GLOBAL EXISTENCE OF WEAK SOLUTION FOR A NONSTRICTLY HYPERBOLIC SYSTEM OF CONSERVATION LAWS

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Abstract. In this paper, we study the Cauchy problem for a class of nonstrictly hyperbolic conservation laws which contain pressureless gas dynamics as a prototypical example. The Riemann solutions include two elementary waves: delta-wave and contact discontinuity. By studying the interaction of these elementary waves and employing improved generalized characteristics approach, we establish the convergence of approximate solutions constructed by piecing together all Riemann solutions, and prove the global existence of measure-value solution to the Cauchy problem. Moreover, the singular set of solution consists of at most denumerable Lipschitz continuous curves.

Key Words: Hyperbolic conservation laws, Generalized potential, Generalized characteristics, Delta wave

1. Introduction

We are concerned with the following nonstrictly hyperbolic conservation laws

$$\begin{cases} \rho_t + (\rho f(u))_x = 0, \\ (\rho u)_t + (\rho u f(u))_x = 0, \end{cases} \tag{1.1}$$

with initial data

$$(\rho, u)(x,0) = (\rho_0(x), u_0(x)),\ \ \rho_0(x) \geq 0,\ \ \rho_0(x), u_0(x) \in \mathrm{BV}(R), \tag{1.2}$$

where $f(u)$ is strictly increasing. System (1.1) has duplicate eigenvalue $\lambda = f(u)$ and the corresponding characteristic field is linearly degenerate. When $f(u) = u$, (1.1) is known as the system of pressureless gases where $\rho \geq 0, \rho u$ denotes the mass density and the momentum, respectively.

One of main difficulties for the mathematical analysis of (1.1) is that the solution is measure, not function. This may cause the product problem of measure and discontinuous function. Another difficulty is that the generalized variational principle introduced by [6] fails here due to the fact that the physical

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1991 *Mathematics Subject Classification.* 35D05, 35L65.
Supported partially by NSFC # 19801039.

velocity u is usually different from characteristical speed $f(u)$, while the two speeds always concide for pressureless gases.

For $f(u) = u$, the global existence of weak solution was first obtained independently by Brenier, Grenier [5] and E, Rykov, Sinai[6]. Since then, Wang, Huang, Ding [20] and Li [15] also considered the problem and generalized their existence results by different approaches. Recently, Boudin [4] has showed that the weak solution can also be obtained as limits of solutions to a viscous pressureless model. In [5,6], the authors found that the most natural entropy condition, i.e. convex entropy condition, is not sufficient for the uniqueness of weak solution. They conjectured that Oleinik's entropy condition should be suitable for uniqueness. In 1997, Wang and Ding [19] established the uniqueness of global weak solution for pressureless gases with Oleinik's entropy condition provided that the initial data $\rho_0(x), u_0(x) \in L^\infty(R)$. Bouchut and James [3] also obtained similar uniqueness results by different approach. More recently, Huang and Wang [12] solved the general uniqueness problem. For more details, we refer the reader to papers [1-6,9,11,12,15,16,18-20]. When $f(u) \neq u$, the system (1.1) is more complicated than pressureless gases because there exist two kinds of speeds. Yang [21] studied the Riemann problem of (1.1) and showed the location, speed, strength of delta wave of (1.1) are usually different from pressureless gases.

The main purpose of this paper is to prove global the existence of weak solution to the Cauchy problem of (1.1). Since all the methods in [5,6,15,120] are based on the special structure of solution of pressureless gases and not easy to be applied for general case, we have to seek a new approach to achieve our goal. The key point of our proof is we construct a sequence of approximate solutions by piecing together all the Riemann solutions with peicewise constant initial data. This idea is from the observation that only one wave is generated when interaction of waves takes place. Namely the total number of waves is time-decreasing. Thus we are able to construct a sequence of weak solutions (ρ_n, u_n) with peicewise constant initial data. By studying the interaction of elementary waves, some uniform estimates of (ρ_n, u_n) are obtained so that we can choose a subsequence of (ρ_n, u_n) (still denoted by (ρ_n, u_n)) such that ρ_n weakly converge to ρ in $\mathfrak{M}$, the space of Radon measure. If there exists a function $u(x,t) \in L^\infty(\mathrm{BV}(R), R)$ such that $u_n\rho_n, f(u_n)\rho_n, u_n f(u_n)\rho_n$ weakly converge to $u\rho, f(u)\rho, uf(u)\rho$, respectively, then our proof is completed. Fortunately, we prove that there indeed exists such $u(x,t)$ by carefully analyzing the structure of approximate solutions with the help of generalized characteristics introduced by Dafermos [7].

To overcome the difficulty caused by the product ρu, we shall use the definition of solution introduced by [20]. Since the first equation of (1.1) is conserved, we introduce generalized potential as follows

$$m(x,t)=\oint_{(0,0)}^{(x,t)}\rho dx-\rho f(u)dt. \tag{1.3}$$

It is easy to see that $m(x,t)$ is independent of integral path and $m_x=\rho$. Therefore (1.1) is changed into

$$\begin{cases} m_t+f(u)m_x=0, \\ (um_x)_t+(uf(u)m_x)_x=0, \end{cases} \tag{1.4}$$

with initial data

$$(m,u)|_{t=0}=(\int_0^x \rho_0(y)dy, u_0(x)) \tag{1.5}$$

We focus our attention to system (1.4), (1.5) instead of (1.1), (1.2).

