-\eta(u_l)\Big)\,dx. \end{aligned} \tag{4.56}$$

Under assumption (4.52), we have $\vartheta_r(u_l,u_r)\leq\vartheta_l(u_l,u_r)$, and the inequality (4.55) becomes conservative, with numerical flux equal either to $\vartheta_r$ or to $\vartheta_l$.

*Remark 4.4* The same formalism works with a space dependent approximate Riemann solver $R_{i+1/2}(x/t,u_l,u_r)$.

**Proposition 4.4** *Let $\mathcal{U}_{stab}$ be a (not necessarily convex) subset of $\mathcal{U}$, and $F^+$, $F^-$ verify (4.4), such that the stability condition (4.6) holds, and assume the CFL conditions*

$$c\Delta t\leq \Delta x/2, \tag{4.57}$$

$$\mathrm{Id}-F^+/c,\quad \mathrm{Id}+F^-/c \quad \textit{are } \eta\textit{-dissipative in } \mathcal{U}_{stab},\quad \eta\in\mathcal{E}, \tag{4.58}$$

*for some $c > 0$. Then the flux vector splitting scheme (4.2)-(4.3) is the Godunov-type scheme associated to the approximate Riemann solver*

$$R(x/t, u_l, u_r) = \begin{cases} u_l \quad \textit{if}\ \dfrac{x}{t} < -c, \\ u_l - [F^-(u_r) - F^-(u_l)]/c \quad \textit{if}\ -c < \dfrac{x}{t} < 0, \\ u_r - [F^+(u_r) - F^+(u_l)]/c \quad \textit{if}\ 0 < \dfrac{x}{t} < c, \\ u_r \quad \textit{if}\ c < \dfrac{x}{t}, \end{cases} \tag{4.59}$$

*which is entropy satisfying under the assumption that the four states in (4.59) lie in $\mathcal{U}_{stab}$.*

**Proof.** Using the half CFL condition (4.57), one can easily check that (4.50)-(4.51) hold, and that the numerical flux in (4.54) is given by $F(u_l, u_r) = F^+(u_l) + F^-(u_r)$, which matches (4.2)-(4.3). Next, we have to check that (4.52) holds, or equivalently that $\vartheta_r \le \vartheta_l$. Indeed, in order to prove that (4.55) becomes the inequality (4.8), we are going to prove more precisely that

$$\vartheta_r(u_l, u_r) \le \vartheta_\eta^+(u_l) + \vartheta_\eta^-(u_r) \le \vartheta_l(u_l, u_r). \tag{4.60}$$

One can check that

$$\begin{aligned} \vartheta_r(u_l, u_r) &= c\,\eta\Big(u_r - F^+(u_r)/c + F^+(u_l)/c\Big) - c\,\eta(u_r) + \vartheta(u_r), \\ \vartheta_l(u_l, u_r) &= -c\,\eta\Big(u_l - F^-(u_r)/c + F^-(u_l)/c\Big) + c\,\eta(u_l) + \vartheta(u_l). \end{aligned} \tag{4.61}$$

Therefore, since $\vartheta_\eta^+ + \vartheta_\eta^- = \vartheta$ and with the notation (2.27), a computation gives

$$\begin{aligned} &\Big[\vartheta_r(u_l, u_r) - \vartheta_\eta^+(u_l) - \vartheta_\eta^-(u_r)\Big]/c \\ &= \eta\Big(u_r - F^+(u_r)/c + F^+(u_l)/c\Big) - \eta(u_r) + \Big[\vartheta_\eta^+(u_r) - \vartheta_\eta^+(u_l)\Big]/c \\ &= D_\eta[F^+/c]\Big(u_r - F^+(u_r)/c + F^+(u_l)/c, u_l\Big) \\ &\quad + D_\eta[\mathrm{Id} - F^+/c]\Big(u_r - F^+(u_r)/c + F^+(u_l)/c, u_r\Big) \\ &\le 0, \end{aligned} \tag{4.62}$$

because of (4.6) and (4.58). Similarly,

$$\begin{aligned} &\Big[\vartheta_\eta^+(u_l) + \vartheta_\eta^-(u_r) - \vartheta_l(u_l, u_r)\Big]/c \\ &= \eta\Big(u_l - F^-(u_r)/c + F^-(u_l)/c\Big) - \eta(u_l) + \Big[\vartheta_\eta^-(u_r) - \vartheta_\eta^-(u_l)\Big]/c \\ &= D_\eta[-F^-/c]\Big(u_l - F^-(u_r)/c + F^-(u_l)/c, u_r\Big) \\ &\quad + D_\eta[\mathrm{Id} + F^-/c]\Big(u_l - F^-(u_r)/c + F^-(u_l)/c, u_l\Big) \\ &\le 0, \end{aligned} \tag{4.63}$$

which yields (4.60). □

*Remark 4.5* The function $u$ in (4.49) is exactly obtained by putting together the approximate solutions (4.59) at each cell interface, and the above proof of entropy consistency corresponds to the decomposition (3.27). Indeed, the expressions computed in (4.62) and (4.63) are exactly the integral in (3.28), for each intermediate value in (4.59). Notice that the stability conditions (4.6) and (4.19) imply (4.58) by addition.

**General kinetic scheme**

The general kinetic scheme (3.5)-(3.6) can also be interpreted as an approximate Riemann solver. One can check that under a CFL condition 1/2 in (3.15), the function $u(t,x)$ defined for all times by (2.7) is built locally through the approximate Riemann solver

$$R(x/t, u_l, u_r) = \int_{x/t<a(\xi)} M(u_l,\xi)\, d\xi + \int_{x/t>a(\xi)} M(u_r,\xi)\, d\xi - k. \qquad (4.64)$$

The consistency relation (4.51) follows from the moment relations (1.5)-(1.6), and the entropy consistency can be obtained easily since by (1.13) and (3.25)

$$\begin{aligned}
&\eta\Big(R(x/t,u_l,u_r)\Big)\\
&= \int G_\eta\Big(R(x/t,u_l,u_r),\xi\Big)\, d\xi - c_\eta\\
&= \int \Bigg[G_\eta\Big(R(x/t,u_l,u_r),\xi\Big) + \eta'\Big(R(x/t,u_l,u_r)\Big)\\
&\quad \times\Big(M\Big(u_l \mathbb{1}_{x/t<a(\xi)} + u_r \mathbb{1}_{x/t>a(\xi)},\xi\Big) - M\Big(R(x/t,u_l,u_r),\xi\Big)\Big)\Bigg] d\xi - c_\eta\\
&\leq \int G_\eta\Big(u_l \mathbb{1}_{x/t<a(\xi)} + u_r \mathbb{1}_{x/t>a(\xi)},\xi\Big)\, d\xi - c_\eta\\
&= \int_{x/t<a(\xi)} G_\eta(u_l,\xi)\, d\xi + \int_{x/t>a(\xi)} G_\eta(u_r,\xi)\, d\xi - c_\eta.
\end{aligned} \qquad (4.65)$$

*Remark 4.6* This subsection is an illustration of a general procedure to get an approximate Riemann solver: we just take the $u$-component of an exact Riemann solution to an entropy compatible relaxation model in time-discrete form.

# 5 Applications

In this section we apply the results of Section 4 to the construction of entropy satisfying relaxation models for the one-dimensional system of conservation laws

$$\partial_t u + \partial_x F(u) = 0, \qquad (5.1)$$

with $u(t,x) \in \mathcal{U}$ a convex subset of $\mathbb{R}^p$, that is endowed with a non empty family $\mathcal{E}$ of convex entropies. A very precise analysis will be given for the gas dynamics system.

