Convergence of Singular Limits for Multi-D Semilinear Hyperbolic Systems to Parabolic Systems

Donatella Donatelli
Scuola Normale Superiore
Piazza Cavalieri 7
56100 Pisa, Italy
donatell@cibs.sns.it

Pierangelo Marcati
Dip. di Matematica Pura ed Applicata
Università degli Studi dell'Aquila
67100 L'Aquila, Italy
marcati@univaq.it

Abstract

In this paper we study the relaxation of multi-D semilinear hyperbolic systems to parabolic systems. The singular limits are studied combining Tartar's and Gérard's generalized compensated compactness and by using the properties of the pseudodifferential symmetrizer of the system.

Key words and phrases: Hyperbolic systems, parabolic systems, pseudodifferential operators, relaxation theory.

Contents

1 Introduction

In this paper we study the following semilinear multidimensional hyperbolic system with a small parameter $\varepsilon > 0$

$$W_t(x,t) + \frac{1}{\varepsilon}A(x,D)W(x,t) = \frac{1}{\varepsilon^2}B(x,W(x,t)) + \frac{1}{\varepsilon}D(W(x,t)), \tag{1.1}$$

where $W = W(x,t)$ takes values in $\mathbb{R}^N$, $x \in \mathbb{R}^d$, $t \geq 0$, $A(x,D)$ is a pseudodifferential operator. System (1.1) includes also the case of a hyperbolic differential operator of the form

$$W_t(x,t) + \frac{1}{\varepsilon}\sum_{j=1}^{d} A_j(x)\partial_j W(x,t) = \frac{1}{\varepsilon^2}B(x,W(x,t)) + \frac{1}{\varepsilon}D(W(x,t)), \tag{1.2}$$

where $A_j(x)$, $j = 1,\dots,d$ are $N \times N$ matrices for any $x \in \mathbb{R}^d$. Our aim is to understand the limiting behaviour of the system (1.1) as ε goes to zero. We will look for structure condition in order that (1.1) will approximate a second order parabolic system. Our interest in this problem is motivated also by a very strong similarity with the limiting structure appearing in the investigation of the hydrodynamic limit for the Boltzmann equation, in particular in the discrete velocity case. The Boltzmann equation describes the evolution of the density $f(x,\xi,t)$ of particles that are at time t in position x with velocity ξ and has the following form

$$\nu f_t + \xi \nabla \cdot f = \frac{1}{\varepsilon}Q(f,f)$$

where ν is the Mach number and ε the Knudsen number. By averaging $f(x,\xi,t)$ in ξ and by using higher order momenta we can define a hierarchy of macroscopic quantities. The investigation of the hydrodynamic limit regards the behaviour of those quantities (actually combined with the closure problem) as the Knudsen number goes to zero. In the case where the Mach number is of the same order of the Knudsen number our limit can be described by the Navier Stokes equation, otherwise when the Mach number is fixed and the Knudsen number tends to zero we end up with the Euler equation. Those limits allow us to understand the differences between relaxation limits of hyperbolic type to parabolic with respect to those one of hyperbolic to hyperbolic type [2], [11], [12], [13], [24], [42]. In particular if we deal with a discrete velocity models, the equivalent " Boltzmann equation" is a semilinear hyperbolic system. The simplest example is given by the Carlemann's equation

$$\begin{cases} u_t + \dfrac{1}{\varepsilon}u_x = \dfrac{1}{\varepsilon^2}(v^2 - u^2) \\[2ex] v_t + \dfrac{1}{\varepsilon}v_x = \dfrac{1}{\varepsilon^2}(u^2 - v^2), \end{cases}$$

where we take $\xi \in \{-1,1\}$ and $u = f(x,1,t)$, $v = f(x,-1,t)$. By rescaling the variable we get that $\rho = u + v$ as $\varepsilon \downarrow 0$ satisfies the nonlinear diffusion equation

$$\rho_t = (\log \rho)_{xx}.$$

This asymptotic problem was first investigated by Kurtz [17] and McKean [32]. Therefore the nonlinear diffusion problem, obtained as the limit of the Cattaneo hyperbolic nonlinear heat conduction equation, was proved by Marcati, Milani and Secchi [28]. The paper of Marcati and Milani [27] concernes the pourous media flow as the limit of the Euler equation in 1-D, later generalized

by Marcati and Rubino [31] to the multi-D case. Relaxation phenomena of the same nature appear in the zero relaxation limits for the Euler-Poisson model for the semiconductor devices and it was investigated by Marcati and Natalini [29], [30] in the 1-D case and by Lattanzio and Marcati [18] in the multi-D case. More recently Lions and Toscani [23] investigated a discrete velocity model leading to the pourous media flow. All of these papers, with exception of [17], [32], make use of the techniques of compensated compactness. Similar ideas have been applied by Marcati and Rubino [31] to show the general theory for 2×2 systems in the 1-D case and to propose a general framework that we are going to investigate here in the semilinear system case. Models of BGK type approximation have been successfully studied by Bouchut, Guarguaglini and Natalini [1] and Lattanzio, Natalini [19] in the case of 1-D systems. The framework of [31] was also investigated in the quasilinear case by Lattanzio and Yong [20] for H^s- smooth solutions. Preliminary results concerning semilinear systems have been obtained in [6], [7], in particular in [7] we considered a 1- D semilinear system with variable coefficients. Already in that case the classical compensated compactness is not sufficient and it is necessary to use a generalization of this theory due to Tartar [38] and P.Gérard [10]. With respect to the previous papers the problem, here, is more complicated, since we deal with the case of a multi- D pseudodifferential system and since the symmetrizers are pseudodifferential operators which make more difficult the proof of the a priori estimates.
The plan of the paper is as follows. In Section 2 we give the basic definitions, the basic notations and we recall the mathematical tools needed in the paper. Moreover we will describe the formal framework of the limiting process carried on in this paper. The Section 3 is devoted to give rigorous proof of the previous formal analysis, moreover we shall give there the structure condition on our system and on its symmetrizer. These assumptions will allow us to provide the a priori estimates uniformly in ε. By using only these informations and by the previous mentioned compactness framework we will be able to achieve our relaxation results. The structure conditions used in the limit process allow us also to satisfy the parabolicity condition for the limit system in the case of system (1.2) . In Section 4 we show how this theory can be greatly simplified in the case of constant coefficients. Finally, in Section 5 we show how it is possible to use the theory developed in the previous sections in order to approximate a given parabolic system by means of greater hyperbolic system.

2 Multidimensional Framework

2.1 Preliminary notions

We start this section introducing the main notations and definitions used in the article. In particular we recall basic facts related to the theory of pseudodifferential operators and we also mentions our main mathematical tools to get strong convergence. Therefore

(a) $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^q$, $(q = 1, 2, ...)$ and $|\cdot|$ the usual norm of $\mathbb{R}^q$ $(q = 1, 2, ...)$,

(b) $(\cdot,\cdot)_2$ denotes the scalar product in $L^2(\mathbb{R}^d)$ and $\|\cdot\|$ the norm in $L^2(\mathbb{R}^d)$,

(c) $\mathcal{D}(\mathbb{R}^d \times \mathbb{R}_+)$ denotes the space of test function $C_0^\infty(\mathbb{R}^d \times \mathbb{R}_+)$, $\mathcal{D}'(\mathbb{R}^d \times \mathbb{R}_+)$ the Schwarz space of distributions and $\langle\cdot,\cdot\rangle$ the duality bracket in $\mathcal{D}'(\mathbb{R}^d \times \mathbb{R}_+)$,

(d) $\mathcal{M}_{m\times n}$ denotes the linear space of $m \times n$ matrices,

(e) H is a separable Hilbert space, $\mathcal{L}(H)$ the space of bounded operators, $\mathcal{K}(H)$ the space of compact operators,

(f) we denote by $H^s_{loc}(\Omega, H)$ the classical local Sobolev space of order s, i.e. $u \in H^s_{loc}(\Omega, H) \iff \forall\, \varphi \in C_0^\infty$, $(\widehat{\varphi u}) \in L^2(\mathbb{R}^n, (1+|\xi|^2)^s d\xi)$.

We will also make use of the notion of parabolicity for systems of equations in various way (see Taylor [41] volume III, [40], Eidel'man [8], Kreiss and Lorenz [16]). Let us consider the system

$$u_t = \sum_{j,k} A^{j,k}(t,x,D'_x u)\partial_j\partial_k u + B(t,x,D'_x u), \tag{2.1}$$

where $u \in \mathbb{R}^p$, $A^{j,k}(t,x,D'_x u) \in \mathcal{M}_{p\times p}$, $B(t,x,D'_x u) \in \mathbb{R}^p$ and D'_x is a differential operator of order not greater than two. The system is said *strongly parabolic* if there exists $c_0 > 0$ such that for all $\xi \in \mathbb{R}^d$ one has

$$\sum_{j,k} A^{j,k}(t,x,D'_x u)\xi_j\xi_k \geq c_0|\xi|^2 I.$$

Namely, if we denote $L(t,x,D'_x u,\xi) = -\sum_{j,k} A^{j,k}(t,x,D'_x u)\xi_j\xi_k$ this condition is equivalent to say $L + L^T$ is a negative definite matrix. Unfortunately this condition is often difficult to be verified then we formulate now a more general notion of parabolicity (which in Taylor [41] volume III, [40] is referred to as *Petrowski parabolicity*).