Definition 1.1 Assume that $m(x,t)$ is locally of bounded variation in x, and $u(x,t)$ is bounded measurable to m_x, $(\rho,u)=(m_x,u)$ is called a weak solution of (1.1), (1.2), if

$$\begin{cases} \int\int \varphi_t m dxdt-\int\int \varphi f(u)dmdt=0, \\ \int\int(\psi_t u+\psi_x uf(u))dmdt=0, \end{cases} \tag{1.6}$$

hold for any $\varphi,\psi\in C_0^\infty(R_+^2)$, where $\int\int\cdots dmdt$ is Lebesgue-Stieltjes integral. And the initial data is understood in the following sense, $\forall\phi(x)\in C_0^\infty(R)$, as $t\to 0$,

$$\int_{-\infty}^{\infty}\phi(x)dm(x,t)\longrightarrow\int_{-\infty}^{\infty}\phi(x)\rho_0(x)dx, \tag{1.7}$$

$$\int_{-\infty}^{\infty}\phi(x)u(x,t)dm(x,t)\longrightarrow\int_{-\infty}^{\infty}\phi(x)\rho_0(x)u_0(x)dx. \tag{1.8}$$

Remark: This definition is a natural extension of the classical weak solution. When $m(x,t)$ is absolutely continuous in x, $\rho=m_x$ is just the classical one. However, if $m(x,t)$ is locally of bounded variation in x, then ρ becomes a Radon measure.

In the sense of definition 1.1, we have the following existence theorem.

Theorem 1 Assume that $\rho_0(x),u_0(x)\in \mathrm{BV}(R)$ with $\rho_0(x)\geq 0$. Then there exists a global weak solution of (1.1), (1.2).

The program of this paper is as follows: in sec.2, we study the Riemann problem and interaction of waves. in sec.3, we construct approximate solutions and introduce generalized characteristics. in sec.4, we first analyze the structure of generalized characteristic and the limits of apprximate solutions, then prove the global existence of weak solution of (1.1), (1.2).

2. Riemann problem

Consider the Riemann problem

$$\begin{cases} \rho_t + (\rho f(u))_x = 0, \\ (\rho u)_t + (\rho u f(u))_x = 0, \\ (\rho, u)|_{t=0} = \begin{cases} (\rho_-, u_-), & x < 0, \\ (\rho_+, u_+), & x > 0. \end{cases} \end{cases} \tag{2.1}$$

It is easy to check that the characteristic equation of (2.1) is

$$\begin{cases} \dfrac{dx}{dt} = f(u), \\ \dfrac{du}{dt} = 0. \end{cases} \tag{2.2}$$

This implies the characteristic is straight line and u keeps constant along each of them. If we seek the self-similar solution

$$(\rho, u)(t, x) = (\rho, u)(\xi), \ \xi = \frac{x}{t},$$

then (1.1) becomes

$$\begin{cases} -\xi \rho_\xi + (\rho f(u))_\xi = 0, \\ -\xi (\rho u)_\xi + (\rho u f(u))_\xi = 0. \end{cases} \tag{2.3}$$

For smooth solution of (2.2), we have

$$\rho(-\xi u_\xi + f(u) u_\xi) = 0. \tag{2.4}$$

This infers either $(\rho, u)(\xi) =$ constant or the vacuum $\rho = 0$.

Therefore the smooth solutions of system (1.1) only contain constant and vacuum states. There is no central rarefaction wave.

For a bounded discontinuity at $\xi = \sigma$, the Rankine-Hugoniot condition holds

$$\begin{cases} -\sigma[\rho] + [\rho f(u)] = 0, \\ -\sigma[\rho u] + [\rho u f(u)] = 0, \end{cases} \tag{2.5}$$

where $[\rho] = \rho_+ - \rho_-$ denotes the jump of ρ across the discontinuity.

Eliminating σ in (2.5), we have

$$\rho_+ \rho_- (u_+ - u_-)(f(u_+) - f(u_-)) = 0. \tag{2.6}$$

This infers $\rho_+\rho_- = 0$ or $u_- = u_+$. For the later, it is a contact discontinuity, denoted by J. It is easy to check that two states (ρ_-, u_-) and (ρ_+, u_+) can be connected by a contact discontinuity J if and only if $u_- = u_+$. Therefore there is only one classically elementary wave, i.e. contact discontinuity. Now we divide the Riemann problem (2.1) into three cases. For simplicity, let $\rho_-, \rho_+ > 0$.

CASE I. If $f(u_-) < f(u_+)$, then the solution is $J + J$, the intermediate state is vacuum.

$$(\rho(x,t), u(x,t)) = \begin{cases} (\rho_-, u_-), & x < f(u_-)t, \\ (0, u(\xi)), & f(u_-)t < x < f(u_+)t, \\ (\rho_+, u_+), & x > f(u_+)t. \end{cases} \tag{2.7}$$

CASE II. If $f(u_-) = f(u_+)$, then the solution is J.

$$(\rho(x,t), u(x,t)) = \begin{cases} (\rho_-, u_-), & x < f(u_-)t, \\ (\rho_+, u_+), & x > f(u_+)t. \end{cases} \tag{2.8}$$

CASE III. If $f(u_-) > f(u_+)$, it is impossible to connect the two states (ρ_-, u_-) and (ρ_+, u_+) by contact discontinuity or vacuum in view of the discussion above. The characteristics from initial data will overlap in the domain $\Omega = \{(x,t) : f(u_+)t < x < f(u_-)t\}$. In view of [16,18-21], a singular wave–delta wave (denoted by δ) must develop in Ω. Analogous to shock wave, the two side states of delta wave must satisfy additional condition, i.e. generalized Rankine-Hugoniot (GRH) condition as pointed out by by Li [15] and Yang [21]. Next, we shall study GRH from definition 1.1.

Generalized Rankine-Hugoniot condition: Suppose that (m, u) is a peicewise smooth solution of (1.1) in the sense of definition 1.1. Let Γ be a smooth curve across which u and m have a jump discontinuity. Let P be any point on Γ and D be a small ball centered at P. Let D_1, D_2 be the components of D which are determined by Γ. Assume Γ is given by $x = x(t)$. Let $\sigma = x'(t)$ and $u_\delta = u(x(t), t)$. $\forall\ \varphi, \psi \in C_0^\infty(D)$, from (1.6), we have

$$\begin{aligned} &\int\int_{D_1} \varphi_t m dx dt - \int\int_{D_1} \varphi f(u) dm dt + \int\int_{D_2} \varphi_t m dx dt \\ &- \int\int_{D_2} \varphi f(u) dm dt - \int_\Gamma \varphi(x(t), t) f(u_\delta)[m] dt = 0, \\ &\int\int_{D_1} (\psi_t u + \psi_x u f(u)) dm dt + \int\int_{D_2} (\psi_t u + \psi_x u f(u)) dm dt \\ &+ \int_\Gamma (\psi_t u_\delta + \psi_x u_\delta f(u_\delta))[m] dt = 0, \end{aligned} \tag{2.9}$$

where $[m](t) = m(x(t) + 0, t) - m(x(t) - 0, t)$.

