A BGK Approximation to Nonlinear Parabolic Initial-Boundary Value Problems *

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Abstract

We present a relaxation semilinear system of conservation laws with source which approximates nonlinear parabolic equations with initial and boundary conditions. We prove the well-posedness of the model, a priori estimates and we obtain the convergence towards the solution of the parabolic problem. Moreover we prove a similar result for a weakly degenerate problem.

Key-words: Parabolic nonlinear conservation laws – degenerate parabolic equations – singular perturbation problems – BGK models

1 Introduction

In this paper we deal with a hyperbolic relaxation approximation for the following parabolic equation

$$\partial_t u + \partial_x A(u) = \partial_{xx}[B(u)], \qquad (x,t) \in \mathbb{R}^+ \times]0,\infty[, \tag{1.1}$$

with initial data

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^+ \ , \tag{1.2}$$

and boundary condition

$$u(0,t) = a_0(t) \qquad t \in (0,T) \ , \tag{1.3}$$

where A, B are locally lipschitz functions, u_0 and a_0 are BV functions and

$$B'(u) \geq \nu > 0 \ . \tag{1.4}$$

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A solution of problem (1.1)-(1.3) is a function $u \in C([0,T]; L^1_{loc}(\mathbb{R}^+)) \cap L^\infty([0,T]; BV(\mathbb{R}^+))$ satisfying the equation in the weak sense and assuming the data in the sense of the traces.

We are interested in the approximation of the above problem by means of a sequence of semilinear systems of conservation laws with source and initial-boundary conditions.

To this aim we introduce the following BGK model

$$\begin{cases} \partial_t f_1^\epsilon + \frac{\theta_1}{\sqrt{\epsilon}} \partial_x f_1^\epsilon = \frac{1}{\epsilon}(M_1(u^\epsilon, \epsilon)) - f_1^\epsilon), \\ \\ \partial_t f_2^\epsilon = \frac{1}{\epsilon}(M_2(u^\epsilon, \epsilon)) - f_2^\epsilon), \\ \\ \partial_t f_3^\epsilon + \frac{\theta_3}{\sqrt{\epsilon}} \partial_x f_3^\epsilon = \frac{1}{\epsilon}(M_3(u^\epsilon, \epsilon)) - f_3^\epsilon) \ , \end{cases} \tag{1.5}$$

with $\theta_1 < 0$, $\theta_3 > 0$ and

$$u^\epsilon(x,t) = \sum_{i=1}^{3} f_i^\epsilon(x,t) \ ; \tag{1.6}$$

the functions M_i (called maxwellian functions) are given by the following expressions

$$\begin{aligned} M_1(u,\epsilon) &= \frac{\sqrt{\epsilon}}{\theta_1 - \theta_3} A(u) + \frac{B(u)}{(\theta_1-\theta_3)\theta_1} \ , \\ \\ M_2(u,\epsilon) &= u + \frac{B(u)}{\theta_1\theta_3} \\ \\ M_3(u,\epsilon) &= -\frac{\sqrt{\epsilon}}{\theta_1 - \theta_3} A(u) - \frac{B(u)}{(\theta_1-\theta_3)\theta_3} \ . \end{aligned} \tag{1.7}$$

We complement the system by the initial conditions

$$f_i^\epsilon(x,0) = M_i(u_0(x),0) \ , \qquad i = 1,2,3 \tag{1.8}$$

and the boundary condition, only for the function satisfying the unique equation in (1.5) with positive transport velocity,

$$f_3^\epsilon(0,t) = M_3(a_0(t),0) \ . \tag{1.9}$$

For uniformity of notations in the following we set $\theta_2 = 0$.

We have the following definition for the weak solution to the relaxation system (1.5).

Definition 1.1 *Let $u_0 \in BV(\mathbb{R}^+)$, $a_0 \in BV((0,T))$. The tern of functions $(f_1^\epsilon, f_2^\epsilon, f_3^\epsilon) \in (BV(\mathbb{R}^+ \times (0,T)) \cap L^\infty(\mathbb{R}^+ \times (0,T)))^3$ is a weak solution to problem (1.5) - (1.9) if* *i) for every* $\phi \in C^1_0(\mathbb{R}^+ \times (0,T))$

$$\int_0^T \int_{\mathbb{R}+} f_i^\epsilon \left(\phi_t + \frac{\theta_i}{\sqrt{\epsilon}} \phi_x \right) + \frac{1}{\epsilon} \left(M_i(u^\epsilon, \epsilon) - f_i^\epsilon \right) \, dxdt = 0 \ , \qquad i = 1,2,3 \ ;$$

ii) $f_i^\epsilon(x,0) = M_i(u_0(x),0)$ *for almost every* $x \in \mathbb{R}^+, i = 1,2,3$;

iii) $f_3^\epsilon(0,t) = M_3(a_0(t),0)$ *for almost every* $t \in (0,T)$.

In this paper we prove comparison and stability results with respect to the data for the weak solutions of problem (1.5)-(1.9), under a further condition on the coefficients θ_1, θ_3 (see (1.10)).

Then we prove the main result of the paper: for the relaxation parameter ϵ going to zero the sequence $\{u^\epsilon\}$ converges strongly in $C([0,T]; L^1_{loc}(\mathbb{R}^+))$ to the unique solution of the parabolic problem (1.1)-(1.4).

The last aim of the paper is to extend the convergence result to a weakly degenerate parabolic initial-boundary value problem. We consider nonnegative solutions of equation (1.1), where we assume that the function $B'(u)$ can be zero when $u = 0$, and we derive a BGK approximation in the case of no degeneracy on the boundary.

It is well known, no matter how smooth A, B and the data are, that in presence of the above degeneration, the equation (1.1) needs not have classical solutions [9, 22]. Generalizations of the notion of solution and existence and uniqueness results can be found in [22, 7, 8, 3, 5] under different assumptions on the functions A, B and on the data.

Here we will prove that the sequence of solutions of the BGK model proposed for the weakly degenerate problem converge strongly to its unique weak solution.