We say that the system (2.1) is *parabolic* if, denoted by $\lambda_k(t,x,D'_x u,\xi)$ the eigenvalues of the matrix $L(t,x,D'_x u,\xi)$, one has there exists $\alpha_0 > 0$ such that, for all $\xi \in \mathbb{R}^d$,

$$Re\lambda_k(t,x,D'_x u,\xi) \leq -\alpha_0|\xi|^2.$$

The latter notion of parabolicity is equivalent to ask the existence of a matrix $P_0(t,x,D'_x u,\xi)$, positive definite (i.e. $P_0 \geq cI > 0$), homogeneous of degree 0 in ξ, such that

$$-(P_0 L + L^* P_0) \geq c|\xi|^2 I > 0.$$

Let us recall the basic notations concerning pseudodifferential operators to be used later on, we refer to [40] for details. Assuming $\rho, \delta \in [0,1]$, $m \in \mathbb{R}$, we denote $S^m_{\rho,\delta}$ the set of C^∞ symbols satisfying

$$\left| D_x^\beta D_\xi^\alpha p(x,\xi) \right| \leq C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}$$

for all α, β, where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$. In such case we say that the associated operator defined by

$$P(x,D)f(x) = \int p(x,\xi)\hat{f}(\xi)e^{ix\xi}d\xi := OP(p(x,\xi))$$

(where$\hat{f}(\xi) = (2\pi)^{-n}\int f(x)e^{-ix\xi}dx$ is the Fourier transform of a function on $\mathbb{R}^n$) belongs to $OPS^m_{\rho,\delta}$. If there are smooth $p_{m-j}(x,\xi)$, homogeneous in ξ of degree $m-j$ for $|\xi| \geq 1$, i.e. $p_{m-j}(x,r\xi) = r^{m-j}p_{m-j}(x,\xi)$ for r, $|\xi| \geq 1$, and if

$$p(x,\xi) \sim \sum_{j\geq 0} p_{m-j}(x,\xi)$$

in the sense that

$$p(x,\xi) - \sum_{j\geq 0}^{N} p_{m-j}(x,\xi) \in S^{m-N}_{1,0}$$

for all N, the we say $p(x,\xi) \in S^m$. The following properties will also be used here.

Theorem 2.1 *([40], Theorem 0.5 A) If $P(x,D) \in OPS^0_{\rho,\delta}$ and $0 \le \delta < \rho \le 1$ then*

$$P(x,D) : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n).$$

Theorem 2.2 *([39], Chapter II, Ex. 8.1) If $P(x,D) \in OPS^0_{\rho,\delta}$ and $\Re e P(x,\xi) \ge c > 0$ then*

$$(P(x,D)u,u)_{L^2(\mathbb{R}^n)} \ge c\|u\|_{L^2(\mathbb{R}^n)}.$$

Proposition 2.3 (Commutator estimate) *([40], Proposition 3.6 B) Given $P(x,D) \in OPS^0_{1,0}$ we have*

$$\|[P,\ f]u\|_{W^{\sigma,p}} \le c\|f\|_{Lip}\|u\|_{W^{\sigma-1,p}} \qquad \text{for } 0 \le \sigma \le 1,$$

in particular if $\sigma = 0$ and $p = 2$

$$\|[P,\ f]u\|_{L^2(\mathbb{R}^n)} \le c\|f\|_{Lip}\|u\|_{H^{-1}}.$$

Finally we state here our main tool to study the convergence of quadratic forms with variable coefficients: generalization of classical Compensated Compactness Theorem due to Tartar and Gérard ([38], [10])). Let H, $H^\sharp$ denote separable Hilbert spaces, $\Omega \in \mathbb{R}^n$, an open set, $m \in \mathbb{N}$. We have the following theorem.

Theorem 2.4 (Compensated Compactness)
Let $P \in OPS^m$ with principal symbol $p(x,\xi)$ and $\{u_k\}$ be a bounded sequence of $L^2_{loc}(\Omega, H)$, such that $u_k \rightharpoonup u$. Assume that there exists a dense subset $D \in H^\sharp$ such that, for any $h \in D$, the sequence $(\langle Pu_k, h\rangle)$ is relatively compact in $H^{-m}_{loc}(\Omega)$. Moreover, let $q \in C(\Omega, \mathcal{K}(H))$.
(i) *If $q = q^*$ and*

$$\forall\, (x,\xi,h) \in \Omega \times S^{n-1} \times H, \qquad (p(x,\xi)h = 0) \;\Rightarrow\; (\langle q(x)h, h\rangle \ge 0)$$

then, for any nonnegative $\varphi \in C_0^\infty(\Omega)$

$$\liminf_{k\to\infty} \int_\Omega \varphi\langle q(x)u_k, u_k\rangle dx \ge \int_\Omega \varphi\langle q(x)u, u\rangle dx$$

(ii) *If*

$$\forall\, (x,\xi,h) \in \Omega \times S^{n-1} \times H, \qquad (p(x,\xi)h = 0) \;\Rightarrow\; (\langle q(x)h, h\rangle = 0)$$

then

$$\langle q(x)u_k, u_k\rangle \quad \text{converges to} \quad \langle q(x)u, u\rangle \qquad \text{in } \mathcal{D}'(\Omega).$$

The previous theorem holds also if instead of a differential operator $P \in OPS^m$ we consider the differential operator

$$Pu(x) = \sum_{|\alpha|\le m} \partial^\alpha(a_\alpha(x)u(x)),$$

where $\alpha \in \mathbb{N}^n$, $|\alpha| \le m$, $a_\alpha \in C(\Omega, \mathcal{L}(H, H^\sharp))$.

2.2 Decoupled system

We will consider the following semilinear system of equations

$$W_t(x,t) + A(x,D)W(x,t) = B(x, W(x,t)) + D(W(x,t)) \tag{2.2}$$

where $t \geq 0$, $x \in \mathbb{R}^d$, $W \in \mathbb{R}^N$. Moreover we assume the following hypotheses hold.

(A.1) $B(x,W), D(W) \in \mathbb{R}^N$,

(A.2) $A(x,D) \in OPS^1$, the system (2.2) is hyperbolic, namely the principal symbol of $A(x,D)$ is the matrix $A(x,\xi) \in \mathcal{M}_{N\times N}$ whose eigenvalues for $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$, $\xi \neq 0$ are pure imaginary.

We point out that (2.2) includes the case of the following hyperbolic semilinear system

$$W_t(x,t) + \sum_{j=1}^{d} A_j(x)\partial_j W(x,t) = B(x, W(x,t)) + D(W(x,t))$$

where $A_j(x) \in \mathcal{M}_{N\times N}$, $j = 0, \dots, d$ and for all nonzero vector $\xi \in \mathbb{R}^d$, the matrix $A(x,\xi) = \sum_{j=1}^{d} \xi_j A_j(x)$ has real eigenvalues.
We denote by $S = span\left\{B(x,W) \mid W \in \mathbb{R}^N\right\}$ and we assume

(A.3) $dim\ S = N - k$, $0 < k < N$.

Let us consider $[\sigma_{k+1}, \dots, \sigma_N]$ a vector basis for S, $[\sigma_1, \dots, \sigma_k]$ an ortonormal vector basis for $S^\perp$. In this way we can construct an invertible matrix $P = \begin{bmatrix} P^I \\ P^{II} \end{bmatrix}$ where the rows of P^I are given by $[\sigma_1, \dots, \sigma_k]$, while the rows of P^{II} are given by the vectors $[\sigma_{k+1}, \dots, \sigma_N]$. Now we set, for any $W \in \mathbb{R}^N$

$$Z^I = P^I W, \qquad Z^{II} = P^{II} W, \qquad Z = \begin{bmatrix} Z^I \\ Z^{II} \end{bmatrix},$$

$$PA_j(x,\xi)P^{-1} = \begin{bmatrix} M^{11}(x,\xi) & M^{12}(x,\xi) \\ \\ M^{21}(x,\xi) & M^{22}(x,\xi) \end{bmatrix}, \qquad M^{ij}(x,D) = OPM^{ij}(x,\xi)\ 1 \leq i,j \leq 2,$$

$$D^I(Z^I, Z^{II}) = P^I D(P^{-1}Z) \qquad D^{II}(Z^I, Z^{II}) = P^{II} D(P^{-1}Z)$$

$$Q(Z^I, Z^{II}) = P^{II} B(P^{-1}Z),$$

hence we can rewrite the system (2.2) in the following way

$$\begin{cases} Z_t^I + M^{11}(x,D)Z^I + M^{12}(x,D)Z^{II} = D^I(Z^I, Z^{II}) \\ \\ Z_t^{II} + M^{21}(x,D)Z^I + M^{22}(x,D)Z^{II} = Q(x, Z^I, Z^{II}) + D^{II}(Z^I, Z^{II}) \end{cases} \tag{2.3}$$

by construction $Z^I = Z^I(x,t) \in \mathbb{R}^k$, $Z^{II} = Z^{II}(x,t) \in \mathbb{R}^{N-k}$ and $M^{11}(x,\xi) \in \mathcal{M}_{k\times k}$, $M^{12}(x,\xi) \in \mathcal{M}_{k\times(N-k)}$, $M^{21}(x,\xi) \in \mathcal{M}_{(N-k)\times k}$, $M^{22}(x,\xi) \in \mathcal{M}_{(N-k)\times(N-k)}$, $D^I(Z^I, Z^{II}) \in \mathbb{R}^k$, $D^{II}(Z^I, Z^{II}) \in \mathbb{R}^{N-k}$, $Q(x, Z^I, Z^{II}) \in \mathbb{R}^{N-k}$. We remark that system (2.3) is still hyperbolic. We also need the following structural assumption.

(A.4) $M^{11}(x,\xi)=0$, for any $(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d$.