Using the fact that (m, u) is C^1 in $D_i, i = 1, 2$, the divergence theorem gives

$$\int_\Gamma \varphi(\sigma - f(u_\delta))[m] dt = 0, \tag{2.10}$$

$$\begin{aligned} &\int_\Gamma \psi([\rho u]\sigma - [\rho u f(u)]) dt + \int_\Gamma (\psi_t + \psi_x \sigma) u_\delta [m] dt \\ &= \int_\Gamma \psi([\rho u]\sigma - [\rho u f(u)] - \frac{d}{dt}(u_\delta [m])) dt = 0. \end{aligned} \tag{2.11}$$

Since φ, ψ are arbitrary, we have

$$\begin{cases} \frac{dx}{dt} = \sigma = f(u_\delta), \\ \dfrac{d}{dt}(u_\delta[m]) = [\rho u]\sigma - [\rho u f(u)]. \end{cases} \tag{2.12}$$

We call $\sigma = x'$ the speed of delta wave. Similarly, choosing φ_x instead of φ in the first equation of (2.9), we have

$$\frac{d[m]}{dt} = [\rho]\sigma - [\rho f(u)]. \tag{2.13}$$

(2.12) and (2.13) are called generalized Rankine-Hugoniot condition. And the entropy condition is

$$u_- > u_\delta > u_+. \tag{2.14}$$

Now let's turn to the Riemann problem (2.1) with $u_- > u_+$. In view of (2.12) and (2.13), we compute

$$[\rho]u_\delta f(u_\delta) - [\rho u]f(u_\delta) - [\rho f(u)]u_\delta + [\rho u f(u)] = 0. \tag{2.15}$$

where we have used the fact that u_δ is a constant because u is self-similar. Let

$$F(u) = [\rho]uf(u) - [\rho u]f(u) - [\rho f(u)]u + [\rho u f(u)], \text{ on } u_+ < u < u_-. \tag{2.16}$$

Since $f'(u) > 0$, by direct computation, we have

$$\begin{aligned} F(u_-) &= \rho_+[u][f(u)] > 0, \\ F(u_+) &= -\rho_-[u][f(u)] < 0, \end{aligned}$$

and

$$\begin{aligned} F'(u) &= [\rho]f(u) + ([\rho]u - [\rho u])f'(u) - [\rho f(u)] \\ &= \rho_+(f(u) - f(u_+)) - \rho_-(f(u) - f(u_-)) \\ &\quad + \rho_+(u - u_+)f'(u) - \rho_-(u - u_-)f'(u) \\ &> 0, \text{ on } u_+ < u < u_-. \end{aligned}$$

It is easy to see that there exists one and only one point u_δ between u_+ and u_- satisfying $F(u_\delta) = 0$.

So the solution of CASE III is δ, i.e.

$$(\rho(x,t), u(x,t)) = \begin{cases} (\rho_-, u_-), & x < f(u_\delta)t, \\ ([m]\delta(x - f(u_\delta)t), u_\delta), & x = f(u_\delta)t, \\ (\rho_+, u_+), & x > f(u_\delta)t, \end{cases} \tag{2.17}$$

where $[m] = ([\rho]f(u_\delta) - [\rho f(u)])t = \int_{x-f(u_-)t}^{x-f(u_+)t} \rho_0(\xi, 0)d\xi > 0$ with $x = f(u_\delta)t$ is called the strength of delta wave.

Remark: Since the system (1.1) is equivalent to

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \rho(u_t + f(u)u_x) = 0, \end{cases} \tag{2.18}$$

for smooth solution, we expect that $f(u_\delta)$ is the shock speed of $u_t+f(u)u_x=0$ if $\rho_+=\rho_-$. For pressureless gases, it is true. However, by (2.16), it is not ture if $f(u)\neq u$. This shows (1.1) is more complicated than pressureless gases.

Interaction of elementary waves: According to the argument above, there are only two elementary waves–contact discontinuity and delta wave. It is easy to see two contact discontinuities will never interact with each other. Either a delta wave interacts with a contact discontinuity or a delta wave. When interaction of waves takes place, we need to solve Riemann problem of (1.1) with initial delta wave. By studying GRH, we have the following estimates.

Lemma 2.1 let $(\rho_-,u_-),(\rho_m,u_m)$ be connected by wave w_1 with speed $\sigma_1(t)$, strength $[m_1](t)$ ($[m_1](t)=0$ if the wave is contact discontinuity) and $(\rho_m,u_m),(\rho_+,u_+)$ be connected by wave w_2 with speed $\sigma_2(t)$, strength $[m_2](t)$. Suppose that the two waves w_1, w_2 interact at the point (x_0,t_0), then there exists a delta wave w originated from the point (x_0,t_0) connecting the states $(\rho_-,u_-),(\rho_+,u_+)$. Moreover, the following estimates hold

$$f(u_-)>\sigma>f(u_+),\ \text{if}\ u_->u_+, \tag{2.19}$$

$$|\frac{d\sigma(t)}{dt}|\leq C\max\{\rho_+,\rho_-\}\frac{(u_+-u_-)^2}{[m](t)}. \tag{2.20}$$

where $\sigma(t)$, $[m](t)$ denote the speed, strength of the new wave w, respectively, and the constant $C>0$ only depends on the initial data.