The paper takes ispiration from the results in [4] and [19]. In the first paper it is shown that the same BGK model with a general number of equations approximates a parabolic, eventually degenerate, initial value problem in $\mathbb{R}^d \times [0,T]$, $T > 0$; in the second one the authors prove the convergence of a similar relaxation system, with transport coefficients independent from ϵ, towards an initial-boundary value problem for a conservation law.

We recall that this kind of discrete velocities approximations was introduced in [18] for conservation laws in $\mathbb{R}^d \times [0,T]$. For diffusive limit of Cauchy problems for discrete velocities hyperbolic systems we refer to [12, 16, 15, 14] and to [10, 17, 6] for numerical results. Finally, relaxation approximations for boundary value problems are studied in [21, 20, 25, 24] (besides [19]) in the setting of hyperbolic equations.

Some properties of the Maxwellian functions (1.7) will be crucial in all the proofs of the paper.

It is easily seen that they verify the following compatibility conditions

(M_1) $\sum_{i=1}^3 M_i(w,\epsilon) = w$ for all $\epsilon \in]0,1]$ and for all $w \in I$;

(M_2) $\sum_{i=1,3} \frac{\theta_i}{\sqrt{\epsilon}} M_i(w,\epsilon) = A(w)$ for all $\epsilon \in]0,1]$ and for all $w \in I$;

(M_3) $\sum_{i=1,3} \theta_i^2 M_i(w,0) = B(w)$ for all $w \in I$,

which are the link between (1.1) and (1.5) and ensure the consistence of the approximation. In fact it is easy to see, formally, that if the sequence $\{u^\epsilon\}$ converges to some limit function u strongly in $C([0,T]; L^1_{loc}(\mathbb{R}^+))$, the function u is a weak solution to equation (1.1) [4].

Moreover we have to impose the following stability condition on the transport velocities of the system

$$\max\{B'(u) : u \in I\} < -\theta_1\theta_3 \qquad \text{for } u \in I\ ; \tag{1.10}$$

This ensures that the maxwellian functions (1.7) are monotone increasing on the interval I for small ϵ. This property is crucial in proving the comparison result for the solutions of the system (1.5) and compactness properties for the sequence $\{f_i^\epsilon\}$. In fact we will show that, under the stability condition (1.10), our approximation (1.5)-(1.9) keeps the monotonicity properties which are tipical of equation (1.1).

We remark that from a numerical point of view our approximation has some good features; in fact it is simple to implement numerical schemes by using high order algorithms, also in more complicate situations (source terms, variable coefficients).

The paper is organized in three further sections: Section 2 is devoted to the comparison and stability results and compactness estimates for the solutions of problem (1.5)-(1.10) and Section 3 to the statement and the proof of our convergence result in the parabolic case. We deal with the weakly degenerate case in the last section where we introduce a relaxation system like (1.5), with slightly modified Maxwellian functions in order to keep the monotonicity properties also in the degeneration regions (under a new stability condition which involves the values of the function A, too); then we prove that solutions of the weakly degenerate problem can be approximated, in $C([0,T]; L^1_{loc}(\mathbb{R}^+))$ by solutions of the system.

2 A priori estimates

The goal of this section is to establish stability, comparison, existence and uniqueness results and some a priori estimates for the solutions of the system (1.5). Some of the proofs follow from ideas in [19].

We will make use of the following lemma [11].

Lemma 2.1 *Let u be a weak solution to the linear Cauchy problem*

$$\begin{cases} \partial_t u + \theta \partial_x u = g(x,t) \ , \\ u(x,0) = u_0(x) \ , \end{cases}$$

for $x \in \mathbb{R}^+$ and $t \in (0,T)$ ($T > 0$), and v the solution of the above problem with a function h in place of g and the initial data v_0 in place of u_0. Then for every $\phi \in C_0^1(\mathbb{R}^+ \times (0,T))$, with $\phi \geq 0$, there holds

$$\int_0^T \int_{\mathbb{R}^+} [u-v]_+ \partial_t \phi + \theta [u-v]_+ \partial_x \phi \ dx \ dt \geq - \int_0^T \int_{\mathbb{R}^+} H(u-v)(g-h) \phi \ dx \ dt \ , \quad (2.1)$$

where H is the Heaviside function.

In the following $f_i^\epsilon(0,t), u^\epsilon(0,t)$ denote the traces of the functions $f_i^\epsilon(x,t), u^\epsilon(x,t)$ on the boundary $x = 0$.

Set $\overline{\epsilon}$ a suitable value such that the maxwellian functions are increasing in u for $\epsilon \in (0, \overline{\epsilon})$. In the following we assume $\epsilon \in (0, \overline{\epsilon})$.

Proposition 2.1 *Assume that $u_0, \overline{u_0} \in BV(\mathbb{R}^+)$, $a_0, \overline{a_0} \in BV((0,T))$ and that condition (1.10) is satisfied on a given interval I. Let $(f_1^\epsilon, f_2^\epsilon, f_3^\epsilon)$ be a solution of problem (1.5)-(1.9) and $u^\epsilon \in I$ for almost every $(x,t) \in \mathbb{R}^+ \times (0,T)$. Let $(\overline{f}_1^\epsilon, \overline{f}_2^\epsilon, \overline{f}_3^\epsilon)$ be another solution corresponding to the initial-boundary conditions $(\overline{u_0}, \overline{a_0})$ and $\overline{u^\epsilon} = \sum_{i=1}^3 \overline{f_i^\epsilon} \in I$ for almost every $(x,t) \in \mathbb{R}^+ \times (0,T)$. Then for every $K \in \mathbb{R}^+$ and for almost every $t \in (0,T)$ the following inequalities hold*

$$\int_0^K \sum_{i=1}^3 [f_i^\epsilon(x,t) - \overline{f_i^\epsilon}(x,t)]_+ dx \leq \int_0^{K + \frac{|\theta_1|}{\sqrt{\epsilon}} t} [u_0(x) - \overline{u_0}(x)]_+ dx + \frac{\theta_3}{\sqrt{\epsilon}} \int_0^t [a_0(s) - \overline{a_0}(s)]_+ ds \ , \quad (2.2)$$

$$\frac{|\theta_1|}{\sqrt{\epsilon}} \int_0^t [f_1^\epsilon(0,s) - \overline{f_1^\epsilon}(0,s)]_+ ds \leq \int_0^{\frac{|\theta_1|}{\sqrt{\epsilon}} t} [u_0(x) - \overline{u_0}(x)]_+ dx + \frac{\theta_3}{\sqrt{\epsilon}} \int_0^t [a_0(s) - \overline{a_0}(s)]_+ ds \ . \quad (2.3)$$