The general principle is the following. At first, we have to select an entropy flux vector splitting

$$F = F^+ + F^-, \qquad (5.2)$$

with

$$F^+,\ -F^- \in \mathcal{M}_+^{\mathcal{E}}, \qquad (5.3)$$

where $\mathcal{M}_+^{\mathcal{E}}$ is defined in (1.17). This can be done easily if $\mathcal{E}$ is a single strictly convex entropy by using the entropy variable, see Remark 4.2. In general, we have to determine the set $\mathcal{M}_+^{\mathcal{E}}$. To be rigorous, we have indeed to check more precisely the stability condition

$$F^+,\ -F^- \quad \text{are } \eta\text{-dissipative in } \mathcal{U}_{stab}, \quad \eta \in \mathcal{E}, \tag{5.4}$$

for some domain $\mathcal{U}_{stab} \subset \mathcal{U}$. Then, we apply our interpretation of flux vector splitting schemes, Proposition 4.2, and obtain the finite velocity relaxation system (4.20), which is entropy satisfying under the condition

$$\mathrm{Id} - (F^+ - F^-)/c \quad \text{is } \eta\text{-dissipative in } \mathcal{U}_{stab}, \quad \eta \in \mathcal{E}. \tag{5.5}$$

For numerical purposes, we are indeed more interested in its time discrete transport-projection version

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x z = \displaystyle\sum_{n=1}^{\infty} \delta(t-t_n)(F(u^{n-}) - v^{n-}), \\ \partial_t z + c^2 \partial_x v = \displaystyle\sum_{n=1}^{\infty} \delta(t-t_n)\Big(c(F^+(u^{n-}) - F^-(u^{n-})) - z^{n-}\Big). \end{cases} \tag{5.6}$$

*Remark 5.1* Before verifying (5.4)-(5.5), a preliminary step is to check the necessary reduced stability conditions

$$F^+,\ -F^-,\ \mathrm{Id} - (F^+ - F^-)/c\ \in \mathcal{M}_+^{\mathcal{E}}, \tag{5.7}$$

see Subsection 2.2.

*Remark 5.2* Because of (4.25), the time numerical diffusion matrix $D$ in (2.13) is given here by

$$D(u) = Q'(u) - F'(u)^2, \qquad Q(u) = c(F^+(u) - F^-(u)). \tag{5.8}$$

If $\mathcal{E}$ contains a strictly convex entropy, its eigenvalues are nonnegative according to [5], and for best precision, one should make them as small as possible.

*Example 5.1* Let us consider the Lax-Friedrichs flux vector splitting given by

$$F^+(u) = \Big(F(u) + cu\Big)/2, \qquad F^-(u) = \Big(F(u) - cu\Big)/2. \tag{5.9}$$

We can take for $\mathcal{E}$ the set of all convex entropies of (5.1), and if we assume that there is at least one that is strictly convex, (5.3) is equivalent to the fact that the eigenvalues of $F'$ are less than $c$ in absolute value by Remark 1.1, and (5.5) is empty. We get $c(F^+(u) - F^-(u)) = c^2 u$, and we observe that obviously, $z = c^2 u$ in (4.20), thus it only remains

$$\partial_t u + \partial_x v = 0, \qquad \partial_t v + c^2 \partial_x u = \Big(F(u) - v\Big)/\varepsilon, \tag{5.10}$$

which is the relaxation system of [25].

## 5.1 The p-system

In this section we consider the p-system

$$\begin{cases} \partial_t \tau - \partial_x u = 0, \\ \partial_t u + \partial_x\, p(\tau) = 0, \end{cases} \tag{5.11}$$

where $\tau$, $u \in \mathbb{R}$, and the nonlinearity $p(\tau)$ satisfies

$$dp/d\tau < 0. \tag{5.12}$$

By applying the general strategy described above, we are going to obtain the relaxation model of Suliciu [42], [43],

$$\begin{cases} \partial_t \tau - \partial_x u = 0, \\ \partial_t u + \partial_x\, \pi = 0, \\ \partial_t \pi + c^2 \partial_x u = \big(p(\tau) - \pi\big)/\varepsilon, \end{cases} \tag{5.13}$$

thus giving a new interpretation of it. This model has been proved to converge to (5.11) in [47] (see also [22]).

Here we take for $\mathcal{E}$ the single entropy

$$\eta(\tau, u) = u^2/2 + e(\tau), \qquad de/d\tau = -p. \tag{5.14}$$

Then

$$\eta' = (-p, u), \qquad \eta'' = \begin{pmatrix} -p' & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.15}$$

and $\eta$ is strictly convex according to (5.12). Therefore, we are able to use the results of Subsection 1.2 to characterize the maxwellians. According to Remark 4.2, in order to derive an entropy flux vector splitting, we have to write a decomposition

$$\psi \equiv pu = \psi^+ + \psi^-, \tag{5.16}$$

where $\psi^+$, $-\psi^-$ are locally convex with respect to $(-p, u)$, and then

$$F^\pm = (-\partial_p \psi^\pm, \partial_u \psi^\pm). \tag{5.17}$$

Let us make the following choice, for which the previous convexity is obvious,

$$pu = (p + cu)^2/4c - (p - cu)^2/4c. \tag{5.18}$$

This gives

$$\begin{gathered} F^+ = \frac{p + cu}{2c}(-1, c), \qquad F^- = \frac{p - cu}{2c}(1, c), \\ F^+ - F^- = (-p/c, cu). \end{gathered} \tag{5.19}$$

Then, in order to verify (5.7), we need the eigenvalues of $(\mathrm{Id} - F^+/c + F^-/c)'$ to be nonnegative. Since

$$\mathrm{Id} - F^+/c + F^-/c = \big(\tau + p(\tau)/c^2, 0\big), \tag{5.20}$$

this gives the necessary reduced stability condition $-p' \leq c^2$. This is indeed sufficient.

**Proposition 5.1** *Let $I$ be an interval such that*

$$-p'(\tau) \le c^2, \qquad \tau \in I, \tag{5.21}$$

*and define*

$$\mathcal{U}_{stab} = I \times \mathbb{R}. \tag{5.22}$$

*Then the entropy compatibility conditions (5.4)-(5.5) are satisfied.*

**Proof.** We can apply Lemma 2.5, but we prefer here to give an explicit proof. According to Definition 2.2, we need to prove that $D_\eta[F^+]$, $D_\eta[-F^-]$, $D_\eta[\mathrm{Id} - F^+/c + F^-/c]$ are nonpositive in $\mathcal{U}_{stab} \times \mathcal{U}_{stab}$. The fluxes $\vartheta_\eta^\pm$ defined by (4.10) are given here (see (1.23)) by

$$\vartheta_\eta^+(\tau, u) = (p + cu)^2/4c, \qquad \vartheta_\eta^-(\tau, u) = -(p - cu)^2/4c. \tag{5.23}$$

Therefore, for any couple of states $(\tau_1, u_1)$, $(\tau_2, u_2)$ in $\mathcal{U}_{stab}$, denoting $p_1 = p(\tau_1)$ and $p_2 = p(\tau_2)$,

$$\begin{aligned}
&D_\eta[F^+]\big((\tau_1, u_1), (\tau_2, u_2)\big)\\
&= \vartheta_\eta^+(\tau_1, u_1) - \vartheta_\eta^+(\tau_2, u_2) + \eta'(\tau_1, u_1)\left[F^+(\tau_2, u_2) - F^+(\tau_1, u_1)\right]\\
&= (p_1 + cu_1)^2/4c - (p_2 + cu_2)^2/4c\\
&\quad + (-p_1, u_1)\left[(p_1 + cu_1)/2c - (p_2 + cu_2)/2c, -(p_1 + cu_1)/2 + (p_2 + cu_2)/2\right]\\
&= -(p_1 + cu_1)^2/4c - (p_2 + cu_2)^2/4c + (p_1 + cu_1)(p_2 + cu_2)/2c\\
&= -(p_1 + cu_1 - p_2 - cu_2)^2/4c\\
&\le 0.
\end{aligned} \tag{5.24}$$

Similarly,

$$D_\eta[-F^-]\big((\tau_1, u_1), (\tau_2, u_2)\big) = -(p_1 - cu_1 - p_2 + cu_2)^2/4c \le 0. \tag{5.25}$$

Finally,

$$\begin{aligned}
&D_\eta[\mathrm{Id} - F^+/c + F^-/c]\big((\tau_1, u_1), (\tau_2, u_2)\big)\\
&= e_1 - p_1^2/2c^2 - e_2 + p_2^2/2c^2 - p_1\left[\tau_2 + p_2/c^2 - \tau_1 - p_1/c^2\right]\\
&\le 0.
\end{aligned} \tag{5.26}$$

Indeed, the derivative of the expression in (5.26) with respect to $\tau_2$ is given by $(p_2 - p_1)(1 + p'(\tau_2)/c^2)$, which has the sign of $\tau_1 - \tau_2$ because of (5.12) and (5.21), and this gives the inequality. □

