Diffusive Kinetic Explicit Schemes for Nonlinear Degenerate Parabolic Systems *

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Abstract

We design numerical schemes for nonlinear degenerate parabolic systems. These schemes are based on discrete BGK models where both characteristic velocities and the source-term depend singularly on the relaxation parameter. General stability conditions are derived, and for scalar equations convergence is proved. Although we restrict this paper to cartesian grids, we show that the method may be also adapted to unstructured meshes.

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1 Introduction

In this paper we study discrete kinetic schemes for systems of conservation laws with possibly degenerate diffusion:

$$\partial_t u_k + \sum_{d=1}^{D} \partial_{x_d}[A_{kd}(u)] = \Delta_x[B_k(u)], \qquad (x,t) \in \mathbb{R}^D \times]0,\infty[, \qquad 1 \le k \le K, \quad (1.1)$$

and with initial data

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^D. \quad (1.2)$$

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We assume that the functions A and B are Lipschitz continuous and satisfy for all u lying in some fixed rectangle I:

- For all $\xi \in \mathbb{R}^D$, $\sum_{d=1}^{D} \xi_d A'_d(u)$ has real eigenvalues and is diagonalizable,
- The real parts of the eigenvalues of $B'(u)$ are non negative.

The theory of such systems is far to be complete, except in the scalar case where the well-posedness has a long story, from the first existence results by O.A. Oleinik and S.N. Kruzkov [36], to the recent uniqueness and stability result by J. Carrillo [8], see also [32] for more results and extension to non homogeneous boundary conditions. The reader may also consult some interesting overviews in the articles of S. Evje and K.H. Karlsen [15] and of M. Espedal and K.H. Karlsen [14]. For the theory of general parabolic systems we refer to the books by Eidel'man [13] and by Taylor [44]. For some recent results on parabolic systems with weaker parabolicity conditions, see [11].

From the numerical viewpoint, there is a new interest in this problem for a few years. For classical and recent schemes in the non degenerate case let us refer just to the recent Morton's book [33] and references therein. A first study of convergence of monotone numerical schemes in the fully degenerate case can be found in [15]. Concerning systems, some results are presented in the paper of Karlsen *et al.* [23] where numerical schemes based on a splitting between the hyperbolic and the parabolic parts of the problem are under consideration. Quite effective central schemes have been recently proposed by Kurganov and E. Tadmor [25].

Here we tray to set a complete multidimensional framework to deal with general nonlinear systems. One feature of our schemes, which seems to be original in this context, is that we do not study separately the hyperbolic and parabolic parts of the system. Our viewpoint is global, in the construction of the schemes as well as in the convergence analysis. In particular it may happen that the hyperbolic numerical fluxes depend on the diffusion function B and a numerical antidissipation depending on B compensates partly the usual numerical diffusion. Our approach is inspired, on one part from relaxation schemes [22], [1], and on the other hand from kinetic approximations of hydrodynamic equations, in the hyperbolic setting, see for instance [37]. Concerning diffusive kinetic models and approximations, we refer to the papers of P.L. Lions and G. Toscani [28], S. Jin, L. Pareschi and G. Toscani [21], S. Jin and H.L. Liu [20], A. De Masi, R. Esposito, and J.L. Lebowitz [12]. A recent survey of the results concerning the Boltzmann equation and its various hydrodynamic limits is given in F. Golse's lectures [16].

In [2] we presented a family of discrete kinetic schemes for general hyperbolic systems of conservation laws. These schemes are based on a class of kinetic models which convergence has been proved in [35]. Here, we study numerical approximations of (1.1) which are based on diffusive kinetic models taking the general form:

$$\begin{cases} \partial_t f_l^\epsilon + \displaystyle\sum_{d=1}^{D} \gamma_{ld}^\epsilon \partial_{x_d} f_l^\epsilon = \frac{1}{\epsilon} \left(M_l(\epsilon, u^\epsilon) - f_l^\epsilon \right), \quad l = 1, \dots, L, \\ u^\epsilon(x,t) = \displaystyle\sum_{l=1}^{L} f_l^\epsilon(x,t), \end{cases} \tag{1.3}$$

with initial condition

$$f_l^\epsilon(x,0) = f_{0l}^\epsilon(x), \quad l = 1, \dots, L. \tag{1.4}$$

Here each f_l^ϵ and M_l take values in $\mathbb{R}^K$, ϵ is a positive parameter, and

$$\gamma_{ld}^\epsilon := \lambda_{ld} + \frac{\theta_{ld}}{\sqrt{\epsilon}} \tag{1.5}$$

for some fixed real constants λ_{ld}, θ_{ld}. In the sequel we denote I a domain of $\mathbb{R}^K$ such that $u(x,t) \in I$ for all (x,t). If (1.1) is scalar, $I = [-\|u_0\|_\infty, \|u_0\|_\infty]$. We make the following assumptions:

(M_1) $\displaystyle\sum_{l=1}^{L} M_l(\epsilon, w) = w$ for all $\epsilon \in]0,1]$ and for all $w \in I$;

(M_2) $\displaystyle\sum_{l=1}^{L} \gamma_{ld}^\epsilon M_l(\epsilon, w) = A_d(w)$ for all $\epsilon \in]0,1]$ and for all $w \in I$;

(M_3) $\displaystyle\sum_{l=1}^{L} \theta_{ld}\theta_{lj} M_l(0, w) = \delta_{dj} B(w)$ for all $w \in I$;

(M_4) $M_l(\epsilon, w) \to M_l(0, w)$ when $\epsilon \to 0$, uniformly for $w \in I$.

If M satisfies the above conditions we call it a local Maxwellian Function for system (1.1) on I. It is easy to see that if the sequence u^ϵ converges to some limit function u in a suitable (strong) topology, then the limit function is a weak solution to equation (1.1). In the purely hyperbolic case, this framework reduces to the one developed in [35] and [2]. The model (1.3) was proposed in [6] where convergence was proved in the scalar case. In [26], the convergence was also proved for a special class of one dimensional strongly parabolic systems. We recall the main results of these papers just below the following definition:

Definition 1.1 *A local Maxwellian function M is a Monotone Maxwellian Function (MMF) (respectively Strictly Monotone Maxwellian Function (SMMF)) if for all $l \in \{1, \dots, L\}$, for all $\epsilon \in]0,1]$ and for all $u \in I$, the real parts of the eigenvalues of $M_l'(u)$ are non negative (respectively positive).*

Theorem 1.1 *[6] Suppose $K = 1$. Let $u_0 \in L^\infty(\mathbb{R}^D) \cap L^1(\mathbb{R}^D)$ be fixed, assume that M is a MMF on the interval $I = \{u \in \mathbb{R}; |u| \leq \|u_0\|_\infty\}$ and set $f_{0i}^\epsilon = M_i(\epsilon, u_0)$. For all $\epsilon > 0$ there exists a unique solution $f^\epsilon \in C([0,\infty[; L^1_{loc}(\mathbb{R}^D)^L) \cap L^\infty(\mathbb{R}^D \times]0,\infty[)^L$ to problem (1.3)-(1.4). There exists a subsequence $\{f^{\epsilon_k}\}$ of $\{f^\epsilon\}$ and a bounded function f such that*

$$f^{\epsilon_k} \to f \qquad \text{in } C([0,\infty); L^1_{loc}(\mathbb{R}^D)) \tag{1.6}$$

as $\epsilon_k \to 0$. Moreover the function $u := \sum_{l=1}^{L} f_l$ is the unique weak entropy solution to problem (1.1)-(1.2) in the following sense [8]: for any $\eta : \mathbb{R} \to \mathbb{R}$ convex Lipschitz, there holds

$$\partial_t[\eta(u)] + \operatorname{div}[A_\eta(u)] - \Delta[B_\eta(u)] \leq 0 \quad \text{in } \mathbb{R}^D \times]0,\infty[, \tag{1.7}$$

with

$$A'_\eta = \eta' A', \qquad B'_\eta = \eta' B', \tag{1.8}$$

and

$$\nabla_x[B(u)] \in L^2(\mathbb{R}^D \times]0,\infty[). \tag{1.9}$$

Theorem 1.2 *[26] Consider a system of the form (1.1) in one space dimension such that $B(u)$ verifies the strong parabolicity condition*

$$\nabla B(u) + (\nabla B(u))^* > 0 \tag{1.10}$$

for any u in a suitable compact domain. Take a model (1.3) where M does not depend on ϵ. Let $u_0 \in L^\infty(\mathbb{R})^K$ be fixed, and set $f_{0i}^\epsilon = M_i(u_0)$. Suppose that there exists a convex compact invariant set I for the system (1.1), containing the values of u_0. Suppose that the eigenvalues of M' are real and positive on I and that the commutators $[M'_l(u), A'(u)]$ and $[M'_l(u), B(u)]$ are zero for any $u \in I$ and l. Suppose also that there exists a function $F : I \to \mathbb{R}^K$ and a positive constant α such that, for any $u \in I$ and l, the matrix $M'_l(u)^T F'(u)$ is symmetric and satisfies $M'_l(u)^T F'(u) \geq \alpha > 0$.

Then the Cauchy problem (1.3)-(1.4) admits an unique global in time solution f^ϵ. If moreover f^ϵ is of finite energy then $u^\epsilon \to u$ in $L^p_{loc}(\mathbb{R} \times \mathbb{R}^+)$ for any $p < +\infty$, and u is a weak solution of (1.1)-(1.2).

From the numerical viewpoint, an advantage of a model like (1.3) is that the nonlinearity inside the derivatives in (1.1) is replaced by a semilinearity: the differential part is linear and moreover diagonal, and all the nonlinearity is concentrated in the source-term. Moreover, the ordinary differential system associated to the source-term owns an exact solution. These features allow us to design Riemann solver free schemes for (1.1) by discretizing (1.3) and taking ϵ small enough. Another nice property is that we deal naturally with the diffusion degeneracy, so that hyperbolic-parabolic systems are treated without additional complication. However, two difficulties appear: both the source-term and the characteristic velocities depend singularly on ϵ. The first difficulty already appears in the purely hyperbolic case and has been treated in [2].

In the more general context of conservation laws with relaxation this problem was addressed by S. Jin [19]. The second difficulty is specifically connected to the parabolic nature of (1.1). In somewhat different contexts, answers have been brought, see for example [21] and references therein. Here we propose two different methods, based on two ways of splitting the system (1.1). When dealing with cartesian meshes, which is the only configuration we look for in the present article, both methods lead to similar schemes. However there exists a natural extension of our second method, without any dimensional splitting, to unstructured grids. This aspect will be treated in a future paper. The constructed numerical schemes depend on three main choices: the particular model (1.3), the space discretization, the time discretization. As far as we are concerned by the choice of the space discretization, we only have to construct schemes for linear hyperbolic systems, so that our method may be thought as a way of transferring, *via* the nonlinear Maxwellian functions, the approximation of linear systems to the one of nonlinear systems.

The plan of the paper is the following: in section 2, we give different examples of kinetic approximations. In particular, we construct models such that the hyperbolic discretization comes from some components of (1.3) and the parabolic one comes from the others. For each model we investigate the monotonicity of the Maxwellian Functions and show that it can be read as a generalized subcharacteristic condition (see Whitham [43], Liu [29]). Related numerical schemes are constructed in section 3. We begin by explicit first order in time discretizations and then reach different time approximations, such as higher order or implicit schemes, see subsection 3.3. Section 4 is devoted to the study of numerical stability. In the scalar case we have convergence to the entropy solution of the problem. Our result has to be related to the S. Evje and K.H. Karlsen's one [15] where an additional hypothesis on the regularity of the data is done to prove the convergence of some particular monotone schemes. Here, in our more general framework, we do not need such an assumption. Numerical experiments are presented in section 5, and comparison are made mainly with the central schemes proposed in [25].

2 Models

In [2], some kinetic models for hyperbolic systems of conservation laws and associated numerical schemes have been presented. In the first subsection we show how these models can be modified in such a way that they enter the diffusive framework as explained in the introduction, and in the purely hyperbolic case they reduce to their form in [2]. The second subsection is devoted to others diffusive models. In each case we discuss the monotonicity condition of definition 1.1.

2.1 Models issued from the hyperbolic kinetic theory.

Consider the hyperbolic system of conservation laws:

$$\partial_t u + \sum_{d=1}^{D} \partial_{x_d} A_d(u) = 0. \tag{2.1}$$

We associate to (2.1) a discrete BGK model:

$$\begin{cases} \partial_t f_l^\epsilon + \displaystyle\sum_{d=1}^{D} \lambda_{ld} \partial_{x_d} f_l^\epsilon = \frac{1}{\epsilon} \left(\mathcal{M}_l(u^\epsilon) - f_l^\epsilon \right), \quad l = 1, \ldots, N, \\ u^\epsilon = \displaystyle\sum_{l=1}^{N} f_l^\epsilon. \end{cases} \tag{2.2}$$

with the compatibility conditions:

$$\sum_{l=1}^{N} \mathcal{M}_l(u) = u, \quad \sum_{l=1}^{N} \lambda_{ld} \mathcal{M}_l(u) = A_d(u), \quad \forall u \in I, \quad d = 1, \ldots, D.$$

For scalar equations, it has been shown in [35] that if each $\mathcal{M}_i$ is a nondecreasing function on $[-\|u_0\|_\infty, \|u_0\|_\infty]$ then u^ϵ converges to the unique entropy solution of (2.1). For general systems a stability condition is established in [2], numerical schemes are constructed and their convergence is studied. We refer to [41], [27] for other convergence results, and to [3] for the compatibility of the models (2.2) with the entropies of (2.1).

A diffusive model for (1.1) can be obtained by adding linear combinations of B to the $\mathcal{M}_l$ together with supplementary equations taking into account the diffusive scaling. More precisely we take $N' \geq D + 1$, $L = N + N'$ and we design the model (1.3) under the form:

$$\begin{cases} \partial_t f_l^\epsilon + \displaystyle\sum_{d=1}^{D} \lambda_{ld} \partial_{x_d} f_l^\epsilon = \frac{1}{\epsilon} \left(M_l(u^\epsilon) - f_l^\epsilon \right), & 1 \leq l \leq N, \\ \partial_t f_{N+m}^\epsilon + \left(\mu + \dfrac{\theta \sqrt{N'}}{\sqrt{\epsilon}} \right) \displaystyle\sum_{d=1}^{D} \sigma_{md} \partial_{x_d} f_{N+m}^\epsilon = \frac{1}{\epsilon} \left(\frac{B(u^\epsilon)}{N' \theta^2} - f_{N+m}^\epsilon \right), & 1 \leq m \leq N', \\ u^\epsilon(x,t) = \displaystyle\sum_{l=1}^{N+N'} f_l^\epsilon(x,t). \end{cases} \tag{2.3}$$

For $1 \leq l \leq N$:

$$M_l(u) = \mathcal{M}_l(u) + b_l B(u), \quad b_l \in \mathbb{R}$$

and the λ_{ld} are unchanged.

For the N' last equations:

$\theta \geq 0$, $\mu \geq 0$, and $\left\{\sigma^{(1)}, \ldots, \sigma^{(D)}\right\}$ is an orthonormal family in $\left\{X \in \mathbb{R}^{N'},\ \sum_{m=1}^{N'} X_m = 0\right\}$.

Note that the Maxwellian functions do not depend anymore on ϵ. For such models conditions $(M_1) - (M_4)$ read:

$$\sum_{l=1}^{N} M_l(u) = u - \frac{B(u)}{\theta^2}\ , \qquad \sum_{l=1}^{N} \lambda_{ld} M_l(u) = A_d(u),\ \ 1 \leq d \leq D. \tag{2.4}$$

Remark that a necessary condition for M to be a MMF is that $B' \leq \theta^2 I$.

Let us now develop concrete examples of this process, in one and two space dimensions. We shall prove that the monotonicity condition may be satisfied: it consists in a generalized subcharacteristic condition.

2.1.1 Example 1: the Diagonal Relaxation Model 1 (DRM1)

We first consider the minimal case $N = D + 1$. We take $\lambda_1, \ldots, \lambda_D > 0$ and

$$\lambda_{jd} = \begin{cases} -\lambda_d & if \qquad j = d\ , \\ \lambda_d & if \qquad j = D+1\ , \\ 0 & otherwise. \end{cases}$$

The Maxwellian function for (2.1) is given by means of the following $D+1$ functions $\mathcal{M}_1, \ldots, \mathcal{M}_{D+1}$:

$$\begin{cases} \mathcal{M}_{D+1}(u) &= \left(u + \displaystyle\sum_{d=1}^{D} \lambda_d^{-1} A_d(u)\right) / (D+1)\ , \\ \mathcal{M}_j(u) &= -\lambda_j^{-1} A_j(u) + \mathcal{M}_{D+1}(u)\ , \qquad j = 1, \ldots, D\ . \end{cases}$$

For a one-dimensional system of conservation laws this formulation coincides with the relaxation approximation of [22]. In fact we can set

$$v^\epsilon := \sum_{i=1}^{L} \lambda_{i1} f_i^\epsilon.$$

Then we have

$$\begin{cases} \partial_t u^\epsilon + \partial_x v^\epsilon = 0\ , \\ \partial_t v^\epsilon + \lambda_1^2 \partial_x u^\epsilon = \frac{1}{\epsilon}(A(u^\epsilon) - v^\epsilon)\ . \end{cases} \tag{2.5}$$

A lot a studies have been devoted to this model, see the references in [2]. However, in several space dimensions there is no diagonal form for the formulation of [22]. In the parabolic case we obtain the following diffusive models:

One-dimensional model

We take $N' = 2$ and:

$$\left\{ \begin{array}{l} \gamma^\epsilon = \left(-\lambda, \lambda, -\dfrac{\mu}{\sqrt{2}} - \dfrac{\theta}{\sqrt{\epsilon}}, \dfrac{\mu}{\sqrt{2}} + \dfrac{\theta}{\sqrt{\epsilon}}\right)^T, \\ M(u) = \dfrac{1}{2}\left(u - \dfrac{A(u)}{\lambda} - \dfrac{B(u)}{\theta^2}, u + \dfrac{A(u)}{\lambda} - \dfrac{B(u)}{\theta^2}, \dfrac{B(u)}{\theta^2}, \dfrac{B(u)}{\theta^2}\right)^T. \end{array} \right. \tag{2.6}$$

Suppose that for all u, $A'(u)$ and $B'(u)$ have a basis of common eigenvectors. This assumption is reasonable in view of the invariant regions theory and Theorem 1.2. We denote $\lambda_k(u)$ and $\theta_k^2(u)$ their respective eigenvalues for $1 \le k \le K$. For simplicity we suppose that the eigenvalues of B' are real. M is a MMF if:

$$\sup_{u \in I} \sup_{1 \le k \le K} \frac{|\lambda_k(u)|}{\lambda} + \frac{\theta_k^2(u)}{\theta^2} \le 1. \tag{2.7}$$

This is a generalized subcharacteristic condition. In the purely hyperbolic case we can take $\theta = 0$ and we retrieve as in [2]:

$$\sup_{u \in I} \sup_{1 \le k \le K} \frac{|\lambda_k(u)|}{\lambda} \le 1, \tag{2.8}$$

while the purely parabolic equation gives:

$$\sup_{u \in I} \sup_{1 \le k \le K} \frac{\theta_k^2(u)}{\theta^2} \le 1. \tag{2.9}$$

In general there is a competition between hyperbolic and parabolic parts. The experience shows that the closer λ is to $|A'(u)|$ the higher is the accuracy of the method. Remark also that there is no condition on μ which could be taken equal to zero as in [6]. Actually the value of this parameter will be fixed later for numerical convenience.