Proof: Without loss of generality, let both of incoming waves be delta waves. We consider the following Riemann problem with initial delta wave

$$(\rho,u)|_{t=t_0}=\begin{cases}(\rho_-,u_-), & x<x_0\\ ([m](t_0)\delta(x-x_0),u_\delta(t_0)), & x=x_0\\ (\rho_+,u_+), & x>x_0\end{cases} \tag{2.21}$$

where $[m](t_0),u_\delta(t_0)$ are given by

$$\begin{cases}[m](t_0)=[m_1](t_0)+[m_2](t_0),\\ [m](t_0)u_\delta(t_0)=[m_1](t_0)u_{\delta_1}(t_0)+[m_2](t_0)u_{\delta_2}(t_0),\end{cases} \tag{2.22}$$

which is derived from (2.9).

Since $u_->u_{\delta_1}(t_0)>u_m>u_{\delta_2}(t_0)>u_+$, it is easy to see

$$[m](t_0)>0,\ u_->u_\delta(t_0)>u_+.$$

Therefore the solution of (1.1), (2.21) is a delta wave. In fact, (2.12), (2.13) yield

$$\begin{cases}[m]=[\rho](x-x_0)-[\rho f(u)](t-t_0)+[m](t_0),\\ [m]u_\delta=[\rho u](x-x_0)-[\rho uf(u)](t-t_0)+[m](t_0)u_\delta(t_0).\end{cases} \tag{2.23}$$

For convenience, let (x_0, t_0) be the origin point $(0,0)$ and $[m](t_0) = m_0, u_\delta(t_0) = v_0$. We compute

$$\begin{aligned} [m](u_- - u_\delta) &= [\rho]u_- x - [\rho f(u)]u_- t + m_0 u_- \\ &\quad -[\rho u]x + [\rho u f(u)]t - m_0 v_0 \qquad (2.24) \\ &= \rho_+(u_- - u_+)(x - f(u_+)t) + m_0(u_- - v_0) > 0, \end{aligned}$$

$$[m](u_+ - u_\delta) = -\rho_-[u](x - f(u_-)t) + m_0(u_+ - u_0) < 0. \qquad (2.25)$$

where $f(u_+)t < x < f(u_-)t$. This implies $u_- > u_\delta > u_+$ and (2.19) holds due to $f' > 0$.

On the other hand, (2.12),(2.13) also infer

$$[m]u_\delta' = \rho_+(\sigma - f(u_+))(u_+ - u_\delta) - \rho_-(\sigma - f(u_-))(u_- - u_\delta). \qquad (2.26)$$

Then, we have

$$|\sigma'| \le C|u_\delta'| \le C\max\{\rho_+, \rho_-\}\frac{(u_+ - u_-)^2}{[m](t)}. \qquad (2.27)$$

Therefore Lemma 2.1 is proved.

In general, the interaction of two waves only generates one wave– delta wave. If we denote the outgoing wave as part of incoming wave, then delta wave will never disappear once it is generated.

3. Construction of Approximate Solutions

In this section, we shall construct a sequence of approximate solutions of system (1.1). Since $\rho_0, u_0 \in \mathrm{BV}(R)$, we choose a sequence of piecewise constant functions $\rho_n(x,0), u_n(x,0)$ with finitely many jump points such that

$$\begin{cases} \rho_n(x,0) > 0 \xrightarrow{\text{a.e.}} \rho_0(x), \\ u_n(x,0) \xrightarrow{\text{a.e.}} u_0(x), \\ \mathrm{T.V.}(u_n(x,0)) \le \mathrm{T.V.}(u_0), \\ |\rho_n(x,0)|_{L^\infty},\ |u_n(x,0)|_{L^\infty} \le M = \max\{|u_0(x)|_{L^\infty},\ |\rho_0(x)|_{L^\infty}\}. \end{cases} \qquad (3.1)$$

For each $n \ge 1$, we construct the approximate solutions as follows: the constructive procedure starts at $t = 0$ with initial data $(\rho_n(x,0), u_n(x,0))$. Suppose that $x_1 < \cdots < x_N$ be the points where $(\rho_n(x,0), u_n(x,0))$ is discontinuous. For each $\alpha = 1, ..., N$, we obtain the Riemann solutions (ρ_n, u_n) until a time t_1 is reached, when the first set of interaction between two or more wave-fronts takes place. After time t_1, the solution (ρ_n, u_n) can be obtained by solving the Riemann problem with initial delta wave which has been studied in (2.21). So the solution (ρ_n, u_n) can then be continued up to a time t_2 where the second set of wave interactions takes place, etc $\cdots$. According to the argument of §2, only one delta wave is generated from the point of interaction of waves.

The number of wave-front is time-decreasing. So the constructive procedure can be continued up to any time $t > 0$. Therefore (ρ_n, u_n) is a measure solution of system (1.1) with piecewise constant initial data $(\rho_n(x,0), u_n(x,0))$. In fact, we can give a explicit formula for the approximate solution (ρ_n, u_n) by generalized characteristics in the later context.

First, we define the generalized characteristics $X_n(\xi, t)$ as follows: For any continuous point ξ of $(\rho_n(x,0), u_n(x,0))$ in x-axis, we introduce the genuinely characteristic L:$x = \xi + tf(u_n(\xi, 0))$ which may interact with a contact discontinuity J or a delta wave δ at time t_0. We define

$$X_n(\xi,t) = \begin{cases} \xi + tf(u_n(\xi,0)), & \text{if } 0 \le t \le t_0, \\ x(t), & \text{if } t \ge t_0, \end{cases}$$

where $x(t)$ represents the location of interacting wave J or δ until the next time when it interacts with other wave. If J or δ continues to interact with other wave at time t_1, we still denote $x(t)$ the location of the outgoing delta wave after t_1, etc $\cdots$. By the same procedure, $x(t)$ can be well defined for any time $t \ge t_0$ because there is one and only one delta wave generated when wave interaction takes place from the interacting point. For convenience, we always understand $x(t)$ in the above sense. If ξ is of discontinuity set of $(\rho_n(x,0), u_n(x,0))$, there are two cases. When $u_n(\xi - 0, 0) > u_n(\xi + 0, 0)$, there is a delta wave originated from $(\xi, 0)$. We define $X_n(\xi, t) = x(t)$, the location of delta wave. When $u_n(\xi - 0, 0) < u_n(\xi + 0, 0)$, there are two contact discontinuities J_1, J_2 and intermediate state is vacuum. We define $X_n(\xi, t) = x(t)$ the location of left contact discontinuity J_1. Thus for any point ξ of x-axis and any time t, we construct the generalized characteristics $X_n(\xi, t)$. For $X_n(\xi, t)$, we have the following properties.