Now, we can apply Proposition 4.2, and it only remains to identify the system (4.20), or equivalently (4.21). We notice that according to (5.19)-(5.20), each maxwellian $M(., \xi)$, $\xi = -1, 0, 1$ given by (4.13)-(4.14) is colinear to a constant vector $K(\xi) = (1, -c\xi)$. Therefore, the functions $f(t, x, -1)$, $f(t, x, 0)$ and $f(t, x, 1)$ in (4.21) also satisfy this property (see (1.8)-(1.9)), and we can write

$$f(t, x, 1) = -g_1(t, x)(1, -c), \qquad f(t, x, -1) = -g_{-1}(t, x)(1, c), \tag{5.27}$$

$$f(t,x,0) = g_0(t,x)(1,0). \tag{5.28}$$

Then, the scalars $g_{-1}$, $g_0$, $g_1$ satisfy

$$\begin{cases} \partial_t g_1 + c\partial_x g_1 = \Big((p(\tau)+cu)/2c^2 - g_1\Big)/\varepsilon, \\ \partial_t g_{-1} - c\partial_x g_{-1} = \Big((p(\tau)-cu)/2c^2 - g_{-1}\Big)/\varepsilon, \\ \partial_t g_0 = \Big(\tau + p(\tau)/c^2 - g_0\Big)/\varepsilon, \end{cases} \tag{5.29}$$

and as in (4.22),

$$\tau = g_0 - g_1 - g_{-1}, \qquad u = cg_1 - cg_{-1}. \tag{5.30}$$

By defining

$$\pi = c^2 g_1 + c^2 g_{-1}, \tag{5.31}$$

we see that (5.29) can be rewritten as (5.13). Also, another way to write (5.29) is

$$\begin{cases} \partial_t(\pi+cu) + c\partial_x(\pi+cu) = \left(p(\tau)+cu-(\pi+cu)\right)/\varepsilon, \\ \partial_t(\pi-cu) - c\partial_x(\pi-cu) = \left(p(\tau)-cu-(\pi-cu)\right)/\varepsilon, \\ \partial_t(\tau+\pi/c^2) = \Big(\tau+p(\tau)/c^2 - (\tau+\pi/c^2)\Big)/\varepsilon. \end{cases} \tag{5.32}$$

Finally, we are able to express the global entropy $\mathcal{H}$ for (5.13) (or (5.32)), which is defined as

$$\mathcal{H}(\tau,u,\pi) = \int H(f(\xi),\xi)\,d\xi, \tag{5.33}$$

where $f(-1)$, $f(0)$, $f(1)$ are related to $\tau$, $u$, $\pi$ via (5.27), (5.28), (5.30) and (5.31). We use the formula (2.39). Here the set $D_{\xi=0}$ of (1.8)-(1.9) is just $D_{\xi=0} = \{g_0\,(1,0)\}_{g_0\in J}$, $J = \{\tau + p(\tau)/c^2\}_{\tau\in I}$, and we obtain

$$\mathcal{H}(\tau,u,\pi) = \varphi(\tau+\pi/c^2) + (\pi+cu)^2/4c^2 + (\pi-cu)^2/4c^2, \tag{5.34}$$

where $\varphi$ is given for any $g_0 \in J$ by

$$\varphi(g_0) = \sup_{\tau\in I}\Big\{e(\tau) - p(\tau)^2/2c^2 - p(\tau)\Big[g_0 - (\tau+p(\tau)/c^2)\Big]\Big\}, \tag{5.35}$$

or equivalently by

$$\varphi(\tau+p(\tau)/c^2) = e(\tau) - p(\tau)^2/2c^2, \qquad \tau\in I. \tag{5.36}$$

The entropy flux of $\mathcal{H}$ is $\int a(\xi)H(f(\xi),\xi)d\xi = \pi u$.

### Euler coordinates

The relaxation system (5.13) can be used to provide a numerical method when (5.11) is transformed into Euler coordinates to the system of isentropic gas dynamics

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) = 0, \end{cases} \tag{5.37}$$

where $\rho > 0$, $u \in \mathbb{R}$ and $dp/d\rho > 0$. The variables $\rho$ and $\tau$ are related through $\rho = 1/\tau$, the entropy is now $\rho(u^2/2 + e)$, and $e$ satisfies $de/d\rho = p/\rho^2$.

In order to obtain an entropy satisfying relaxation system for (5.37), as in [12] we just transform (5.13) in Euler coordinates, which yields

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \pi) = 0, \\ \partial_t(\rho \pi) + \partial_x(\rho \pi u + c^2 u) = \big(\rho p(\rho) - \rho \pi\big)/\varepsilon. \end{cases} \tag{5.38}$$

The global entropy is now $\rho\mathcal{H}$, with entropy flux $\rho\mathcal{H}u + \pi u$. We have to observe that (5.38) can be diagonalized, as

$$\begin{cases} \partial_t(\pi + cu) + (u + c/\rho)\partial_x(\pi + cu) = \left(p(\rho) + cu - (\pi + cu)\right)/\varepsilon, \\ \partial_t(\pi - cu) + (u - c/\rho)\partial_x(\pi - cu) = \left(p(\rho) - cu - (\pi - cu)\right)/\varepsilon, \\ \partial_t(1/\rho + \pi/c^2) + u\partial_x(1/\rho + \pi/c^2) = \Big(1/\rho + p(\rho)/c^2 - (1/\rho + \pi/c^2)\Big)/\varepsilon. \end{cases} \tag{5.39}$$

Even if the principal part is nonlinear, we see under this form that all the eigenvalues are linearly degenerate, and this enables to solve the Riemann problem easily.

### Practical implementation

In practical calculations for solving either (5.11) or (5.37), we use the time discrete forms of the relaxation systems (5.13) or (5.38) (similarly to (5.6)), and we proceed as follows. We start with initial data piecewise constant over a mesh of size $\Delta x$, and we complete the data by setting $\pi$ to equilibrium $\pi = p(\tau)$ (or $\pi = p(\rho)$). We then solve local Riemann problems for (5.13) or (5.38) without right-hand side, and determine $c$ such that the initial data and the intermediate states in the approximate Riemann solution remain in $U_{stab}$. For (5.38), we have also to take $c$ large enough so that these Riemann problems have a classical piecewise constant Riemann solution, which is exactly equivalent to the property that $\tau$ in (5.13) remains positive. Finally we determine the timestep $\Delta t$ by a CFL 1/2 condition determined by the maximum and minimum speeds in the Riemann problems (for (5.13) it is just $c\Delta t \leq \Delta x/2$). At the end of the timestep the value of $\pi$ is just forgotten, while the conservative variables are averaged over a cell. Since the local approximate Riemann solvers are entropy satisfying and do not interact, the final scheme will be entropy satisfying. We refer to Proposition 5.4 for an explicit computation of the intermediate states.

*Remark 5.3* In the above procedure, our analysis does not ensure that the timestep will not tend to 0.

*Remark 5.4* In (5.38), the density remains automatically nonnegative. However, in order to get $\tau > 0$ in (5.13), one has to take $c$ large enough (but this is commonly obtained once $\tau$ satisfies (5.21)).

*Remark 5.5* For better precision, one uses variable speeds $c_{i+1/2}$, one at each cell interface. Since the local approximate Riemann solvers do not interact, this still gives an entropy satisfying scheme (see Remark 4.4).

## 5.2 Full gas dynamics

We now consider the full gas dynamics system in lagrangian coordinates

$$\begin{cases} \partial_t \tau - \partial_x u = 0, \\ \partial_t u + \partial_x p = 0, \\ \partial_t (u^2/2 + e) + \partial_x (pu) = 0, \end{cases} \tag{5.40}$$

with $\tau > 0$, $u \in \mathbb{R}$, $e, p > 0$. The convex set of states $(\tau, u, u^2/2 + e)$ satisfying these constraints will be denoted by $\mathcal{U}$. We only consider the gamma law

$$p = (\gamma - 1)e/\tau, \qquad \gamma > 1, \tag{5.41}$$

but general state laws should work also. By applying our abstract construction (5.2)-(5.6), we are going to recover the scheme of [14], [12] (see also [19]). Its basic properties are that specific volume $\tau$ and internal energy $e$ remain nonnegative, it is compatible with all entropy inequalities (which ensures the maximum principle on the specific entropy), and it preserves contact discontinuities. It means more explicitly that if initial data are such that the velocity $u$ and the pressure $p$ are constant, then the whole solution is kept to its initial value exactly. Moreover, it does not need the costly computations involved in the Godunov solver. Such schemes have been recently designed by different authors in [14], [4], [12], [19], [20]. We refer to [46] for an introduction to schemes for fluid dynamics. It is now quite well understood, see [23], [16], that flux vector splitting schemes in Euler variable cannot satisfy all the above properties. However, as our construction shows it is possible for flux vector splitting schemes in Lagrange coordinates. The scheme can then be converted into an Euler scheme, but it is no longer in flux vector splitting form. A general conversion procedure from a Lagrange approximate Riemann solver to an Euler one is provided in [17], [19].