Two-dimensional model

We take $N' \ge D + 1 = 3$, $\theta > 0$, $\mu \ge 0$. Let $\{\sigma^{(1)}, \sigma^{(2)}\}$ be orthonormal in H.

$$\gamma_1^\epsilon = \begin{pmatrix} \lambda_1(-1,0,1)^T \\ \left(\mu + \dfrac{\theta\sqrt{N'}}{\sqrt{\epsilon}}\right)\sigma^{(1)} \end{pmatrix}, \quad \gamma_2^\epsilon = \begin{pmatrix} \lambda_2(0,-1,1)^T \\ \left(\mu + \dfrac{\theta\sqrt{N'}}{\sqrt{\epsilon}}\right)\sigma^{(2)} \end{pmatrix}. \tag{2.10}$$

The Maxwellian functions are:

$$M(u) = \frac{1}{3}\begin{pmatrix} u - 2\dfrac{A_1(u)}{\lambda_1} + \dfrac{A_2(u)}{\lambda_2} \\ u + \dfrac{A_1(u)}{\lambda_1} - 2\dfrac{A_2(u)}{\lambda_2} \\ u + \dfrac{A_1(u)}{\lambda_1} + \dfrac{A_2(u)}{\lambda_2} \\ (0,\ldots,0)^T \end{pmatrix} + \frac{B(u)}{\theta^2}\begin{pmatrix} -\dfrac{1}{3}(1,1,1)^T \\ \dfrac{1}{N'}(1,\ldots,1)^T \end{pmatrix}. \tag{2.11}$$

For a scalar equation ($K = 1$), M is a MMF if the following condition is satisfied on I:

$$\max\left(\frac{2A_1'}{\lambda_1} - \frac{A_2'}{\lambda_2}, \frac{-A_1'}{\lambda_1} + \frac{2A_2'}{\lambda_2}, -\frac{A_1'}{\lambda_1} - \frac{A_2'}{\lambda_2}\right) \leq 1 - \frac{B'}{\theta^2}. \tag{2.12}$$

In the purely hyperbolic case we can take $\theta = 0$ and we retrieve the subcharacteristic condition of [2], while the purely parabolic condition is (2.9).

2.1.2 Example 2: the Diagonal Relaxation Model 2 (DRM2)

In the preceding model one can replace λ by two distinct values λ_m, λ_p. For simplicity we write the model only for $K = D = 1$. Again we take $N' = 2$. The model is:

$$\begin{cases} \gamma^\epsilon = \left(\lambda_m, \lambda_p, -\frac{\mu}{\sqrt{2}} - \frac{\theta}{\sqrt{\epsilon}}, \frac{\mu}{\sqrt{2}} + \frac{\theta}{\sqrt{\epsilon}}\right)^T, \\ M_1(u) = \frac{1}{\lambda_p - \lambda_m}\left(\lambda_p\left(u - \frac{B(u)}{\theta^2}\right) - A(u)\right), \\ M_2(u) = \frac{1}{\lambda_p - \lambda_m}\left(-\lambda_m\left(u - \frac{B(u)}{\theta^2}\right) + A(u)\right), \\ M_3(u) = M_4(u) = \frac{B(u)}{2\theta^2}. \end{cases} \tag{2.13}$$

Here we suppose that $\lambda_m < \lambda_p$. The function M is monotone if:

$$\forall u \in I, \quad \lambda_m(1 - \frac{B'(u)}{\theta^2}) \leq A'(u) \leq \lambda_p(1 - \frac{B'(u)}{\theta^2}). \tag{2.14}$$

This condition shows that the characteristic velocities of the kinetic model approximate more sharply A' than the above choice $\lambda_m = -\lambda$, $\lambda_p = \lambda$, letting hope that the related numerical approximation is better. In practice, condition (2.14) implies that

$$\forall u \in I, \quad 1 - \frac{B'(u)}{\theta^2} > 0. \tag{2.15}$$

Thus we fix $\alpha \in]0, 1[$ and we first take θ such that

$$0 < \alpha \leq 1 - \frac{\sup_{u\in I} B'(u)}{\theta^2} \leq 1. \tag{2.16}$$

This parameter being fixed, it remains to take

$$\lambda_m = \inf_{u\in I} \frac{A'(u)}{1 - B'(u)/\theta^2}, \quad \lambda_p = \sup_{u\in I} \frac{A'(u)}{1 - B'(u)/\theta^2}. \tag{2.17}$$

2.1.3 Example 3: Flux Decomposition Method (FDM)

In this example, in view of a more accurate approximation, we take a greater number of equations, $N = 2D + 1$ and, following an idea due to Y. Brenier [7], we decompose the Jacobians of the fluxes in positive and negative part. Denoting by C_d the diagonal matrix of eigenvalues of A'_d and by Q_d the associated matrix of the right eigenvectors we set

$$(A'_d)_\pm = Q_d \, (C_d)_\pm \, Q_d^{-1} \; .$$

$$A'_d = (A'_d)_+ - (A'_d)_- \, , \quad |A'_d| = (A'_d)_+ + (A'_d)_-$$

Then we can define

$$(A_d)_\pm \, (u) = \int_0^1 (A'_d)_\pm \, (tu + (1-t)u_0).(u - u_0) dt \; ,$$

with $A_d(u_0) = 0$. Set now

$$\lambda_{id} = \begin{cases} \lambda_d \delta_{id} & if \quad i = 1, \ldots, D \; , \\ 0 & if \quad i = D+1 \; , \\ -\lambda_{i-(D+1)} \delta_{i-(D+1),i} & if \quad i = D+2, \ldots, 2D+1 \; . \end{cases}$$

for some $\lambda_d > 0$ and

$$\begin{cases} \mathcal{M}_d(u) = (A_d)_+ \, (u)/\lambda_d \quad \text{if } d = 1, \ldots, D \; , \\ \\ \mathcal{M}_{D+1}(u) = u - \left(\displaystyle\sum_{d=1}^{D} \lambda_d^{-1} \left[(A_d)_+ \, (u) + (A_d)_- \, (u) \right] \right) \; , \\ \\ \mathcal{M}_d(u) = \left(A_{d-(D+1)} \right)_- (u)/\lambda_{d-(D+1)} \quad \text{if } d = D+2, \ldots, 2D+1 \; . \end{cases}$$

For a convex flux and an appropriate (first–order) discretization, this choice corresponds, in the relaxation limit $\epsilon \to 0$, to the Engquist-Osher numerical scheme [7] for (2.1).

In the parabolic case we obtain the following diffusive models:

One-dimensional model

We take $N' = 2$ and:

$$\begin{cases} \gamma^\epsilon = \left(\lambda, 0, -\lambda, -\dfrac{\mu}{\sqrt{2}} - \dfrac{\theta}{\sqrt{\epsilon}}, \dfrac{\mu}{\sqrt{2}} + \dfrac{\theta}{\sqrt{\epsilon}} \right)^T \; , \\ M = \left(\dfrac{A_+}{\lambda}, u - \dfrac{A_+ + A_-}{\lambda} - \dfrac{B}{\theta^2}, \dfrac{A_-}{\lambda}, \dfrac{B}{2\theta^2}, \dfrac{B}{2\theta^2} \right)^T . \end{cases} \tag{2.18}$$

M is a MMF if (2.7) is satisfied.

Two-dimensional model

Here again we put the diffusion term on the null velocity:

$$\gamma_1^\epsilon = \begin{pmatrix} \lambda_1(1,0,0,-1,0)^T \\ \left(\mu + \dfrac{\theta\sqrt{N'}}{\sqrt{\epsilon}}\right)\sigma^{(1)} \end{pmatrix}, \quad \gamma_2^\epsilon = \begin{pmatrix} \lambda_2(0,1,0,0,-1)^T \\ \left(\mu + \dfrac{\theta\sqrt{N'}}{\sqrt{\epsilon}}\right)\sigma^{(2)} \end{pmatrix}. \tag{2.19}$$

The Maxwellian functions are:

$$M = \begin{pmatrix} \dfrac{(A_1)_+}{\lambda_1} \\ \dfrac{(A_2)_+}{\lambda_2} \\ u - \dfrac{(A_1)_+ + (A_1)_-}{\lambda_1} - \dfrac{(A_2)_+ + (A_2)_-}{\lambda_2} - \dfrac{B}{\theta^2} \\ \dfrac{(A_1)_-}{\lambda_1} \\ \dfrac{(A_2)_-}{\lambda_2} \\ \dfrac{B}{N'\theta^2}(1,\ldots,1)^T \end{pmatrix}. \tag{2.20}$$

M is a MMF if the following condition is satisfied:

$$\forall u \in I, \quad \frac{|A_1'(u)|}{\lambda_1} + \frac{|A_2'(u)|}{\lambda_2} + \frac{B'(u)}{\theta^2} \leq 1. \tag{2.21}$$

In the purely hyperbolic case we can take $\theta = 0$ and we retrieve the subcharacteristic condition of [2]:

$$\forall u \in I, \quad \frac{|A_1'(u)|}{\lambda_1} + \frac{|A_2'(u)|}{\lambda_2} \leq 1$$

while the parabolic condition is (2.9).

It is an easy calculation to derive diffusive models from the orthogonal velocity method developed in [2].

2.2 Others models

In the preceding subsection all the Maxwellian functions are independent from ϵ. But one can also construct models with dependence on ϵ in the functions M. The following examples come from [6].

2.2.1 Example 4: a 3 velocities model [6].

Let $\overline{N} = 1$, $\lambda_1 = \theta_1 = 0$, $\lambda_2 = -\lambda_3 = \lambda > 0$, $\theta_2 = -\theta_3 = \theta > 0$; then we have

$$M_1(\epsilon, u) = u - \frac{B(u)}{\theta^2},$$

$$M_2(\epsilon,u) = \frac{\left(\lambda + \frac{\theta}{\sqrt{\epsilon}}\right)}{2\left|\lambda + \frac{\theta}{\sqrt{\epsilon}}\right|^2} A(u) + \frac{B(u)}{2\theta^2},$$

and

$$M_3(\epsilon,u) = -\frac{\left(\lambda + \frac{\theta}{\sqrt{\epsilon}}\right)}{2\left|\lambda + \frac{\theta}{\sqrt{\epsilon}}\right|^2} A(u) + \frac{B(u)}{2\theta^2}.$$

If the equation (1.1) does not degenerates or some particular conditions linking A' and B' hold, like

$$B'(u) \geq |A'(u)|, \tag{2.22}$$

the monotonicity conditions are verified for ϵ small enough if $\theta^2 > \max_I B'$.

2.2.2 Example 5

In [20] the authors prove that the problem

$$(E_\epsilon) \qquad \begin{cases} u_t^\epsilon + v_x^\epsilon = 0, \\ \epsilon^2 v_t^\epsilon + a u_x^\epsilon = A(u^\epsilon) - v^\epsilon, \quad (x,t) \in \mathbb{R}\times]0,\infty[, \quad a > 0 \end{cases}$$

as $\epsilon \to 0$ reduces to the problem

$$(E_0) \qquad \begin{cases} u_t + A(u)_x = a u_{xx}, \\ v = A(u) - a u_x, \qquad (x,t) \in \mathbb{R}\times]0,\infty[, \quad a > 0; \end{cases}$$

more precisely they investigate the diffusion limit of system (E_ϵ), with initial data prescribed around a traveling wave of (E_ϵ).

Here we want to show that problem (E_ϵ) can be obtained by our relaxation scheme for the parabolic equation in (E_0), by a suitable choice of parameters λ_i, θ_i and functions M_i, satisfying conditions $(M_1) - (M_4)$ and then we have convergence of $\{u^\epsilon\}$ to u for initial data in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Set $N = 2$,

$$\lambda_1 = \lambda_2 = 0, \qquad \theta_1 = \sqrt{a}, \qquad \theta_2 = -\sqrt{a},$$

and

$$M_1(\epsilon,u) = \frac{u}{2} + \frac{\sqrt{\epsilon}A(u)}{2\sqrt{a}}, \quad M_2(\epsilon,u) = \frac{u}{2} - \frac{\sqrt{\epsilon}A(u)}{2\sqrt{a}}.$$

It is readily seen that conditions $(M_1) - (M_4)$ are immediately verified for ϵ small enough and we obtain the following relaxation system

$$\begin{cases} f_{1_t}^\epsilon + \frac{\sqrt{a}}{\sqrt{\epsilon}} f_{1_x}^\epsilon &= \frac{1}{\epsilon}\left(\frac{u^\epsilon}{2} + \frac{\sqrt{\epsilon}A(u^\epsilon)}{2\sqrt{a}} - f_1^\epsilon\right), \\ \\ f_{2_t}^\epsilon - \frac{\sqrt{a}}{\sqrt{\epsilon}} f_{2_x}^\epsilon &= \frac{1}{\epsilon}\left(\frac{u^\epsilon}{2} + \frac{\sqrt{\epsilon}A(u^\epsilon)}{2\sqrt{a}} - f_2^\epsilon\right). \end{cases}$$

Setting

$$u^\epsilon = f_1^\epsilon + f_2^\epsilon, \qquad v^\epsilon = \frac{|a|}{\sqrt{\epsilon}}(f_1^\epsilon - f_2^\epsilon),$$

we obtain the system (E_ϵ).

3 Numerical schemes

In this section we use the diffusive kinetic models introduced in section 2 to design numerical schemes for the convection-diffusion equation (1.1). The general idea is to construct a discretization of (1.3) in order to obtain, for sufficiently small ϵ, a scheme for (1.1). The problem is challenging because we have to deal with characteristic velocities which become infinite when ϵ tends to zero, together with a singular right-hand side. The singularity of the source-term has been treated in many works because it is also present in the purely hyperbolic case. See [19], [22], [1], [2] for example. The singularity of the velocities is linked to the parabolicity of the problem. It makes difficult the design of a scheme which converges uniformly with respect to ϵ, at least when a kinetic splitting method is used as in [2]. This problem has already been pointed out in others works, as for example in [21]. We overcome this difficulty in two different ways. First we use a kinetic splitting and we connect ϵ with the discretization parameter Δx so that this process provides consistent approximations of (1.1) (subsection 3.1). Another method, available for a continuous range $[0, +\infty[$ of values of ϵ, and inspired by the ideas of [21], is presented in the second subsection.

Let us first introduce some notations. In this paper we restrict ourselves to cartesian uniform grids:

$$\mathbb{R}^D = \bigcup_{\alpha \in \mathbb{Z}^D} \mathrm{I}_\alpha, \quad [0,T] = \bigcup_{0 \leq n \leq N-1} [t_n, t_{n+1}].$$

Set $\alpha = (\alpha_d)_{1 \leq d \leq D} \in \mathbb{Z}^D$ and let e_d be the canonical d^{th} vector in $\mathbb{R}^D$.

As usual we denote by x_α the center of I_α, Δx_d the length of I_α in the direction d, $\Delta t = t_{n+1} - t_n$, $\Delta x = (\Delta x_d)_{1 \leq d \leq D}$. Finally we set

$$f_\Delta^\epsilon(x,t) = \sum_{n=0}^{N-1} \sum_{\alpha \in \mathbb{Z}^D} f_\alpha^{\epsilon,n} \mathbb{1}_{\mathrm{I}_\alpha}(x) \mathbb{1}_{]t_n, t_{n+1}[}(t),$$
$$f_\Delta^{\epsilon,n}(x) = \sum_{\alpha \in \mathbb{Z}^D} f_\alpha^{\epsilon,n} \mathbb{1}_{\mathrm{I}_\alpha}(x),$$
$$f^{\epsilon,n} = (f_\alpha^{\epsilon,n})_{\alpha \in \mathbb{Z}^D}.$$

If

$$u_\alpha^0 = (vol(\mathrm{I}_\alpha))^{-1} \int_{\mathrm{I}_\alpha} u_0(x) dx, \tag{3.1}$$

then f_0^ϵ is approximated by

$$f_\alpha^{\epsilon,0} = M(u_\alpha^0). \tag{3.2}$$

We use for u the same notation as for f here above. When F is a function of u, we denote

$$F_{\alpha}^{\epsilon,n} = F(u_{\alpha}^{\epsilon,n}). \tag{3.3}$$

Our goal is to obtain a stable and consistent approximation of (1.1) in the conservation form:

$$u_{\alpha}^{n+1} = u_{\alpha}^{n} - \sum_{d=1}^{D} \frac{\Delta t}{\Delta x_d} \left(\phi_{\alpha+e_d/2}^{n} - \phi_{\alpha-e_d/2}^{n}\right), \tag{3.4}$$

where $\phi_{\alpha+e_d/2}^{n}$ is consistent with $A_d(u) - \partial_{x_d}[B(u)]$, or more particularly:

$$u_{\alpha}^{n+1} = u_{\alpha}^{n} - \sum_{d=1}^{D} \frac{\Delta t}{\Delta x_d} \left(\mathcal{A}_{\alpha+e_d/2}^{n} - \mathcal{A}_{\alpha-e_d/2}^{n}\right) + \sum_{d=1}^{D} \frac{\Delta t}{\Delta x_d^2} \left(\mathcal{B}_{\alpha+e_d,d}^{n} - 2\mathcal{B}_{\alpha,d}^{n} + \mathcal{B}_{\alpha-e_d,d}^{n}\right), \tag{3.5}$$

with

$$\mathcal{A}_{\alpha+e_d/2}^{n} = \mathcal{A}_d\left(u_{\alpha-a+e_d}^{n}, \ldots, u_{\alpha+a}^{n}\right), \quad a \in (\mathbb{N}^*)^D, \tag{3.6}$$

where the function $\mathcal{A}_d$ of $2a_d.\Pi_{j\neq d}(2a_j+1)$ variables is Lipschitz continuous and satisfies $\mathcal{A}_d\left(u, \ldots, u\right) = A_d(u)$ for all u, and

$$\mathcal{B}_{\alpha,d}^{n} = \mathcal{B}_d\left(u_{\alpha-b}^{n}, \ldots, u_{\alpha+b}^{n}\right), \quad b \in \mathbb{N}^D, \tag{3.7}$$

where the function $\mathcal{B}_d$ of $\Pi_d(2b_d+1)$ variables is Lipschitz continuous and satisfies $\mathcal{B}_d\left(u, \ldots, u\right) = B(u)$. We will focus our attention on the models described in subsection 2.1, in the general formulation (2.3). For the numerical approximation of models described in subsection 2.2, see the end of subsection 3.1.