Proof. Setting $w_i = f_i^\epsilon - \overline{f_i^\epsilon}$ and using inequality (2.1) we obtain

$$\int_0^T \int_{\mathbb{R}^+} \sum_{i=1}^3 [w_i]_+ \left(\phi_t + \frac{\theta_i}{\sqrt{\epsilon}} \phi_x \right) dx dt$$

$$\geq -\frac{1}{\epsilon} \int_0^T \int_{\mathbb{R}^+} \sum_{i=1}^3 H(w_i) (M_i(u^\epsilon, \epsilon) - M_i(\overline{u^\epsilon}, \epsilon) - w_i) \ \phi \ dx dt \ ,$$

for every $\phi \in C_0^1(\mathbb{R}^+ \times (0,T))$, $\phi \geq 0$. If $H(w_i) = 1$ for $i = 1,2,3$ or $H(w_i) = 0$ for $i = 1,2,3$ then the right hand side of the above inequality is zero. In order to prove that it is always nonnegative set $J_+ = \{i : H(w_i) = 1\}$; then, thanks to (M_1) and (1.10)

$$\sum_{J_+} (M_i(u^\epsilon,\epsilon) - M_i(\overline{u^\epsilon},\epsilon) - w_i) \leq \left(\sum_{J_+} M_j'(\eta,\epsilon) - 1\right) \sum_{J_+} w_j \leq 0 ,$$

where η is a suitable intermediate value.

Hence

$$\int_0^T \int_{\mathbb{R}^+} \sum_{i=1}^3 [f_i^\epsilon - \overline{f_i^\epsilon}]_+ \left(\phi_t + \frac{\theta_i}{\sqrt{\epsilon}}\phi_x\right) dxdt \geq 0 .$$

Now, choosing a sequence of test functions approximating the characteristic function of the set $\{(x,s) \in \mathbb{R}^+ \times (0,t) : 0 \leq x \leq K + \frac{|\theta_1|}{\sqrt{\epsilon}}(t-s)\}$ we obtain

$$\int_0^K \sum_{i=1}^3 [f_i^\epsilon(x,t) - \overline{f_i^\epsilon}(x,t)]_+ dx \leq$$

$$\int_0^{K+\frac{|\theta_1|}{\sqrt{\epsilon}}t} [M_i(u_0,\epsilon) - M_i(\overline{u_0},\epsilon)]_+ dx + \int_0^t \sum_{i=1}^3 \frac{\theta_i}{\sqrt{\epsilon}} [f_i^\epsilon(0,s) - \overline{f_i^\epsilon}(0,s)]_+ ds$$

$$+ \int_0^t \sum_{i=1}^3 \left(-|\frac{\theta_1}{\sqrt{\epsilon}}| - \frac{\theta_i}{\sqrt{\epsilon}}\right) [f_i^\epsilon(K + \frac{|\theta_1|}{\sqrt{\epsilon}}(t-s),s) - \overline{f_i^\epsilon}(K + \frac{|\theta_1|}{\sqrt{\epsilon}}(t-s),s)]_+ ds$$

and (2.2) follows since $\theta_1 < 0$ and $\theta_3 > 0$.

In order to prove inequality (2.3) we introduce a sequence of test functions approximating the characteristic function of the set $\{(x,s) \in \mathbb{R}^+ \times (0,t) : 0 \leq x \leq \frac{|\theta_1|}{\sqrt{\epsilon}}(t-s)\}$ and, with the same technique used in the first part of the proof, we obtain the claim. □

As a consequence of the previous proposition we have the following two results.

Corollary 2.1 *Assume that* $u_0, \overline{u_0} \in BV(\mathbb{R}^+)$, $a_0, \overline{a_0} \in BV((0,T))$ *and that condition (1.10) is satisfied on a given interval I. Let* $(f_1^\epsilon, f_2^\epsilon, f_3^\epsilon)$ *be a solution of problem (1.5)-(1.9) and* $u^\epsilon \in I$ *for almost every* $(x,t) \in \mathbb{R}^+ \times (0,T)$*. Let* $(\overline{f}_1^\epsilon, \overline{f}_2^\epsilon, \overline{f}_3^\epsilon)$ *be another solution corresponding to the initial-boundary conditions* $(\overline{u_0}, \overline{a_0})$ *and* $\overline{u^\epsilon} = \sum_{i=1}^3 \overline{f_i^\epsilon} \in I$ *for almost every* $(x,t) \in \mathbb{R}^+ \times (0,T)$*. Assume that*

$$u_0 \leq \overline{u}_0 \qquad \textit{almost everywhere in } \mathbb{R}^+ ;$$

$$a_0 \leq \overline{a}_0 \qquad \textit{almost everywhere in } (0,T) \ .$$

Then

$$f_i^\epsilon \leq \overline{f}_i^\epsilon \qquad \textit{almost everywhere in } \mathbb{R}^+ \times (0,T) \ \ \textit{for } i = 1,2,3.$$

Corollary 2.2 *Assume that condition (1.10) is satisfied on a given interval I, that $u_0 \in BV(\mathbb{R}^+)$, $a_0 \in BV((0,T))$ and $u_0, a_0 \in I$. Let $(f_1^\epsilon, f_2^\epsilon, f_3^\epsilon)$ be a solution of problem (1.5)-(1.9) and $u^\epsilon \in I$ for almost every $(x,t) \in \mathbb{R}^+ \times (0,T)$. Then for almost every $(x,t) \in \mathbb{R}^+ \times (0,T)$*

$$M_i(\min\{ess\inf(u_0), ess\inf(a_0)\}) \leq f_i^\epsilon \leq M_i(\max\{ess\sup(u_0), ess\sup(a_0)\})$$

for $i = 1,2,3$.