Let us define the variable $w$ by

$$w = \tau p^{1/\gamma}. \tag{5.42}$$

As is well known, the function

$$\eta_\phi = \phi(w) \tag{5.43}$$

is an entropy of (5.40) (with vanishing entropy flux) for any nonlinearity $\phi$, and it is convex in the conservative variables if and only if

$$\phi' \leq 0 \qquad \text{and} \qquad \phi'' \geq 0. \tag{5.44}$$

We choose for $\mathcal{E}$ the family of all $\eta_\phi$ with $\phi$ satisfying (5.44).

Let us now explain why the positivity of the specific volume $\tau$ (or of the density $\rho = 1/\tau$ in Euler variables) and the whole family of entropy inequalities imply the minimum principle on $w$ and the nonnegativity of the internal energy. This is quite easy in Lagrange coordinates, hence let us rather explain it for Euler coordinates. Assume that

$$\partial_t \rho\phi(w) + \partial_x \vartheta_\phi \leq 0 \tag{5.45}$$

for some flux $\vartheta_\phi$, for any $\phi$ satisfying (5.44). Integrating over all $x$ gives

$$\frac{d}{dt}\int \rho\phi(w)\,dx \leq 0. \tag{5.46}$$

Then, by taking $\phi(w) = (k-w)_+$, we get since $\rho > 0$ that

$$w^0(x) \geq k \text{ for all } x \quad \Rightarrow \quad w(t,x) \geq k \text{ for all } x, \tag{5.47}$$

which is the minimum principle. The limit case $k = 0$ gives that $w$ remains nonnegative, and thus $e$ also. For numerical schemes this needs to be justified more precisely, but the argument indicates that it should hold. This will be checked in our particular scheme.

In order to proceed with (5.2)-(5.6), we need to determine the sets $\mathcal{M}^\mathcal{E}$ and $\mathcal{M}^\mathcal{E}_+$. We express them in the variables $(p,u,w)$.

**Theorem 5.2** *The space $\mathcal{M}^\mathcal{E}$ consists of all functions $M = (M_0, M_1, M_2)$ of the form*

$$\begin{aligned}
&M_0 = -\partial_p\psi(p,u) + \iota(w)p^{-1/\gamma},\\
&M_1 = \partial_u\psi(p,u),\\
&M_2 = p\,\partial_p\psi(p,u) + u\,\partial_u\psi(p,u) - \psi(p,u) + \iota(w)p^{1-1/\gamma}/(\gamma-1),
\end{aligned} \tag{5.48}$$

*where $\psi(p,u)$ and $\iota(w)$ are arbitrary functions, and $(\psi,\iota)$ is defined up to a constant times $(p^{1-1/\gamma}, 1-1/\gamma)$. The associated flux $G_\phi$, defined by $G'_\phi = \eta'_\phi M'$, is given by*

$$G_\phi = \Upsilon(w), \qquad \Upsilon'(w) = \phi'(w)\iota'(w). \tag{5.49}$$

*The identity* Id *is obtained for $\psi(p,u) = u^2/2$, $\iota(w) = w$, and the flux $F$ is obtained for $\psi(p,u) = pu$, $\iota(w) = 0$.*
*Moreover, the function $M$ lies in $\mathcal{M}^\mathcal{E}_+$ if and only if*

$$\iota'(w) \geq 0, \qquad \begin{pmatrix} \partial^2_{pp}\psi + \iota(w)p^{-(1+1/\gamma)}/\gamma & \partial^2_{pu}\psi \\ \partial^2_{pu}\psi & \partial^2_{uu}\psi \end{pmatrix} \geq 0. \tag{5.50}$$

**Proof.** We have to write that the bilinear forms $(M')^t\eta''_\phi$ are symmetric for all $\eta_\phi \in \mathcal{E}$ (respectively symmetric nonnegative for $\mathcal{M}^\mathcal{E}_+$). We recall that prime denotes differentiation with respect to the conservative variables, $(\tau, u, u^2/2+e)$ here. Denoting by $v = (p,u,w)$ the new variable, we use as in [5] the formula giving the matrix of the bilinear form in the new basis,

$$\begin{aligned}
\text{matrix}\Big((M')^t\eta''_\phi\Big) &= \Big(\partial_v M\Big)^t\Big(\partial_v\eta'_\phi\Big)\\
&= \partial_v\Big(\eta'_\phi\partial_v M\Big) - \eta'_\phi\partial^2_{vv}M.
\end{aligned} \tag{5.51}$$

In order to compute the last expression, we just need to differentiate with respect to $v$ the product $\eta'_\phi\partial_v M$, and to drop the second order derivatives of $M$. Since $dp = -\frac{p}{\tau}d\tau - (\gamma-1)\frac{u}{\tau}du + \frac{\gamma-1}{\tau}d(u^2/2+e)$, we have

$$\eta'_\phi = (1-1/\gamma)\frac{\phi'(w)}{p^{1-1/\gamma}}\,(p,-u,1), \tag{5.52}$$

$$\eta'_\phi \partial_v M = (1-1/\gamma)\frac{\phi'(w)}{p^{1-1/\gamma}}\Big(p\,\partial_v M_0 - u\,\partial_v M_1 + \partial_v M_2\Big), \tag{5.53}$$

and according to (5.51), we get

$$\begin{aligned}(M')^t\eta''_\phi = {} & (1-1/\gamma)\frac{\phi''(w)}{p^{1-1/\gamma}}\Big\{p\,dM_0 - u\,dM_1 + dM_2\Big\}\otimes dw \\ & + (1-1/\gamma)\frac{\phi'(w)}{p^{2-1/\gamma}}\Big\{\frac{p}{\gamma}dM_0\otimes dp + (1-1/\gamma)\,u\,dM_1\otimes dp \\ & \qquad - (1-1/\gamma)\,dM_2\otimes dp - p\,dM_1\otimes du\Big\}.\end{aligned} \tag{5.54}$$

Let us define

$$M_3 = pM_0 - uM_1 + M_2. \tag{5.55}$$

Then

$$dM_3 = p\,dM_0 - u\,dM_1 + dM_2 + M_0\,dp - M_1\,du, \tag{5.56}$$

and we obtain

$$\begin{aligned}(M')^t\eta''_\phi = {} & (1-1/\gamma)\frac{\phi''(w)}{p^{1-1/\gamma}}\Big\{dM_3 - M_0\,dp + M_1\,du\Big\}\otimes dw \\ & + (1-1/\gamma)\frac{\phi'(w)}{p^{2-1/\gamma}}\Big\{p\,dM_0\otimes dp - (1-1/\gamma)\,dM_3\otimes dp \\ & \qquad - (1-1/\gamma)(M_1\,du - M_0\,dp)\otimes dp - p\,dM_1\otimes du\Big\}.\end{aligned} \tag{5.57}$$

In order that (5.57) be symmetric for all $\phi$, we need the first expression between braces to be colinear to $dw$, and the second to be symmetric. The first condition gives

$$\partial_p M_3 = M_0, \qquad \partial_u M_3 = -M_1, \tag{5.58}$$

and the second gives

$$\begin{aligned}& p\,\partial_w M_0 - (1-1/\gamma)\,\partial_w M_3 = 0, \qquad \partial_w M_1 = 0, \\ & p\,\partial_u M_0 - (1-1/\gamma)\Big(\partial_u M_3 + M_1\Big) = -p\,\partial_p M_1.\end{aligned} \tag{5.59}$$