Lemma 3.1 $X_n(\xi, t)$ is increasing in ξ and Lipschitz continuous in t. In particular, $f(u_n(X_n(\xi,t),t)) = \frac{\partial}{\partial t} X_n(\xi,t)$ holds except finitely many points of t. If $X_n(\xi_1, t_0) = X_n(\xi_2, t_0)$ for some time t_0, where $\xi_1 \ne \xi_2$, then $X_n(\xi_1, t) = X_n(\xi_2, t)$ holds for any time $t \ge t_0$.

In view of our definition of generalized characteristics, it is easy to check Lemma 3.1 holds. We omit the proof here.

Lemma 3.2 For any $\xi_1 < \xi_2$,

$$X_n'(\xi_2, t) - X_n'(\xi_1, t) \le \frac{1}{t}(X_n(\xi_2, t) - X_n(\xi_1, t)). \tag{3.2}$$

holds for any time $t > 0$.

Proof: Since the right term of (3.2) is always nonnegative by Lemma 3.1, we only need to study which factor causes the growth of $X_n'(\xi,t)$. Accoding to our constructive procedure, (ρ_n,u_n) consists of a finite number of wave fronts. If $X_n(\xi_1,t)$ and $X_n(\xi_2,t)$ are located two sides of a delta wave or a contact discontinuity, then $X_n'(\xi_2,t)-X_n'(\xi_1,t)$ is nonpositive. Only does there exist vacuum between $X_n(\xi_1,t)$ and $X_n(\xi_2,t)$ as shown in CASE I, $X_n'(\xi_2,t)-X_n'(\xi_1,t)$ is positive. Without loss of generality, we consider the Riemann problem with $u_- < u_+$. Suppose that $X_n(\xi_1,t)$ and $X_n(\xi_2,t)$ are located outside of vacuum. We compute

$$\begin{aligned}
X_n'(\xi_2,t)-X_n'(\xi_1,t) &= f(u_+)-f(u_-)\\
&= \tfrac{1}{t}(f(u_+)t-f(u_-)t)\\
&\le \tfrac{1}{t}(X_n(\xi_2,t)-X_n(\xi_1,t)).
\end{aligned}$$

Thus Lemma 3.2 is proved.

Since $X_n(\xi,t)$ is increasing in ξ, we define the inverse function as

$$\xi_n(x,t)=\sup\{\xi; X_n(\xi,t)<x\} \tag{3.3}$$

$$\eta_n(\xi,t)=\inf\{\xi; X_n(\xi,t)>x\} \tag{3.4}$$

By (3.3),(3.4), it is easy to see that the following properties hold.

Lemma 3.3

(i) $\xi_n(x,t),\eta_n(x,t)$ are increasing in x.

(ii) If $\xi_n(x,t)<\eta_n(x,t)$, then $x=X_n(\xi,t)$, for any $\xi_n(x,t)<\xi<\eta_n(x,t)$.

(iii) $\xi_n(x,t)=\xi_n(x-0,t),\eta_n(x,t)=\xi_n(x+0,t)$. Furthermore,

$$X_n(\xi_n(x,t)-0,t)\le x\le X_n(\xi_n(x,t)+0,t), \tag{3.5}$$

$$x-Nt\le \xi_n(x,t)\le x+Nt. \tag{3.6}$$

where $N=\max_{|u|\le M}|f(u)|$.

Define

$$\begin{aligned}
m_n(x,t) &= \int_0^{\xi_n(x,t)} \rho_n(\xi,0)d\xi,\\
q_n(x,t) &= \int_0^{\xi_n(x,t)} \rho_n(\xi,0)b(X_n'(\xi,t))d\xi,\\
\theta_n(x,t) &= \int_0^{\xi_n(x,t)} \rho_n(\xi,0)X_n'(\xi,t)d\xi,\\
e_n(x,t) &= \int_0^{\xi_n(x,t)} \rho_n(\xi,0)b(X_n'(\xi,t))X_n'(\xi,t)d\xi.
\end{aligned}$$

where $b(u)$ is the inverse function of $f(u)$.

Lemma 3.4

$$\begin{aligned}
&\rho_n = m_{nx},\\
&dq_n(x,t) = u_n(x,t)dm_n(x,t),\\
&d\theta_n(x,t) = f(u_n(x,t))dm_n(x,t),\\
&de_n(x,t) = u_n(x,t)f(u_n(x,t))dm_n(x,t)
\end{aligned}$$

hold in the sense of Radon-Nikodym derivative in x.

Proof: Since (ρ_n, u_n) is constructed by piecing together a finite number of Riemann solutions, we divide our proof into three parts. First, for any given point (x,t), if (x,t) is located in vacuum, then $m_n(x,t)$ is constant in the area because all generalized characteristic $X_n(\xi,t)$ are away from the Vacuum. This case is trivial. Then, we consider the case that (x,t) lies in some constant area V_c. It is easy to see that there exists a characteristic straight line $X_n(\xi_0,\tau) = \xi_0 + f(u_n(x,t))\tau$ through the point (x,t). On the other hand, there is a small enough neighborhood N of (x,t). For any $(y,t) \in N$, the characteristic belonging to (y,t) is denoted by: $X_n(y_0,\tau) = y_0 + f(u_n(x,t))\tau, \tau \le t$ along which (ρ_n, u_n) keeps constant. We compute

$$\begin{aligned}
&\xi_n(x,t) = x - f(u_n(x,t))t,\\
&\xi_n(y,t) = y - f(u_n(x,t))t.
\end{aligned}$$