In subsection 3.3 we construct high order in time and implicit schemes based on the same space discretization.

3.1 A kinetic splitting method

We begin by recalling two well–known results which will be useful in the sequel.

Lemma 3.1 *Consider the scalar transport equation:*

$$\partial_t f + \sum_{d=1}^{D} \gamma_d \partial_{x_d} f = 0, \tag{3.8}$$

and a related linear $D \times 5$ points discretization:

$$\begin{cases} f_{\alpha}^{n+1} = f_{\alpha}^{n} - \displaystyle\sum_{d=1}^{D} \gamma_d \frac{\Delta t}{\Delta x_d} \left(F_{\alpha+e_d/2}^{n} - F_{\alpha-e_d/2}^{n}\right), \\ F_{\alpha+e_d/2}^{n} = \displaystyle\sum_{k=-1}^{2} a_{kd} f_{\alpha+ke_d}^{n}. \end{cases} \tag{3.9}$$

The scheme (3.9) is monotone and consistent with (3.8) if and only if for all d:

$$\gamma_d a_{0d} \geq \gamma_d a_{-1d} \geq 0 \geq \gamma_d a_{2d} \geq \gamma_d a_{1d}, \quad \sum_{k=-1}^{2} a_{kd} = 1, \tag{3.10}$$

and

$$\sum_{d=1}^{D} \gamma_d (a_{0d} - a_{1d}) \frac{\Delta t}{\Delta x_d} \leq 1. \tag{3.11}$$

In that case, the following discrete entropy inequality holds: for all convex function S:

$$\begin{cases} S(f_\alpha^{n+1}) \leq S(f_\alpha^n) - \displaystyle\sum_{d=1}^{D} \gamma_d \frac{\Delta t}{\Delta x_d} \left(S^n_{\alpha+e_d/2} - S^n_{\alpha-e_d/2} \right), \\ S^n_{\alpha+e_d/2} = \displaystyle\sum_{k=-1}^{2} a_{kd} S(f^n_{\alpha+ke_d}). \end{cases} \tag{3.12}$$

Lemma 3.2 *Consider the scalar nonlinear equation:*

$$\partial_t u - \Delta B(u) = 0, \quad B'(u) \geq 0, \tag{3.13}$$

and a related linear $D \times 5$ points discretization:

$$\begin{cases} u_\alpha^{n+1} = u_\alpha^n + \displaystyle\sum_{d=1}^{D} \frac{\Delta t}{\Delta x_d^2} \left(\mathcal{B}^n_{\alpha-e_d,d} - 2\mathcal{B}^n_{\alpha,d} + \mathcal{B}^n_{\alpha+e_d,d} \right), \\ \mathcal{B}^n_{\alpha,d} = \displaystyle\sum_{k=-1}^{1} c_{kd} B(u^n_{\alpha+ke_d}). \end{cases} \tag{3.14}$$

The scheme (3.14) is consistent with (3.13), monotone and symmetric, if and only if $\mathcal{B}^n_{\alpha,d}$ can be put under the form:

$$\mathcal{B}^n_{\alpha,d} = m_d B(u^n_{\alpha-e_d}) + (1-2m_d) B(u^n_\alpha) + m_d B(u^n_{\alpha+e_d}), \quad m_d \in [0; 1/4], \tag{3.15}$$

and Δt satisfies the time step restriction:

$$\Delta t \sum_{d=1}^{D} \frac{2(1-3m_d) \max_{u \in I} B'(u)}{\Delta x^2} \leq 1. \tag{3.16}$$

After these preliminaries, in order to motivate our choices we first detail the one-dimensional case. Then we turn to the general case.

3.1.1 One space dimension

We consider the one-dimensional version of (2.3):

$$\begin{cases} \partial_t f_l^\epsilon + \lambda_l \partial_x f_l^\epsilon = \dfrac{1}{\epsilon}\left(M_l(u^\epsilon) - f_l^\epsilon\right), & 1 \le l \le N, \\ \partial_t f_{N+1}^\epsilon - \left(\dfrac{\mu}{\sqrt{2}} + \dfrac{\theta}{\sqrt{\epsilon}}\right)\partial_x f_{N+1}^\epsilon = \dfrac{1}{\epsilon}\left(\dfrac{B(u^\epsilon)}{2\theta^2} - f_{N+1}^\epsilon\right), & \\ \partial_t f_{N+2}^\epsilon + \left(\dfrac{\mu}{\sqrt{2}} + \dfrac{\theta}{\sqrt{\epsilon}}\right)\partial_x f_{N+2}^\epsilon = \dfrac{1}{\epsilon}\left(\dfrac{B(u^\epsilon)}{2\theta^2} - f_{N+2}^\epsilon\right), & \\ u^\epsilon(x,t) = \displaystyle\sum_{l=1}^{N+2} f_l^\epsilon(x,t). & \end{cases} \tag{3.17}$$

We split (3.17) into its hyperbolic diagonal linear part and an ordinary differential system: for a given $f_\Delta^{\epsilon,n}$, the function $f_\Delta^{\epsilon,n+1/2}$ is an approximate solution at time t_{n+1} of the solution of the problem

$$\partial_t f + \Gamma^\epsilon \partial_x f = 0, \tag{3.18}$$

$$f(t_n) = f_\Delta^{\epsilon,n} . \tag{3.19}$$

Here

$$\Gamma^\epsilon = diag\left(\lambda_1, \ldots, \lambda_N, -\frac{\mu}{\sqrt{2}} - \frac{\theta}{\sqrt{\epsilon}}, \frac{\mu}{\sqrt{2}} + \frac{\theta}{\sqrt{\epsilon}}\right).$$

As the system is diagonal, we may consider each equation separately. We suppose that the scheme is stable and can be put in conservation form:

$$f_\alpha^{\epsilon,n+1/2} = f_\alpha^{\epsilon,n} - \Gamma^\epsilon \frac{\Delta t}{\Delta x}\left(f_{\alpha+\frac{1}{2}}^{\epsilon,n} - f_{\alpha-\frac{1}{2}}^{\epsilon,n}\right). \tag{3.20}$$

The N first equations are approximated by any high order (for example MUSCL type) scheme:

$$f_{\alpha+\frac{1}{2},l}^{\epsilon,n} = F_l\left(f_{\alpha-k_l+1,l}^{\epsilon,n}, \ldots, f_{\alpha+k_l,l}^{\epsilon,n}\right), \quad F_l(g,\ldots,g) = g, \quad 1 \le l \le N, \tag{3.21}$$

with F_l Lipschitz continuous. The last two equations are discretized by a linear monotone scheme. We detail here five points schemes and we suppose that the corresponding numerical flux functions F_l do not depend on ϵ and satisfy the following symmetry property:

$$\forall g \in \mathbb{R}^4, \quad F_{N+1}\left(g_{-1}, \ldots, g_2\right) = F_{N+2}\left(g_2, \ldots, g_{-1}\right). \tag{3.22}$$

This assumption means that the discretization of transport to the left is symmetric to the one of transport to the right. We can write:

$$\begin{cases} f_{\alpha+\frac{1}{2},N+2}^{\epsilon,n} = F_{N+2}\left(f_{\alpha-1,N+2}^{\epsilon,n}, \ldots, f_{\alpha+2,N+2}^{\epsilon,n}\right), \\ F_{N+2}\left(g_{-1}, \ldots, g_2\right) = \displaystyle\sum_{k=-1}^{2} a_k g_k \end{cases} \tag{3.23}$$

with the conditions (3.10). As a particular case we recover three points monotone schemes for which $a_{-1} = a_2 = 0$, $a_0 \geq 0 \geq a_1$ and $a_0 + a_1 = 1$.

For these last two equations the CFL condition (3.11)reads:

$$\left(\frac{\mu}{\sqrt{2}} + \frac{\theta}{\sqrt{\epsilon}}\right)(a_0 - a_1)\frac{\Delta t}{\Delta x} \leq 1. \tag{3.24}$$

In the following, the scheme on the linear part will be referred to as (HS) (*homogeneous scheme*) and the associated evolution operator will be denoted H^ϵ_Δ:

$$f_\Delta^{\epsilon,n+1/2} = H^\epsilon_\Delta(\Delta t) f_\Delta^{\epsilon,n} \ .$$

To take into account the contribution of the singular perturbation term on the right–hand side, we solve on $[t_n, t_{n+1}]$ the ordinary differential system:

$$\begin{cases} F' = \frac{1}{\epsilon}\left(M(u) - F\right), \\ u = \displaystyle\sum_{l=1}^{L} F_l \end{cases} \tag{3.25}$$

with initial data

$$F(t_n) = f_\alpha^{\epsilon,n+1/2}$$

for all $\alpha \in \mathbb{Z}$. We obtain by (M_1):

$$u' = 0 \ ,$$

so that the solution of (3.25) with data $F(t_n) = G$ can be explicitly obtained as

$$\begin{cases} S^\epsilon(t, t_n, G) = M\left(u\right) + \exp\left(-\frac{t-t_n}{\epsilon}\right)\left[G - M\left(u\right)\right], \\ u = \displaystyle\sum_{l=1}^{L} G_l. \end{cases}$$

Hence

$$f_\alpha^{\epsilon,n+1} = M(u_\alpha^{\epsilon,n+1/2}) + \exp\left(-\frac{\Delta t}{\epsilon}\right)\left[f_\alpha^{\epsilon,n+1/2} - M(u_\alpha^{\epsilon,n+1/2})\right].$$

In terms of the macroscopic variable u, we have:

$$u_\alpha^{\epsilon,n+1} = u_\alpha^{\epsilon,n+1/2} = u_\alpha^{\epsilon,n} - \frac{\Delta t}{\Delta x}\sum_{l=1}^{N+2}\gamma_l^\epsilon\left(f^{\epsilon,n}_{\alpha+\frac{1}{2},l} - f^{\epsilon,n}_{\alpha-\frac{1}{2},l}\right).$$

In this formula the fluxes depend explicitly on ϵ and $f_\alpha^{\epsilon,n}$. Clearly ϵ cannot be put to 0 here, so that by this way we do not obtain a relaxed scheme for (1.1). Therefore we consider ϵ as a discretization parameter which has to be connected with Δx and we put $\epsilon = 0$ only in the second step:

$$f_\alpha^{\epsilon,n+1} = S^0\left(t_{n+1}, t_n, (H^\epsilon_\Delta(\Delta t) f_\Delta^{\epsilon,n})_\alpha\right) \ . \tag{3.26}$$

Hence $u_\alpha^{\epsilon,n+1}$ can be expressed only in terms of $u^{\epsilon,n}$ and we have:

$$u_\alpha^{\epsilon,n+1} = u_\alpha^{\epsilon,n} - \frac{\Delta t}{\Delta x}\left(\mathcal{F}_{\alpha+1/2}^{\epsilon,n} - \mathcal{F}_{\alpha-1/2}^{\epsilon,n}\right), \tag{3.27}$$

where

$$\mathcal{F}_{\alpha+1/2}^{\epsilon,n} = \sum_{l=1}^{N+2} \gamma_l^\epsilon F_l\left(M_l\left(u_{\alpha-k_l+1}^{\epsilon,n}\right), \ldots, M_l\left(u_{\alpha+k_l}^{\epsilon,n}\right)\right). \tag{3.28}$$

Note that for fixed ϵ (3.27) is a consistent discretization of the hyperbolic conservation law (2.1): when $\Delta x \to 0$, u_Δ^ϵ converges towards a solution of (2.1). To obtain a consistent, conservation form written, discretization of (1.1) we have to make both ϵ and Δx go to zero.

From now on we use the notation (3.3) for $F = M$, $F = B$. We are going to show that for a certain value $\epsilon = \epsilon(\Delta x)$, we can find functions $\mathcal{A}$ and $\mathcal{B}$ such that:

$$\sum_{l=1}^{N+2} \gamma_l^\epsilon F_l\left(M_{\alpha-k_l+1,l}^{\epsilon,n}, \ldots, M_{\alpha+k_l,l}^{\epsilon,n}\right) = \mathcal{A}_{\alpha+1/2}^n - \frac{\mathcal{B}_{\alpha+1}^n - \mathcal{B}_\alpha^n}{\Delta x}. \tag{3.29}$$

Actually we have:

$$\begin{aligned} \mathcal{F}_{\alpha+1/2}^{\epsilon,n} &= \sum_{l=1}^{N} \lambda_l F_l\left(M_{\alpha-k_l+1,l}^{\epsilon,n}, \ldots, M_{\alpha+k_l,l}^{\epsilon,n}\right) \qquad (3.30)\\ &+ \frac{\gamma_{N+2}^\epsilon}{2\theta^2}\left[F_{N+2}\left(B_{\alpha-s+1}^{\epsilon,n}, \ldots, B_{\alpha+s}^{\epsilon,n}\right) - F_{N+2}\left(B_{\alpha+s}^{\epsilon,n}, \ldots, B_{\alpha-s+1}^{\epsilon,n}\right)\right]. \end{aligned}$$

We can define the numerical flux function associated to A as:

$$\mathcal{A}_{\alpha+1/2}^n = \sum_{l=1}^{N} \lambda_l F_l\left(M_{\alpha-k_l+1,l}^{\epsilon,n}, \ldots, M_{\alpha+k_l,l}^{\epsilon,n}\right), \tag{3.31}$$

which is consistent by (2.4). It remains to find $\mathcal{B}_\alpha^n$ and ϵ such that:

$$\begin{aligned} \mathcal{B}_{\alpha+1}^n - 2\mathcal{B}_\alpha^n + \mathcal{B}_{\alpha-1}^n = \frac{\gamma_{N+2}^\epsilon \Delta x}{2\theta^2}\Big[&B_{\alpha-2}^n(-a_2 + a_{-1}) + B_{\alpha-1}^n(a_2 - a_1 - a_{-1} + a_0)\\ &-2B_\alpha^n(a_0 - a_1) \qquad (3.32)\\ &+ B_{\alpha+1}^n(a_2 - a_1 - a_{-1} + a_0) + B_{\alpha+2}^n(-a_2 + a_{-1})\Big]. \end{aligned}$$

Realizing (3.32) means that we reach a family of linear central five points discretizations of $\partial_{xx}B$. It is easy to prove the following result.

Proposition 3.1 *Given $\{a_k,\ -1 \le k \le 2\}$ satisfying (3.10), there exists $\mathcal{B}_\alpha^n$ and $\epsilon > 0$ such that (3.32) is satisfied. $\mathcal{B}_\alpha^n$ is given by:*

$$\begin{cases} \mathcal{B}_\alpha^n = mB_{\alpha-1}^n + (1-2m)B_\alpha^n + mB_{\alpha+1}^n, \\ m = \dfrac{-a_2 + a_{-1}}{a_0 - a_1 - 3a_2 + 3a_{-1}}. \end{cases} \tag{3.33}$$

and ϵ is such that:

$$\left(\frac{\mu}{\sqrt{2}} + \frac{\theta}{\sqrt{\epsilon}}\right)\frac{\Delta x}{2\theta^2} = \frac{1-3m}{a_0 - a_1}\,. \tag{3.34}$$

Of course μ must be small enough but this is always possible, since there is no other condition on this parameter. As neither the final scheme nor the time step do not depend on μ, we may take for example $\mu = 0$. The obtained ϵ is positive and of order Δx^2. Observe also that conditions (3.10) insure that $m \in [0; 1/4]$. As recalled in Lemma 3.1, this last condition is necessary for the monotonicity of the related first order time discretization of $\partial_t u - \partial_{xx} B(u)$, together with the time step restriction (3.16). The CFL condition (3.24) may be expressed as:

$$\Delta t \leq \frac{\Delta x^2}{2(1-3m)\theta^2}. \tag{3.35}$$

As seen in section 2, θ^2 measures $B'(u)$, hence condition (3.35) is the same as (3.16) in the purely parabolic case, while it is more restrictive in the case where $A' \neq 0$, see (2.7).

Finally, we have designed an approximation of (1.1) under the form (3.5). The hyperbolic flux $\mathcal{A}^n_{\alpha+1/2}$ depends only on the (possibly high order) discretization of the N first equations of (3.17) and is given by (3.31). Note that even for $l \leq N$, M_l depends on B, so that also $\mathcal{A}^n_{\alpha+1/2}$ does. The parabolic function $\mathcal{B}$ depends on the choice of the coefficients $\{a_k,\ -1 \leq k \leq 2\}$ but we always obtain a central linear approximation of $\partial_{xx} B$. The time step restriction for this scheme is the one imposed by the CFL condition on H^ϵ_Δ:

$$\Delta t \leq \min\left\{\frac{\Delta x^2}{2\theta^2(1-3m)},\ \frac{\Delta x}{|\lambda_l|}\ ,\ 1 \leq l \leq N\right\} \leq 1. \tag{3.36}$$

The following proposition gives an important reciprocal: by our method, we are able to reach all monotone linear central five points discretizations of $\partial_{xx} B$.