Now, obviously, the second line in (5.59) is a consequence of (5.58), and we can forget about it. Thus we can take (5.58) as a definition of $M_0$ and $M_1$, and the first line in (5.59) gives two equations on $M_3$, namely

$$\partial^2_{uw} M_3 = 0, \qquad p\,\partial^2_{pw} M_3 = (1-1/\gamma)\,\partial_w M_3. \tag{5.60}$$

The general solution $\partial_w M_3$ to (5.60) is obviously

$$\partial_w M_3 = \frac{\gamma}{\gamma-1}p^{1-1/\gamma}\,\iota'(w), \tag{5.61}$$

for an arbitrary function $\iota'(w)$. Integrating with respect to $w$, we obtain

$$M_3 = \frac{\gamma}{\gamma-1} p^{1-1/\gamma} \iota(w) - \psi(p,u), \tag{5.62}$$

for an arbitrary function $\psi(p,u)$. We finally get $M_0$, $M_1$ by (5.58) and $M_2$ by (5.55), which leads to (5.48). In order to write the nonnegativity conditions, one can compute with (5.48)

$$\begin{aligned}(M')^t \eta''_\phi = \;& \phi''(w)\iota'(w)\, dw \otimes dw \\ & - (1-1/\gamma)\frac{\phi'(w)}{p^{1-1/\gamma}} \Big\{ \Big( \partial^2_{pp}\psi + \frac{\iota(w)}{\gamma p^{1+1/\gamma}} \Big)\, dp \otimes dp \\ & \qquad + \partial^2_{pu}\psi\,(du \otimes dp + dp \otimes du) + \partial^2_{uu}\psi\, du \otimes du \Big\},\end{aligned} \tag{5.63}$$

and this gives obviously condition (5.50). In order to obtain $G_\phi$, just write $\partial_v G_\phi = \eta'_\phi \partial_v M$, and with (5.53) one easily gets (5.49). The remaining assertions of the theorem are left to the reader. □

*Remark 5.6* By taking $\psi(p,u) = u^2/2$, $\iota(w) = w$, we get $M = Id$, and formula (5.63) gives the value of $\eta''_\phi$. This enables to prove that $\eta''_\phi \geq 0$ if and only if (5.44) holds.

*Remark 5.7* The formula (5.48) says that the space $\mathcal{M}^{\mathcal{E}}$ of maxwellians corresponding to the *whole set of entropies* $\eta_\phi$ contains two families. The first, obtained for general $\psi(p,u)$, and $\iota(w) = 0$, corresponds exactly to the space of maxwellians for the isentropic gas dynamics, but with a *single entropy* given by the energy. The last component, $M_2$ in (5.48), is nothing but the isentropic flux (see (1.20) and (1.23)). This family is also the space of maxwellians which have vanishing fluxes $G_\phi$. They are fully characterized by the equation $\eta'_\phi M' = 0$. The second family, generated by $\iota$, is almost trivial, and gives essentially Id. This very special structure, which is related to the results of [15], does not occur in Euler coordinates, for which the space of maxwellians was obtained in [5].

*Remark 5.8* By comparing (5.48) to the formula for Euler coordinates obtained in [5], we see that there is a one to one correspondence between the Lagrange maxwellians and the Euler ones. If $M = (M_0, M_1, M_2)$ is a Lagrange maxwellian, its Euler version is given by

$$\Big( -\rho M_0\,,\; -\rho u M_0 + M_1\,,\; -\rho(u^2/2 + e) M_0 + M_2 \Big). \tag{5.64}$$

Now, in order to obtain a flux vector splitting (5.2)-(5.3), it is enough to consider maxwellians $M$ in Theorem 5.2 with $\iota = 0$, hence we need a decomposition $\psi \equiv pu = \psi^+ + \psi^-$, with $\psi^+$, $-\psi^-$ locally convex in $(p,u)$. Therefore, we can make the same choice (5.18) as in the isentropic case (see Remark 5.7), and we obtain with (5.19) and (5.23)

$$F^+ = \frac{p+cu}{2c}\Big(-1, c, \frac{p+cu}{2}\Big), \qquad F^- = \frac{p-cu}{2c}\Big(1, c, -\frac{p-cu}{2}\Big), \tag{5.65}$$

which is the flux vector splitting of [14]. According to (5.49), the fluxes $\vartheta_\phi^\pm$ vanish identically. Now the conservation of contact discontinuities is obvious since $F^\pm$ only depend on $p$ and $u$. In order to write the stability conditions we compute the maxwellian associated to $\xi = 0$,

$$M(0) = \mathrm{Id} - (F^+ - F^-)/c = (\tau + p/c^2, 0, e - p^2/2c^2). \tag{5.66}$$

We need that $M(0)$ lies in $\mathcal{M}_+^{\mathcal{E}}$, and since $\mathcal{E}$ contains strictly convex entropies, this is equivalent to check the sign of the eigenvalues of $M(0)'$, which are given by 0, 1, $1 - \gamma p/\tau c^2$. Thus we obtain the reduced stability condition

$$\gamma\, p/\tau \le c^2. \tag{5.67}$$

We are now able to prove the entropy dissipation conditions (5.4)-(5.5).

**Proposition 5.3** *Let $\kappa > 0$, and define*

$$\mathcal{U}_{stab} = \left\{(\tau, u, u^2/2 + e) \in \mathcal{U}\ ;\ w^\gamma = \tau^\gamma p \le \kappa,\ \frac{\gamma\kappa}{\tau^{\gamma+1}} \le c^2\right\}. \tag{5.68}$$

*Then $F^+$, $-F^-$, $\mathrm{Id} - (F^+ - F^-)/c$ are $\eta_\phi$-dissipative in $\mathcal{U}_{stab}$ for any $\eta_\phi \in \mathcal{E}$.*

**Proof.** Let us denote $U_1 = (\tau_1, u_1, u_1^2/2 + e_1)$, and $U_2 = (\tau_2, u_2, u_2^2/2 + e_2)$. Since $\vartheta_\phi^\pm$ vanish, we have according to (5.52)

$$\begin{aligned}
&D_{\eta_\phi}[F^+](U_1, U_2)\\
&= \eta_\phi'(U_1)\Big(F^+(U_2) - F^+(U_1)\Big)\\
&= -(1 - 1/\gamma)\frac{\phi'(w_1)}{p_1^{1-1/\gamma}}(-p_1, u_1, -1)\Big(F^+(U_2) - F^+(U_1)\Big)\\
&= (1 - 1/\gamma)\frac{\phi'(w_1)}{p_1^{1-1/\gamma}}(p_1 + cu_1 - p_2 - cu_2)^2/4c\\
&\le 0,
\end{aligned} \tag{5.69}$$

indeed the computation is the same as in (5.24). Similarly,

$$D_{\eta_\phi}[-F^-](U_1, U_2) = (1 - 1/\gamma)\frac{\phi'(w_1)}{p_1^{1-1/\gamma}}(p_1 - cu_1 - p_2 + cu_2)^2/4c \le 0, \tag{5.70}$$

and in these computations, $\mathcal{U}_{stab}$ was not necessary. Next, we have

$$\begin{aligned}
&D_{\eta_\phi}[M(0)](U_1, U_2)\\
&= \phi(w_1) - \phi(w_2) + \eta_\phi'(U_1)\Big(M(U_2, 0) - M(U_1, 0)\Big)\\
&= \phi(w_1) - \phi(w_2) - (1 - 1/\gamma)\frac{\phi'(w_1)}{p_1^{1-1/\gamma}}\Big(-p_1(\tau_2 + p_2/c^2 - \tau_1 - p_1/c^2)\\
&\qquad\qquad + e_1 - p_1^2/2c^2 - e_2 + p_2^2/2c^2\Big).
\end{aligned} \tag{5.71}$$

Since $\phi$ is convex, we can use the inequality $\phi(w_1) - \phi(w_2) \leq \phi'(w_1)(w_1 - w_2)$, which yields after some computations

$$
\begin{aligned}
&D_{\eta_\phi}[M(0)](U_1, U_2) \\
&\leq (1 - 1/\gamma)\frac{\phi'(w_1)}{p_1^{1-1/\gamma}}\Big\{p_1\tau_2 + \frac{\tau_2 p_2}{\gamma - 1} - \frac{\gamma}{\gamma - 1}p_1^{1-1/\gamma}\tau_2 p_2^{1/\gamma} - (p_2 - p_1)^2/2c^2\Big\}.
\end{aligned}
\tag{5.72}
$$