Proof. Observe that for every $p \in \mathbb{R}$ the vector $(M_1(p,\epsilon), M_2(p,\epsilon), M_3(p,\epsilon))$ is a weak solution to system (1.5)-(1.7) with initial and boundary data identically equal to p. Then the claim follows by Corollary 2.1.□

In the following we set $I := \{u \in \mathbb{R}; \min\{ess\inf(u_0), ess\inf(a_0)\} \leq u \leq \max\{ess\sup(u_0), ess\sup(a_0)\}$.

By using standard regularity properties, the continuation principle, Corollary 2.2 and proceeding as in [18] we obtain the following result for the solutions of problem (1.5)-(1.9) (see [23]).

Theorem 2.1 *Let $u_0 \in BV(\mathbb{R}^+), a_0 \in BV((0,T))$ and assume that condition (1.10) is satisfied on the interval I. Then there exists a unique global weak solution $(f_1^\epsilon, f_2^\epsilon, f_3^\epsilon)$ to problem (1.5)-(1.9) and $u^\epsilon \in I$. Moreover, when the data u_0, a_0 are in the class C^k with $u_0(0) = a_0(0)$, then $f_i^\epsilon \in Lip(\mathbb{R}^+ \times [0,T)) \cup C^k(\mathbb{R}^+ \times (0,T) \setminus \gamma_3)$, $i = 1,2,3$, where $\gamma_3 := \{(x,t) \in \mathbb{R}^+ \times [0,T) : x = \frac{\theta_3}{\sqrt{\epsilon}} t\}$*

Now we prove some estimates for the solutions of problem (1.5) - (1.10) in the case of smooth data, which can be extended to the general BV case thanks to Proposition 2.1.

Proposition 2.2 *Let $u_0 \in C^2(\overline{\mathbb{R}^+}) \cap BV(\mathbb{R}^+)$ and $a_0 \in C^2([0,T)) \cap BV((0,T))$ with $u_0(0) = a_0(0)$. Let $(f_1^\epsilon, f_2^\epsilon, f_3^\epsilon)$ be the solution of problem (1.5) - (1.10). Then for every $t \in [0,T)$ the following estimate holds for $i = 1,2,3$*

$$\int_{\mathbb{R}^+} |\partial_t f_i^\epsilon(x,t)| dx \leq \frac{max\{|\theta_1|, \theta_3\} TV(u_0) + \theta_3 TV(a_0)}{\sqrt{\epsilon}} \ . \tag{2.4}$$

Proof. We fix $L, t \in \mathbb{R}^+$ and we set $p_i = \partial_t f_i^\epsilon$. Now we first differentiate the three equations with respect to t and multipy each equation by the corresponding $sgn(p_i)$, then taking the sum for $i = 1, 2, 3$ and integrating on the domain $D = \{(x, s) \in \mathbb{R}^+ \times (0, t) : 0 \leq x \leq L + \frac{|\theta_1|}{\sqrt{\epsilon}}(t - s)\}$, we have

$$\int\int_D \partial_t \left(\sum_{i=1}^{3} |p_i(x,t)| \right) + \partial_x \left(\sum_{i=1}^{3} \frac{\theta_i}{\sqrt{\epsilon}} |p_i(x,t)| \right) dxdt = \frac{1}{\epsilon} \int\int_D F(x,t) dxdt$$

where

$$F(x,t) := \sum_{i=1}^{3} \left(\sum_{j=1}^{3} M_j'(u^\epsilon(x,t)) sgn(p_j) - sgn(p_i) \right) p_i(x,t) \ .$$

Since $F(x,t) \leq 0$ (thanks to condition (M_1) and (1.10)), using the divergence theorem on the domain D we obtain

$$\int_0^L \sum_{i=1}^{3} |p_i(x,s)| dx \leq \int_0^{L + \frac{|\theta_1|}{\sqrt{\epsilon}}} \sum_{i=1}^{3} |p_i(x,0)| dx + \int_0^t \sum_{i=1}^{3} \frac{\theta_i}{\sqrt{\epsilon}} |p_i(0,s)| ds \ ;$$

then, using the initial and the boundary conditions, we have the claim.□

The estimate proved in the above proposition is not uniform in ϵ, however it allows to obtain the following uniform estimate for the BV norm of the traces of f_1^ϵ and f_2^ϵ on the boundary. We remark that the assumption of parabolicity for equation (1.1) is crucial in the proof of inequality (2.6) in the next proposition.

Proposition 2.3 *Let $u_0 \in C^2(\overline{\mathbb{R}^+}) \cap BV(\mathbb{R}^+)$ and $a_0 \in C^2([0,T]) \cap BV((0,T))$ with $u_0(0) = a_0(0)$. Let $(f_1^\epsilon, f_2^\epsilon, f_3^\epsilon)$ be the solution of problem (1.5) - (1.10). Then there exists a constant $K_1 = K_1(\nu, \theta_1, \theta_3, TV(a_0), TV(u_0))$, not depending on ϵ, such that for every $h > 0$ small enough*

$$\int_0^{T-h} |f_1^\epsilon(0, t+h) - f_1^\epsilon(0,t)| dt \leq K_1 h \ , \tag{2.5}$$

$$\int_0^{T-h} |f_2^\epsilon(0, t+h) - f_2^\epsilon(0,t)| dt \leq \epsilon |f_2^\epsilon(0,h) - f_2^\epsilon(0,0)| + K_1 h \ . \tag{2.6}$$

Proof. For $i = 1, 2, 3$ we set $w_i(x,t) = f_i^\epsilon(x, t+h) - f_i^\epsilon(x,t)$ and

$$C_i(x,t) := \frac{M_i(u^\epsilon(x,t+h)) - M_i(u^\epsilon(x,t))}{u^\epsilon(x,t+h) - u^\epsilon(x,t)} \ .$$