Proposition 3.2 *We suppose that the flux function $\mathcal{A}$ is given by (3.31). Consider a central five points discretization of $\partial_{xx} B(u)$ under the form $\dfrac{\mathcal{B}_{\alpha+1} - 2\mathcal{B}_\alpha + \mathcal{B}_{\alpha-1}}{\Delta x^2}$ with (3.15). There exist $\epsilon > 0$ and $\{a_k,\ -1 \leq k \leq 2\}$ satisfying (3.10), such that the scheme takes the form (3.5). Moreover the CFL condition can be expressed independently of the a_k as (3.36).*

Proof of proposition 3.2. We could try to look for $\{a^m_k,\ -1 \leq k \leq 2,\ m = 1, 2\}$ such that

$$F_{N+m}\left(g_{-1}, \ldots, g_2\right) = \sum_{k=-1}^{2} a^m_k g_k, \quad m = 1, 2$$

and

$$\begin{cases} -\gamma^{\epsilon}_{N+1}\dfrac{\Delta t}{\Delta x}\left(f^n_{\alpha+1/2,N+1} - f^n_{\alpha-1/2,N+1}\right) = \dfrac{\Delta t}{\Delta x^2}\left(\mathcal{B}^n_{\alpha+1} - \mathcal{B}^n_{\alpha}\right), \\ -\gamma^{\epsilon}_{N+2}\dfrac{\Delta t}{\Delta x}\left(f^n_{\alpha+1/2,N+2} - f^n_{\alpha-1/2,N+2}\right) = \dfrac{\Delta t}{\Delta x^2}\left(\mathcal{B}^n_{\alpha} - \mathcal{B}^n_{\alpha-1}\right). \end{cases}$$

Unfortunately this method does not lead to monotone schemes for the transport equations $N+1$ and $N+2$, except if $m = 0$.

Let us define

$$\rho = \frac{\gamma^{\epsilon}_{N+2}\Delta x}{2\theta^2}, \quad b_k = \rho a_k, \quad -1 \leq k \leq 2. \tag{3.37}$$

We look for ρ and the b_k with ρ not depending on ϵ. The b_k must satisfy:

$$\begin{cases} -b_2 + b_{-1} = m, \\ b_2 - b_1 - b_{-1} + b_0 = 1 - 4m, \\ -b_1 + b_0 = 1 - 3m, \\ b_{-1} + b_0 + b_1 + b_2 = \rho, \end{cases} \tag{3.38}$$

which set of solution is

$$\left\{ b_2 = m + \frac{1}{2}(\rho - 1) - b_1, \;\; b_0 = 1 - 3m + b_1, \;\; b_{-1} = 2m + \frac{1}{2}(\rho - 1) - b_1, \;\; b_1 \in \mathbb{R} \right\}.$$

Conditions (3.10) are satisfied if

$$\max\left[\frac{5}{2}m + \frac{1}{4}(\rho - 3), m + \frac{1}{2}(\rho - 1)\right] \leq b_1 \leq \min\left[\frac{1}{2}m + \frac{1}{4}(\rho - 1), 2m + \frac{1}{2}(\rho - 1)\right]. \tag{3.39}$$

If $m = 1/6$, b_1 exists for all ρ. Else, b_1 exists as soon as ρ is such that:

$$0 < \rho \leq |-6m + 1|. \tag{3.40}$$

As a consequence the a_k exist and ϵ is given by (3.37).

Moreover the CFL condition (3.24) may be written:

$$1 \geq (a_0 - a_1)\frac{\gamma^{\epsilon}_{N+2}\Delta t}{\Delta x} = 2\theta^2(a_0 - a_1)\rho\frac{\Delta t}{\Delta x^2} = 2\theta^2(b_0 - b_1)\frac{\Delta t}{\Delta x^2} = 2\theta^2(1 - 3m)\frac{\Delta t}{\Delta x^2}$$

which ends the proof of the proposition.□

Two particular cases are of interest: $m = 0$ and $m = 1/4$.

For $m = 0$, we have $a_{-1} = a_2 = 0$. Therefore all monotone linear three points discretization of the last two equations of (3.17) lead to the three points discretization

$$\partial_{xx} B \sim \frac{B_{\alpha-1} - 2B_{\alpha} + B_{\alpha+1}}{\Delta x^2}. \tag{3.41}$$

For $m = 1/4$ we have

$$\partial_{xx} B \sim \frac{B_{\alpha-2} - 2B_\alpha + B_{\alpha+2}}{4\Delta x^2}. \tag{3.42}$$

In the proof of proposition 3.2 we can take $\rho = 1/2$ and $a_1 = a_2 = 0$, $a_{-1} = a_0 = 1/2$. In general $\Delta B(u)$ is approximated by

$$\Delta_{m,\alpha} B := \frac{m(B_{\alpha-2} + B_{\alpha+2}) + (1-4m)(B_{\alpha-1} + B_{\alpha+1}) - 2B_\alpha(1-3m)}{\Delta x^2}. \tag{3.43}$$

Remark 3.1 *If one intends to put a second order MUSCL type correction on equations $N+1$, $N+2$, the scheme is no longer efficient because this correction is like $\gamma_{N+2}^{\epsilon(\Delta x)}\Delta x$ which is of order 1 and not of order Δx. The numerical tests confirm this viewpoint, see figure 1.*

3.1.2 Higher space dimensions

One may develop the same method as in 1-D: approximate the transport part by a high order scheme on the N first equations and a linear scheme on the N' last ones (see Lemma 3.1), then solve exactly the ODEs associated to the RHS and put $\epsilon = 0$ in this last step. One obtains:

$$u_\alpha^{\epsilon,n+1} = u_\alpha^{\epsilon,n} - \sum_{d=1}^{D} \frac{\Delta t}{\Delta x_d}\left(\mathcal{F}_{\alpha+e_d/2}^{\epsilon,n} - \mathcal{F}_{\alpha-e_d/2}^{\epsilon,n}\right). \tag{3.44}$$

We retrieve the numerical flux function associated to A with the N first equations:

$$\mathcal{A}_{d,\alpha+e_d/2}^n = \sum_{l=1}^{N} \lambda_{ld} F_{ld}\left(M_l\left(u_{\alpha-k_l+e_d}^{\epsilon,n}\right), \ldots, M_l\left(u_{\alpha+k_l}^{\epsilon,n}\right)\right), \quad d = 1, \ldots, D \tag{3.45}$$

with obvious notations. Concerning the diffusion, we write the generalization of (3.32) and, for example, we find that $\mathcal{B}_{\alpha,d}^n = B_\alpha^n$ if ϵ is such that:

$$\gamma^\epsilon := \mu + \frac{\theta\sqrt{N'}}{\sqrt{\epsilon}} = \frac{N'\theta^2}{s_d \Delta x_d (a_{0d} - a_{1d})} \tag{3.46}$$

where $s_d = \sum_{m,\sigma_{md}>0} \sigma_{md}$. This shows that there is an essential difference with the one dimensional case: an arbitrary choice of the a_{jd} is not possible because γ^ϵ must not depend on d. Nevertheless, and this is the most important, we are able to give a generalization of proposition 3.2.

Proposition 3.3 *We make the following choice:*

$$N' = 2D \text{ and } \sigma_{md} = \begin{cases} -1/\sqrt{2} & \text{if } m = 2d-1, \\ 1/\sqrt{2} & \text{if } m = 2d, \\ 0 & \text{else.} \end{cases} \tag{3.47}$$

Consider a monotone central five points discretization of $\Delta B(u)$ under the form $\sum_{d=1}^{D} \frac{\mathcal{B}_{\alpha+e_d,d} - 2\mathcal{B}_\alpha + \mathcal{B}_{\alpha-e_d,d}}{\Delta x_d^2}$ *with*

$$\mathcal{B}^n_{\alpha,d} = m_d B^n_{\alpha-e_d} + (1-2m_d)B^n_\alpha + m_d B^n_{\alpha+e_d}, \quad m_d \in [0, 1/4]. \tag{3.48}$$

There exist $\epsilon > 0$ and $\{a_{kd},\ -1 \le k \le 2,\ 1 \le d \le D\}$ satisfying (3.10), such that the numerical scheme (3.44) takes the form (3.5). Moreover the CFL condition can be expressed independently of the a_{kd} as:

$$\Delta t \le \min\left\{ \left(\sum_{d=1}^{D} \frac{|\lambda_{ld}|}{\Delta x_d} \right)^{-1}, \left(\sum_{d=1}^{D} \frac{2D(1-3m_d)\theta^2}{\Delta x_d^2} \right)^{-1} \right\}. \tag{3.49}$$

Proof of proposition 3.3. Due to the particular form (3.47), we can follow the proof of proposition 3.2 because for each d, only the two equations $N+2d-1$ and $N+2d$ are involved in the computation. So we define ρ_d by

$$\rho_d = \frac{\gamma^\epsilon \Delta x_d}{2D\theta^2\sqrt{2}}, \quad \gamma^\epsilon = \mu + \frac{\theta\sqrt{2D}}{\sqrt{\epsilon}}. \tag{3.50}$$

It must be such that for all d one of the two following properties is true:

$$m_d \neq 1/6 \text{ and } \rho_d \le |-6m_d + 1|, \tag{3.51}$$

$$m_d = 1/6. \tag{3.52}$$

and this is the case as soon as ϵ is large enough. □

3.1.3 Application to some models.

We give for $D = 1$ the explicit formulation of the schemes obtained for some kinetic models described in section 2. As soon as the model is under the form (3.17), the approximation of $\partial_{xx}B(u)$ is only determinated by the choice of m in formula (3.43), so that we keep the notation of this formula.

1. The parabolic DRM.

We apply a first order upwind scheme on the two first equations of (3.17) with (2.13). The time step restriction is given by (3.36). The parameters λ_m, λ_p and θ

must satisfy the generalized subcharacteristic condition (2.14) or a similar one for systems. If we can take $0 \le \lambda_m < \lambda_p$, then the scheme reads as an upwind scheme:

$$u_\alpha^{n+1} = u_\alpha^n - \frac{\Delta t}{\Delta x}\left(A_\alpha^n - A_{\alpha-1}^n\right) + \Delta t\, \Delta_{m,\alpha} B^n.$$

An upwind formula holds also if $\lambda_m < \lambda_p \le 0$. In general, the sign of A' or its eigenvalues varies: $\lambda_m \le 0 \le \lambda_p$ and:

$$\begin{aligned} u_\alpha^{n+1} = u_\alpha^n - & \frac{\Delta t}{\Delta x}\frac{-\lambda_m A_{\alpha+1}^n + (\lambda_m + \lambda_p)A_\alpha^n - \lambda_p A_{\alpha-1}^n}{\lambda_p - \lambda_m} \\ & + \frac{\Delta t}{\Delta x}\left|\frac{\lambda_m \lambda_p}{\lambda_p - \lambda_m}\right| \left(u_{\alpha+1}^n - 2u_\alpha^n + u_{\alpha-1}^n\right) \\ & - \frac{\Delta t}{\Delta x}\left|\frac{\lambda_m \lambda_p}{\theta^2(\lambda_p - \lambda_m)}\right| \left(B_{\alpha+1}^n - 2B_\alpha^n + B_{\alpha-1}^n\right) \\ & + \Delta t\, \Delta_{m,\alpha} B^n. \end{aligned} \tag{3.53}$$

For the DRM1, we put $\lambda_m = -\lambda$, $\lambda_p = -\lambda$, $\lambda > 0$, and we obtain:

$$\begin{aligned} u_\alpha^{n+1} = u_\alpha^n - & \frac{\Delta t}{\Delta x}\frac{A_{\alpha+1}^n - A_{\alpha-1}^n}{2\Delta x} + \lambda\frac{\Delta t}{2\Delta x}\left(u_{\alpha+1}^n - 2u_\alpha^n + u_{\alpha-1}^n\right) \\ & - \lambda\frac{\Delta t}{2\Delta x}\frac{B_{\alpha+1}^n - 2B_\alpha^n + B_{\alpha-1}^n}{\theta^2} \\ & + \Delta t\, \Delta_{m,\alpha} B^n. \end{aligned} \tag{3.54}$$

The first line is the Lax-Friedrichs type scheme which can be obtained by the purely hyperbolic model [2]. The numerical anti-diffusion on second line comes from the B term of the Maxwellian functions M_1 and M_2, while the third line is the discretization of the parabolic part. We can interpret formulas (3.53) and (3.54) by saying that the numerical dissipation here is proportional to $u - B(u)/\theta^2$, which is a natural generalization of the Lax-Friedrichs scheme. The necessary condition (2.15) implies that this is actually a diffusion, and the diffusion coefficient is measured by α in (2.16).

Due to the fact that the schemes are linear, A and B parts are clearly decoupled and one can also integrate the second line of (3.54) into the parabolic part. If a second order MUSCL correction is added in the two first equations, then must also add to (3.53) or (3.54) some correction terms where A and B parts are not decoupled, and therefore we have a really new way of approximating (1.1), a way that cannot be reduced to a splitting between hyperbolic and parabolic parts of the equation.

Let us write these supplementary terms. For $l = 1, 2$, the numerical flux can be written:

$$f_{\alpha+\frac{1}{2},l}^n = f_{\alpha+\frac{1}{2}(1-\mathrm{sgn}\lambda_l),l}^n + \mathrm{sgn}\lambda_l.\frac{\Delta x}{2}\left(1 - |\lambda_l|\frac{\Delta t}{\Delta x}\right)\sigma_{\alpha+\frac{1}{2}(1-\mathrm{sgn}\lambda_l),l}^n \tag{3.55}$$

where the slope of the solution in the cell I_α, σ_β^n, is evaluated through a nonlinear function:

$$\sigma_{\beta,l}^n = \phi\left(\frac{f_{\beta+1,l}^n - f_{\beta,l}^n}{\Delta x}, \frac{f_{\beta,l}^n - f_{\beta-1,l}^n}{\Delta x}\right) \text{ and } f_{\beta,l}^n = M_l(u_\beta^n).$$

The functions M_l, defined in (2.6) and (2.13), involve A and B. Thus, there is no way to decouple the "A" and "B" parts in the scheme. We want to point out that at first order, we could avoid the implementation of the Maxwellian functions and just write the scheme under the final form (3.53) or (3.54). This is not possible for higher order. Note that nevertheless the implementation remains always easy.

2. The parabolic FDM.

Here again we apply a first order upwind scheme on the three first equations of (3.17) with (2.18). The time step restriction is given by (3.36). The parameters λ and θ must satisfy the generalized subcharacteristic condition (2.7) or a similar one for systems. The scheme is:

$$u_\alpha^{n+1} = u_\alpha^n - \frac{\Delta t}{\Delta x}\left[A_+(u_\alpha^n) + A_-(u_\alpha^n) - A_+(u_{\alpha-1}^n) - A_-(u_{\alpha+1}^n)\right] + \Delta t\,\Delta_{m,\alpha} B^n. \quad (3.56)$$

If A is convex scalar, we recover the hyperbolic numerical flux of Engquist-Osher scheme. It has already been noted that this scheme has a kinetic interpretation, see [7]. There are important differences with the schemes obtained by the DRM. Firstly, B does not appear in the hyperbolic flux, neither at first order nor at second. This is due to the fact that B is not in M_1 nor in M_3. Consequently, the scheme is decoupled. Secondly, the parameters λ and θ do not appear in the above formula, they are necessary only to the computation of the time step and in higher order approximations. In the case where the eigenvalues of A' all have the same sign, the first order discretization of the DRM2 coincide with the above formula. The schemes obtained by the discretization of the DRM2 seem to be a good compromise between facility and efficiency, because they are close to the ones obtained by the FDM but they do not require the complete diagonalization of A' but only the evaluation of its eigenvalues.

Remark 3.2 *This kinetic splitting method can also be used for others models such as those in subsection 2.2. For the 3 velocities one dimensional model, we obtain, at first order, the following scheme:*

$$u_\alpha^{n+1} = u_\alpha^n - \frac{A(u_{\alpha+1}^n) - A(u_{\alpha-1}^n)}{2\Delta x} + \Delta t\,\Delta_{m,\alpha} B^n$$

Here we see clearly the coherence between the monotonicity condition which imposes the non degeneracy of the diffusion and the related discretization which becomes unstable in this case.

3.2 A semi-conservative splitting method

Here, we adopt a more usual viewpoint, which consists in constructing a scheme for the kinetic model (1.3) and then making ϵ tend to zero to obtain a so-called relaxed scheme for (1.1). As seen in the previous section, this is not possible when keeping the kinetic variables.