Let us define an auxiliary function

$$
\zeta(p) = p_1\tau_2 + \frac{\tau_2 p}{\gamma - 1} - \frac{\gamma}{\gamma - 1}p_1^{1-1/\gamma}\tau_2 p^{1/\gamma} - (p - p_1)^2/2c^2. \tag{5.73}
$$

We want to prove that $\zeta(p_2) \geq 0$ (so that (5.72) becomes nonpositive), and we have $\zeta(p_1) = 0$. Thus it is sufficient to prove that

$$
(p_2 - p_1)\zeta'(p) \geq 0 \quad \text{for any } p \in [p_1, p_2]. \tag{5.74}
$$

We have

$$
\begin{aligned}
\zeta'(p) &= \frac{\tau_2}{\gamma - 1} - \frac{\tau_2}{\gamma - 1}\frac{p_1^{1-1/\gamma}}{p^{1-1/\gamma}} - (p - p_1)/c^2 \\
&= \frac{\tau_2}{\gamma p^{1-1/\gamma}}\left(\frac{p^{1-1/\gamma}}{1 - 1/\gamma} - \frac{p_1^{1-1/\gamma}}{1 - 1/\gamma} - (p - p_1)\frac{\gamma}{c^2}\frac{p^{1-1/\gamma}}{\tau_2}\right) \\
&= \frac{\tau_2}{\gamma p^{1-1/\gamma}}(p - p_1)\left\{\widetilde{p}^{-1/\gamma} - \frac{\gamma}{c^2}\frac{p^{1-1/\gamma}}{\tau_2}\right\},
\end{aligned}
\tag{5.75}
$$

for some $\widetilde{p} \in [p_1, p]$. Therefore, it is enough for (5.74) to hold that the last expression between braces in (5.75) be nonnegative, or

$$
\gamma\frac{p^{1-1/\gamma}\widetilde{p}^{1/\gamma}}{\tau_2} \leq c^2. \tag{5.76}
$$

This is a consequence of the inequality

$$
\gamma\frac{\max(p_1, p_2)}{\tau_2} \leq c^2, \tag{5.77}
$$

which obviously holds true whenever $U_1, U_2 \in \mathcal{U}_{stab}$, since $p_1 = w_1^\gamma/\tau_1^\gamma$. $\square$

*Remark 5.9* The reduced condition (5.67) alone is not sufficient to get the entropy dissipation. Indeed, if $\phi(w) = -w$, (5.72) becomes an equality, the term between braces depends only on $p_1$ (not on $\tau_1$), and tends to $-\infty$ as $p_1 \to \infty$. Therefore, we need a bound on $p$ in $\mathcal{U}_{stab}$, which is not ensured by (5.67) alone. This situation can occur because the image of $\eta'_\phi$ and of $M(0)$ are not convex (see Remark 2.3).

We are now going to give explicitly the relaxation system built in Proposition 4.2. We start from the diagonal form (4.21). We have to give some sets

$D_\xi$, $\xi = -1, 0, 1$, where $f(-1)$, $f(0)$, $f(1)$ lie respectively, as stated in (1.8)-(1.9). These sets have to be convex and contain the image of the maxwellians $M(-1) = -F^-/c$, $M(0) = \mathrm{Id} - (F^+ - F^-)/c$, $M(1) = F^+/c$ respectively. According to the expressions (5.65)-(5.66), we can take

$$\begin{aligned} f(-1) &= (-g_{-1}, -cg_{-1}, h_{-1}), & h_{-1} - c^2 g_{-1}^2 &\geq 0, \\ f(1) &= (-g_1, cg_1, h_1), & h_1 - c^2 g_1^2 &\geq 0, \\ f(0) &= (g_0, 0, h_0), & g_0 \geq 0,\ h_0 &\geq 0. \end{aligned} \tag{5.78}$$

The last condition $h_0 \geq 0$ is put because the last component $e - p^2/2c^2$ in (5.66) is nonnegative for states satisfying (5.67). Then, we define the conserved quantities by the sum over $\xi$ of $f(\xi)$ in (5.78),

$$\tau = g_0 - g_1 - g_{-1}, \qquad u = cg_1 - cg_{-1}, \qquad u^2/2 + e = h_0 + h_1 + h_{-1}. \tag{5.79}$$

As in the isentropic case, we define also

$$\pi = c^2 g_1 + c^2 g_{-1}, \tag{5.80}$$

and the variables $g_0$, $g_1$, $g_{-1}$ can be replaced by $\tau$, $u$, $\pi$ since we have the inverse relations

$$g_0 = \tau + \pi/c^2, \qquad g_1 = \frac{\pi + cu}{2c^2}, \qquad g_{-1} = \frac{\pi - cu}{2c^2}. \tag{5.81}$$

This gives

$$\begin{cases} \partial_t \tau - \partial_x u = 0, \\ \partial_t u + \partial_x \pi = 0, \\ \partial_t \pi + c^2 \partial_x u = (p - \pi)/\varepsilon, \end{cases} \tag{5.82}$$

where $p$ stands for $p(\tau, e)$. The system is completed by the equations on $h_0$, $h_1$, and $h_{-1}$

$$\begin{cases} \partial_t h_1 + c\partial_x h_1 = \Big((p + cu)^2/4c^2 - h_1\Big)/\varepsilon \\ \partial_t h_{-1} - c\partial_x h_{-1} = \Big((p - cu)^2/4c^2 - h_{-1}\Big)/\varepsilon \\ \partial_t h_0 = \Big(e - p^2/2c^2 - h_0\Big)/\varepsilon. \end{cases} \tag{5.83}$$

We can replace the last equation of (5.83) by that on the energy

$$\partial_t(u^2/2 + e) + \partial_x(ch_1 - ch_{-1}) = 0. \tag{5.84}$$

On this system one can check easily that if initially $p$ and $u$ are constant, there is no evolution. We can prove the nonnegativity of $e$ with (5.78),

$$e = h_0 + h_1 + h_{-1} - u^2/2 \geq h_0 + c^2 g_1^2 + c^2 g_{-1}^2 - u^2/2 = h_0 + \pi^2/2c^2 \geq 0. \tag{5.85}$$

For the kinetic entropies, we can take $H_\phi(f, \pm 1) = 0$ for any $f \in D_{\pm 1}$, this is enough for (1.18)-(1.19) to be satisfied, because $G_\phi(U, \pm 1) = 0$ (see (4.36)) and for any $f \in D_{\pm 1}$, $\eta'_\phi(U)(f - M(U, \pm 1)) \le 0$, the computation is very close to (5.69)-(5.70). However, there is no simple formula for $H_\phi(f, 0)$, $f \in D_0$, except (2.39). Thus we obtain the global entropy $\mathcal{H}_\phi = \int H_\phi(f(\xi), \xi)\, d\xi = \varphi_\phi(g_0, h_0)$ for some nonlinear function $\varphi_\phi$. Its flux is $\int a(\xi) H_\phi(f(\xi), \xi)\, d\xi = 0$.

An important simplification occurs for the time discrete transport-projection model. In this case, as was used in Sections 2 and 3, even nonconvex sets are preserved during the transport step. This implies that

$$h_1 = c^2 g_1^2 = (\pi + cu)^2/4c^2, \qquad h_{-1} = c^2 g_{-1}^2 = (\pi - cu)^2/4c^2, \tag{5.86}$$

and therefore

$$h_0 = e - \pi^2/2c^2, \qquad ch_1 - ch_{-1} = \pi u. \tag{5.87}$$

Thus we obtain the system

$$\begin{cases} \partial_t \tau - \partial_x u = 0, \\ \partial_t u + \partial_x \pi = 0, \\ \partial_t (u^2/2 + e) + \partial_x (\pi u) = 0, \\ \partial_t \pi + c^2 \partial_x u = \sum\limits_{n=1}^{\infty} \delta(t - t_n) \Big( p(\tau^{n-}, e^{n-}) - \pi^{n-} \Big). \end{cases} \tag{5.88}$$

We can observe that the energy equation is nonlinear, but it can be replaced by the linear one on $h_0$,

$$\partial_t (e - \pi^2/2c^2) = \sum_{n=1}^{\infty} \delta(t - t_n) \Big( e^{n-} - (p^{n-})^2/2c^2 - (e^{n-} - (\pi^{n-})^2/2c^2) \Big). \tag{5.89}$$