Without this assumption in the relaxation process we wouldn't have any nontrivial dynamics for Z^I, namely the violation of this condition would prevent the system to relax on a parabolic system. In order to understand better what happens in this situation we can study, at a formal level, the following 1-D example. Let us consider this 2×2 linear system

$$\begin{cases} u_t - \dfrac{1}{\varepsilon}u_x - \dfrac{1}{\varepsilon}\bar{v}_x = 0 \\ \\ \bar{v}_t - \dfrac{1}{\varepsilon}u_x = -\dfrac{1}{\varepsilon^2}\bar{v} \end{cases} \tag{2.4}$$

with $u=u(x,t)\in\mathbb{R}$, $\bar{v}=\bar{v}(x,t)\in\mathbb{R}$, $(x,t)\in\mathbb{R}\times\mathbb{R}_+$. Setting $\bar{v}=\varepsilon v$ we get

$$u_t - \frac{1}{\varepsilon}u_x - v_x = 0 \qquad \varepsilon^2 v_t - u_x + v = 0. \tag{2.5}$$

Now we look for a solution (u,v) of the following form

$$u(x,t)=\varphi_0(x,t)+\varepsilon\varphi_1(x,t)+o(\varepsilon^2) \qquad\qquad v(x,t)=\psi_0(x,t)+\varepsilon\psi_1(x,t)+o(\varepsilon^2)$$

where we have to find $\varphi_0,\ \varphi_1,\ \psi_0,\ \psi_1$. Since (u,v) have to solve (2.5), we get

$$\begin{cases} (\varphi_0+\varepsilon\varphi_1)_t - \dfrac{1}{\varepsilon}(\varphi_0+\varepsilon\varphi_1)_x - (\psi_0+\varepsilon\psi_1)_x = 0 \\ \\ \varepsilon^2(\psi_0+\varepsilon\psi_1)_t - (\varphi_0+\varepsilon\varphi_1)_x + (\psi_0+\varepsilon\psi_1) = 0 \end{cases} \tag{2.6}$$

Now equating the coefficients of $\varepsilon^{-1},\ \varepsilon,\ \varepsilon^2,\ \varepsilon^3$, we get the following relations

$$\begin{array}{ll} \varphi_{0,x}=0 & \\ \varphi_{0,t}-\varphi_{1,x}-\psi_{0,x}=0 & \qquad\qquad \psi_0-\varphi_{0,x}=0 \\ \varphi_{1,t}-\psi_{1,x}=0 & \qquad\qquad \psi_1-\varphi_{1,x}=0 \\ \psi_{0,t}=0 & \qquad\qquad \psi_{1,t}=0. \end{array}$$

Combining them we get

$$\varphi_{0,x}=\varphi_{0,t}=\varphi_{1,t}=\varphi_{1,x}=0 \qquad\qquad \psi_0=\psi_1=0$$

and finally

$$u=constant \qquad\qquad v=0.$$

So we get no other solution for (2.5) except the trivial one.

2.3 Formal analysis of the singular limit

We will analyse the relaxation process of the following system

$$W_t+\frac{1}{\varepsilon}A(x,D)W=\frac{1}{\varepsilon^2}B(x,W)+\frac{1}{\varepsilon}D(W). \tag{2.7}$$

Following the construction of the previous paragraph we rewrite (2.7) in this way

$$\begin{cases} Z^I_t + \dfrac{1}{\varepsilon}M^{12}(x,D)Z^{II} = \dfrac{1}{\varepsilon}D^I(Z^I,Z^{II}) \\ Z^{II}_t + \dfrac{1}{\varepsilon}M^{21}(x,D)Z^I + \dfrac{1}{\varepsilon}M^{22}(x,D)Z^{II} = \dfrac{1}{\varepsilon^2}Q(x,Z^I,Z^{II}) + \dfrac{1}{\varepsilon}D^{II}(Z^I,Z^{II}) \end{cases} \tag{2.8}$$

Since we are interested in the relaxation problem for any $\varepsilon > 0$ we set

$$Z^I(x,t) = U^I(x,t), \qquad\qquad Z^{II}(x,t) = \varepsilon U^{II}(x,t). \tag{2.9}$$

The previous scaling has an equivalent interpretation as a scaling of the time variable setting $\partial_\tau = \varepsilon\partial_t$, for more details see [31]. With previous position the system (2.8) transforms into

$$\begin{cases} U^I_t + M^{12}(x,D)U^{II} = \dfrac{1}{\varepsilon}D^I(U^I,\varepsilon U^{II}) \\ \\ \varepsilon^2 U^{II}_t + M^{21}(x,D)U^I + \varepsilon M^{22}(x,D)U^{II} = \dfrac{1}{\varepsilon}Q(x,U^I,\varepsilon U^{II}) + D^{II}(U^I,\varepsilon U^{II}). \end{cases} \tag{2.10}$$

If we denote by (U^{I0},U^{II0}) the limit profile as $\varepsilon \downarrow 0$, formally we obtain that the system (2.10) relaxes to the system

$$\begin{cases} U^{I0}_t + M^{12}(x,D)U^{II0} = D^I_\nu(U^{I0},0)U^{II0} \\ \\ M^{21}(x,D)U^{I0} = Q_\nu(x,U^{I0},0)U^{II0} + D^{II}(U^{I0},0), \end{cases} \tag{2.11}$$

where $Q_\nu,\ D^I_\nu$, denote the derivative respect to the variable Z^{II}respectively of Q and D^I. Assuming suitable additional hypotheses on our data, system (2.11) transforms into

$$\begin{cases} U^{I0}_t + M^{12}(x,D)U^{II0} = D^I_\nu(U^{I0},0)U^{II0} \\ \\ U^{II0} = Q^{-1}_\nu(x,U^{I0},0)\left[M^{21}(x,D)U^{I0} - D^{II}(U^{I0},0)\right] \end{cases}$$

That is equivalent to (setting $U = U^{I0}$)

$$\begin{aligned} U_t + M^{12}(x,D)Q^{-1}_\nu(x,U,0)M^{21}(x,D)U &= M^{12}(x,D)Q^{-1}_\nu(x,U,0)D^{II}(U,0) \\ &+ D^I_\nu(U,0)Q^{-1}_\nu(x,U,0)\left[M^{21}(x,D)U - D^{II}(U,0)\right]. \end{aligned} \tag{2.12}$$

In the next sextion we will find sufficient conditions in order to justify rigorously this formal analysis.

3 Estimates and convergence

In this part we consider our rescaled system

$$\begin{cases} U^I_t + M^{12}(x,D)U^{II} = \dfrac{1}{\varepsilon}D^I(U^I,\varepsilon U^{II}) \\ \varepsilon^2 U^{II}_t + M^{21}(x,D)U^I + \varepsilon M^{22}(x,D)U^{II} = \dfrac{1}{\varepsilon}Q(x,U^I,\varepsilon U^{II}) + D^{II}(U^I,\varepsilon U^{II}). \end{cases} \tag{3.1}$$

and we are going to develop the rigorous theory in order to get the relaxed system (2.11). We want to show that, as $\varepsilon \downarrow 0$, the weak solutions of the rescaled system satisfy

$$\begin{aligned} U^I &\longrightarrow U^{I0} && \text{strongly in } L^2_{loc}(\mathbb{R}^d \times \mathbb{R}_+), \\ U^{II} &\rightharpoonup U^{II0} && \text{weakly in } L^2(\mathbb{R}^d \times [0,T]), \\ \varepsilon U^{II} &\longrightarrow 0 && \text{strongly in } L^2_{loc}(\mathbb{R}^d \times \mathbb{R}_+). \end{aligned}$$

The basic idea used in this section is the assumption of the existence of a symmetrizer for the system (2.3) whose symbol has a suitable block structure. In the definition of the symmetrizer and on its properties we follow Taylor [39], [40] but we want to remark that if our system is strictly hyperbolic we have automatically the existence of a symmetrizer following the construction of Kreiss [15] or of Majda and Osher in the case of boundary conditions [26]. The block structure follows also by using a deep result of Métivier [33] in the case of constant multiplicity, which includes many physical examples.
Now we are going to rewrite the system (2.3) into a more compact form in order to obtain "energy" type estimate by using pseudodifferential symmetrizer. Taking into account the pseudodifferential notations and setting

$$M(x,D) = OP\left\{-\begin{pmatrix} 0 & M^{12}(x,\xi) \\ M^{21}(x,\xi) & M^{22}(x,\xi)\end{pmatrix}\right\} \tag{3.2}$$

we can rewrite the system (2.3) in this way

$$Z_t - M(x,D)Z = Q(x, Z^I, Z^{II}) + D(Z^I, Z^{II}). \tag{3.3}$$

3.1 A priori estimates

In this section we establish a priori estimates, independent of ε, for the solution of the system (3.1). We make on (3.1) the following hypotheses

(B.1) $U^I(x,0) \in \left[L^2(\mathbb{R}^d)\right]^k$, $U^{II}(x,0) \in \left[L^2(\mathbb{R}^d)\right]^{N-k}$,

(B.2) $M^{12}(x,\xi), M^{21}(x,\xi), M^{22}(x,\xi) \in S^1(\mathbb{R}^d)$,

(B.3) $det\left[\left(M^{21}(x,\xi)\right)^T M^{21}(x,\xi)\right] \neq 0$,

(B.4) $D = (D^I(Z^I,Z^{II}), D^{II}(Z^I,Z^{II})) \in C^1(\mathbb{R}^N; \mathbb{R}^k \times \mathbb{R}^{N-k})$, $D^I(Z^I,0) = 0$, $D^{II}(0,0) = 0$, and D^I_ν is bounded in (Z^I, Z^{II}), D^{II} is a α-lipschitz function in (Z^I, Z^{II}),

(B.5) there exists a symmetrizer $R(x,D) \in OPS^0_{1,0}$ for the operator $M(x,D)$ whose symbol has the following form

$$R(x,\xi) = \begin{pmatrix} R_{11}(x,\xi) & 0 \\ 0 & R_{22}(x,\xi)\end{pmatrix}$$

where $R_{11}(x,\xi) \in \mathcal{M}_{k\times k}$, $R_{22}(x,\xi) \in \mathcal{M}_{(N-k)\times(N-k)}$ are symmetric positive definite matrices and $R(x,D)M(x,D) + (R(x,D)M(x,D))^* \in OPS^0_{1,0}$.