Let us consider a multidimensional model under the form (2.3) with $N' = D+1$. We transform this system in a linear way, keeping the unknowns $f_1^\epsilon, \dots, f_N^\epsilon$ and replacing $f_{N+1}^\epsilon, \dots, f_{N+N'}^\epsilon$ by

$$u^\epsilon = \sum_{l=1}^{N+N'} f_l^\epsilon \quad \text{and} \quad v_j^\epsilon = \sum_{l=1}^{N+N'} \gamma_{lj}^\epsilon f_l^\epsilon, \quad 1 \le j \le D.$$

This transformation is regular because the matrix

$$\begin{pmatrix} \frac{1}{\sqrt{D+1}} & \dots & \frac{1}{\sqrt{D+1}} \\ \sigma_{11} & \dots & \sigma_{D+1,1} \\ \dots & \dots & \dots \\ \sigma_{1D} & \dots & \sigma_{D+1,D} \end{pmatrix}$$

is orthogonal. We obtain

$$\left\{ \begin{array}{lll} \partial_t f_l^\epsilon & + & \displaystyle\sum_{d=1}^D \lambda_{ld} \partial_{x_d} f_l^\epsilon = \frac{1}{\epsilon}\left(M_l(u^\epsilon) - f_l^\epsilon\right), \quad 1 \le l \le N, \\ \partial_t u^\epsilon & + & \displaystyle\sum_{d=1}^D \partial_{x_d} v_d^\epsilon = 0, \\ \partial_t v_j^\epsilon & + & \displaystyle\sum_{d=1}^D \left[\sum_{l=1}^N \lambda_{lj}\lambda_{ld}\partial_{x_d} f_l^\epsilon + \mu^2 \sum_{m=1}^{N'} \sigma_{mj}\sigma_{md}\partial_{x_d} f_{N+m}^\epsilon\right] = \frac{1}{\epsilon}\left(A_j - v_j\right) \\ & - & \displaystyle\frac{1}{\epsilon}(\sqrt{\epsilon}2\mu\theta\sqrt{N'} + \theta^2 N') \sum_{d=1}^D \sum_{m=1}^{N'} \sigma_{mj}\sigma_{md}\partial_{x_d} f_{N+m}^\epsilon, \quad 1 \le j \le D. \end{array} \right. \tag{3.57}$$

The numerical approximation is constructed as follows: we fix the initial data $U^0 = (f_1^0, \dots, f_N^0, u^0, v^0)$ by (3.1)(3.2) and denote U^n the approximate solution at time t_n. We split (3.57) into its LHS and RHS. The LHS is a linear first order system which can be written under the form:

$$\partial_t U + \sum_{d=1}^D C^{(d)} \partial_{x_d} U = 0 \tag{3.58}$$

where the $C^{(d)}$ are matrices defined by blocks:

$$C^{(d)} = \begin{pmatrix} Diag(\lambda_{ld}) & 0 \\ D^{(d)} & \Sigma^{(d)} \end{pmatrix}. \tag{3.59}$$

As we show below, (3.58) is hyperbolic diagonalizable if and only if $\mu \neq 0$. As a consequence we may discretize it by any method for hyperbolic linear systems including finite volumes on unstructured meshes. Anyway for the sake of simplicity we continue in this paper with cartesian grids and applying a stable, high order, conservation form written method, we obtain $U^{n+1/2}$. In particular we have:

$$u_\alpha^{n+1/2} = u_\alpha^n - \sum_{d=1}^{D} \frac{\Delta t}{\Delta x_d} \left(v^n_{d,\alpha+e_d/2} - v^n_{d,\alpha-e_d/2} \right). \tag{3.60}$$

The numerical solution U^{n+1} is then an approximation of the limit when ϵ tends to zero of the solution of the following system:

$$\begin{cases} \partial_t f_l^\epsilon = \dfrac{1}{\epsilon}\left(M_l(u^\epsilon) - f_l^\epsilon\right), \quad 1 \le l \le N, \\ \partial_t u^\epsilon = 0, \\ \partial_t v_j^\epsilon = \dfrac{1}{\epsilon}\left(A_j - v_j - (\sqrt{\epsilon} 2\mu\theta\sqrt{N'} + \theta^2 N') \displaystyle\sum_{d=1}^{D}\sum_{m=1}^{N'} \sigma_{mj}\sigma_{md}\partial_{x_d} f^\epsilon_{N+m} \right) \end{cases} \tag{3.61}$$

with initial condition $U^{n+1/2}$. The following lemma is proven in subsections 3.2.1 and 3.2.2:

Lemma 3.3 *Denote $U^\epsilon(t) = R^\epsilon(t, t_n, U^{n+1/2})$ the solution of (3.61) with $U^\epsilon(t_n) = U^{n+1/2}$ and:*

$$\overline{U^{n+1}} = \lim_{\epsilon\to 0} U^\epsilon(t_{n+1}).$$

$\overline{U^{n+1}}$ is given by:

$$\overline{u^{n+1}} = u^{n+1/2}, \quad \overline{f_l^{n+1}} = M_l(\overline{u^{n+1}}), \quad 1 \le l \le N,$$

and

$$\overline{v_j^{n+1}} = A_j(\overline{u^{n+1}}) - \partial_{x_j} B(\overline{u^{n+1}}), \quad 1 \le j \le D.$$

As a consequence if $U^{n+1} = \overline{U^{n+1}}$ we should have:

$$u_\alpha^{n+1} = u_\alpha^n - \sum_{d=1}^{D} \frac{\Delta t}{\Delta x_d} \left[\left(A_d(u^n) - \partial_{x_d} B(u^n)\right)_{\alpha+e_d/2} - \left(A_d(u^n) - \partial_{x_d} B(u^n)\right)_{\alpha-e_d/2} \right] \tag{3.62}$$

This equality insures that if converging, the method converges to the right solution. In practice we have to define an approximate value for $\partial_{x_d} B(u^n)_\alpha$. Here we choose a central difference formula:

$$\partial_{x_d} B(u^n)_\alpha = \frac{B(u^n_{\alpha+e_d}) - B(u^n_{\alpha-e_d})}{2\Delta x_d}. \tag{3.63}$$

Hence we define:

$$\begin{cases} u_\alpha^{n+1} = u_\alpha^{n+1/2}, & \\ f_{\alpha,l}^{n+1} = M_l(u_\alpha^{n+1}), & 1 \le l \le N, \\ v_{\alpha,j}^{n+1} = A_j(u_\alpha^{n+1}) - \dfrac{B(u^n_{\alpha+e_d}) - B(u^n_{\alpha-e_d})}{2\Delta x_d}, & 1 \le j \le D. \end{cases} \tag{3.64}$$

It remains to prove the hyperbolicity of the system (3.58): for all $\xi \in \mathbb{R}^D$, $|\xi| = 1$, and for all $\mu > 0$, let us denote

$$C_\mu(\xi) = \sum_{d=1}^{D} \xi_d C^{(d)} = \begin{pmatrix} \Lambda(\xi) & 0 \\ D_\mu(\xi) & \Sigma_\mu(\xi) \end{pmatrix}. \tag{3.65}$$

The following two results are proved in subsection 3.2.1 for $D = 1$ and 3.2.2 for $D = 2$.

Lemma 3.4 *For all $\mu > 0$ and for all $\xi \in \mathbb{R}^D$ such that $|\xi| = 1$, $\Sigma_\mu(\xi)$ has real eigenvalues and is diagonalizable. Moreover there exists an orthogonal matrix $P(\xi)$ such that the matrix $\Pi(\mu, \xi)$ of right eigenvectors of $\Sigma_\mu(\xi)$ can be written:*

$$\Pi_{1j}(\mu, \xi) = P_{1j}(\xi)\frac{\sqrt{D+1}}{\mu}, \quad \Pi_{kj}(\mu, \xi) = P_{kj}(\xi) \text{ for } k = 2, \ldots, D+1, \quad j = 1, \ldots, D+1. \tag{3.66}$$

Proposition 3.4 *For all $\mu > 0$, the system (3.58) is hyperbolic diagonalizable. For all $\xi \in \mathbb{R}^2$ such that $|\xi| = 1$ the matrix of eigenvectors for $C_\mu(\xi)$ takes the form:*

$$Q(\mu, \xi) = \begin{pmatrix} I & 0 \\ R & \Pi(\mu, \xi) \end{pmatrix} \tag{3.67}$$

where I is the identity matrix in $\mathbb{R}^N$, $\Pi(\mu, \xi)$ is defined in (3.66), and

$$R_{1,l} = 1, \quad R_{1+d,l} = \lambda_{ld}, \quad 1 \le l \le N, \quad 1 \le d \le D.$$

If $\mu = 0$ the system is not hyperbolic.

For the sake of simplicity we explicit the obtained scheme (3.60)–(3.64) only in one space dimension, higher space dimensions requiring more notations and rather long computations. Nevertheless, the particular structure of the matrix $Q(\mu, \xi)$ lets see that in all cases we have a numerical scheme where the hyperbolic numerical flux is the same as the one obtained by the kinetic splitting method.

3.2.1 One space dimension.

System (3.57) takes the simpler form:

$$\begin{cases} \partial_t f_l^\epsilon + & \lambda_l \partial_x f_l^\epsilon = \dfrac{1}{\epsilon}\left(M_l(u^\epsilon) - f_l^\epsilon\right) \quad 1 \le l \le N, \\ \partial_t u^\epsilon + & \partial_x v^\epsilon = 0, \\ \partial_t v^\epsilon + & \displaystyle\sum_{l=1}^{N} (\lambda_l^2 - \frac{\mu^2}{2})\partial_x f_l^\epsilon + \frac{\mu^2}{2}\partial_x u \\ & = \dfrac{1}{\epsilon}\left(A - v - (\theta\mu\sqrt{2}\sqrt{\epsilon} + \theta^2)\partial_x(f_{N+1}^\epsilon + f_{N+2}^\epsilon)\right). \end{cases} \tag{3.68}$$

Let us first prove Lemma 3.3:
Proof of Lemma 3.3 for D=1. We have to solve the system:

$$\left\{\begin{array}{ll} \partial_t f_l^\epsilon & = \dfrac{1}{\epsilon}\left(M_l(u^\epsilon) - f_l^\epsilon\right) \quad 1 \le l \le N, \\ \partial_t u^\epsilon & = 0, \\ \partial_t v^\epsilon & = \dfrac{1}{\epsilon}\Big(A(u^\epsilon) - v^\epsilon - (\theta\mu\sqrt{2}\sqrt{\epsilon} + \theta^2)\partial_x(f_{N+1}^\epsilon + f_{N+2}^\epsilon)\Big). \end{array}\right. \tag{3.69}$$

Here

$$f_{N+1}^\epsilon + f_{N+2}^\epsilon = u^\epsilon - \sum_{l=1}^{N} f_l^\epsilon. \tag{3.70}$$

For $t \in [t_n, t_{n+1}]$ and $x \in]x_{i-1/2}, x_{i+1/2}[$,

$$\left\{\begin{array}{l} u^\epsilon(x,t) = u_i^{n+1/2}, \\ f_l^\epsilon(x,t) = f_{l,i}^{n+1/2} e^{-(t-t_n)/\epsilon} + \left(1 - e^{-(t-t_n)/\epsilon}\right) M_l(u_i^{n+1/2}). \end{array}\right. \tag{3.71}$$

So we are able to define the functions u^ϵ and f_l^ϵ on $[t_n, t_{n+1}]$ and also their spatial derivatives. Using (3.70) we obtain

$$f_{N+1}^\epsilon(.,t) + f_{N+2}^\epsilon(.,t) = \left(1 - e^{-(t-t_n)/\epsilon}\right)\frac{B^{n+1/2}}{\theta^2} + e^{-(t-t_n)/\epsilon}\left(u^{n+1/2} - \sum_{l=1}^{N} f_l^{n+1/2}\right).$$

Hence the third equation of (3.69) owns an analytical solution and the end of the proof is straightforward.□

A direct computation shows that as soon as $\mu \neq 0$ the left hand side of system (3.68) is hyperbolic diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_l, -\mu/\sqrt{2}, \mu/\sqrt{2}$, while for $\mu = 0$ it is not diagonalizable. The matrix $P(\xi)$ is

$$P(\xi) = \begin{pmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{pmatrix}$$

and the diagonal variables are:

$$\left\{\begin{array}{l} f_1, \ldots, f_N, \\ y = \dfrac{1}{2}\left[\mu\left(u - \displaystyle\sum_{l=1}^{N} f_l\right) - \sqrt{2}\left(v - \displaystyle\sum_{l=1}^{N}\lambda_l f_l\right)\right] \\ z = \dfrac{1}{2}\left[\mu\left(u - \displaystyle\sum_{l=1}^{N} f_l\right) + \sqrt{2}\left(v - \displaystyle\sum_{l=1}^{N}\lambda_l f_l\right)\right] \end{array}\right.$$

and reciprocally:

$$u = \sum_{l=1}^{N} f_l + (y+z)/\mu, \quad v = \sum_{l=1}^{N} \lambda_l f_l + (-y+z)/\sqrt{2}. \tag{3.72}$$

Let us detail the obtained numerical schemes. We begin by applying a high order, conservation form written scheme on the N first diagonal variables, and a first order upwind scheme on y, z. Formulas (3.64) give:

$$\begin{aligned} u_\alpha^{n+1} = u_\alpha^n \quad & - \quad \frac{\Delta t}{\Delta x} \sum_{l=1}^{N} \lambda_l (f_{\alpha+1/2,l}^n - f_{\alpha-1/2,l}^n) \\ & + \quad \frac{\mu \Delta t}{2\sqrt{2}\theta^2 \Delta x} (B_{\alpha+1}^n - 2B_\alpha^n + B_{\alpha-1}^n) \\ & + \quad \frac{\Delta t}{4\Delta x^2} (B_{\alpha+2}^n - 2B_\alpha^n + B_{\alpha-2}^n). \end{aligned} \tag{3.73}$$

We retrieve the same hyperbolic numerical flux as when using the kinetic splitting:

$$\mathcal{A}_{\alpha+1/2}^n = \sum_{l=1}^{N} \lambda_l f_{\alpha+1/2,l}^n$$

to which we have to add a supplementary numerical diffusion depending on μ (second line). However, the only requirement on μ is its positivity, so that we can take it very small:

$$\mu = 2\sqrt{2}\theta^2\eta, \quad \eta << 1.$$

The diffusion term in the third line is $\Delta_{1/4,\alpha} B^n$, with the notation (3.43).

We remark that the time step restriction is to be given a posteriori: the resolution of the LHS requires only a hyperbolic CFL condition but it is clear that we must impose a parabolic condition. In the limit $\mu \to 0$, we retrieve some of the schemes obtained by the kinetic splitting, and as soon as we keep first order on y, z, we can even put $\mu = 0$ in (3.73). We can also take a higher order scheme on these two variables. Hence, it seems, and the numerical experiments confirm that the only advantage of this technique is the fact that it is applicable to non cartesian meshes. We shall explore this possibility in a forthcoming paper.

As an alternative to the central difference evaluation of $(\partial_x B(u^n))_\alpha$, one can remark that only the evaluation of $(\partial_x B(u^n))_{\alpha+1/2}$ is needed. If at time t_n we have obtained u^n, $f_l^n = M_l(u^n)$ for $l \leq N$ and $v^n = A(u^n) - \partial_x B(u^n)$ then:

$$\left\{ \begin{aligned} y^n &= \frac{1}{2}\left[\mu \frac{B(u^n)}{\theta^2} + \sqrt{2}\partial_x B(u^n)\right] \\ z^n &= \frac{1}{2}\left[\mu \frac{B(u^n)}{\theta^2} - \sqrt{2}\partial_x B(u^n)\right] \end{aligned} \right. \tag{3.74}$$

Consequently we propose to put:

$$(\partial_x B(u^n))_{\alpha+1/2} = \frac{B(u^n_{\alpha+1}) - B(u^n_\alpha)}{\Delta x} \tag{3.75}$$

and

$$\left\{ \begin{aligned} y^n_{\alpha+1/2} &= \frac{1}{2}\left[\mu \frac{B(u^n_{\alpha+1})}{\theta^2} + \sqrt{2}(\partial_x B(u^n))_{\alpha+1/2}\right] \\ z^n_{\alpha+1/2} &= \frac{1}{2}\left[\mu \frac{B(u^n_{\alpha})}{\theta^2} - \sqrt{2}(\partial_x B(u^n))_{\alpha+1/2}\right]. \end{aligned}\right. \tag{3.76}$$

We obtain an approximation of $\Delta B(u)$ as $\Delta_{0,\alpha}B$ with notation (3.43).

3.2.2 Two space dimensions.

Let us first introduce some notations:

$$s_{ijk} = \sum_{m=1}^{3} \sigma_{mi}\sigma_{mj}\sigma_{mk},$$

$$s_{ij}(\xi) = s_{1ij}\xi_1 + s_{2ij}\xi_2 \quad \text{for } \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$$

Proof of Lemma 3.3 for D=2. Again we have (3.71). We then have to solve the 2×2 system of PDE for v which can be written:

$$\begin{aligned} \partial_t v^\epsilon_j \quad &+ \frac{a^\epsilon}{\sqrt{\epsilon}} \sum_{d=1}^{2}\sum_{k=1}^{2} s_{djk}\partial_{x_k} v^\epsilon_k \\ = \frac{1}{\epsilon} \quad &\left[-v^\epsilon_j + A_j(u^{n+1/2}) - \partial_{x_j} B(u^{n+1/2}) + \rho^\epsilon_j\sqrt{\epsilon} + e^{-(t-t_n)/\epsilon}(p^\epsilon_j + q^\epsilon_j\sqrt{\epsilon})\right], \end{aligned} \tag{3.77}$$

for $j = 1, 2$. Here a^ϵ, $\rho^\epsilon_j(x)$, $p^\epsilon_j(x)$, $q^\epsilon_j(x)$ are uniformly bounded for $\epsilon \in [0, 1]$. By Fourier transform in the space variable we can write the solution and make $\epsilon \to 0$, so that we obtain the result.□

Proof of proposition 3.4 for D=2. The matrix $D_\mu(\xi)$ in (3.65) takes the form:

$$\left\{ \begin{aligned} D_{1l}(\xi) = \quad & 0, \\ D_{2l}(\xi) = \quad & (\lambda^2_{l1} - \mu^2/3 - \mu(s_{111}\lambda_{l1} + s_{112}\lambda_{l2}))\,\xi_1 \\ & + (\lambda_{l1}\lambda_{l2} - \mu(s_{211}\lambda_{l1} + s_{212}\lambda_{l2}))\,\xi_2, \\ D_{3l}(\xi) = \quad & (\lambda_{l1}\lambda_{l2} - \mu(s_{211}\lambda_{l1} + s_{212}\lambda_{l2}))\,\xi_1 \\ & + (\lambda^2_{l2} - \mu^2/3 - \mu(s_{221}\lambda_{l1} + s_{222}\lambda_{l2}))\,\xi_2, \end{aligned}\right. \tag{3.78}$$

for $1 \leq l \leq N$. The matrix $\Sigma_\mu(\xi)$ in (3.65) takes the form:

$$\Sigma_\mu(\xi) = \begin{pmatrix} 0 & \xi_1 & \xi_2 \\ \xi_1\mu^2/3 & \mu s_{11}(\xi) & \mu s_{12}(\xi) \\ \xi_2\mu^2/3 & \mu s_{12}(\xi) & \mu s_{22}(\xi) \end{pmatrix}. \tag{3.79}$$

The eigenvalues of $C_\mu(\xi)$ are: $\Lambda_l(\xi) = \xi_1\lambda_{l1} + \xi_2\lambda_{l2}$, $l = 1, \ldots, N$, and the eigenvalues of $\Sigma_\mu(\xi)$. We can take as eigenvector for $\Lambda_l(\xi)$:

$$(f, u, v_1, v_2) = (e_l, 1, \lambda_{l1}, \lambda_{l2}), \tag{3.80}$$

where e_l is the l^{th} canonical vector of $\mathbb{R}^N$. Consequently $C_\mu(\xi)$ is diagonalizable if and only if $\Sigma_\mu(\xi)$ is diagonalizable. To end the proof it remains to prove Lemma 3.4.$\square$

Proof of Lemma 3.4. Let $F(\xi, \mu, \alpha) = det(\Sigma_\mu(\xi) - \alpha)$. Dividing each column of $F(\xi, \mu, \alpha)$ by μ, we remark that there exists a function $G(\xi, x)$, not depending on μ such that $F(\xi, \mu, \alpha) = \mu^3 G(\xi, x = \alpha/\mu)$. For $\mu = \sqrt{3}$, the roots x of $G(\xi, x) = 0$ are the eigenvalues of a symmetric matrix. Therefore there exists three real roots $x_1(\xi), x_2(\xi), x_3(\xi)$ for this equation. This proves that $\Sigma_\mu(\xi)$ has three real eigenvalues $\mu x_1(\xi) = \beta_1(\xi), \mu x_2(\xi) = \beta_2(\xi), \mu x_3(\xi) = \beta_3(\xi)$.