The system can be transformed into Euler variables, which gives

$$\begin{cases} \partial_t \rho + \partial_x (\rho u) = 0, \\ \partial_t (\rho u) + \partial_x (\rho u^2 + \pi) = 0, \\ \partial_t (\rho (u^2/2 + e)) + \partial_x (\rho (u^2/2 + e) u + \pi u) = 0, \\ \partial_t (\rho \pi) + \partial_x (\rho \pi u + c^2 u) = \sum\limits_{n=1}^{\infty} \delta(t - t_n) \rho^{n-} \Big( p(\tau^{n-}, e^{n-}) - \pi^{n-} \Big). \end{cases} \tag{5.90}$$

Again the energy equation can be replaced by one on $\rho(e - \pi^2/2c^2)$ that we shall not write. We can write the diagonal form of (5.90) (without the right-hand sides to simplify),

$$\begin{cases} \partial_t (\pi + cu) + (u + c/\rho) \partial_x (\pi + cu) = 0, \\ \partial_t (\pi - cu) + (u - c/\rho) \partial_x (\pi - cu) = 0, \\ \partial_t (\tau + \pi/c^2) + u\, \partial_x (\tau + \pi/c^2) = 0, \\ \partial_t (e - \pi^2/2c^2) + u\, \partial_x (e - \pi^2/2c^2) = 0. \end{cases} \tag{5.91}$$

It is again a linearly degenerate system since $u - c\tau = (\pi + cu)/c - c(\tau + \pi/c^2)$, $u = (\pi + cu)/2c - (\pi - cu)/2c$, $u + c\tau = -(\pi - cu)/c + c(\tau + \pi/c^2)$.

The final implementation goes the same as in the isentropic case. We can give the explicit solution of the Riemann problems.

**Proposition 5.4** *The solution to the Riemann problem for the Lagrange system (5.88) without right-hand side has two intermediate states $U_l^*$, $U_r^*$ between the speeds $-c, 0$ and $0, c$ respectively, given with obvious notations by*

$$\begin{aligned}
\tau_l^* &= \tau_l + (u_r - u_l)/2c - (\pi_r - \pi_l)/2c^2,\\
\tau_r^* &= \tau_r + (u_r - u_l)/2c + (\pi_r - \pi_l)/2c^2,\\
u_l^* &= u_r^* = (u_l + u_r)/2 - (\pi_r - \pi_l)/2c,\\
\pi_l^* &= \pi_r^* = (\pi_l + \pi_r)/2 - c(u_r - u_l)/2,\\
e_l^* &= e_l + \Big[\Big((\pi_l + \pi_r)/2 - c(u_r - u_l)/2\Big)^2 - \pi_l^2\Big]/2c^2,\\
e_r^* &= e_r + \Big[\Big((\pi_l + \pi_r)/2 - c(u_r - u_l)/2\Big)^2 - \pi_r^2\Big]/2c^2.
\end{aligned} \tag{5.92}$$

*The solution to the Riemann problem for the Euler system (5.90) without right-hand side has the same two intermediate states under the assumption that $\tau_l > 0$, $\tau_r > 0$, $\tau_l^* > 0$ and $\tau_r^* > 0$, but the speeds are*

$$\lambda_1 = u_l - c/\rho_l, \qquad \lambda_2 = u_l^* = u_r^*, \qquad \lambda_3 = u_r + c/\rho_r. \tag{5.93}$$

**Proof.** The energy equation in (5.88) can be replaced by $\partial_t(e - \pi^2/2c^2) = 0$, thus we have a linear system. Denoting by $\tau^0(x)$, $u^0(x)$, $\pi^0(x)$, $e^0(x)$ the initial data, we get obviously the value of the Riemann invariants,

$$\begin{aligned}
&\pi(t,x) + cu(t,x) = \pi^0(x - ct) + cu^0(x - ct),\\
&\pi(t,x) - cu(t,x) = \pi^0(x + ct) - cu^0(x + ct),\\
&\tau(t,x) + \pi(t,x)/c^2 = \tau^0(x) + \pi^0(x)/c^2,\\
&e(t,x) - \pi(t,x)^2/2c^2 = e^0(x) - \pi^0(x)^2/2c^2,
\end{aligned} \tag{5.94}$$

and this gives easily (5.92). For the Euler system, since we have a full set of Riemann invariants in (5.91) and the system is linearly degenerate, we write that while crossing a discontinuity at speed $\lambda_i$, only the $i^{th}$ Riemann invariant can jump. This yields

$$\begin{gathered}
\pi_l^* + cu_l^* = \pi_r^* + cu_r^* = \pi_l + cu_l,\\
\pi_l^* - cu_l^* = \pi_r^* - cu_r^* = \pi_r - cu_r,\\
\tau_l^* + \pi_l^*/c^2 = \tau_l + \pi_l/c^2, \qquad \tau_r^* + \pi_r^*/c^2 = \tau_r + \pi_r/c^2,\\
e_l^* - \pi_l^{*2}/2c^2 = e_l - \pi_l^2/2c^2, \qquad e_r^* - \pi_r^{*2}/2c^2 = e_r - \pi_r^2/2c^2.
\end{gathered} \tag{5.95}$$

We deduce the intermediate states (5.92), and the eigenvalues $\lambda_1 = u_l - c\tau_l = u_l^* - c\tau_l^*$, $\lambda_2 = u_l^* = u_r^*$, $\lambda_3 = u_r^* + c\tau_r^* = u_r + c\tau_r$, which gives (5.93). In order that the solution is valid we need to check that $\lambda_1 < \lambda_2 < \lambda_3$, and this is the case since

$$\lambda_2 - \lambda_1 = c\tau_l^*, \qquad \lambda_3 - \lambda_2 = c\tau_r^*, \tag{5.96}$$

which are assumed to be positive. □

*Remark 5.10* In the final scheme, the value of $\pi$ is unnecessary, while the intermediate values of the conserved quantities can be computed via (4.59), which match (5.92) by construction.

*Remark 5.11* From the last line of (5.94) we can again deduce the nonnegativity of $e$, since $e(t,x) \geq e^0(x) - \pi^0(x)^2/2c^2$ which is nonnegative when $\pi^0(x) = p(\tau^0(x), e^0(x))$ and the initial state $U^0(x)$ satisfies (5.67). On the contrary, there is no reason for the pseudo pressure $\pi$ to remain nonnegative.

*Remark 5.12* If we do not assume $\tau_l^*$ and $\tau_r^*$ to be positive for the Euler Riemann problem, one should look for a delta-wave solution instead of a piecewise constant one. However, then there is no reason for (5.86) to hold, and we have to go back to the Euler version of (5.82)-(5.83). We could also use a piecewise constant approximate Riemann solver instead of the exact solution in this case.

Finally, we wish to explicit the conditions that need to be enforced, via the choice of $c$ in the Riemann problem between two cells, so that the scheme is entropy satisfying. Theoretically, according to Proposition 4.4, we should find some $\kappa$ and $c$ in such a way that the intermediate values remain in $\mathcal{U}_{stab}$ defined in (5.68). However, we can use a condition that is a bit less restrictive, and simpler to deal with.