which infers $\rho_n = m_{nx}$ and

$$\begin{aligned}
&\lim_{x_2,x_1 \longrightarrow x\pm 0} \frac{q_n(x_2,t) - q_n(x_1,t)}{m_n(x_2,t) - m_n(x_1,t)}\\
&= \lim_{x_2,x_1 \longrightarrow x\pm 0} \frac{\int_{\xi_n(x_1,t)}^{\xi_n(x_2,t)} \rho_n(\xi,0) b(X_n'(\xi,t))d\xi}{\int_{\xi_n(x_1,t)}^{\xi_n(x_2,t)} \rho_n(\xi,0)d\xi}\\
&= \lim_{x_2,x_1 \longrightarrow x\pm 0} \frac{\int_{x_1 - f(u_n(x,t))t}^{x_2 - f(u_n(x,t))t} \rho_n(x,t)u_n(x,t)d\xi}{\int_{x_1 - f(u_n(x,t))t}^{x_2 - f(u_n(x,t))t} \rho_n(x,t)d\xi}\\
&= u_n(x,t).
\end{aligned} \tag{3.7}$$

Finaly, we consider the case that (x,t) lies on a delta wave or a contact discontinuity. For the later, it is similar to the second part. So we only need to study the case of delta wave. Suppose that a generalized characteristic $X_n(\xi,\tau)$ travels through the point (x,t), then there must exist $\xi_1 < \xi_2$ such that $X_n(\xi,t) = x$, for any $\xi_1 < \xi < \xi_2$. This implies $\xi_n(x,t) \neq \eta_n(x,t)$ and $X_n'(\xi,t) = f(u_n(x,t))$, $\forall \xi_n(x,t) < \xi < \eta_n(x,t)$. By (2.17) and (2.22), we have $\rho_n(]x,x]) = \int_{\xi_n(x,t)}^{\eta_n(x,t)} \rho_n(\xi,0)d\xi = [m_n]$. So

$$\lim_{x_2,x_1 \longrightarrow x\pm 0} \frac{q_n(x_2,t) - q_n(x_1,t)}{m_n(x_2,t) - m_n(x_1,t)} = \frac{\int_{\xi_n(x,t)}^{\eta_n(x,t)} \rho_n(\xi,0) b(X_n'(\xi,t))d\xi}{\int_{\xi_n(x,t)}^{\eta_n(x,t)} \rho_n(\xi,0)d\xi} = u_n(x,t). \tag{3.8}$$

By (3,7) and (3.8), we know that $\frac{dq_n}{dm_n} = u_n(x,t)$ in the sense of the Radon-Nikodym derivative. In the same fashion, we also have

$$\frac{d\theta_n}{dm_n} = f(u_n(x,t)), \frac{de_n}{dm_n} = u_n(x,t)f(u_n(x,t)).$$

This infers Lemma 3.4.

On the other hand, since (ρ_n, u_n) is a weak solution of (1.1) with the initial data $\rho_n(x,0), u_n(x,0)$, by Lemma 3.4,

$$\begin{cases} \int\int \varphi_t m_n dxdt - \int\int \varphi f(u_n) dm_n dt \\ = \int\int \varphi_t m_n dxdt - \int\int \varphi d\theta_n dt = 0, \\ \int\int (\psi_t u_n + \psi_x u_n f(u_n)) dm_n dt = \int\int \psi_t dq_n + \psi_x de_n dt = 0, \end{cases} \tag{3.9}$$

hold for any $\varphi, \psi \in C_0^\infty(R_+ \times R)$.

4. Existence of Generalized Solution

Proof of theorem 1: To prove theorem 1, we need to study the convergence of $X_n(\xi,t)$. Since $X_n(\xi,t)$ is increasing in ξ and $|X_n(\xi,t_2) - X_n(\xi,t_1)| \leq M(t_2 - t_1)$, we can choose a subsequence of $X_n(\xi,t)$ (still denoted by $X_n(\xi,t)$) such that $X_n(\xi,t)$ converge to some increasing function $X(\xi,t)$ by diagonal method. Moreover, $X(\xi,t)$ is Lipschitz continuous in t.

For $X(\xi,t)$, we have the following properties.

Lemma 4.1 (1) $t^{-1}(X(\xi_2,t) - X(\xi_1,t))$ is time-decreasing for any $\xi_1 < \xi_2$.

(2) If $X(\xi_1,t_0) = X(\xi_2,t_0)$ for some $t_0 \geq 0$, then

$$X(\xi_1,t) = X(\xi_2,t) \tag{4.1}$$

holds for any $t \geq t_0$.

Proof of (1): Since $X_n(\xi,t)$ converge to $X(\xi,t)$, we only need to show that $t^{-1}(X_n(\xi_2,t) - X_n(\xi_1,t))$ is decreasing in t for any $\xi_1 < \xi_2$. From (3.2), we have

$$\begin{aligned} &\frac{d}{dt}(t^{-1}(X_n(\xi_2,t) - X_n(\xi_1,t)) \\ &= t^{-2}(X_n(\xi_1,t) - X_n(\xi_2,t) + t(X_n'(\xi_2,t) - X_n'(\xi_1,t))) \\ &\leq 0. \end{aligned} \tag{4.2}$$

Let $n \to \infty$, we get (1).

Proof of (2):(2) is easy from

$$t^{-1}(X_n(\xi_2,t)-X_n(\xi_1,t))\le t_0^{-1}(X_n(\xi_2,t_0)-X_n(\xi_1,t_0))=0. \tag{4.3}$$

Similar to the construction of $\xi_n(x,t)$ and $\eta_n(x,t)$, we define

$$\xi(x,t)=\sup\{\xi; X(\xi,t)<x\} \tag{4.4}$$

$$\eta(\xi,t)=\inf\{\xi; X(\xi,t)>x\} \tag{4.5}$$

For $\xi(x,t),\eta(x,t)$, we still have
(i) $\xi(x,t),\eta(x,t)$ are increasing in x,
(ii) $x=X(\xi,t)$, $\forall\xi(x,t)<\xi<\eta(x,t)$,
(iii) $\xi(x,t)=\xi(x-0,t),\eta(x,t)=\xi(x+0,t)$,
(iv) $x-Nt\le\xi(x,t)\le x+Nt$.