Now for $\mu = \sqrt{3}$, $\Sigma_\mu(\xi)$ is diagonalizable and there exists an orthogonal matrix $P(\xi)$ which columns are eigenvectors for it. It is easy to see that for any $\mu > 0$ the matrix $\Pi(\mu, \xi)$ defined by (3.66) is the regular matrix of right eigenvectors we are looking for. This ends the proof of the lemma.$\square$

3.3 Higher order in time, implicit schemes

To construct higher order in time approximations we remark that the we have obtained functions $\mathcal{A}_d$ and $\mathcal{B}_d$ such that formula (3.5) may be viewed as a first order discretization of the differential equation:

$$u'_\alpha = u_\alpha - \sum_{d=1}^{D} \frac{\overline{\mathcal{A}}_{\alpha+e_d/2} - \overline{\mathcal{A}}_{\alpha-e_d/2}}{\Delta x_d} + \sum_{d=1}^{D} \frac{\mathcal{B}_{\alpha+e_d,d} - 2\mathcal{B}_{\alpha,d} + \mathcal{B}_{\alpha-e_d,d}}{\Delta x_d^2}\,. \tag{3.81}$$

Here $\overline{\mathcal{A}}_{\alpha+e_d/2}$ is the limit of $\mathcal{A}_d\left(u_{\alpha-a+e_d}, \ldots, u_{\alpha+a}\right)$ when $\Delta t \to 0$ and Δx is fixed, and $\mathcal{B}_{\alpha,d} = \mathcal{B}_d\left(u_{\alpha-b}, \ldots, u_{\alpha+b}\right)$. Therefore, one may discretize (3.81) with any high order, for example Runge-Kutta method, or make an implicit treatment of the parabolic part of the equation. The stability condition on the hyperbolic part remains identical. As usual the order in time has to be related with order in space, see [39].

For example, if a first order scheme is used on the N first equations of the kinetic model, then $\mathcal{A} = \overline{\mathcal{A}}$ and $\mathcal{B} = \overline{\mathcal{B}}$. If a second order scheme like (3.55) is used, then we obtain

$$\overline{f_{\alpha+\frac{1}{2},l}} = f_{\alpha+\frac{1}{2}(1-\mathrm{sgn}\lambda_l),l} + \mathrm{sgn}\lambda_l.\frac{\Delta x}{2}\sigma_{\alpha+\frac{1}{2}(1-\mathrm{sgn}\lambda_l),l},$$

$$\sigma_{\beta,l} = \phi\left(\frac{f_{\beta+1,l} - f_{\beta,l}}{\Delta x}, \frac{f_{\beta,l} - f_{\beta-1,l}}{\Delta x}\right), \quad f_{\beta,l} = M_l(u_\beta),$$

and

$$\overline{\mathcal{A}}_{\alpha+1/2} = \sum_{l=1}^{N} \lambda_l \overline{f_{\alpha+\frac{1}{2},l}}.$$

We have still $\mathcal{B} = \overline{\mathcal{B}}$. The TVD properties of the numerical solution are kept by dividing the CFL condition by 2.

Remark that (1.3) and the numerical choices above can be viewed like a "flux function manufacture" furnishing $\mathcal{A}_d$ and $\mathcal{B}_d$, which can then be mixed with any (stable) discretization in time.

4 Stability and convergence for a scalar equation

Diagonality and quasimonotonicity in the sense of [17] are two important features of models (1.3). These properties are essential for proving the convergence of u^ϵ towards u in [6], as well as in the proof of numerical convergence for the hyperbolic case in [2]. Here the kinetic splitting method preserves both diagonality and monotonicity so that, roughly speaking, the scheme (3.5) obtained by this method owns the same stability properties as H^ϵ_Δ, including the cell entropy inequality. The semi-conservative splitting method destroys these features and to show convergence seems more difficult. However, we have already pointed out that for a large class of discretizations, the scheme (3.60)(3.64) may be obtained by the kinetic splitting method. Therefore, in this paragraph we study the stability properties of the scheme obtained by kinetic splitting on the system (2.3) with the choice (3.47), namely the scheme (3.5) with (3.45), (3.48) and (3.49). This analysis applies to others discretizations by kinetic splitting, such as the one in remark 3.2. As a consequence of our results, and thanks to a result of Bouchut [5], we have a convergence theorem for a class of monotone flux splitting schemes.

In the sequel, ϵ depends on Δx through relation (3.34) or proposition 3.3 and we suppress the superscript ϵ in the notations. We call DKS the Discrete Kinetic Scheme given by (3.1),(3.5) with (3.45), (3.48), (3.49), and a model under the form (2.3)(3.47). According to the time step restriction (3.49), Δx and Δt tend to zero in such a way that for all d, $\xi_d := \Delta t/\Delta x_d^2$ is constant and $\xi_d \leq 1$. We denote $\mathcal{V}$ the volume of each cell:

$$\mathcal{V} = \Delta x_1 \dots \Delta x_D.$$

Convergence to a weak solution will be proved under the following hypothesis:

(H1) $u_0 \in \mathrm{L}^\infty(\mathbb{R}^D) \cap \mathrm{L}^1(\mathbb{R}^D) \cap \mathrm{BV}(\mathbb{R}^D)$ and M is a MMF on $I = [-\|u_0\|_\infty, \|u_0\|_\infty]$.

(H2) H_Δ preserves the extrema:

$$\forall f = (f_{\alpha,l})_{1\leq l\leq L,\alpha\in\mathbb{Z}^D},\ \ \forall l,\ \ \forall\alpha,\ \ \min_{\alpha'\in\mathbb{Z}^D} f_{\alpha',l} \leq (H_\Delta(\Delta t)f)_{\alpha,l} \leq \max_{\alpha'\in\mathbb{Z}^D} f_{\alpha',l}.$$

If the scheme H_Δ is monotone then we have convergence to the entropy solution of the problem and it is only necessary to suppose that

(H1') $u_0 \in \mathrm{L}^\infty(\mathbb{R}^D) \cap \mathrm{L}^1(\mathbb{R}^D)$ and M is a MMF on $I = [-\|u_0\|_\infty, \|u_0\|_\infty]$.

We recall that:

$$\|u_\Delta^0\|_\infty \le \|u_0\|_\infty,\ \ \|u_\Delta^0\|_1 \le \|u_0\|_1,\ \ \mathrm{TV}(u_\Delta^0) \le \mathrm{TV}(u_0), \tag{4.1}$$

and

$$\|u_\Delta^0 - u_0\|_1 \le \|\Delta x\|\mathrm{TV}(u_0). \tag{4.2}$$

Proposition 4.1 *Under hypothesis (H1),(H2), the numerical scheme DKS is L^∞ stable:*

$$\|u_\Delta^n\|_\infty \le \|u_0\|_\infty. \tag{4.3}$$

Proof of proposition 4.1. The numerical scheme DKS may be summarized as:

$$f_{\alpha,l}^n = M_l(u_\alpha^n),\ \ u_\alpha^{n+1} = \sum_{l=1}^{N+N'} f_{\alpha,l}^{n+1/2},\ \ n \ge 0 \tag{4.4}$$

where $f_{\alpha,l}^{n+1/2}$ is computed by formula (3.20) to (3.23) or their multidimensional analogue. We know that (4.3) is true for $n = 0$. Take now $n \ge 1$ and suppose that $u_\alpha^{n-1} \in [-\|u_0\|_\infty, \|u_0\|_\infty]$. By (H1) and (4.4):

$$M_l(-\|u_0\|_\infty) \le f_{\alpha,l}^{n-1} \le M_l(\|u_0\|_\infty),$$

and by (H2):

$$M_l(-\|u_0\|_\infty) \le f_{\alpha,l}^{n-1/2} \le M_l(\|u_0\|_\infty), \tag{4.5}$$

and summing on l we find (4.3). □

Remark that the preceding proof also shows that $\|f_{\Delta,l}^n\|_\infty \le \|M_l(u_0)\|_\infty$.

We study now the BV and L^1 stability of the scheme.

Proposition 4.2 *Under hypothesis (H1),(H2), if H_Δ is TVD, then DKS is TVD and the two following estimates hold:*

$$\forall t \in [0, +\infty[,\ \ \|u_\Delta(t)\|_1 \le \|u_0\|_1 + C_1.t.\mathrm{TV}(u_0). \tag{4.6}$$

For all $T \ge 0$ and all compact set K, there exists a constant $C = C(T, K, u_0)$ such that

$$\forall t, t' \in [0, T],\ \ \|u_\Delta(t) - u_\Delta(t')\|_{L^1(K)} \le C(|t - t'| + \Delta t)^{1/3}. \tag{4.7}$$

In the purely hyperbolic case the Lipschitz continuity of the numerical flux functions or, for monotone schemes, the L^1-contraction property, are used to prove that:

$$\|u_\Delta^{n+1} - u_\Delta^n\|_1 \le C.\Delta t,$$

and one concludes that:

$$\|u_\Delta(t') - u_\Delta(t)\|_1 \le C.(|t - t'| + \Delta t).$$

In our case such a method leads to:

$$\|u_\Delta^{n+1} - u_\Delta^n\|_1 \leq C.\sqrt{\Delta t},$$

and this is not enough. Instead we are going to use Kruzkov's Lemma [24] to prove (4.7).

Proof of proposition 4.2. By (4.1), $\mathrm{TV}(u_\Delta^0) \leq \mathrm{TV}(u_0)$. For $n \geq 1$, suppose that $\mathrm{TV}(u_\Delta^{n-1}) \leq \mathrm{TV}(u_0)$. We have:

$$\mathrm{TV}(u_\Delta^n) = \sum_{\alpha\in\mathbb{Z}^D} \sum_{d=1}^{D} \frac{\mathcal{V}}{\Delta x_d} \left| \sum_{l=1}^{N+N'} \left(f_{\alpha+e_d,l}^{n-1/2} - f_{\alpha,l}^{n-1/2} \right) \right| .$$

Denoting $\mathrm{TV}(f_\Delta) = \sum_{l=1}^{N+N'} \mathrm{TV}(f_{l,\Delta})$ and using the fact that (HS) is TVD:

$$\mathrm{TV}(u_\Delta^n) \leq \mathrm{TV}(f_\Delta^{n-1}) = \sum_{\alpha\in\mathbb{Z}^D} \sum_{d=1}^{D} \frac{\mathcal{V}}{\Delta x_d} \sum_{l=1}^{N+N'} \left| M_l(u_{\alpha+e_d}^{n-1}) - M_l(u_\alpha^{n-1}) \right| .$$

By (H1) and the assumption (M_1), we conclude that

$$\mathrm{TV}(u_\Delta^n) \leq \mathrm{TV}(f_\Delta^{n-1}) \leq \mathrm{TV}(u_0), \tag{4.8}$$

and DKS is TVD.

In order to prove (4.6) we write u_α^{n+1} as:

$$u_\alpha^{n+1} = \sum_{l=1}^{N} M_l(u_\alpha^n) - \sum_{d=1}^{D} \frac{\Delta t}{\Delta x_d} \left(\mathcal{A}_{\alpha+e_d/2}^n - \mathcal{A}_{\alpha-e_d/2}^n \right) + \sum_{l=N+1}^{N+N'} f_{\alpha,l}^{n+1/2}.$$

The numerical flux functions $\mathcal{A}$ are Lipschitz continuous because the F_{ld} and the Maxwellian functions are. Moreover the scheme (HS) is monotone on the N' last equations:

$$\|f_{\Delta,l}^{n+1/2}\|_1 \leq \|f_{\Delta,l}^n\|_1 = \|M_l(u_\Delta^n)\|_1.$$

Consequently, denoting C_1 the Lipschitz constant of $\mathcal{A}$:

$$\|u_\Delta^{n+1}\|_1 \leq \sum_{l=1}^{N+N'} \|M_l(u_\Delta^n)\|_1 + C_1 \Delta t \mathrm{TV}(u_0).$$

Hypothesis (H1), (M_1) and L^1 stability give:

$$\|u_\Delta^{n+1}\|_1 \leq \|u_\Delta^n\|_1 + C_1 \Delta t \mathrm{TV}(u_0)$$

which implies (4.6).

Let us now demonstrate (4.7). We use here Kruzkov's method, see Lemma 1 and 5 in [24]. We first prove that for all integers $m \geq 0$, $p \geq 1$ with $(m+p)\Delta t \leq T$, there exists a constant $C(T, u_0) > 0$ such that:

$$\forall \phi \in \mathcal{C}_0^2(\mathbb{R}^D), \quad \left| \int_{\mathbb{R}^D} \left(u_\Delta^{m+p} - u_\Delta^m\right) \phi dx \right| \leq C(T, u_0).p\Delta t.\|\phi\|_{\mathcal{C}_0^2}. \tag{4.9}$$

For all $\phi \in \mathcal{C}_0^2(\mathbb{R}^D)$ we denote

$$\phi_\alpha = \frac{1}{\mathcal{V}} \int_{\mathrm{I}_\alpha} \phi dx.$$

We use the scheme DKS under the form (3.5). We can write:

$$\begin{aligned}
\left| \int_{\mathbb{R}^D} \left(u_\Delta^{m+p} - u_\Delta^m\right) \phi dx \right| &= \mathcal{V} \left| \sum_{\alpha \in \mathbb{Z}^D} \left(u_\alpha^{m+p} - u_\alpha^m\right) \phi_\alpha \right| \\
&\leq \mathcal{V} \sum_{\alpha \in \mathbb{Z}^D} \sum_{n=m}^{m+p-1} \sum_{d=1}^{D} \frac{\Delta t}{\Delta x_d} \left| \left(\mathcal{A}_{\alpha+e_d/2}^n - \mathcal{A}_{\alpha-e_d/2}^n\right) \phi_\alpha \right| \\
&\quad + \mathcal{V} \sum_{\alpha \in \mathbb{Z}^D} \sum_{n=m}^{m+p-1} \sum_{d=1}^{D} \frac{\Delta t}{\Delta x_d^2} |(\mathcal{B}_{\alpha+e_d,d}^n - 2\mathcal{B}_{\alpha+e_d,d}^n + \mathcal{B}_{\alpha+e_d,d}^n)\phi_\alpha| \\
&= H + P.
\end{aligned} \tag{4.10}$$

The Lipschitz continuity of the numerical flux functions and of the MMF implies that $\mathcal{A}$ and $\mathcal{B}$ are Lipschitz continuous. Using Green's formula we obtain

$$\begin{aligned}
H &\leq \mathcal{V} \sum_{\alpha \in \mathbb{Z}^D} \sum_{n=m}^{m+p-1} \sum_{d=1}^{D} \frac{\Delta t}{\Delta x_d} \left| \mathcal{A}_{\alpha+e_d/2}^n (\phi_\alpha - \phi_{\alpha-e_d}) \right| \\
&\leq \Delta t \sum_{n=m}^{m+p-1} \sum_{d=1}^{D} \sup_\alpha \left| \frac{\phi_\alpha - \phi_{\alpha-e_d}}{\Delta x_d} \right| .C.\|u_\Delta^n\|_1 \\
&\leq C(T, u_0).p\Delta t \|\phi\|_{\mathcal{C}_0^1}.
\end{aligned} \tag{4.11}$$

In the same way:

$$P \leq C(T, u_0).p\Delta t \|\phi\|_{\mathcal{C}_0^2}, \tag{4.12}$$

which ends the proof of (4.9).