We consider now the equations verified by the functions w_i

$$\partial_t w_i(x,t)+\frac{\theta_i}{\sqrt{\epsilon}}\partial_x w_i(x,t) = \frac{1}{\epsilon}\left((C_i(x,t)-1)w_i(x,t) + C_i(x,t)\sum_{j\neq i} w_j(x,t)\right) \ , \qquad i=1,3$$

$$\partial_t w_2(x,t) = \frac{1}{\epsilon}\left((C_2(x,t)-1)w_2(x,t) + C_2(x,t)\sum_{j\neq 2} w_j(x,t)\right) \ ,$$

and we multiply each of them by the correspondig $sgn(w_i)$; summing up the equations and integrating on the domain $D := \{(x,t) \in \mathbb{R}^+ \times (0,T) : 0 < x < \frac{|\theta_1|}{\sqrt{\epsilon}}(T-t)\}$ we obtain the equality

$$\int\int_D \partial_t \left(\sum_{i=1}^3 |w_i|\right) + \partial_x \left(\sum_{i=1}^3 \frac{\theta_i}{\sqrt{\epsilon}}|w_i|\right) dxdt = \frac{1}{\epsilon}\int\int_D F dxdt$$

where

$$F(x,t) := \sum_{i=1}^3 \left(\sum_{j=1}^3 C_j(x,t)sgn(w_j) - sgn(w_i)\right) w_i(x,t) \ .$$

We observe that the conditions (M_1) and (1.10) imply that $0 \le C_i(x,t) \le 1$ for $i = 1,2,3$ and $\sum_{i=1}^3 C_i(x,t) = 1$ and then that $F(x,t) \le 0$; therefore, using the divergence theorem on the domain D we obtain

$$\int_0^t |f_1^\epsilon(0,t+h) - f_1^\epsilon(0,t)|dt \le$$

$$\frac{\sqrt{\epsilon}}{|\theta_1|}\int_0^{\frac{|\theta_1|}{\sqrt{\epsilon}}t} \sum_{i=1}^3 |f_i^\epsilon(x,h) - f_i^\epsilon(x,0)|dx + \int_0^t \frac{\theta_3}{|\theta_1|}|M_3(a_0(t+h)) - M_3(a_0(t))|dt \ ,$$

and the first claim follows using (2.4).

In order to prove the estimate for f_2^ϵ we consider the equation verified by w_2; then we have

$$w_2(0,t) = e^{\frac{-\int_0^t (1-M_2'(\eta_h(\tau),\epsilon))d\tau}{\epsilon}} w_2(0,0)$$

$$+\frac{1}{\epsilon}\int_0^t e^{\frac{-\int_s^t (1-M_2'(\eta_h(\tau),\epsilon))d\tau}{\epsilon}} M_2'(\eta_h(t),\epsilon)(w_1(0,s)+w_3(0,s))ds \ ,$$

where $\eta_h(t)$ is a suitable intermediate value; then it follows that

$$\int_0^T |w_2(0,t)|dt \le |w_2(0,0)| \int_0^T e^{\frac{-\int_0^t (1-M_2'(\eta_h(\tau),\epsilon))d\tau}{\epsilon}} \ dt$$

$$+\frac{1}{\epsilon}\int_0^T \left((|w_1(0,s)| + |w_3(0,s)|) \int_s^T e^{\frac{-\int_s^t (1-M_2'(\eta_h(\tau),\epsilon))d\tau}{\epsilon}} M_2'(\eta_h(t),\epsilon) \ dt\right) ds$$

and in view of the expression of M_2 we deduce that

$$\int_0^T |w_2(0,t)|dt \le \frac{\epsilon}{\nu}\left(1-e^{-\frac{\nu T}{\epsilon}}\right)|w_2(0,0)| + \int_0^T \frac{1-e^{-\frac{\nu(T-s)}{\epsilon}}}{\nu}(|w_1(0,s)|+|w_3(0,s)|)ds$$

and thanks to estimate (2.5) we can conclude the proof.□

In the following proposition we shall prove a further uniform estimate holding for the sequence $\{f_i^\epsilon\}$ which will be useful to derive compactness properties.

Proposition 2.4 *Let $u_0 \in C^2(\overline{\mathbb{R}^+}) \cap BV(\mathbb{R}^+)$ and $a_0 \in C^2([0,T)) \cap BV((0,T))$ with $u_0(0) = a_0(0)$. Let $(f_1^\epsilon, f_2^\epsilon, f_3^\epsilon)$ be the solution of problem (1.5) - (1.10). Then there exists a constant $K_2 = K_2(\nu, \theta_1, \theta_3, TV(a_0), TV(u_0))$, not depending on ϵ, such that for every $L > 0$ and for every $h > 0$ small enough*

$$\int_0^L |\partial_x f_i^\epsilon(x,t)|dx \le K_2 \ , \qquad \textit{for } i = 1,2,3 \ . \tag{2.7}$$

Proof. We fix $L, t > 0$ and we set $m_i = \partial_x f_i^\epsilon$; applying the technique of the previous proofs we establish that

$$\int_0^L \sum_{i=1}^3 |m_i(x,t)|dx \le \int_0^{L+\frac{|\theta_1|}{\sqrt{\epsilon}}} \sum_{i=1}^3 |m_i(x,0)|dx + \int_0^t \sum_{i=1}^3 \frac{\theta_i}{\sqrt{\epsilon}}|m_i(0,s)|ds \ .$$

The last integral can be estimate using (2.5) and (2.6) since

$$\frac{\theta_1}{\sqrt{\epsilon}}|m_1(0,s)| + \frac{\theta_3}{\sqrt{\epsilon}}|m_3(0,s)| \le \left|\frac{\theta_1}{\sqrt{\epsilon}}m_1(0,s) + \frac{\theta_3}{\sqrt{\epsilon}}m_3(0,s)\right| = |u_t^\epsilon(0,s)|$$

and the proof is complete.□

Now we state a proposition which gives an estimate of the deviation from the equilibrium in the L^1 norm (see [4] for the proof). The result will be crucial in the proof of the last uniform estimate in this section and in the proof of consistency of our relaxation approximation in the next section.