Now we define

$$\begin{aligned}
m(x,t)&=\int_0^{\xi(x,t)}\rho(\xi,0)d\xi,\\
q(x,t)&=\int_0^{\xi(x,t)}\rho(\xi,0)b(X'(\xi,t))d\xi,\\
\theta(x,t)&=\int_0^{\xi(x,t)}\rho(\xi,0)X'(\xi,t)d\xi,\\
e(x,t)&=\int_0^{\xi(x,t)}\rho(\xi,0)b(X'(\xi,t))X'(\xi,t)d\xi.
\end{aligned}$$

It is easy to see $m(x,t),q(x,t),\theta(x,t),e(x,t)$ are locally of bounded variation.

Let

$$C_t=\{x;\ \xi(x,t)=\eta(x,t)\} \tag{4.6}$$

Fix t, for any point $x\in R/C_t$, $\xi(x,t)\ne\eta(x,t)$, there is a characteristic triangle belonging to (x,t) with the corresponding interval $(\xi(x,t),\eta(x,t))$ in x-axis. Therefore R/C_t is a zero measure set for every t.

Lemma 4.2 For any point $x\in C_t$, $\xi_n(x,t)$ converges to $\xi(x,t)$ as $n\to\infty$.

Proof: By way of contradiction, suppose that the conclusion does not hold, then there exists a positive constant δ_0 and a subsequence n_i of n such that

$$|\xi_{n_i}(x,t)-\xi(x,t)|\ge\delta_0. \tag{4.7}$$

It is easy to see there are infinitely many n_i satisfying $\xi_{n_i}(x,t)\le\xi(x,t)-\delta_0$ or $\xi_{n_i}(x,t)\ge\xi(x,t)+\delta_0$. Without loss of generality, assume that the first holds. Choose any $\xi_0\in(\xi(x,t)-\delta_0,\xi(x,t))$, then $X_{n_i}(\xi_0,t)$ must be larger than x by the definition of $\xi_{n_i}(x,t)$. Let $n_i\to\infty$, then $X(\xi_0,t)\ge x$. This contradicts with the definition of $\xi(x,t)$. Therefore Lemma 4.2 is proved.

To obtain the convergence of $m_n(x,t), q_n(x,t), \theta_n(x,t), e_n(x,t)$, some estimates of $\rho_n(\xi,0)X_n'(\xi,t)$ are necessary. We have the following lemma.

Lemma 4.3 For any $-\infty < a < b < +\infty$, we have

$$\text{T.V.}(\rho_n(\xi,0)X_n'(\xi,t)) \leq C, \tag{4.8}$$

$$\int_a^b \rho_n(\xi,0)|X_n'(\xi,t_2) - X_n'(\xi,t_1)|d\xi \leq C(t_2 - t_1), \tag{4.9}$$

where C only depends on the initial data.

Proof: If $X_n(\xi,t)$ is a geniuely characteristic or a contact discontinuity, then $u_n(\xi,t)$ keeps constant along each of them. So $X_n'(\xi,t_2) - X_n'(\xi,t_1)$ is zero. Now let's consider the case that $X_n(\xi,t)$ is a delta wave. Since there only exist a finite number of delta waves for every n originated from the interval (a,b), we denote them $X_{n_i}(\xi,t), i = 1,2,\cdots,m$. For each delta wave $X_{n_i}(\xi,t)$, it has a characteristic interval $(\xi_{n_i}(t), \eta_{n_i}(t))$. Fix time t, all these characteristic intervals never interact each other. We divide the interval (t_1,t_2) into many small subintervals $(s_j, s_{j+1}), j = 0,1,\cdots,l-1$, such that $X_{n_i}(\xi,t)$ is a smooth curve for every (s_j,s_{j+1}). We note that it is indeed possible becasue the generalized characteristic is piecewise smooth curve. Of couse, m,l depend on the index n. By Lemma 2.1, $\forall t \in (s_j,s_{j+1})$, the following holds

$$|u_n'(X_{n_i}(\xi,t),t)| \leq 2CM^2 \frac{|u_n(X_{n_i}(\xi,t)+0,t) - u_n(X_{n_i}(\xi,t)-0,t)|}{[m](t)}, \tag{4.10}$$

where $[m](t) = \int_{\xi_{n_i}(t)}^{\eta_{n_i}(t)} \rho_n(\xi,0)d\xi$.

Since $X_n'(\xi,t) = f(u_n(X_n(\xi,t),t))$, we have

$$\begin{aligned}
&\int_a^b \rho_n(\xi,0)|X_n'(\xi,t_2) - X_n'(\xi,t_1)|d\xi \\
&\leq C\sum_{i=1}^{m}\sum_{j=0}^{l-1}\int_{s_j}^{s_{j+1}}\int_{\xi_{n_i}(t)}^{\eta_{n_i}(t)} \rho_n(\xi,0)|u_n'|d\xi\ dt \\
&\leq C\sum_{j=0}^{l-1}\int_{s_j}^{s_{j+1}} \text{T.V.}u_n(\cdot,t)dt \\
&\leq C(t_2-t_1).
\end{aligned}$$

With the help of the BV compact principle, there exists a subsequence of $\rho_n(\xi,0)X_n'(\xi,t)$ (still denoted by $\rho_n(\xi,0)X_n'(\xi,t)$) such that $\rho_n(\xi,0)X_n'(\xi,t)$ converge almost everywhere to some function $g(\xi,t)$ in $R_+ \times R$. On the other hand, it is easy to check that $\rho_n(\xi,0)X_n'(\xi,t)$ weakly converge to $\rho_0(\xi)X'(\xi,t)$ by (3.1) and the fact that $X_n(\xi,t) \to X(\xi,t)$, as $n \to \infty$. This implies $g(\xi,t) = \rho_0(\xi)X'(\xi,t)$ a.e. in R_+, i.e. except a zero set $\mathfrak{O}$ of t, $g(\xi,t) = \rho_0(\xi)X'(\xi,t)$

holds for almost all $\xi \in R$. Furthermore, we also have

$$\rho_n(\xi,0)u_n(\xi,t) \longrightarrow \rho_0(\xi)b(X'(\xi,t)), \text{ a.e. in } R_+,$$
$$\rho_n(\xi,0)u_n(\xi,t)f(u_n(\xi,t)) \longrightarrow \rho_0(\xi)b(X'(\xi,t))X'(\xi,t), \text{ a.e. in } R_+.$$