We remark that since $M^{21}(x,\xi) \in \mathcal{M}_{(N-k)\times k}$ from elementary linear algebra we deduce condition (B.3) is violated whenever $k > \frac{N}{2}$. Finally we state the assumptions concerning the nonhomogeneous term $Q(x, Z^I, Z^{II})$.

(D) Dissipativity condition
$Q(x, Z^I, Z^{II})$ has the following form

$$Q(x, Z^I, Z^{II}) = Q_0(x, Z^I, Z^{II}) + Q_1(x, Z^I, Z^{II})$$

and $Q(x, Z^I, 0) = 0$ for any $(x, Z^I) \in \mathbb{R}^d \times \mathbb{R}^k$ moreover

(d1) $Q_0(x, Z^I, Z^{II}) \in C^1(\mathbb{R}^{N+d}; \mathbb{R}^{N-k})$, $Q_{0\nu}(x, Z^I, Z^{II})$ is bounded in (x, Z^I, Z^{II}), $[Q_{0\nu}, R_{22}^{1/2}(x, D)] = 0$. There exists $\lambda_0 > 0$ such that for any $x \in \mathbb{R}^d$, $(Z^I, Z^{II}) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, $Q_{0\nu}(x, Z^I, Z^{II}) \leq -\lambda_0 I$,

(d2) $Q_1(x, Z^I, Z^{II}) \in C^1(\mathbb{R}^{N+1}; \mathbb{R}^{N-k})$, $Q_{1\nu}(x, Z^I, Z^{II})$ is bounded in (x, Z^I, Z^{II}) and the operator $R_{22}(x, D)Q_{1\nu}$ has norm $\|R_{22}(x, D)Q_{1\nu}\|_{\mathcal{L}(L^2)} \leq \lambda_1$, $\lambda_1 > 0$, $\lambda_1 \leq \lambda_0/2$.

Remark 3.1 We have decomposed the nonhomogeneous term $Q(x, Z^I, Z^{II})$ in two parts that take into account the different dissipativity nature of Q. The first term $Q_0(x, Z^I, Z^{II})$ cares about the dissipativity of Q, and it commutes with $R_{22}(x, D)Q_{1\nu}$. The second one, $Q_1(x, Z^I, Z^{II})$ does not commutes but defines with $R_{22}(x, D)$ a bounded operator that doesn't affect the dissipativity of $Q_0(x, Z^I, Z^{II})$. ■

Remark 3.2 The class of dissipativity terms defined in (D) is not empty. In fact it is sufficient to take Q of the following form

$$Q(x, Z^I, Z^{II}) = C(x)Z^{II} \tag{3.4}$$

with $C(x) \in \mathcal{M}_{(N-k)\times(N-k)}$ for any $x \in \mathbb{R}^d$, $[R_{22}^{1/2}(x, D), C(x)] = 0$ and there exists $\gamma > 0$ such that for any $x \in \mathbb{R}^d$, $C(x) \leq -\gamma I$. ■

Taking advantage of the block structure of the symmetrizer and on the dissipativity condition (D) we can prove the following result

Theorem 3.3 *Let us consider the solution $\{U^I\}$, $\{U^{II}\}$ of the Cauchy problem for system (3.1). Assume that the hypotheses (B.1), (B.2), (B.3), (B.4), (B.5), (D) hold. Then for ε small enough, one has*

(i) *there exists $T > 0$, $M(T) > 0$, independent from ε, such that* $\|U^{II}\|_{L^2(\mathbb{R}^d\times[0,T])} \leq M(T)$ *and* $\sup_{t\geq 0} \|\varepsilon U^{II}(\cdot, t)\| \leq M(T)$,

(ii) $\{\varepsilon^2 U_t^{II}\}$ *is relatively compact in* $H^{-1}_{loc}(\mathbb{R}^d \times \mathbb{R}_+)$,

(iii) $\{U^I\}$ *is uniformely bounded, with respect to ε, in* $L^\infty\left(\mathbb{R}_+, L^2(\mathbb{R}^d)\right)$, *namely there exists $T > 0$, $M(T) > 0$, independent from ε, such that* $\sup_{t\geq 0} \|U^I(\cdot, t)\| \leq M(T)$.

Proof. To simplify our notation we set

$$U = \begin{bmatrix} U^I \\ U^{II} \end{bmatrix}, \qquad U^\varepsilon = \begin{bmatrix} U^I \\ \varepsilon U^{II} \end{bmatrix}, \qquad F = \begin{bmatrix} \frac{1}{\varepsilon} D^I(U^I, \varepsilon U^{II}) \\ \frac{1}{\varepsilon} Q(x, U^I, \varepsilon U^{II}) + D^{II}(U^I, \varepsilon U^{II}) \end{bmatrix}$$

$$M^\varepsilon(x,D) = OP\left\{ - \begin{pmatrix} 0 & M^{12}(x,\xi) \\ M^{21}(x,\xi) & \varepsilon M^{22}(x,\xi) \end{pmatrix} \right\}$$

We compute

$$\begin{aligned} \frac{d}{dt}(R(x,D)U^\varepsilon, U^\varepsilon)_2 &= (R(x,D)U^\varepsilon_t, U^\varepsilon)_2 + (R(x,D)U^\varepsilon, U^\varepsilon_t)_2 \\ &= (R(x,D)(M^\varepsilon(x,D) + F), U)_2 + (R(x,D)U, M^\varepsilon(x,D)U + F)_2 \\ &= ((R(x,D)M^\varepsilon(x,D) + (R(x,D)M^\varepsilon(x,D))^*)U, U)_2 + 2(R(x,D)F, U)_2 \\ &= I_1 + I_2. \end{aligned}$$

Using the block structure of $R(x,D)$ we estimate separately I_1 and I_2. Applying Theorem 2.1 and using hypothesis (B.5) for I_1 we get

$$I_1 \leq 2\|U^{II}\|\|U^I\| + \varepsilon\|U^{II}\|^2.$$

Now we focus our attention on I_2.

$$\begin{aligned} I_2 &= (R(x,D)D(U^I, \varepsilon U^{II}), U)_2 + (R(x,D)\begin{pmatrix} 0 \\ \varepsilon^{-1}Q(x, U^I, \varepsilon U^{II}) \end{pmatrix}, U)_2 \\ &= (\varepsilon^{-1}D^I(U^I, \varepsilon U^{II}), R_{11}(x,D)U^I)_2 + (D^{II}(U^I, \varepsilon U^{II}), R_{22}(x,D)U^{II})_2 \\ &+ (\varepsilon^{-1}R_{22}(x,D)Q(x, U^I, \varepsilon U^{II}), U^{II})_2 \\ &= I_{2,1} + I_{2,2} + I_{2,3}. \end{aligned}$$

To estimate $I_{2,1}$ and $I_{2,2}$ we use hypothesis (B.4) to have

$$I_{2,1} + I_{2,2} \leq \alpha\|U^{II}\|\|U^I\| + \alpha\varepsilon\|U^{II}\|^2.$$

Now it remains to estimate $I_{2,3}$. Denoting by $\bar{Q}_\nu = \int_0^1 Q_\nu(x, U^I, \varepsilon\theta U^{II})d\theta$ we can rewrite $I_{2,3}$ in the following way

$$I_{2,3} = (R_{22}(x,D)\bar{Q}_{0\nu}U^{II}, U^{II})_2 + (R_{22}(x,D)\bar{Q}_{1\nu}U^{II}, U^{II})_2 = I_{2,31} + I_{2,32}.$$

From (d1) it follows

$$I_{2,31} = (R_{22}^{1/2}(x,D)\bar{Q}_{0\nu}U^{II}, R_{22}^{1/2}(x,D)U^{II})_2 = (\bar{Q}_{0\nu}R_{22}^{1/2}(x,D)U^{II}, R_{22}^{1/2}(x,D)U^{II})_2 \leq -\lambda_0\|U^{II}\|^2.$$

Using (d2) we write

$$I_{2,32} = (R_{22}^{1/2}(x,D)\bar{Q}_{1\nu}U^{II}, R_{22}^{1/2}(x,D)U^{II})_2 \leq \lambda_1\|U^{II}\|^2 \leq \frac{\lambda_0}{2}\|U^{II}\|^2$$

And finally

$$I_{2,3} = -\frac{\lambda_0}{2}\|U^{II}\|^2.$$

Summing I_1, I_2 we obtain for all $\delta > 0$

$$\begin{aligned}\frac{d}{dt}(R(x,D)U^\varepsilon, U^\varepsilon)_2 &\leq (2+\alpha)\|U^{II}\|\|U^I\| + \varepsilon(\alpha+1)\|U^{II}\|^2 - \frac{\lambda_0}{2}\|U^{II}\|^2 \\ &\leq \delta\|U^{II}\|^2 + \frac{2+\alpha}{\delta}\|U^I\|^2 + \varepsilon(\alpha+1)\|U^{II}\|^2 - \frac{\lambda_0}{2}\|U^{II}\|^2.\end{aligned}$$

Choosing $\varepsilon < \dfrac{\lambda_0}{8(\alpha+1)}$ and $\delta < \dfrac{\lambda_0}{8}$ it follows that

$$\frac{d}{dt}(R(x,D)U^\varepsilon, U^\varepsilon)_2 \leq -\frac{\lambda_0}{4}\|U^{II}\|^2 + c\|U^I\|^2. \tag{3.5}$$

Now we set

$$E(t) = \int_{\mathbb{R}^d} \varepsilon^2 |U^{II}(x,t)|^2 dx + \int_{\mathbb{R}^d} \left|U^I(x,t)\right|^2 dx. \tag{3.6}$$