Proposition 2.5 *Let $u_0 \in C^2(\overline{\mathbb{R}^+}) \cap BV(\mathbb{R}^+)$ and $a_0 \in C^2([0,T)) \cap BV((0,T))$ with $u_0(0) = a_0(0)$. Let $(f_1^\epsilon, f_2^\epsilon, f_3^\epsilon)$ be the solution of problem (1.5) - (1.10). Then there exists a constant $K_3 = K_3(\nu, \theta_1, \theta_3, TV(a_0), TV(u_0))$, not depending on ϵ, such that for every $L > 0$*

$$\sum_{i=1}^3 \int_0^L |f_i^\epsilon(x,t) - M_i(u^\epsilon, \epsilon)|dx \le \sqrt{\epsilon} K_3 \ .$$

The following proposition completes the set of estimates necessary to prove the convergence result in the next section.

Proposition 2.6 *Let $u_0 \in C^2(\overline{\mathbb{R}^+}) \cap BV(\mathbb{R}^+)$ and $a_0 \in C^2([0,T)) \cap BV((0,T))$ with $u_0(0) = a_0(0)$. Let $(f_1^\epsilon, f_2^\epsilon, f_3^\epsilon)$ be the solution of problem (1.5) - (1.10). Then for every $L > 0$ there exist a positive constant h_0 and a continuous nondecreasing function $w : [0, h_0] \to \mathbb{R}$, not depending on ϵ, with $w(0) = 0$, such that, for $t \in (0, T-h)$ and $h \in (0, h_0)$*

$$\int_0^L |u^\epsilon(x, t+h) - u^\epsilon(x,t)| dx \leq w(h) \ .$$

Proof. We use Lemma 5 in [11]; in view of Proposition 2.4 we must only prove that there exists a constant C_L such that

$$\left| \int_0^L \partial_t u^\epsilon(x,t) \varphi(x) dx \right| \leq C_L \|\varphi\|_{C^1} \qquad \text{for every function } \varphi \in C_0^1((0,L)) \ .$$

Using the equations for f_i^ϵ and the condition (M_2) we establish that

$$\left| \int_0^t \partial_t u^\epsilon(x,t) \varphi(x) dx \right| \leq \frac{\|\varphi\|_{C^1}}{\sqrt{\epsilon}} \int_0^L \max\{|\theta_i|\} \sum_{i=1}^3 |f_i^\epsilon(x,t) \ - M_i(u^\epsilon, \epsilon)| dx$$

$$+ \|\varphi\|_{C^1} \int_0^L |A(u^\epsilon)| dx$$

and using Proposition 2.5 we conclude the proof.□

Now the above propositions in combination with the comparison results (2.2)-(2.3) imply the following theorem.

Theorem 2.2 *Let $u_0 \in BV(\mathbb{R}^+)$ and $a_0 \in BV((0,T))$. Let $(f_1^\epsilon, f_2^\epsilon, f_3^\epsilon)$ be the solution of problem (1.5) - (1.10) in $(\mathbb{R}^+ \times (0,T))$. Then for every $L, h > 0$ there exist two positive constants $h_0, C_1 = C_1(\nu, \theta_1, \theta_3, TV(a_0), TV(u_0))$ and a continuous nondecreasing function $w : [0, h_0] \to \mathbb{R}$, not depending on ϵ, with $w(0) = 0$, such that*

$$TV(f_i^\epsilon(\cdot, t), (0, L)) \leq C_1 \qquad \textit{for } t \in (0, T] \ ;$$

$$\int_0^L |f_i^\epsilon(x,t) - M_i(u^\epsilon(x,t), \epsilon)| dx \leq C_1 \sqrt{\epsilon} \qquad \textit{for } t \in (0,T] \ ;$$

$$\int_0^{T-h} |f_1^\epsilon(0, t+h) - f_1^\epsilon(0,t)| dt \leq C_1 h \ ;$$

$$\int_0^L |u^\epsilon(x, t+h) - u^\epsilon(x,t)| dx \leq w(h) \qquad \textit{for } t \in (0, T-h) \ , h \in (0, h_0) \ .$$

3 Convergence theorem

Here we prove the convergence result of the sequence $\{u^\epsilon\}$ towards the unique solution of the problem (1.1)-(1.4).

Theorem 3.1 *Let $u_0 \in BV(\mathbb{R}^+)$ and $a_0 \in BV((0,T))$. Let $(f_1^\epsilon, f_2^\epsilon, f_3^\epsilon)$ be the solution of problem (1.5) - (1.10) in $(\mathbb{R}^+ \times (0,T])$, $u^\epsilon = \sum_{i=1}^3 f_i^\epsilon$ and let u be the solution of the problem (1.1) -(1.4) in $(\mathbb{R}^+ \times (0,T])$. Then*

$$\lim_{\epsilon \to 0} u^\epsilon = u \qquad \text{in } C([0,T]; L^1_{loc}(\mathbb{R}^+)) \ .$$

Proof. Thanks to results in the previous section, for every $L > 0$ we can establish that there exists a subsequence $\{u^{\epsilon_n}\}$ converging to some function $\overline{u}$ in $C([0,T]; L^1((0,L)))$; moreover $\{M_i(u^{\epsilon_n}, \epsilon_n)\}$ converge to $M_i(\overline{u}, 0)$ in the same norm.

Since $\|M_i(u^\epsilon(\cdot,t)) - f_i^\epsilon(\cdot,t)\|_{L^1((0,L))}$ goes to zero as ϵ goes to zero, uniformly in $t \in [0,T]$ (see Theorem 2.2), we can conclude that also the subsequences $\{f_i^{\epsilon_n}\}$ converge to some limit functions $\overline{f}_i = M_i(\overline{u}, 0)$ in $C([0,T]; L^1((0,L)))$.