Thus, by Lemma 4.2 and Lemma 4.3, as $n \to \infty$,

$$\begin{aligned}
m_n(x,t) &= \int_0^{\xi_n(x,t)} \rho_n(\xi,0)d\xi \xrightarrow{\text{a.e.}} \int_0^{\xi(x,t)} \rho_0 d\xi = m(x,t) \\
q_n(x,t) &= \int_0^{\xi_n(x,t)} \rho_n(\xi,0)b(X_n'(\xi,t))d\xi \\
&\xrightarrow{\text{a.e.}} \int_0^{\xi(x,t)} \rho_0 b(X'(\xi,t))d\xi = q(x,t), \\
\theta_n(x,t) &= \int_0^{\xi_n(x,t)} \rho_n(\xi,0)X_n'(\xi,t)d\xi \\
&\xrightarrow{\text{a.e.}} \int_0^{\xi(x,t)} \rho_0 X'(\xi,t)d\xi = \theta(x,t), \\
e_n(x,t) &= \int_0^{\xi_n(x,t)} \rho_n(\xi,0)b(X_n'(\xi,t))X_n'(\xi,t)d\xi \\
&\xrightarrow{\text{a.e.}} \int_0^{\xi(x,t)} \rho_0 b(X'(\xi,t))X'(\xi,t)d\xi = e(x,t).
\end{aligned} \tag{4.11}$$

hold for any $t \in R/\mathfrak{D}$. Let $n \to \infty$ in (3.9), then

$$\begin{cases} \displaystyle\int\int \varphi_t m dx dt - \int\int \varphi d\theta dt = 0 \\ \displaystyle\int\int \psi_t dq + \psi_x de dt = 0 \end{cases} \tag{4.12}$$

hold for any $\varphi, \psi \in C_0^\infty(R_+ \times R)$.

Define

$$u(x,t) = \begin{cases} b(X'(\xi(x,t),t)), & \text{if } \xi(x,t) = \eta(x,t), \\ b(X'(\xi,t)), & \forall \xi \in (\xi(x,t),\eta(x,t)), \text{ if } \xi(x,t) \neq \eta(x,t). \end{cases}$$

To prove theorem 1, it is sufficient to prove that, $\forall t \in R_+/\mathfrak{D}$,

$$\begin{aligned} &\frac{dq}{dm} = u(x,t), \frac{d\theta}{dm} = f(u(x,t)), \\ &\frac{de}{dm} = u(x,t)f(u(x,t)) \end{aligned} \tag{4.13}$$

hold for almost everywhere x in the sense of measure m_x.

If there exists a neighborhood $\mathfrak{N}$ of (x,t) such that $\xi(x,t)$ is constant in $\mathfrak{N}$, then the functions $m(x,t), q(x,t), \theta(x,t)$ and $e(x,t)$ are also constants, this case is trivial. Suppose that $\xi(x,t)$ is not always a constant in any neighborhood of (x,t).

Let

$$\Gamma_t = \{x : \xi(x,t) = \eta(x,t)\}. \tag{4.14}$$

Choose any $x_1 < x < x_2$, let $y_1 = \xi(x_1,t)$, $y_2 = \xi(x_2,t)$, then $y_1 < y_2$, and

$$y_1 \to \xi(x,t),\ y_2 \to \eta(x,t),\ \text{as } x_1, x_2 \to x. \tag{4.15}$$

Since $X'(\xi,t)$ is continuous except at most countable points in x-axis, we compute

$$\begin{aligned}\lim_{x_2,x_1\longrightarrow x\pm 0} \frac{q(x_2,t)-q(x_1,t)}{m(x_2,t)-m(x_1,t)} &= \lim_{x_2,x_1\longrightarrow x\pm 0} \frac{\int_{y_1}^{y_2}\rho_0(\xi)b(X'(\xi,t))d\xi}{\int_{y_1}^{y_2}\rho_0(\xi)d\xi} \\ &= u(x,t) \text{ a.e. } dm \text{ in } \Gamma_t.\end{aligned} \tag{4.16}$$

If $\xi(x,t) < \eta(x,t)$, then $x = X(\xi,t)$ and $u(X(\xi,t),t) = u(x,t)$ holds for all $\xi \in (\xi(x,t), \eta(x,t))$. So

$$\begin{aligned}\lim_{x_2,x_1\longrightarrow x\pm 0} \frac{q(x_2,t)-q(x_1,t)}{m(x_2,t)-m(x_1,t)} &= \lim_{x_2,x_1\longrightarrow x\pm 0} \frac{\int_{y_1}^{y_2}\rho_0(\xi)b(X'(\xi,t))d\xi}{\int_{y_1}^{y_2}\rho_0(\xi)d\xi} \\ &= \frac{\int_{\xi(x,t)}^{\eta(x,t)}\rho_0(\xi)u(x,t)d\xi}{\int_{\xi(x,t)}^{\eta(x,t)}\rho_0(\xi)d\xi} = u(x,t).\end{aligned} \tag{4.17}$$

In the same way, (4.13) also holds for $\theta(x,t), e(x,t)$. Therefore theorem 1 is proved.

Remark: From the definition of $\xi(x,t), \eta(x,t)$, the discontinuity of $m(x,t)$ only takes place on set S, where $S = \{(x,t); \xi(x,t) \neq \eta(x,t), t > 0\}$. It is easy to see that S consists of at most countable Lipschitz continuous curves.

Acknowledgment

The author wishes to thank Prof. Xiaqi Ding for his kindly suggesting this problem and Prof. Tong Yang for his kind hospitality. The research was partially supported by NSFC Grant. No. 19801039 and City University of Hong Kong.

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