Integrating (3.5) on $[0,t]$ we obtain the energy $E(t)$ satisfies

$$E(t) \leq E(0) - \frac{\lambda_0}{4}\int_0^t\int_{\mathbb{R}^d} |U^{II}(x,t)|^2 dxds + c\int_0^t E(s)ds \tag{3.7}$$

with c constant, $c > 0$. Applying Gronwall's lemma we obtain

$$E(t) \leq E(0)e^{ct} \tag{3.8}$$

and the following estimate

$$\int_0^t\int_{\mathbb{R}^d} |U^{II}|^2 dxds \leq E(0)\left(e^{ct}+2\right). \tag{3.9}$$

We can conclude that there exists $T > 0$, $M(T) > 0$, indipendent from ε, such that

$$\|U^{II}\|_{L^2(\mathbb{R}^d\times[0,T])} \leq M(T), \qquad \sup_{t\geq 0}\|U^{II}(\cdot,t)\| \leq M(T), \qquad \sup_{t\geq 0}\|U^I(\cdot,t)\| \leq M(T).$$

In this way we proved (i) and (iii). Let us consider ω relatively compact in $\mathbb{R}^d\times\mathbb{R}_+$, then

$$\begin{aligned}\|\varepsilon^2 U_t^{II}\|_{H^{-1}(\omega)} &= \sup_{\|\phi\|_{H_0^1(\omega)}=1} |\langle\varepsilon^2 U_t^{II}, \phi\rangle| = \sup_{\|\phi\|_{H_0^1(\omega)}=1}\left|\iint \varepsilon^2 U^{II}\phi_t dxdt\right| \\ &\leq \varepsilon^2 \sup_{\|\phi\|_{H_0^1(\omega)}=1}(\|U^{II}\|\|\phi_t\|) \leq M(T)\varepsilon^2.\end{aligned}$$

■

Remark 3.4 In the case we can control $\|Q_\nu\|_{Lip}$ we don't need to make assumption (D) but it is sufficient to have that for any $(x, Z^I, Z^{II}) \in \mathbb{R}^d\times\mathbb{R}^k\times\mathbb{R}^{N-k}$, $Q_\nu(x, Z^I, Z^{II}) \leq -\lambda I$, $\lambda > 0$ to estimate $I_{2,3}$. In fact writing $I_{2,3}$ in the following way

$$\begin{aligned}I_{2,3} &= (R_{22}^{1/2}(x,D)\bar{Q}_\nu U^{II}, R_{22}^{1/2}(x,D)U^{II})_2 \\ &= (\bar{Q}_\nu R_{22}^{1/2}(x,D)U^{II}, R_{22}^{1/2}(x,D)U^{II})_2 + ([R_{22}^{1/2}(x,D), \bar{Q}_\nu]U^{II}, U^{II})_2,\end{aligned}$$

taking into account Proposition 2.3 and the dissipative condition on $Q_\nu(x,U^I,\varepsilon U^{II})$ we have

$$I_{2,3} \le -\frac{\lambda}{2}\|U^{II}\|^2 + c\|U^{II}\|^2_{H^{-1}}.$$

Considering the second equation of the system (3.1) and the smoothness of the coefficients we gain the following bound

$$\|U^{II}\|^2_{H^{-1}} \le c\sqrt{E(t)}$$

and finally the energy inequality (3.7). ■

3.2 Strong convergence

We begin with a simple consequence of (i) and (ii) of Theorem (3.3).

Theorem 3.5 *Let us consider the solution $\{U^{II}\}$ of the Cauchy problem for system (3.1). Assume the hypotheses (B.1), (B.2), (B.3), (B.4), (B.5), (D) hold. Then there exists $U^{II0} \in [L^2(\mathbb{R}^d \times [0,T])]^{N-k}$, such that, as $\varepsilon \downarrow 0$, one has (extracting eventually subsequences)*

$$U^{II} \rightharpoonup U^{II0} \qquad \textit{weakly in } L^2(\mathbb{R}^d \times [0,T]) \tag{3.10}$$

$$\varepsilon U^{II} \longrightarrow 0 \qquad \textit{strongly in } L^2_{loc}(\mathbb{R}^d \times \mathbb{R}_+) \tag{3.11}$$

$$\{\varepsilon^2 U^{II}_t\} \longrightarrow 0 \qquad \textit{in } H^{-1}_{loc}(\mathbb{R}^d \times \mathbb{R}_+). \tag{3.12}$$

Our next step is to prove strong convergence for the sequence $\{U^I\}$ in $L^2_{loc}(\mathbb{R}^d \times \mathbb{R}_+)$. To this end we will use only the estimate obtained in the previous paragraph but our main tool in the limit process will be Gérard's and Tartar's Theorem 2.4.

Theorem 3.6 *Let us consider the solution $\{U^I\}$, of the Cauchy problem for system (3.1). Assume the hypotheses (B.1), (B.2), (B.3), (B.4), (B.5), (D) hold. Then there exists $U^{I0} \in [L^2(\mathbb{R}^d \times \mathbb{R}_+)]^k$, such that, as $\varepsilon \downarrow 0$, one has (extracting eventually subsequences)*

$$U^I \longrightarrow U^{I0} \qquad \textit{strongly in } L^2_{loc}(\mathbb{R}^d \times \mathbb{R}_+) \tag{3.13}$$

Proof. By using the hypothesis (D), $\bar{Q}_\nu(x,U^I,\varepsilon U^{II})$ is uniformely bounded in L^∞ then $\varepsilon^{-1}Q_\nu(x,U^I,\varepsilon U^{II}) = \bar{Q}_\nu(x,U^I,\varepsilon U^{II})U^{II}$ is uniformely bounded in L^2, therefore $\varepsilon^{-1}Q_\nu(x,U^I,\varepsilon U^{II})$ is relatively compact in H^{-1}_{loc}. In a similar way, thanks to the conditions on the function $D(U^I,\varepsilon U^{II})$, we get $\varepsilon^{-1}D^I(U^I,\varepsilon U^{II})$ and $D^{II}(U^I,\varepsilon U^{II})$ relatively compact in H^{-1}_{loc}. Now using the relations

$$\begin{aligned} U^I_t + M^{12}(x,D)U^{II} &= \frac{1}{\varepsilon}D^I(U^I,\varepsilon U^{II}) \\ M^{21}(x,D)U^I &= \frac{1}{\varepsilon}Q(x,U^I,\varepsilon U^{II}) + D^{II}(U^I,\varepsilon U^{II}) - \varepsilon^2 U^{II}_t - \varepsilon M^{22}(x,D)U^{II} \end{aligned}$$

and taking into account the a priori estimates and the smoothness of the coefficients of the system we can conclude

$$\begin{pmatrix} U^I_t + M^{12}(x,D)U^{II} \\ M^{21}(x,D)U^I \end{pmatrix} \qquad \text{is relatively compact in } (H^{-1}_{loc})^2.$$

In order to fit onto the framework of Theorem 2.4 we set

$$P\begin{bmatrix}U^I\\U^{II}\end{bmatrix}=\begin{bmatrix}I_{k\times k}&0\\0&0\end{bmatrix}\partial_t\begin{bmatrix}U^I\\U^{II}\end{bmatrix}+\begin{bmatrix}0&M^{12}(x,D)\\M^{21}(x,D)&0\end{bmatrix}\begin{bmatrix}U^I\\U^{II}\end{bmatrix}$$

the principal symbol of P is given by

$$p(x,\xi)=\begin{bmatrix}I_{k\times k}&0\\0&0\end{bmatrix}\xi_0+\begin{bmatrix}0&M^{12}(x,\xi')\\M^{21}(x,\xi')&0\end{bmatrix},$$

for $\xi=(\xi',\xi_0)\in\mathbb{R}^{d+1}$, $|\xi|=1$. We notice that

$$p(x,\xi)\begin{bmatrix}\lambda\\\mu\end{bmatrix}=0\Longleftrightarrow\left\{\begin{array}{l}\xi_0\lambda+M^{12}(x,\xi')\mu=0\\M_j^{21}(x,\xi')\lambda=0\end{array}\right.$$

for all $\lambda\in\mathbb{R}^k$, $\mu\in\mathbb{R}^{N-k}$.
Now if $\xi'=0$ then $\xi_0\neq 0$ and so $\lambda=0$, otherwise if $\xi'\neq 0$ using hypotheses (B.3) we get $\lambda=0$. Therefore

$$\left\{(x,\xi,\lambda,\mu)\quad\text{such that}\quad p(x,\xi)\begin{bmatrix}\lambda\\\mu\end{bmatrix}=0\right\}\subset\{\lambda\mid\lambda=0\}.$$

We take now

$$q(x)=\begin{bmatrix}I_{k\times k}&0\\0&0\end{bmatrix}$$

and for all $\xi\neq 0$, $\xi=(\xi_0,\xi')$ we have

$$p(x,\xi)\begin{bmatrix}\lambda\\\mu\end{bmatrix}=0\qquad\text{implies}\qquad\langle q(x)\begin{bmatrix}\lambda\\\mu\end{bmatrix},\begin{bmatrix}\lambda\\\mu\end{bmatrix}\rangle=0$$

for all $\lambda\in\mathbb{R}^k$, $\mu\in\mathbb{R}^{N-k}$.
Now we can apply Theorem (2.4) of Gérard and we conclude that for any $\varphi\in\mathcal{D}(\Omega)$

$$\int_0^{+\infty}\int_{\mathbb{R}^d}\left|U^I(x,t)\right|^2\varphi(x,t)dxdt\longrightarrow\int_0^{+\infty}\int_{\mathbb{R}^d}\left|U^{I0}(x,t)\right|^2\varphi(x,t)dxdt$$

where U^{I0} denotes, in view of Theorem (3.3) the weak limit of U^I in $L^2(\mathbb{R}^d\times\mathbb{R}_+)$. In this way we obtain

$$U^I\longrightarrow U^{I0}\qquad\text{strongly in } L^2_{loc}(\mathbb{R}^d\times\mathbb{R}_+).$$