The next step in the proof is to verify that $\overline{u}$ is the solution of problem (1.1)- (1.3). For $i = 1,2,3$, for every $\varphi \in C_0^{2,1}(\mathbb{R}^+ \times (0,T))$ the functions f_i^ϵ satisfy

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^+} \left(f_i^\epsilon(x,t)\varphi_t(x,t) + \frac{\theta_i}{\sqrt{\epsilon}} f_i^\epsilon(x,t)\varphi_x(x,t) \right) dxdt = \\ &\qquad -\frac{1}{\epsilon} \int_0^T \int_{\mathbb{R}^+} (M_i(u^\epsilon(x,t) - f_i^\epsilon(x,t))\varphi(x,t) dxdt \ . \end{aligned} \tag{3.1}$$

Now we consider $\psi \in C_0^{2,1}(\mathbb{R}^+ \times (0,T))$ and we set $\varphi = \psi_x$; multiplying each of the above equalities by the corresponding $\sqrt{\epsilon}\theta_i$, and summing up for $i = 1,2,3$ we obtain

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^+} \sum_{i=1}^3 \frac{\theta_i}{\sqrt{\epsilon}} f_i^\epsilon(x,t)\psi_x(x,t) dxdt \\ &= \int_0^T \int_{\mathbb{R}^+} \left(A(u^\epsilon(x,t))\psi_x(x,t) + \sum_{i=1}^3 \theta_i^2 M_i(u^\epsilon(x,t),\epsilon)\psi_{xx} \right) dxdt \\ &+\sqrt{\epsilon} \int_0^T \int_{\mathbb{R}^+} \sum_{i=1}^3 \theta_i f_i^\epsilon(x,t)\psi_{xt}(x,t) dxdt \\ &+ \int_0^T \int_{\mathbb{R}^+} \sum_{i=1}^3 \theta_i^2 (f_i^\epsilon(x,t) - M_i(u^\epsilon(x,t),\epsilon))\psi_x(x,t) dxdt \ . \end{aligned}$$

Summing the equalities (3.1) for the subsequences $\{f^{\epsilon_n}\}$ and using the above equality, letting ϵ_n go to zero we have

$$\int_0^T \int_{\mathbb{R}^+} (\overline{u}(x,t)\psi_t(x,t) + A(\overline{u}(x,t)\psi_x(x,t) + B(\overline{u}(x,t))\psi_{xx}(x,t))\, dxdt = 0$$

for every $\psi \in C_0^{2,1}(\mathbb{R}^+ \times (0,T))$.

Therefore $\overline{u}$ is a weak solution to equation (1.1) verifying the initial condition (1.2). In view of uniqueness of weak solution of problem (1.1)-(1.4) [13], it remains only to prove that $\overline{u}$ satisfies the boundary condition (1.3), then the whole sequence $\{u^\epsilon\}$ will converge.

We recall that thanks to Theorem 2.2 there exists a subsequence of $\{f_1^{\epsilon_n}(0,t)\}$ converging to some function $w_1(t)$ in $L^1((0,T))$; moreover we set $w_3(t) = M_3(a_0(t),0)$.

For every $\varphi \in C_0^{2,1}([0,+\infty) \times (0,T])$, proceeding as before, we can establish that

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^+} (\overline{u}(x,t)\varphi_t(x,t) + A(\overline{u}(x,t))\varphi_x(x,t) + B(\overline{u}(x,t))\varphi_{xx}(x,t))\, dxdt \\ &- \int_0^T \sum_{i=1,3} \theta_i^2 w_i(t)\varphi_x(0,t)dt = \lim_{\epsilon_n \to 0} \int_0^T \sum_{i=1}^3 \frac{\theta_i}{\sqrt{\epsilon}} f_i^\epsilon(0,t)\varphi(x,t)dt \ , \end{aligned} \tag{3.2}$$

where $\overline{u}(0,t)$ denotes the trace on the boundary of the function $u(x,t)$.

As a consequence of the above equality we obtain that

$$\sum_{i=1,3} \theta_i w_i(t) = 0 \ . \tag{3.3}$$

In (3.2) we can choose $\varphi(x,t) = \int_0^x \xi(t) max\{0, 1 - \frac{y}{\eta}\}dy, \eta \in \mathbb{R}^+, \xi \in C_0^1(0,T)$ and letting η go to zero we obtain the further condition on the boundary values of the function $\overline{u}$

$$B(\overline{u}(0,t)) = \sum_{i=1,3} \theta_i^2 w_i(t) \ . \tag{3.4}$$

Finally, in view of the expression of the Maxwellian functions we also know that

$$\sum_{i=1,3} \theta_i M_i(\overline{u}(0,t), 0) = 0 \ . \tag{3.5}$$

The combination of the three conditions (3.3)-(3.5) implies that

$$M_3(\overline{u}(0,t),0)(\theta_3^2 - \theta_3\theta_1) = M_3(a_0(t),0)(\theta_3^2 - \theta_3\theta_1) \ ;$$

since $\theta_1 \neq \theta_3$ and $M_3(\cdot, 0)$ is increasing, due to the positivity of B', we can conclude that $\overline{u}(0,t) = a_0(t)$ for $t \in (0,T)$. Then $\overline{u}$ is the unique solution of problem (1.1)-(1.3). $\square$

4 Weakly degenerate problem

In this section we consider nonnegative solutions of the initial-boundary value problem

$$\begin{cases} \partial_t u + \partial_x A(u) = \partial_{xx}[B(u)], & (x,t) \in \mathbb{R}^+ \times]0,\infty[, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^+ \ , \\ u(0,t) = a_0(t), & t \in (0,T) \ , \end{cases} \tag{4.1}$$

where

$$u_0(x) \geq 0 \text{ for } x \in (0,+\infty) \ , \quad a_0(t) \geq \alpha > 0 \text{ for } t \in (0,T) \ , \tag{4.2}$$