We follow now the proof of Lemma 5 in [24]. Let K be a compact set and take $X > 0$ such that $K \subset B(0, X/2) := \{x \in \mathbb{R}^D, \ |x| < X/2\}$, $\rho \in]0, X/2[$. Put

$$\psi(x) = \begin{cases} \operatorname{sgn}(u_\Delta^{m+p}(x) - u_\Delta^m(x)) & \text{if } |x| \leq X - \rho \\ 0 & \text{else.} \end{cases}$$

A smooth approximation of ψ is constructed with a standard smoothing function $\xi_\rho(x) = \rho^{-D}\xi(x/\rho)$ where $\xi \in \mathcal{C}_0^\infty(\mathbb{R}^D)$, $\xi \geq 0$, $\xi(x) = 0$ for $|x| > 1$ and $\int_{\mathbb{R}^D} \xi dx = 1$: the function $\psi_\rho = \xi_\rho * \psi$ is $\mathcal{C}_0^\infty$ with support in $B(0, X)$ and

$$\|\psi_\rho\|_{\mathcal{C}_0^2(\mathbb{R}^D)} \leq C_2.(\rho^{-1} + \rho^{-2}). \tag{4.13}$$

We have

$$\begin{aligned}
\int_K |u_\Delta^{m+p}(x) - u_\Delta^m(x)|dx &\leq \int_{B(0,X-\rho)} (u_\Delta^{m+p}(x) - u_\Delta^m(x))\psi_\rho(x)dx \\
&\quad + \int_{B(0,X-\rho)} (u_\Delta^{m+p}(x) - u_\Delta^m(x))(\psi(x) - \psi_\rho(x))dx \\
&= I_1 + I_2.
\end{aligned} \tag{4.14}$$

Using (4.9) and (4.13):

$$|I_1| \leq C(T, u_0).p\Delta t(\rho^{-1} + \rho^{-2}) + 2\|u_0\|_\infty X^{D-1}\rho.$$

To estimate I_2 we follow exactly the proof of Lemma 1 of [24] and we find

$$\begin{aligned}
|I_2| &\leq 2 \sup_{|z|\leq\rho} \int_{\mathbb{R}^D} |(u_\Delta^{m+p}(x) - u_\Delta^m(x)) - (u_\Delta^{m+p}(x-z) - u_\Delta^m(x-z))|dx \\
&\leq C_3.\rho.\mathrm{TV}(u_0).
\end{aligned} \tag{4.15}$$

Choosing $\rho = (p\Delta t)^{1/3}$, which is always possible by taking X large enough, gives

$$\|u_\Delta^{m+p} - u_\Delta^m\|_{L^1(K)} \leq C(T, X, u_0).(p\Delta t)^{1/3}$$

and (4.7) follows. $\square$

Proposition 4.3 *Under hypothesis (H1), if H_Δ is monotone, then DKS is monotone,* L^1*-contracting, TVD and (4.7) hold. The estimate (4.6) can be improved by*

$$\forall t \in [0, +\infty[, \quad \|u_\Delta(t)\|_1 \leq \|u_0\|_1. \tag{4.16}$$

Proof of proposition 4.3. The monotonicity is clear. Let us consider $\overline{u}_0$ with $\|\overline{u}_0\|_\infty \leq \|u_0\|_\infty$. For all $n \geq 0$:

$$\|u_\Delta^{n+1} - \overline{u}_\Delta^{n+1}\|_1 \leq \sum_{l=1}^{N+N'} \|f_{l,\Delta}^{n+1/2} - \overline{f}_{l,\Delta}^{n+1/2}\|_1.$$

The scheme H_Δ is monotone and therefore L^1-contracting:

$$\sum_{l=1}^{N+N'} \|f_{l,\Delta}^{n+1/2} - \overline{f}_{l,\Delta}^{n+1/2}\|_1 \leq \sum_{l=1}^{N+N'} \|f_{l,\Delta}^n - \overline{f}_{l,\Delta}^n\|_1.$$

Condition (M_1), L^∞ stability and the fact that M is a MMF on $[-\|u_0\|_\infty, \|u_0\|_\infty]$ imply that

$$\begin{aligned}\sum_{l=1}^{N+N'} \|f^n_{l,\Delta} - \overline{f}^n_{l,\Delta}\|_1 &= \sum_{l=1}^{N+N'} \|M_l(u^n_\Delta) - M_l(\overline{u}^n_\Delta)\|_1 \\ &= \|\sum_{l=1}^{N+N'} M_l(u^n_\Delta) - M_l(\overline{u}^n_\Delta)\|_1 = \|u^n_\Delta - \overline{u}^n_\Delta\|_1, \end{aligned} \tag{4.17}$$

which proves that the scheme is L^1-contracting.

If H_Δ is monotone then it is TVD so that proposition 4.2 applies. $\square$

Proposition 4.4 *Suppose the hypothesis (H1),(H2) are satisfied and that a discrete entropy inequality holds for each component of H_Δ, i.e. for all convex function S, for all $l = 1, \ldots, N$, there exists a Lipschitz continuous function R_l such that:*

$$\frac{S(f^{n+1/2}_{\alpha,l}) - S(f^n_{\alpha,l})}{\Delta t} + \sum_{d=1}^{D} \lambda_{ld} \frac{R^n_{\alpha+e_d/2,l} - R^n_{\alpha-e_d/2,l}}{\Delta x_d} \leq 0, \tag{4.18}$$

with

$$R^n_{\alpha+e_d/2,l} = R_{ld}\left(f^n_{\alpha-s_l+e_d,l}, \ldots, f^n_{\alpha+s_l,l}\right), \quad R_{ld}\left(f_l, \ldots, f_l\right) = S(f_l).$$

Then a discrete entropy inequality holds for DKS: for all convex entropy η with the related entropy functions defined in Theorem 1.1, there exist two Lipschitz continuous functions Φ and Π such that:

$$\frac{\eta(u^{n+1}_\alpha) - \eta(u^n_\alpha)}{\Delta t} + \sum_{d=1}^{D} \left(\frac{\Phi^n_{\alpha+e_d/2} - \Phi^n_{\alpha-e_d/2}}{\Delta x_d} - \frac{\Pi^n_{\alpha+e_d,d} - 2\Pi^n_{\alpha,d} + \Pi^n_{\alpha-e_d,d}}{\Delta x_d^2} \right) \leq 0, \tag{4.19}$$

with

$$\Phi^n_{\alpha+e_d/2} = \Phi_d\left(u^n_{\alpha-s+e_d}, \ldots, u^n_{\alpha+s}\right), \quad \Phi_d\left(u, \ldots, u\right) = A_{\eta,d}(u)$$

and

$$\Pi^n_{\alpha,d} = \Pi_d\left(u^n_{\alpha-e_d}, u^n_\alpha, u^n_{\alpha+e_d}\right), \quad \Pi_d\left(u,u,u\right) = B_\eta(u).$$

For the $2D$ last equations we know that a discrete entropy inequality holds because we use a monotone scheme, and the discrete entropy flux is given by (3.12). As far as we are concerned with the N first equations, we point out that it is difficult to find a high order, conservation form written scheme which satisfies (4.18) for a conservation law: this assumption implies that H_Δ is first order, even if it approximates "only" scalar linear transport equations.

In order to prove this proposition we recall some results of [6]. Let η be a convex entropy for (1.1). If η is a Kruzkov entropy: $\eta(u) = |u-k| - |k|$, then the kinetic entropies are defined by:

$$S_{l,\eta}(f) = |f - M_l(k)| - |M_l(k)|, \quad l = 1, \ldots, N + N'.$$

For a convex $\mathcal{C}^2$ entropy η we define:

$$S_{l,\eta}(f) = \int_{\mathbb{R}} \frac{1}{2} \left(|f - M_l(k)| - |M_l(k)| \right) \eta''(k) dk + \frac{1}{2} f \left(\eta'(-\infty) + \eta'(+\infty) \right). \tag{4.20}$$

The kinetic entropy $S_{l,\eta}$ is convex. If M is strictly monotone on I and if $\eta \in \mathcal{C}^2(\mathbb{R})$, then $S_{l,\eta} \in \mathcal{C}^2([M_l(-\|u_0\|_\infty), M_l(\|u_0\|_\infty)])$ and we have for all $w \in I$:

$$S'_{l,\eta}(M_l(w)) = \eta'(w), \tag{4.21}$$

$$S''_{l,\eta}(M_l(w)) = \eta''(w)\, M'(w)^{-1}. \tag{4.22}$$

Otherwise, the identity (4.21) holds in the sense of subdifferentials: for all $f \in [M_l(-\|u_0\|_\infty), M_l(\|u_0\|_\infty)]$, for all $w \in I$:

$$S_{l,\eta}(M_l(w)) - S_{l,\eta}(f) - \eta'(w)(M_l(w) - f) \leq 0. \tag{4.23}$$

We have also:

$$\sum_{l=1}^{N+N'} S_{l,\eta}(M_l(w)) = \eta(w) - \eta(0), \tag{4.24}$$

$$\sum_{l=1}^{N+N'} \gamma_{ld} S_{l,\eta}(M_l(w)) = A_{\eta,d}(w), \quad \sum_{l=1}^{N+N'} \theta_{ld}\theta_{lj} S_{l,\eta}(M_l(w)) = \delta_{jd} B_\eta(w)$$

where the notations of (1.3) and of Theorem 1.1 have been used. Due to the particular form of the kinetic model, these two last identities can be expressed as:

$$\sum_{l=1}^{N} \lambda_{ld} S_{l,\eta}(M_l(w)) = A_{\eta,d}(w), \quad S_{N+m,\eta}(M_{N+m}(w)) = \frac{B_\eta(w)}{2D\theta^2}, \text{ for } m = 1, \ldots, 2D. \tag{4.25}$$

Proof of proposition 4.4. Let η be a convex entropy. We denote

$$E^n_{\alpha,l} = S_{l,\eta}(f^{n+1}_{\alpha,l}) - S_{l,\eta}(f^n_{\alpha,l}) + \Delta t \sum_{d=1}^{D} \gamma_{ld} \frac{R^n_{\alpha+e_d/2,l} - R^n_{\alpha-e_d/2,l}}{\Delta x_d}.$$

We apply (4.18) with $S = S_{l,\eta}$ for each l:

$$\begin{aligned} E^n_{\alpha,l} &= E^n_{\alpha,l} + S_{l,\eta}(f^{n+1/2}_{\alpha,l}) - S_{l,\eta}(f^{n+1/2}_{\alpha,l}) \\ &\leq S_{l,\eta}(f^{n+1}_{\alpha,l}) - S_{l,\eta}(f^{n+1/2}_{\alpha,l}) = S_{l,\eta}(M_l(u^{n+1}_\alpha)) - S_{l,\eta}(f^{n+1/2}_{\alpha,l}). \end{aligned} \tag{4.26}$$

Moreover, by (4.21) (or (4.23)), (4.4), assumption (M_1) and by the convexity of $S_{l,\eta}$:

$$\begin{aligned} \sum_{l=1}^{N+N'} E^n_{\alpha,l} &\leq \sum_{l=1}^{N+N'} \left[S_{l,\eta}(M_l(u^{n+1}_\alpha)) - S_{l,\eta}(f^{n+1/2}_{\alpha,l}) - \eta'(u^{n+1}_\alpha)(M_l(u^{n+1}_\alpha) - f^{n+1/2}_{\alpha,l}) \right] \\ &\leq 0 \end{aligned} \tag{4.27}$$

This inequality is the one we are looking for. Indeed, using (4.24):

$$\sum_{l=1}^{N+N'} E_{\alpha,l}^n = \eta(u_\alpha^{n+1}) - \eta(u_\alpha^n) + \Delta t \sum_{d=1}^{D} \sum_{l=1}^{N+N'} \gamma_{ld} \frac{R_{\alpha+e_d/2,l}^n - R_{\alpha-e_d/2,l}^n}{\Delta x_d}.$$

Moreover:

$$\sum_{l=1}^{N} \gamma_{ld} R_{\alpha+e_d/2,l}^n = \sum_{l=1}^{N} \lambda_{ld} R_{ld}(M_l(u_{\alpha-s_l+e_d}^n), \ldots, M_l(u_{\alpha+s_l}^n) = \Phi_d(u_{\alpha-s_l+e_d}^n, \ldots, u_{\alpha+s_l}^n),$$

and

$$\Phi_d(u, \ldots, u) = \sum_{l=1}^{N} \lambda_{ld} S_{l,\eta}(M_l(u)) = A_{\eta,d}(u).$$

For $l = N+m$, by Lemma 3.1 and by (4.25):

$$\begin{aligned}
\sum_{l=N+1}^{N+N'} \gamma_{ld} R_{\alpha+e_d/2,l}^n &= \sum_{m=1}^{2D} \gamma^\epsilon \sigma_{md} \sum_{k=-1}^{2} a_{kd}^m S_{N+m,\eta}(M_{N+m}(u_{\alpha+ke_d}^n)) \\
&= \frac{\gamma^\epsilon}{2D\theta^2\sqrt{2}} \sum_{k=-1}^{2} (a_{kd} - a_{2-k-1,d}) B_\eta(u_{\alpha+ke_d}^n)
\end{aligned}$$

with $\gamma^\epsilon = \mu + \dfrac{\theta\sqrt{2D}}{\sqrt{\epsilon}}$. Consequently we can reproduce the same calculation as in section 3.1, and we find:

$$\Pi_{\alpha,d}^n = m_d B_\eta(u_{\alpha-e_d}^n) + (1-2m_d) B_\eta(u_\alpha^n) + m_d B_\eta(u_{\alpha+e_d}^n). \tag{4.28}$$

□

The following estimate allows us to prove that (1.9) holds.

Proposition 4.5 *We make the same assumptions as in proposition 4.4. For all compact set $K \subset \mathbb{R}^D$ there exists a constant $C > 0$ and real numbers $a_1, \ldots, a_D$, such that for all $N \geq 0$:*

$$\mathcal{V}\Delta t \sum_{\mathrm{I}_\alpha \subset K} \sum_{n=0}^{N} \sum_{d=1}^{D} \left| 2a_d \frac{B_{\alpha+2e_d}^n - B_{\alpha-2e_d}^n}{4\Delta x_d} + (1-2a_d) \frac{B_{\alpha+e_d}^n - B_{\alpha-e_d}^n}{2\Delta x_d} \right|^2 \leq C \sup_{0 \leq n \leq N} \|u_\Delta^n\|_1. \tag{4.29}$$

We recall that we have an estimate of the solution in L^1 by (4.6) for the general case and by proposition 4.3 for monotone schemes.

Proof of proposition 4.5. We choose here the entropy $\eta(u) = u^2/2$. As in the proof of proposition 4.4, we have again the inequality (4.27) but here we have to be more precise.

Let us first suppose that $M_l' > 0$ for all l. The inequality (4.5) and the continuity of M_l insure that $[M_l(u_\alpha^{n+1}), f_{\alpha,l}^{n+1/2}] \subset M_l\left([-\|u_0\|_\infty, \|u_0\|_\infty]\right)$ and

$$\sum_{l=1}^{N+N'} E_{\alpha,l}^n \le -\frac{1}{2}\sum_{l=1}^{N+N'} \left(M_l(u_\alpha^{n+1}) - f_{\alpha,l}^{n+1/2}\right)^2 \min\left\{S_{l,\eta}''(g),\ \ g \in [M_l(u_\alpha^{n+1}), f_{\alpha,l}^{n+1/2}]\right\}.$$

Applying (4.22) gives:

$$\sum_{l=1}^{N+N'} E_{\alpha,l}^n \le -\frac{1}{2}\sum_{l=1}^{N+N'} \left(M_l(u_\alpha^{n+1}) - f_{\alpha,l}^{n+1/2}\right)^2. \tag{4.30}$$

Let us integrate this inequality and divide by Δt:

$$\begin{aligned}\frac{1}{2}\mathcal{V}\Delta t\sum_{n=0}^{N}\sum_{\alpha\in\mathbb{Z}^D}\sum_{l=1}^{N+N'}\left|\frac{M_l(u_\alpha^{n+1}) - f_{\alpha,l}^{n+1/2}}{\sqrt{\Delta t}}\right|^2 &\le -\mathcal{V}\sum_{\alpha\in\mathbb{Z}^D}\left(\eta(u_\alpha^N) - \eta(u_\alpha^0)\right)\\ &\le C.\sup_{0\le n\le N}\|u_\Delta^n\|_1. \end{aligned}\tag{4.31}$$

On the other hand, we remark that for all l:

$$\frac{f_{\alpha,l}^{n+1} - f_{\alpha,l}^n}{\Delta t} + \sum_{d=1}^{D}\gamma_{ld}\frac{f_{\alpha+e_d/2,l}^n - f_{\alpha-e_d/2,l}^n}{\Delta x_d} = \frac{1}{\Delta t}\left(M_l(u_\alpha^{n+1}) - f_{\alpha,l}^{n+1/2}\right) \tag{4.32}$$

with $f_{\alpha,l}^n = M_l(u_\alpha^n)$. Consequently, subtracting equations $N+2d-1$ and $N+2d$, we have:

$$\begin{aligned}&\frac{\gamma^\epsilon}{2D\theta^2\sqrt{2}\Delta x_d}\left(a_d(B_{\alpha+2e_d}^n - B_{\alpha-2e_d}^n) + (1-2a_d)(B_{\alpha+e_d}^n - B_{\alpha-e_d}^n)\right)\\ &= \frac{1}{\Delta t}\left((M_{N+2d-1}(u_\alpha^{n+1}) - f_{\alpha,N+2d-1}^{n+1/2}) - (M_{N+2d}(u_\alpha^{n+1}) - f_{\alpha,N+2d}^{n+1/2})\right)\end{aligned}$$

where we have put $a_d = a_{-1,d} + a_{2,d}$ and used (3.10). Now we put $\rho_d = \frac{\gamma^\epsilon\Delta x_d}{2D\theta^2\sqrt{2}}$ as in (3.50). According to the proof of proposition 3.3, for all d, one of the two properties (3.51), (3.52) is true, so ρ_d is a constant not depending on Δx. We multiply the above equality by $\mathcal{V}\Delta t\sqrt{\Delta t}$, sum over n and α, use (4.31) and we find:

$$\mathcal{V}\Delta t\rho_d^2\xi_d\sum_{\alpha\in\mathbb{Z}^D}\sum_{n=0}^{N}|2a_d\frac{B_{\alpha+2e_d}^n - B_{\alpha-2e_d}^n}{4\Delta x_d} + (1-2a_d)\frac{B_{\alpha+e_d}^n - B_{\alpha-e_d}^n}{2\Delta x_d}|^2 \le 2C.\sup_{0\le n\le N}\|u_\Delta^n\|_1.$$

Here $\xi_d = \Delta t/\Delta x_d^2$ is constant. Remark that this estimate does not depend on the compact set K.