■

Corollary 3.7 *Assume that the hypotheses of Theorems (3.5), (3.6) hold, then* (U^{I0},U^{II0}) *verifies, in the sense of distributions, the following system*

$$\left\{\begin{array}{l}U_t^{I0}+M^{12}(x,D)U^{II0}=D_\nu^I(U^{I0},0)U^{II0}\\[2ex]M^{21}(x,D)U^{I0}=Q_\nu(x,U^{I0},0)U^{II0}+D^{II}(U^{I0},0).\end{array}\right.\tag{3.14}$$

Proof. Using the mean value theorem we have $\varepsilon^{-1}Q(x,U^I,\varepsilon U^{II}) = Q_\nu(x,U^I,\varepsilon\theta U^{II})U^{II}$, $0 \le \theta \le 1$, taking into account the regularity hypothesis on Q, the strong convergence of $\{U^I,\varepsilon U^{II}\}$ in L^2_{loc} and the weak convergence of U^{II} we get

$$\varepsilon^{-1}Q_\nu(x,U^I,\varepsilon U^{II}) \rightharpoonup Q_\nu(x,U^{I0},0)U^{II0} \qquad \text{weakly in } L^2_{loc}(\mathbb{R}^d \times \mathbb{R}_+).$$

In a similar way we have

$$\varepsilon^{-1}D^I(U^I,\varepsilon U^{II}) \rightharpoonup D^I_\nu(U^{I0},0)U^{II0} \qquad \text{weakly in } L^2_{loc}(\mathbb{R}^d \times \mathbb{R}_+).$$

Now we can easly pass to the limit in the other terms and conclude with relaxation result. ■

3.3 Parabolicity

In the case we consider a differential operator of the form

$$W_t + \frac{1}{\varepsilon}\sum_{j=1}^d A_j(x)\partial_j W = \frac{1}{\varepsilon^2}B(x,W) + \frac{1}{\varepsilon}D(W),$$

(3.14) becomes

$$\begin{cases} U^{I0}_t + \sum_{j=1}^d M^{12}_j(x)\partial_j U^{II0} = D^I_\nu(U^{I0},0)U^{II0} \\ \sum_{j=1}^d M^{21}_j(x)\partial_j U^{I0} = Q_\nu(x,U^{I0},0)U^{II0} + D^{II}(U^{I0},0). \end{cases} \tag{3.15}$$

An equivalent way to write (3.15) is given by (where we set $U = U^{I0}$)

$$\begin{aligned} U_t + \sum_{j=1}^d M^{12}_j(x)\partial_j\left(Q_\nu^{-1}(x,U,0)\sum_{k=1}^d M^{21}_k(x)\partial_k U\right) &= \sum_{j=1}^d M^{12}_j(x)\partial_j(Q_\nu^{-1}(x,U,0)D^{II}(U,0)) \\ &+ D^I_\nu(U,0)Q_\nu^{-1}(x,U,0)\left[\sum_{k=1}^d M^{21}_k(x)\partial_k U - D^{II}(U,0)\right]. \end{aligned} \tag{3.16}$$

After some calculation we have, taking into account the notations of Section 2.1,

$$L = M^{12}(x,\xi)Q_\nu^{-1}(x,U,0)M^{21}(x,\xi) \in S^2_{1,0}$$

where $M^{12}(x,\xi) = i\sum_{j=1}^d M^{12}_j(x)\xi_j$ and $M^{21}(x,\xi) = i\sum_{j=1}^d M^{12}_k(x)\xi_k$. Now we are going to show the existence of a positive matrix P_0 such that $P_0 L + L^* P_0$ is negative definite. From the definition of symmetrizer for the system (2.3) follow these relations between the blocks of $R(x,\xi)$ and the coefficients of the system (2.3) :

$$R_{11}(x,\xi)M^{12}(x,\xi) = (M^{21}(x,\xi))^T R_{22}(x,\xi), \quad R_{22}(x,\xi)M^{21}(x,\xi) = (M^{12}(x,\xi))^T R_{11}(x,\xi). \tag{3.17}$$

Now we set

$$P_0 = R_{11}(x,\xi)$$

we have to prove

$$P_0 L + L^* P_0 \leq -\beta I \qquad \beta > 0.$$

Using the relations (3.17) and the condition (D) we get

$$\begin{aligned}
((P_0 L + L^* P_0)\eta, \eta) &= \big(R_{22}(x,\xi) M^{12}(x,\xi) Q_\nu^{-1}(x,U,0) M^{21}(x,\xi)\eta, \eta\big) \\
&+ \big(M^{21}(x,\xi)\big)^T (Q_\nu^{-1}(x,U,0))^T (M^{12}(x,\xi))^T R_{11}(x,\xi)\eta, \eta\big) \\
&= \big((M^{21}(x,\xi))^T R_{22}(x,\xi) Q_\nu^{-1}(x,U,0) M^{21}(x,\xi)\eta, \eta\big) \\
&+ \big((M^{21}(x,\xi))^T (Q_\nu^{-1}(x,U,0))^T R_{22}(x,\xi) M^{21}(x,\xi)\eta, \eta\big) \\
&= \big((M^{21}(x,\xi))^T (R_{22}(x,\xi) Q_\nu^{-1}(x,U,0) + (Q_\nu^{-1}(x,U,0))^T R_{22}(x,\xi)) M^{21}(x,\xi)\eta, \eta\big) \\
&\leq -\lambda_0 \left| M^{21}(x,\xi)\eta \right|^2 \leq -\beta \left|\eta\right|^2
\end{aligned}$$

with $\beta > 0$, taking into account hypotheses B.3. That is exactly the parabolicity in the sense of Section 2.1.

4 Outline of constant coefficients

In this section we want to give an over view of which are the difference and the simplification when we deal with a differential semilinear system with constant coefficient. First of all we want to remark that in this case we don't need to use pseudodifferential theory because of the constant coefficient we can handle them with classical methods. We point out that also in this case we will assume the existence of a symmetrizer $R(x, D)$ with block structure but since the coefficients are constant the principal symbol of $R(x, D)$ depends only on the variable ξ more exactly it is a radial function homogeneous of degree zero of ξ, $R(\xi) = R\left(\frac{\xi}{|\xi|}\right)$.

4.1 Sketch on a priori estimate

Here we consider the following system

$$\begin{cases}
U_t^I + \displaystyle\sum_{j=1}^d M_j^{12} \partial_j U^{II} = \frac{1}{\varepsilon} D^I(U^I, \varepsilon U^{II}) \\
\varepsilon^2 U_t^{II} + \displaystyle\sum_{j=1}^d M_j^{21} \partial_j U^I + \varepsilon \sum_{j=1}^d M_j^{22} \partial_j U^{II} = \frac{1}{\varepsilon} Q(x, U^I, \varepsilon U^{II}) + D^{II}(U^I, \varepsilon U^{II}).
\end{cases} \tag{4.1}$$

with the same hypotheses (B.1), (B.2), (B.3), (B.4), (B.5), (D) of Section 3.1. Then we have also in this case the following theorem

Theorem 4.1 *Let us consider the solution $\{U^I\}$, $\{U^{II}\}$ of the Cauchy problem for system (4.1). Assume that the hypotheses (B.1), (B.2), (B.3), (B.4), (B.5), (D) hold. Then for ε small enough, one has*

(i) *there exists $T > 0$, $M(T) > 0$, independent from ε, such that* $\|U^{II}\|_{L^2(\mathbb{R}^d \times [0,T])} \leq M(T)$ *and* $\sup_{t \geq 0} \|\varepsilon U^{II}(\cdot, t)\| \leq M(T)$,

(ii) $\{\varepsilon^2 U_t^{II}\}$ *is relatively compact in* $H^{-1}_{loc}(\mathbb{R}^d \times \mathbb{R}_+)$,

(iii) $\{U^I\}$ *is uniformely bounded, with respect to* ε, *in* $L^\infty\left(\mathbb{R}_+, L^2(\mathbb{R}^d)\right)$, *namely there exists* $T > 0$, $M(T) > 0$, *independent from* ε, *such that* $\sup_{t\geq 0} \|U^I(\cdot,t)\| \leq M(T)$.

Proof. We apply the Fourier transform to the system (4.1) and we multiply in $L^2(\mathbb{R}^d)$ by $R(\xi)\widehat{U}$ obtaining the following inequality

$$\frac{d}{dt}\left\{\frac{\varepsilon^2}{2}|R_{22}^{1/2}(\xi)\widehat{U^{II}}|^2 + |R_{11}^{1/2}(\xi)\widehat{U^I}|^2\right\} = (R_{11}(\xi)\widehat{U^I}, \varepsilon^{-1}\widehat{D^I}(U^I, \varepsilon U^{II})_2$$
$$+ (R_{22}(\xi)\widehat{U^{II}}, \varepsilon^{-1}\widehat{Q}(U^I, \varepsilon U^{II}))_2 + (R_{22}(\xi)\widehat{U^{II}}, \widehat{D^{II}}(U^I, \varepsilon U^{II}))_2.$$

Defining the energy as in (3.6), taking into account the hypotheses, the properties of the symmetrizer and Plancharel theorem it yields the standard energy inequality (3.7). The remaining part of the proof follows exactly the same arguments used in the previous section so it is omitted. ∎

4.2 Basic ideas on strong convergence

Finally we prove strong convergence for our relaxing sequence. It will be shown following the same path of Section 3.2 with a difference. Since in this case we have constant coefficients we can apply the ideas of Tartar' s compensated compactness [36], [37], [34] (see also [3]) that we briefly recall in the following theorem

Theorem 4.2 *Let us consider*

- *a bounded open set* $\Omega \subset \mathbb{R}^n$;
- *a sequence* $\{l^\nu\}_{\nu=1}^\infty$, $l^\nu : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$;
- *a symmetric matrix* $M : \mathbb{R}^m \longrightarrow \mathbb{R}^m$;
- *constants* $a^i_{jk} \in \mathbb{R}$, $i = 1, \dots, q$, $j = 1, \dots, m$, $k = 1, \dots, n$.