A, B are locally lipschitz functions, $A(0) = B(0) = 0$ and

$$B'(u) \geq 0 \ , B'(u) > 0 \text{ for } u \neq 0. \tag{4.3}$$

We intend to approximate the above problem by means of the BGK model (1.5), (1.6), (1.8), (1.9) where the Maxwellian functions are modified as follows

$$\begin{aligned} M_1(u,\epsilon) &= \sqrt{\epsilon}\left(\theta_3 u + \tfrac{A(u)}{\theta_1-\theta_3}\right) + \tfrac{B(u)}{(\theta_1-\theta_3)\theta_1} \ , \\ M_2(u,\epsilon) &= \left[1 - \sqrt{\epsilon}(\theta_3-\theta_1)\right] u + \tfrac{B(u)}{\theta_1\theta_3} \ , \\ M_3(u,\epsilon) &= \sqrt{\epsilon}\left(-\theta_1 u - \tfrac{A(u)}{\theta_1-\theta_3}\right) - \tfrac{B(u)}{(\theta_1-\theta_3)\theta_3} \ . \end{aligned} \tag{4.4}$$

In this way it is possible to introduce a condition on the coefficients θ_1, θ_3 which ensures the nondecreasing property for the functions $M_i, i = 1,2,3$, also in the degeneracy regions (this was not allowed for the functions (1.7) without assumptions on the function A).

So, in place of condition (1.10) in the parabolic case, now we ask that

$$\begin{cases} \max\{B'(u) : u \in I\} < -\theta_1\theta_3 \ , \\ \\ \theta_1(\theta_3-\theta_1) \leq A'(u) \leq \theta_3(\theta_3-\theta_1) & \text{for } u \in I \ , \end{cases} \tag{4.5}$$

where $I = \{u \in \mathbb{R}; \min\{ess\inf(u_0), ess\inf(a_0)\} \leq u \leq \max\{ess\sup(u_0), ess\sup(a_0)\}$.

Due to the above assumptions the maxwellian functions (4.4) are nondecreasing in u for $\epsilon \in (0,\overline{\epsilon})$. Moreover they satisfy the compatibility conditions $(M_1)-(M_3)$. Therefore comparison results of Proposition 2.1, Corollaries 2.1 and 2.2, and existence and uniqueness result of Theorem 2.1 hold also for system (1.5), (1.6), (4.4), (1.8), (1.9), (4.5), by the same proofs.

Obviously, in case of parabolic equations, the second condition in (4.5) can be removed.

In order to prove the convergence of the solutions of the relaxation system we would like to obtain compactness estimates as in Theorem 2.2. As we already remarked, in the previuos section the assumption $B'(u) \geq \nu > 0$ came into play in the proof of the estimate (2.6); more precisely we used the bound $B'(u^\epsilon(0,t)) \geq \nu > 0$. Now, due to hypothesys (4.2) we can derive the following result.

Proposition 4.1 *Let $u_0 \in C^2(\overline{\mathbb{R}^+}) \cap BV(\mathbb{R}^+)$ and $a_0 \in C^2([0,T)) \cap BV((0,T))$ with $u_0(0) = a_0(0)$. Assume that conditions (4.2) hold. Let $(f_1^\epsilon, f_2^\epsilon, f_3^\epsilon)$ be the solution of problem (1.5), (1.6), (4.4), (1.8), (1.9), (4.5). Then there exists a constant $K_1 = K_1(\alpha, \theta_1, \theta_3, TV(a_0), TV(u_0))$, not depending on ϵ, such that for every $h > 0$ small enough*

$$\int_0^{T-h} |f_2^\epsilon(0,t+h) - f_2^\epsilon(0,t)|dt \leq \epsilon |f_2^\epsilon(0,h) - f_2^\epsilon(0,0)| + K_1 h \ . \tag{4.6}$$

Proof. Observe that $f_3^\epsilon(0,t) = M_3(a_0(t), \epsilon) \geq M_3(\alpha, \epsilon)$; moreover, by comparison result, $f_i^\epsilon(x,t) \geq 0$ for $i = 1,2,3$. Then we can conclude that there exists a constant ν_b, not depending on ϵ, such that $B'(u^\epsilon(0,t) \geq \nu_b$ for $t \in [0,T]$. Due to the last inequality, we obtain (4.6) proceeding as in Proposition 2.3. $\square$

Thanks to the above proposition, the solutions of problem (1.5), (1.6), (4.4), (1.8), (1.9), (4.5) verify the estimates in Theorem 2.2, with the dependence on ν replaced by the dependence on α. Therefore we are able to prove the compactness of the sequences $\{f_i^\epsilon\}$ and $\{u^\epsilon\}$ proceeding as in Section 3.

Now we are going to introduce the set of hypothesys assumed in [8] to prove existence and uniqueness results for the weak solutions of problem (4.1)-(4.3). Thanks to this we are able to identify the limit function of the sequence $\{u^\epsilon\}$.

We assume that

$$\begin{aligned} &u_0 \text{ is a bounded and continuous function on } [0,\infty) \ , \\ &a_0 \text{ is a continuous function on } [0,T] \ , \\ &u_0(0) = a_0(0) \ ; \end{aligned} \tag{4.7}$$

$$\begin{aligned} &A, B \in C([0,\infty)) \cap C^2(0,\infty) \ , \\ &A'', B'' \text{ are locally Holder continuous on } (0,\infty) \ ; \end{aligned} \tag{4.8}$$

$$wB''(w), wB'(w)A'(w) \in L^1(0,\delta) \text{ for any } \delta > 0 \ . \tag{4.9}$$

Now we can state the convergence theorem; the proof is the same of the parabolic case.

Theorem 4.1 *Let $u_0 \in BV(\mathbb{R}^+)$ and $a_0 \in BV((0,T))$. Assume that conditions (4.2), (4.7)-(4.9) hold. Let $(f_1^\epsilon, f_2^\epsilon, f_3^\epsilon)$ be the solution of problem (1.5), (1.6), (4.4), (1.8), (1.9), (4.5) in $(\mathbb{R}^+ \times (0,T])$, $u^\epsilon = \sum_{i=1}^3 f_i^\epsilon$ and let u be the unique weak solution of problem (4.1)-(4.3) in $(\mathbb{R}^+ \times (0,T])$. Then*

$$\lim_{\epsilon \to 0} u^\epsilon = u \qquad in\ C([0,T]; L^1_{loc}(\mathbb{R}^+)) \ .$$

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