**Proposition 5.5** *The solution to the Riemann problem of Proposition 5.4 with initial data $\pi_l = p_l \equiv p(\tau_l, e_l)$, $\pi_r = p_r \equiv p(\tau_r, e_r)$ gives an entropy satisfying approximate Riemann solver for (5.40)-(5.41) if $c$ is chosen such that*

$$\begin{aligned} \gamma\, p_l/\tau_l \leq c^2, \qquad & \gamma\, p_r/\tau_r \leq c^2, \\ \gamma\, \tau_l^\gamma p_l/\tau_l^{*\gamma+1} \leq c^2, \qquad & \gamma\, \tau_r^\gamma p_r/\tau_r^{*\gamma+1} \leq c^2. \end{aligned} \tag{5.97}$$

**Proof.** In (5.97), we implicitly assume that $\tau_l$, $\tau_r$, $\tau_l^*$, $\tau_r^*$ are all positive. According to Remark 5.11, this implies that the intermediate states have nonnegative internal energy. In proving inequalities (4.62) and (4.63), it is not necessary that all terms are nonpositive, only the sum needs to be (see Remarks 3.3 and 4.5). Since here $\vartheta_{\eta_\phi}^{\pm} = 0$ and $\eta_\phi = \phi(w)$ with $\phi$ nonincreasing, we need only to prove that

$$w_l^* \geq w_l, \qquad w_r^* \geq w_r, \tag{5.98}$$

the minimum principle. Assume by contradiction that $w_l^* < w_l$. Then we can define $\kappa = w_l^\gamma$, and (5.97) ensures that $U_l$ and $U_l^*$ lie in $\mathcal{U}_{stab}$ defined in (5.68). Therefore, according to Proposition 5.3, $D_{\eta_\phi}[M(0)](U_l^*, U_l) \leq 0$, and since $D_{\eta_\phi}[M(-1)] \leq 0$, $D_{\eta_\phi}[M(1)] \leq 0$ everywhere according to (5.69)-(5.70), we conclude by addition in (4.63) that $\eta_\phi(U_l^*) - \eta_\phi(U_l) \leq 0$, i.e. $w_l^* \geq w_l$, giving a contradiction. The argument goes the same for $U_r^*$ and $U_r$. □

*Remark 5.13* In the eulerian case, the CFL condition writes $\Delta t \max(|u_l - c/\rho_l|, |u_l^*|, |u_r + c/\rho_r|) \leq \Delta x$, thus the scheme is not able to treat the expansion into vacuum, this would give $\Delta t = 0$.

# 6 Parabolic problems

In this section we wish to give some extensions of the interpretation of flux vector splitting methods by three velocity BGK models to some one-dimensional parabolic/hyperbolic problems

$$\partial_t u + \partial_x F(u) - \partial^2_{xx} B(u) = 0, \tag{6.1}$$

with $u(t,x) \in \mathcal{U}$, a convex subset of $\mathbb{R}^p$. We assume that the system has a nonempty family $\mathcal{E}$ of convex entropies $\eta$, in the sense that there exists an entropy flux $\vartheta$ satisfying

$$\vartheta' = \eta' F', \tag{6.2}$$

and

$$(B')^t \eta'' \text{ is symmetric nonnegative.} \tag{6.3}$$

We define $B_\eta$ by

$$B'_\eta = \eta' B', \tag{6.4}$$

and we look for weak solutions to (6.1) that satisfy the entropy inequalities

$$\partial_t \eta(u) + \partial_x \vartheta(u) - \partial^2_{xx} B_\eta(u) \leq 0. \tag{6.5}$$

Following [6] and [2], a scheme for (6.1) can be derived from a transport-projection method with piecewise constant data over a mesh of size $\Delta x$ as described in Section 3, except that

$$a_{\Delta x}(\xi),\ M_{\Delta x}(u,\xi) \quad \text{depend on } \Delta x, \tag{6.6}$$

but the assumptions (such as the moment relations (1.5)-(1.6)) remain unchanged. The scheme thus takes the flux vector splitting form

$$u_i^{n+1} - u_i^n + \frac{\Delta t}{\Delta x}(F^n_{i+1/2} - F^n_{i-1/2}) = 0, \tag{6.7}$$

$$F^n_{i+1/2} = F^+_{\Delta x}(u_i^n) + F^-_{\Delta x}(u^n_{i+1}), \tag{6.8}$$

where

$$\begin{aligned} F^+_{\Delta x}(u) &= \int \Big(a_{\Delta x}(\xi)\Big)_+ M_{\Delta x}(u,\xi)\, d\xi, \\ F^-_{\Delta x}(u) &= -\int \Big(-a_{\Delta x}(\xi)\Big)_+ M_{\Delta x}(u,\xi)\, d\xi \end{aligned} \tag{6.9}$$

satisfy

$$F^+_{\Delta x} + F^-_{\Delta x} = F. \tag{6.10}$$

In order to get consistency with (6.1), we can ask that as $\Delta x \to 0$,

$$\Delta x\, a_{\Delta x}(\xi) \longrightarrow b(\xi), \qquad M_{\Delta x}(u,\xi) \longrightarrow M_0(u,\xi), \tag{6.11}$$

with

$$\int b(\xi) M_0(u,\xi)\, d\xi = 0, \qquad \int \frac{1}{2}|b(\xi)| M_0(u,\xi)\, d\xi = B(u), \tag{6.12}$$

or equivalently

$$\int b(\xi)_+ M_0(u,\xi)\,d\xi = B(u), \qquad \int \Big(-b(\xi)\Big)_+ M_0(u,\xi)\,d\xi = B(u). \qquad (6.13)$$

Obviously, (6.9), (6.11), (6.13) imply that

$$\begin{aligned} &\Delta x F^+_{\Delta x}(u) \longrightarrow B(u) \qquad \text{as } \Delta x \to 0,\\ &\Delta x F^-_{\Delta x}(u) \longrightarrow -B(u) \qquad \text{as } \Delta x \to 0, \end{aligned} \qquad (6.14)$$

and therefore, with (6.10),

$$F^+_{\Delta x}(u_i^n) + F^-_{\Delta x}(u_{i+1}^n) = F(u_{i+1}^n) - \Big(\Delta x\, F^+_{\Delta x}(u_{i+1}^n) - \Delta x\, F^+_{\Delta x}(u_i^n)\Big)/\Delta x, \quad (6.15)$$

which gives that the flux $F^n_{i+1/2}$ is weakly consistent with $F(u) - \partial_x B(u)$. By directly applying Theorem 3.1 at fixed $\Delta x$, we get the discrete entropy inequalities

$$\eta(u_i^{n+1}) - \eta(u_i^n) + \frac{\Delta t}{\Delta x}(\vartheta^n_{i+1/2} - \vartheta^n_{i-1/2}) \le 0, \qquad (6.16)$$

with

$$\vartheta^n_{i+1/2} = \vartheta^+_{\Delta x}(u_i^n) + \vartheta^-_{\Delta x}(u^n_{i+1}), \qquad (6.17)$$

$$(\vartheta^\pm_{\Delta x})' = \eta'(F^\pm_{\Delta x})'. \qquad (6.18)$$

Of course, we have to assume the dissipativity condition (3.25), and the CFL condition (3.15).

Conversely, if we are given a scheme (6.7)-(6.8) where $F^\pm_{\Delta x}$ satisfy (6.10) and (6.14), then it is consistent with (6.1), and discrete entropy inequalities (6.16)-(6.18) can be obtained via the three velocity kinetic interpretation of Theorem 4.1, Propositions 4.2 and 4.4. We need to assume the entropy dissipativity

$$F^+_{\Delta x},\ -F^-_{\Delta x} \quad \text{are } \eta\text{-dissipative in } \mathcal{U}_{stab}, \quad \eta \in \mathcal{E}, \qquad (6.19)$$

and the CFL condition

$$c \le \Delta x/\Delta t, \qquad (6.20)$$

$$\mathrm{Id} - (F^+_{\Delta x} - F^-_{\Delta x})/c \quad \text{is } \eta\text{-dissipative in } \mathcal{U}_{stab}, \quad \eta \in \mathcal{E}, \qquad (6.21)$$

for some $c > 0$.

Indeed there is a natural choice of such $F^\pm_{\Delta x}$. Starting from purely hyperbolic entropy flux vector splitting

$$F = F^+ + F^-, \qquad (6.22)$$

$$F^+,\ -F^- \quad \eta\text{-dissipative in } \mathcal{U}_{stab}, \quad \eta \in \mathcal{E}, \qquad (6.23)$$

we can take

$$F^+_{\Delta x} = F^+ + B/\Delta x, \qquad F^-_{\Delta x} = F^- - B/\Delta x, \qquad (6.24)$$

under the assumption that

$$B \quad \text{is } \eta\text{-dissipative in } \mathcal{U}_{stab}, \quad \eta \in \mathcal{E}, \qquad (6.25)$$

which is natural in view of (6.3). Condition (6.21) becomes then

$$\mathrm{Id} - (F^+ - F^-)/c - 2B/c\Delta x \quad \text{is } \eta\text{-dissipative in } \mathcal{U}_{stab}, \quad \eta \in \mathcal{E}, \qquad (6.26)$$

and since $c$ is of the order of $\Delta x/\Delta t$, this is a parabolic type CFL condition.

*Remark 6.1* The conditions (6.11)-(6.13) are not really necessary, only (6.14) is needed for consistency.

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