Let us define

$$f(\alpha) = \langle M\alpha, \alpha\rangle, \quad \text{for all } \alpha \in \mathbb{R}^m;$$
$$\Lambda = \left\{\lambda \in \mathbb{R}^m : \exists\xi \in \mathbb{R}^n \setminus \{0\}, \sum_{j,k} a^i_{jk}\lambda_j\xi_k = 0, i = 1, \dots, q\right\}.$$

Assume that

(a) *there exists* $\widetilde{l} \in L^2_m(\Omega)$ *such that* $l^\nu \rightharpoonup \widetilde{l}$ *in* $L^2_m(\Omega)$ *as* $\nu \uparrow \infty$;

(b) $\mathcal{A}^i l^\nu = \sum_{j,k} a^i_{jk}\frac{\partial l^\nu_j}{\partial x_k}$, $i = 1, \dots q$ *are relatively compact in* $H^{-1}_{loc}(\Omega)$;

(c) $f_{|\Lambda} \equiv 0$;

(d) *there exists* $\widetilde{f} \in \mathbb{R}$ *such that* $f(l) \rightharpoonup \widetilde{f}$ *in the sense of measures* $\mathcal{M}(\Omega)$.

Then we have $\widetilde{f} = f(\widetilde{l})$.

Now we can state our convergence result

Theorem 4.3 *Let us consider the solutions $\{U^I, U^{II}\}$ of the Cauchy problem for system (4.1). Assume the hypotheses (B.1), (B.2), (B.3), (B.4), (B.5), (B.6) hold. Then there exists $U^{I0} \in [L^2(\mathbb{R}^d \times [0,T])]^k$ and $U^{II0} \in [L^2(\mathbb{R}^d \times [0,T])]^{N-k}$, such that, as $\varepsilon \downarrow 0$, one has (extracting eventually subsequences)*

$$U^{II} \rightharpoonup U^{II0} \qquad \text{weakly in } L^2(\mathbb{R}^d \times [0,T]) \tag{4.2}$$

$$\varepsilon U^{II} \longrightarrow 0 \qquad \text{strongly in } L^2_{loc}(\mathbb{R}^d \times \mathbb{R}_+) \tag{4.3}$$

$$U^{I} \longrightarrow U^{I0} \qquad \text{strongly in } L^2_{loc}(\mathbb{R}^d \times \mathbb{R}_+) \tag{4.4}$$

and the limit profile (U^{I0}, U^{II0}) verifies in the sense of distribution the system

$$\begin{cases} U^{I0}_t + \displaystyle\sum_{j=1}^d M_j^{12} \partial_j U^{II0} = D^I_\nu(U^{I0}, 0) U^{II0} \\ \displaystyle\sum_{j=1}^d M_j^{21} \partial_j U^{I0} = Q_\nu(U^{I0}, 0) U^{II0} + D^{II}(U^{I0}, 0). \end{cases} \tag{4.5}$$

Proof. (4.2), (4.3) follow from (i) of Theorem 4.1. From the energy estimate it follows that

$$\begin{pmatrix} U^I_t + \displaystyle\sum_{j=1}^d M_j^{12} \partial_j U^{II} \\ \displaystyle\sum_{j=1}^d M_j^{21} \partial_j U^{I} \end{pmatrix} \qquad \text{is relatively compact in } (H^{-1}_{loc})^2.$$

In order to fit into the framework of Theorem 4.2 we set $l^\varepsilon = (U^I, U^{II})$, then the characteristic manifold Λ is given by

$$\Lambda = \left\{ (\lambda, \mu) \in \mathbb{R}^k \times \mathbb{R}^{N-k} \mid \exists \xi \in \mathbb{R}^{d+1} \setminus \{0\}, B(\xi, \lambda) = 0 \right\}$$

where

$$B(\xi, \lambda) = \begin{pmatrix} \xi_0 \lambda + \left(\displaystyle\sum_{j=1}^d M_j^{22} \xi_j \right) \mu \\ \left(\displaystyle\sum_{j=1}^d M_j^{21}(x) \xi_j \right) \lambda \end{pmatrix}.$$

Using hypotheses (B.4) and defining $M = \frac{1}{2} \left(\begin{smallmatrix} I_{k\times k} & 0 \\ 0 & 0 \end{smallmatrix} \right)$, we have $f(\lambda) = \lambda^T M \lambda$ and, of course, $f_{|\Lambda} \equiv 0$. We apply Theorem 4.2 to conclude that

$$(U^I)^2 \rightharpoonup (U^{I0})^2 \qquad \text{in the sense of measure}$$

and finally

$$U^I \longrightarrow U^{I0} \qquad \text{strongly in } L^2_{loc}(\mathbb{R}^d \times \mathbb{R}_+).$$

So we can pass to the limit into the nonlinear terms of the system (4.1) and conclude with the relaxation result. ∎

Remark 4.4 An equivalent form of system (4.5) is given by (setting $U^{I0} = U$)

$$U_t + \sum_{j=1}^{d} M_j^{12} \partial_j \left(Q_\nu^{-1}(x, U, 0) \sum_{k=1}^{d} M_k^{21} \partial_k U \right) = \sum_{j=1}^{d} M_j^{12} \partial_j (Q_\nu^{-1}(x, U, 0) D^{II}(U, 0))$$
$$+ D_\nu^I(U^I, 0) Q_\nu^{-1}(x, U, 0) \left[\sum_{k=1}^{d} M_k^{21} \partial_k U - D^{II}(U, 0) \right]. \tag{4.6}$$

The parabolicity of (4.6) follows with the same arguments of the variable coefficients case. ■

5 Approximation of given parabolic systems

In this last section we want to apply the theory of the previous ones. In fact here we reconstruct a parabolic system by means of a suitable larger semilinear hyperbolic system that relaxes on it. We will consider a linear parabolic system and a quasilinear parabolic system.

5.1 Linear case

Consider the following system

$$U_t = \sum_{j,k=1}^{d} A_{j,k}(x) \partial_j \partial_k U \tag{5.1}$$

where $x \in \mathbb{R}^d$, $t \in \mathbb{R}_+$, $U = U(x,t) \in \mathbb{R}^k$, $k = \frac{N}{2}$. We make the following hypothesis.

(C.1) $A_{j,k}(x) \in C^\infty(\mathbb{R}^d; \mathcal{M}_{k \times k})$, for any $j, k = 1, \dots d$ and $\sum_{j,k=1}^{d} A_{j,k}(x) \lambda_j \lambda_k \geq c_0 |\lambda|^2 I$, for any $\lambda \in \mathbb{R}^d$ (strong parabolicity).

Now we denote by

$$A(x, \xi) = -\sum_{j,k} A_{j,k}(x) \xi_j \xi_k, \tag{5.2}$$

$\xi \in \mathbb{R}^d$, the principal symbol of (5.1) and by $A(x, D) = OPA(x, \xi)$. Now we set

$$B(x, D) = OP[(-A(x, \xi))^{1/2}] \tag{5.3}$$

We have the following theorem.

Theorem 5.1 *Let us consider the system (5.1), suppose that hypothesis (C.1) holds, then the solution of the system*

$$\begin{cases} Z_t^I + \frac{1}{\varepsilon} B(x, D) Z^{II} = 0 \\ Z_t^{II} - \frac{1}{\varepsilon} B(x, D) Z^I = -\frac{1}{\varepsilon^2} Z^{II} \end{cases} \tag{5.4}$$

where $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$, $Z^I = Z^I(x,t) \in \mathbb{R}^k$, $Z^{II} = Z^{II}(x,t) \in \mathbb{R}^{N-k}$, *approximate the system (5.1) in the sense of the Theorem 3.5 and the Theorem 3.6.*

Proof. Rescaling the variables as in (2.9) system (5.1) transforms into

$$\begin{cases} U_t^I + B(x,D)U^{II} = 0 \\ \\ \varepsilon^2 U_t^{II} - B(x,D)U^I = -U^{II} \end{cases} \tag{5.5}$$

Using (C.1) we have that (5.5) is an hyperbolic system. Let us denote by $M^{12}(x,D) = B(x,D)$, $M^{21}(x,D) = -B(x,D)$, by (C.1) hypotheses (B.2) and (B.3) are satisfied. It can also be easily verified that the symmetrizer of (5.4) is given by the matrix $I_{2k\times 2k}$. Finally we have to verify the dissipativity condition (D). Setting $Q(x, Z^I, Z^{II}) = -Z^{II}$ we see that (D) is automatically satisfied. Now we can apply Theorem 3.5 and Theorem 3.6 and we obtain that the solution of (5.5) satisfy as $\varepsilon \downarrow 0$ (setting $U^{I0} = U$)

$$U_t - A(x,D)U = 0$$

that is exactly (5.1) ■

5.2 Quasilinear case

Now we consider the following quasilinear system

$$U_t = \sum_{j,k=1}^{d} \partial_k \left(B(U)\partial_j U + F(U)\right) \tag{5.6}$$

where $x \in \mathbb{R}^d$, $t \in \mathbb{R}_+$, $U = U(x,t) \in \mathbb{R}^k$, $k = \frac{N}{2}$. We make the following hypothesis.

(E.1) $B(U) \in C^1(\mathbb{R}^d; \mathcal{M}_{k\times k})$, and $\sum_{j,k} B(U)\lambda_j\lambda_k \geq c_0|\lambda|^2 I$ for any $\lambda \in \mathbb{R}^d$ (strong parabolicity), moreover $(B^{-1}(U))_\nu$ is bounded on U,

(E.2) $F(U) \in C^1(\mathbb{R}^d; \mathbb{R}^k)$, $F(0) = 0$, $(F(U))_\nu$ is bounded on U.

We have the following theorem

Theorem 5.2 *Let us consider the system (5.6), suppose that hypothesis (E.1) holds, then the solution of the system*

$$\begin{cases} Z_t^I + \dfrac{c}{\varepsilon}\displaystyle\sum_{j=1}^{d} \partial_j Z^{II} = 0 \\ Z_t^{II} + \dfrac{c}{\varepsilon}\displaystyle\sum_{j=1}^{d} \partial_j Z^{I} = -\dfrac{1}{c^2\varepsilon^2} B^{-1}(Z^I)Z^{II} - c^2 B^{-1}(Z^I)F(Z^I) \end{cases} \tag{5.7}$$

where $c \in \mathbb{R}$, $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$, $Z^I = Z^I(x,t) \in \mathbb{R}^k$, $Z^{II} = Z^{II}(x,t) \in \mathbb{R}^{N-k}$, approximate the system (5.6) in the sense of the Theorem 4.3.

Proof. By construction (5.7) is hyperbolic. Rescaling the variables as in (2.9) system (5.7) transforms into

$$\begin{cases} U_t^I + c\displaystyle\sum_{j=1}^{d} \partial_j U^{II} = 0 \\ \varepsilon^2 U_t^{II} + c\displaystyle\sum_{j=1}^{d} \partial_j U^{I} = -\dfrac{1}{c^2} B^{-1}(U^I)U^{II} - c^2 B^{-1}(U^I)F(U^I). \end{cases} \tag{5.8}$$

Let us denote by $M_j^{12} = cI_{k\times k}$, $M_j^{21} = cI_{k\times k}$, for any $j = 1, \dots, d$, then conditions (B.2), (B.3) of Section 3 are satisfied. Now setting $D^{II}(Z^I, Z^{II}) = -c^2 B^{-1}(Z^I)F(Z^I)$ we can easily verify (B.4). We can observe the symmetrizer is given by the matrix $I_{2k\times 2k}$. Denoting by $Q(x, Z^I, Z^{II}) = -B^{-1}(Z^I)Z^{II}$ we have obviously condition (D). Now we can apply Theorem 4.3 and we obtain that the solutions to (5.7) satisfy as $\varepsilon \downarrow 0$, (setting $U^{I0} = U$)

$$U_t - \sum_{k=1}^{d} \partial_k \left(B(U) \sum_{j=1}^{d} \partial_j U \right) = \sum_{k=1}^{d} \partial_k \left(B(U) B^{-1}(U) F(U) \right)$$

that is exactly

$$U_t = \sum_{j,k=1}^{d} \partial_k \left(B(U) \partial_j U + F(U) \right).$$

■

References

[1] F. Bouchut, F. Guarguaglini and R. Natalini. Diffusive BGK approximation for nonlinear multidimensional parabolic equations, Technical report, IAC-CNR, 1999.

[2] G.-Q. Chen, Levermore C.D., and T.-P. Liu. Hyperbolic conservation laws with stiff relaxation terms and entropy. *Comm. Pure Appl. Math.*, 47:787–830, 1994.

[3] B. Dacorogna. *Weak Continuity and Weak Lower Semicontinuity of Nonlinear Functional.* Springer–Verlag, 1982.

[4] R.J. DiPerna. Convergence of approximate solutions to conservation laws. *Arch. Rational Mech. Anal.*, 82:27–70, 1983.

[5] R.J. DiPerna. Compensated compactness and general systems of conservation laws. *Trans. Amer. Math. Soc.*, 292:383–420, 1985.

[6] D. Donatelli, P. Marcati. Relaxation of semilinear hyperbolic systems with variable coefficients. *Ricerche di Matematica*, Vol XLVIII, Supplemento, 295-310, 1999.

[7] D. Donatelli, P. Marcati. $1-\mathcal{D}$ Relaxation from hyperbolic to parabolic systems with variable coefficients. (to appear in *Rendiconti dell' Istituto di Matematica dell'Università di Trieste*, Vol XXXI, Supplemento, 2000).

[8] S.D. Eidel'man. *Parabolic Systems.* North-Holland Publishing Company, 1969.

[9] L.C. Evans. *Weak Convergence Methods for Nonlinear Partial Differential Equations.* CBMS Regional Conference series in Mathematics of AMS, 74, 1990.

[10] P. Gérard. Microlocal defect measures. *Comm. Partial Differential Equations*,vol. 16,n.11,pag. 1761-1794.

[11] M.A. Katsoulakis and A.E. Tzavaras. Contractive relaxation systems and interacting particles for scalar conservation laws. *C. R. Acad. Sci. Paris Sr. I Math.* 323 (1996), no. 8, 865–870.

[12] M.A Katsoulakis and A.E. Tzavaras. Contractive relaxation systems and the scalar multidimensional conservation law. *Comm. Partial Differential Equations 22* (1997), no. 1-2, 195–233. (Reviewer: Christophe Cheverry) 35L65

[13] M.A Katsoulakis and A.E. Tzavaras. Multiscale analysis for interacting particles: relaxation systems and scalar conservation laws. *J. Statist. Phys.* 96 (1999), no.

[14] L.Hörmander. *The analysis of linear partial differential operators*, vol.I-IV, Springer, 1983 e 1985.

[15] H.-0. Kreiss. Initial–boundary Value Problems for Hyperbolic systems, *Comm. on Pure and Applied Math,*, vol. XXIII, 277-298,1970

[16] H.-0. Kreiss and J. Lorenz. *Initial–boundary Value Problems and the Navier-Stokes Equations.* Academic Press, 1989.

[17] T.G. Kurtz. Convergence of sequences of semigroups of nonlinear operators with an application to gas kinetics. *Trans. Amer. Math. Soc.*, 186:259–272, 1973.

[18] C. Lattanzio, P. Marcati. The zero relaxation to the drift-diffusion system for the 3-D isentropic Euler-Poisson model for semiconductors. *Discrete Contin. Dynam. Systems*, 5,1999, no.2, 449-455.

[19] C. Lattanzio, R. Natalini. Convergence of diffusive BGK approximation for parabolic systems. Technical report, IAC-CNR, 2000.

[20] C. Lattanzio, W.-A. Yong. Hyperbolic-Parabolic singular limits for first order nonlinear systems. Technical report, Universität Heidelberg, 1999.

[21] P.D. Lax. Shock waves and entropy. In E.A. Zarantonello, editor, *Contributions to Nonlinear Functional Analysis*, pages 603–634. Academic Press, 1971.

[22] J.L. Lions. *Perturbations singulieres dans les problèmes aux limites et en controle optimale*, volume 323. Springer–Verlag, Berlin, 1973.

[23] P.L. Lions and G. Toscani. Diffusive limit for finite velocity Boltzmann kinetic models. *Rev. Mat. Iberoamericana*,13, 473-513, 1997.

[24] T.-P. Liu. Hyperbolic conservation laws with relaxation. *Comm. Math. Phys.*,vol. 108, 153-175, 1987.

[25] A. Majda. *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Dimensions*, volume 53 of *Appl. Math. Sci.* Springer–Verlag, 1984.

[26] A. Majda and S. Osher. Initial–boundary Value Problems for hyperbolic Equations with Uniformly Characteristics Boundary , *Comm. on Pure and Applied Math,*, vol. XXVIII, 607-675, 1975

[27] P. Marcati and A. Milani. The one-dimensional Darcy's law as the limit of a compressible Euler flow. *J. Differential Equations*, 13:129–147, 1990.

[28] P. Marcati, A. Milani, and P. Secchi. Singular convergence of weak solutions for a quasilinear nonhomogeneous hyperbolic system. *Manuscripta Math.*, 60:49–69, 1988.

[29] P. Marcati and R. Natalini. Weak solutions to a hydrodynamic model for semiconductors and relaxation to the drift-diffusion equation. *Arch. Rational Mech. Anal.*, 129:129–145, 1995.

[30] P. Marcati and R. Natalini. Weak solutions to a hydrodynamic model for semiconductors: the Cauchy problem. *Proc. Roy. Soc. Edinburgh*, 125A:115–131, 1995.

[31] P. Marcati, B. Rubino. Hyperbolic to parabolic relaxation theory for quasilinear first order systems, *J. Differential Equations*, Vol 162, No. 2, April 2000, pp. 359-399.

[32] H.P. McKean. The central limit theorem for Carleman's equation. *Israel J. Math.*, 21:54–92, 1975.

[33] G. Métivier. The block structure condition for symmetric hyperbolic systems. Technical report, 1999.

[34] F. Murat. Compacité par compensation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 5:489–507, 1978.

[35] B. Rubino. Weak solutions to quasilinear wave equations of Klein-Gordon or sine-Gordon type and relaxation to reaction-diffusion equations. *Nonlinear Differential Equations and Appl.*, 4, No.4, 439-457 (1997).

[36] L. Tartar. *Compensated compactness and applications to partial differential equations*, volume 39, pages 136–210. Research Notes in Math., 1979.

[37] L. Tartar. The compensated compactness method applied to partial differential equations. In J.M. Ball, editor, *Systems of Nonlinear Partial Differential Equations*, Reidel, Dordrecht, 1983. NATO ASI.

[38] L. Tartar. H-measures, a new approach for studying homogenization and concentration effects in partial differential equations, *Proceedings of the Royal Society of Edinburg*, 115A, 193-230, 1990.

[39] M.E. Taylor. *Pseudodifferential Operators* , volume 34 of *Princeton mathematical series,*. Princetin University Press, Princeton New Jersey, 1981.

[40] M.E. Taylor. *Pseudodifferential Operators and Nonlinear PDE*, volume 100 of *Progress in Mathematics.* Birkäuser, 1991.

[41] M.E. Taylor. *Partial Differential Equations I, II, III*, volume 115-117 of *Appl. Math. Sciences.* Springer-Verlag, 1996.

[42] W.-A. Yong. Singular perturbations of first-order hyperbolic systems with stiff source terms. *J. Differential Equations*, Vol.155, 89-132, 1999.