We consider now the degenerate case where one of the $M_l' = 0$ on a subset of $M_l^{-1}([M_l(u_\alpha^{n+1}), f_{\alpha,l}^{n+1/2}])$. If this occurs on isolated points, the preceding proof remains valid. Otherwise the inequality (4.30) is more delicate. We remark first that by (4.32) and proposition 4.2:

$$\lim_{\Delta x\to 0}\|M_l(u_\Delta^{n+1})-f_{\Delta,l}^{n+1/2}\|_1=0.$$

Therefore, $M_l(u_\Delta^{n+1})-f_{\Delta,l}^{n+1/2}\to 0$ uniformly on K. Suppose for example that $f_{\alpha,l}^{n+1/2}>M_l(u_\alpha^{n+1})$. For Δx small enough:

$$M_l^{-1}(\{M_l(u_\alpha^{n+1})\})=[u_{\alpha,l}^-,u_{\alpha,l}^+],\quad M_l^{-1}(\{f_{\alpha,l}^{n+1/2}\})=[v_{\alpha,l}^-,v_{\alpha,l}^+],$$

and $M_l'>0$ on $]u_{\alpha,l}^+,v_{\alpha,l}^-[$. Hence we have:

$$\begin{aligned}S_{l,\eta}(M_l(u_\alpha^{n+1}))-S_{l,\eta}(f_{\alpha,l}^{n+1/2})= &\quad S_{l,\eta}(M_l(u_{\alpha,l}^+)-S_{l,\eta}(M_l(v_{\alpha,l}^-))\\ \leq &\quad -\eta'(u_{\alpha,l}^+)(f_{\alpha,l}^{n+1/2}-M_l(u_\alpha^{n+1}))\\ &-(M_l(u_\alpha^{n+1})-f_{\alpha,l}^{n+1/2})^2\min\{S_{l,\eta}''(M_l(w)),\ w\in]u_{\alpha,l}^+,v_{\alpha,l}^-[\}.\end{aligned}$$

As η' is nondecreasing:

$$\eta'(u_{\alpha,l}^+)\geq\eta'(u_\alpha^{n+1})$$

and we retrieve (4.30). The end of the proof is the same, except we consider the integrals over K. □

These results allow us to prove the following convergence theorem:

Theorem 4.1 *(i) Suppose that (H1)(H2) are satisfied and that H_Δ is TVD. When $\Delta x\to 0$ and the $\Delta t/\Delta x_d^2$ are kept constant, for all $T>0$ the numerical solution u_Δ given by DKS converges in $\mathrm{L}^\infty([0,T],\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^D))$ towards a weak solution $u\in C([0,T],\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^D))\cap\mathrm{L}^\infty([0,T]\times\mathbb{R}^D)$ of (1.1)(1.2).*

(ii) Suppose that H_Δ is monotone and that (H1') is satisfied. Then the result of (i) holds. Moreover there is convergence in $\mathrm{L}^\infty([0,T],\mathrm{L}^1(\mathbb{R}^D))$, the limit $u\in C([0,T],\mathrm{L}^1(\mathbb{R}^D))\cap\mathrm{L}^\infty([0,T]\times\mathbb{R}^D)$ and u is the unique weak entropy solution of the problem in the sense of Carrillo, i.e. (1.7) holds with (1.8), (1.9).

Proof of Theorem 4.1. The convergence is obtained as in the purely hyperbolic case, see for example [10]: for all $t\geq 0$, $\{u_\Delta(t),\quad \Delta t>0\}$ is bounded in $\mathrm{L}^1\cap\mathrm{L}^\infty$. Moreover, by proposition 4.2, we can apply Fréchet-Kolmogorov theorem and obtain a relatively compact set of $\mathrm{L}^1_{\mathrm{loc}}$. Using the equicontinuity property (4.7) as in the proof of Ascoli-Arzela's Theorem we obtain convergence in $\mathrm{L}^\infty(0,T;\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}))$ and a.e. towards a function u. Then we can proceed as in the proof of Lax-Wendroff's Theorem: the only difference is of course that we have diffusion terms but the computation is straightforward.

Suppose now that H_Δ is monotone and that $u_0\in\mathrm{L}^1\cap\mathrm{L}^\infty$. As in [10], we can approximate u_0 by a sequence of elements of $\mathrm{L}^1\cap\mathrm{L}^\infty\cap\mathrm{BV}$ with compact support. We obtain the convergence and $u\in C([0,T],\mathrm{L}^1(\mathbb{R}^D))\cap\mathrm{L}^\infty([0,T]\times\mathbb{R}^D)$. The discrete entropy inequality (4.19) holds and thanks to (4.16), the bound in (4.29) is uniform. Here again we may use the same method as in the hyperbolic case and we obtain the entropy inequality (1.7). It remains to prove (1.9). We denote τ_e the translation operator.

$$\tau_e B(u_\Delta)(x) = B(u_\Delta(x+e)).$$

As u_Δ converges to u in $\mathrm{L}^\infty(0,T;\mathrm{L}^1(\mathbb{R}))$,

$$\frac{\tau_{2e_d}B(u_\Delta) - \tau_{-2e_d}B(u_\Delta)}{4\Delta x_d} \to \partial_{x_d}B(u) \text{ weakly}$$

and

$$\frac{\tau_{e_d}B(u_\Delta) - \tau_{-e_d}B(u_\Delta)}{2\Delta x_d} \to \partial_{x_d}B(u) \text{ weakly}$$

in $\mathbb{R}^D \times [0,T]$. Applying proposition (4.5), we conclude. □

Thanks to a recent result of F. Bouchut [5], we have an important corollary which completes this result.

Theorem 4.2 *Here $K=1$ and $D=1$. Consider a monotone scheme for (1.1)(1.2) which is written under the form (3.5) with $\mathcal{B}^n_\alpha$ given by a linear formula (3.15) and:*

$$\mathcal{A}^n_{\alpha+1/2} = \mathcal{A}^{n,+}_\alpha - \mathcal{A}^{n,-}_{\alpha+1}, \tag{4.33}$$

where $\mathcal{A}^{n,\pm}_\alpha = \mathcal{A}^\pm(u^n_\alpha)$, $\mathcal{A}^{\pm\prime} \geq 0$ for all $u \in I$. Suppose that $u_0 \in \mathrm{L}^\infty(\mathbb{R}) \cap \mathrm{L}^1(\mathbb{R})$. Then u^n_α converges in $\mathrm{L}^\infty([0,T],\mathrm{L}^1(\mathbb{R}))$ towards the entropy weak solution $u \in C([0,T],\mathrm{L}^1(\mathbb{R})) \cap \mathrm{L}^\infty([0,T]\times\mathbb{R})$ of (1.1)(1.2).

Proof of Theorem 4.2. As F. Bouchut remarked is the purely hyperbolic case, this scheme can be obtained by applying the kinetic splitting method to the model:

$$\begin{cases} \gamma^\epsilon = \left(\lambda, 0, -\lambda, -\dfrac{\mu}{\sqrt{2}} - \dfrac{\theta}{\sqrt{\epsilon}}, \dfrac{\mu}{\sqrt{2}} + \dfrac{\theta}{\sqrt{\epsilon}}\right)^T, \\ M = \left(\dfrac{\mathcal{A}^+}{\lambda}, u - \dfrac{\mathcal{A}^+ + \mathcal{A}^-}{\lambda} - \dfrac{B}{\theta^2}, \dfrac{\mathcal{A}^-}{\lambda}, \dfrac{B}{2\theta^2}, \dfrac{B}{2\theta^2}\right)^T. \end{cases} \tag{4.34}$$

M is a MMF if

$$\sup_{u\in I} \frac{\mathcal{A}^{+\prime}(u) + \mathcal{A}^{-\prime}(u)}{\lambda} + \frac{B'(u)}{\theta^2} \leq 1. \tag{4.35}$$

It is always possible to find such λ and θ. It remains to precise (H_Δ). On the two last equations, it is determined by proposition 3.2. On the three first ones, we take the upwind scheme:

$$\mathcal{A}^n_{\alpha+1/2} = \lambda\left(f^n_{\alpha+1/2,1} - f^n_{\alpha+1/2,3}\right) = \mathcal{A}^{n,+}_\alpha - \mathcal{A}^{n,-}_\alpha. \quad \square$$

A similar viewpoint is adopted in [4], using Maxwellian functions depending on ϵ and a 3×3 kinetic model.

5 Numerical results

In this section, we shall present the numerical results performed with the diffusive BGK approximations. We also make detailed comparison between our BGK approximation and the existing nice scheme SD2 of A. Kurganov and E. Tadmor [25]. The section ends with some further numerical simulations with high-order PPM method and high-dimensional example.

5.1 Comparison between kinetic and semi-conservative splitting schemes

For a numerical scheme for degenerate parabolic equations (1.1), one needs to specify

- a diffusive BGK model;
- a splitting method for this model (kinetic or semi-conservative);
- a scalar hyperbolic problem solver;
- a time integration method.

In all our 1-D simulations, the BGK model applied is the DRM1 (2.6), the DRM2 (2.13) or the FDM (2.18). Both kinetic and semi-conservative splitting are envisaged. A specific choice yields a system (3.5), or in its semi-discrete form (3.81).

This system is then solved by a second-order Runge-Kutta (mid-point) method. That is, we combine two first order (in time) Euler forward steps into a second-order time step. Each Euler forward step contains two sub-steps, namely, a convection step for the homogeneous linear hyperbolic system (3.18) or (3.58), and a projection step for the source terms. Each equation of the linear hyperbolic system is solved by second order MUSCL type scheme with minmod limiter, or first order upwind scheme, as described in the previous sections. In the projection step, on the other hand, a 'relaxed' scheme is always used as we directly project f_l^ϵ to corresponding equilibrium state. Though we could solve explicitly the ODE (3.25) or the PDE (3.69) for non-zero ϵ, it turns out that this 'relaxed' scheme always provides less numerical smearing effect.

More specifically, we shall perform numerical simulations with three particular numerical schemes based on diffusive BGK models. Depending on the splitting method adopted, they are

- KS1: kinetic splitting method (3.17) with $a_0 = 1, a_1 = 0, m = 0$, and ϵ determined by (3.34). The first N homogeneous equations are solved by MUSCL method, whereas the last two by upwind method.

- KS2: kinetic splitting method (3.17) with $a_0 = 1, a_1 = 0, m = 0$, and ϵ determined by (3.34) but we add also a second order correction on the last two equations, so that all homogeneous equations are solved by MUSCL method.

- SCS: semi-conservative splitting method (3.68). All homogeneous equations are solved by MUSCL method.

Three other schemes are also used for comparison.

- Naive: direct decoupling of the hyperbolic part and diffusion part for the original equation(s) (1.1). The quasilinear hyperbolic part is solved by MUSCL type scheme with minmod limiter, and the parabolic part by centered difference.

- Jin's: with splitting technique devised by S. Jin, L. Pareschi and G. Toscani [21].

- SD2: second order central semi-discretized scheme proposed by A. Kurganov and E. Tadmor [25]. Note that our SD2 code adopts mid-point for time integration, rather than DUMKA3 devised by A. Medovikov in the original paper [25].

We use $\theta = \lambda$ in all cases, and $\mu = \sqrt{2}\lambda$ unless otherwise mentioned. The MMF condition (2.7) is not strictly satisfied in our codes. In fact, we take $\lambda = \max|A'(u)|$ unless otherwise mentioned. We obtain satisfactory numerical results, though a more rigorous study of this point is yet to be made. We remark here that this indifference to the violation of MMF condition has also been reported for purely hyperbolic systems [2].

First we demonstrate the numerical solutions at time $t = 0.7$ with these schemes for fine grids $\triangle x = 0.01$. We compute the 1-D Burgers type equation

$$u_t + (u^2)_x = 0.1(\nu(u)u_x)_x, \tag{5.1}$$

with strongly degenerate diffusion

$$\nu(u) = \begin{cases} 0, & |u| \leq 0.25, \\ 1, & |u| > 0.25, \end{cases}$$

and initial data

$$u(x,0) = \begin{cases} 1, & -\frac{1}{\sqrt{2}} - 0.4 < x < -\frac{1}{\sqrt{2}} - 0.4, \\ -1, & \frac{1}{\sqrt{2}} - 0.4 < x < \frac{1}{\sqrt{2}} - 0.4, \\ 0, & \text{otherwise}. \end{cases}$$

We use the flux decomposition model (2.18). Because of symmetry, we only depict half branch of the solution in Figure 1. As remarked before, the kinetic splitting with MUSCL method yields solution with wrong speed. The other codes, on the other hand, reach essentially the same solution. In fact, solution with SCS is the same as that with Jin's scheme. We note that our semi-conservative splitting method is a natural extension of Jin's technique. They coincide for the simple case here. We also observed that the solution given by KS1 is identical with that by naive splitting scheme. This fact may be proved by writing explicitly the numerical flux for the last two equations with upwind scheme, just like what has been done in (3.56). The functions M_1 and M_3 do not depend on $B(u)$, and M_2 gives no effect as f_2 is stationary. This leads to the independence of the scheme to parameter θ.

In these numerical tests, e.g. the kinetic splitting scheme, the CFL condition (3.24) amounts to $\triangle t \leq 0.0000125$. Yet numerical experiments are satisfactory at much bigger time step size. We use $\triangle t = 0.00025$ for the two kinetic splitting schemes and naive splitting scheme, and $\triangle t = 0.0001$ for SD2. The scheme SCS and Jin's scheme are used with even bigger time step size $\triangle t = 0.0005$. We notice that small wiggles are still observed around the degenerate points for SD2, though the time step size is already the smallest one.

Secondly, for coarse grids, we make comparison among the KS1, Jin's, SCS and SD2. We take the numerical solution by the SCS with fine grid as the 'exact' solution. The results are shown in Figures 2 to 4. The semi-conservative splitting again gives same result as Jin's scheme. Around the border of degeneracy, SD2 produces some oscillations. We summarize the L^1 errors in the following table. It turns out that all these schemes are comparable in accuracy.

	SCS	KS1	Jin's	SD2
$\triangle x = 0.02, \triangle t = 0.0005$	1.8394×10^{-2}	1.7158×10^{-2}	1.8394×10^{-2}	2.4556×10^{-2}
$\triangle x = 0.04, \triangle t = 0.001$	2.7105×10^{2}	2.5528×10^{-2}	2.7105×10^{-2}	3.4870×10^{-2}
$\triangle x = 0.08, \triangle t = 0.005$	5.6389×10^{-2}	5.1421×10^{-2}	5.6389×10^{-2}	7.0031×10^{-2}

Thirdly, as the diffusive BGK model includes the parameter μ, we examine its effect on the accuracy for kinetic and semi-conservative splitting schemes. We denote $\mu = \sqrt{2}\alpha\lambda$, and fix discretization parameters $\triangle x = 0.04, \triangle t = 0.001$. When using KS1, the numerical experiments indicate that the solution is virtually not influenced by μ. The L^1 error remains at 2.5528×10^{-2} over $\alpha \in [0.001, 8]$. A detailed analysis shows that this is a result of the choice

$$\left(\frac{\mu}{\sqrt{2}} + \frac{\theta}{\sqrt{\epsilon}}\right) = \frac{2\theta^2}{\triangle x}.$$

On the other hand, for SCS scheme, the L^1 error varies from 2.7035×10^{-2} to 2.7950×10^{-2} over $\alpha \in [0.001, 8]$. Smaller μ improves, yet with limitation when the error is mainly caused by discretization factors. The solutions for $\alpha = 0.001$ and $\alpha = 8$ are almost indiscernible, as shown in Figure 5. The following table gives quantitative illustration of the L^1 errors.

α	8	4	2	1	0.5
Error($\times 10^{-2}$)	2.7590	2.7425	2.7186	2.7105	2.7071
α	0.25	0.125	0.0625	0.03125	0.001
Error($\times 10^{-2}$)	2.7052	2.7043	2.7039	2.7037	2.7035

5.2 A three phase flow model

This example is taken in the article of K.H. Karlsen *et al* [23]. Here we deal with a one dimensional 2×2 system modeling a multi-phase flow in a porous medium:

$$\begin{cases} \partial_t u + \partial_x f(u) = \epsilon \partial_{xx} d(u), \\ \partial_t v + \partial_x g(u,v) = \epsilon \partial_{xx} d(u) \end{cases} \tag{5.2}$$

with

$$(u_0(x), v_0(x)) = (0.4, 0.6) \text{ if } x < 1, \ (0,0) \text{ else.}$$

The unknown functions u and v are phase saturations (gas and water) and then take values in $[0,1]$. The flux function $A(u,v) = (f(u), g(u,v))$ is defined by:

$$\begin{cases} f(u) = \dfrac{u^2}{u^2 + (1-u)^2/10} \\ g(u,v) = \dfrac{(1-u)^2 + u^2/10}{10u^2 + (1-u)^2} f(v) = \phi(u) f(v), \end{cases}$$

and the diffusion function $B(u,v) = (\epsilon d(u), \epsilon d(v))$ is such that:

$$d'(w) = 4w(1-w), \quad \text{and we take } \epsilon = 0.1$$

For all (u,v), $A'(u,v)$ is a triangular matrix while $B'(u,v)$ is diagonal, hence the assumptions of Theorem 1.2 are not satisfied. Nevertheless we keep the requirement of monotonicity on the Maxwellian functions. First, we may remark that for all $(u,v) \in [0,1]^2$, the eigenvalues of $A'(u,v)$ are positive. Therefore, for the DRM2, M has positive eigenvalues if for all $(u,v) \in I$, the parameters λ_m, λ_p and θ satisfy:

$$\begin{cases} 1 - \dfrac{d'(u)}{\theta^2} \geq \alpha, \\ 0 \leq \lambda_m \leq \dfrac{f'(u)}{1 - \dfrac{d'(u)}{\theta^2}} \leq \lambda_p, \end{cases} \qquad \begin{cases} 1 - \dfrac{d'(v)}{\theta^2} \geq \alpha, \\ 0 \leq \lambda_m \leq \dfrac{\phi(u) f'(v)}{1 - \dfrac{d'(v)}{\theta^2}} \leq \lambda_p, \end{cases} \tag{5.3}$$

where $\alpha \in]0,1[$ is fixed. These parameters are evaluated at each time step. As for the scalar case, our experiments show that the accuracy is higher when α is close to 1, so that we have taken $\alpha = 0.9$. The solution is computed by the kinetic splitting method, with a second order discretization in space and a first order one in time. The value of the parameter m in the approximation of $\partial_{xx}d$ is 1/4 (see formula (3.43)).

As we have pointed out in paragraph 3.1.3, as far as we deal with first order (in space) schemes, the DRM2 and the FDM coincide. It is not the case, *a priori*, for second order schemes. However, in this particular case the numerical results are the same. We present the results obtained with both DRM1 and DRM2 (equivalently FDM) models, a time $t = 0.5$. Figure 7 represents the comparison between the two models for a space step $\Delta x = 0.01$. Here, the results are very good and there are almost no differences between them. In figure 8, we have taken a larger space step $\Delta x = 0.05$. As expected, the DRM2 is better than the DRM1 where the characteristic velocities are $-\lambda$, λ instead of $0 \leq \lambda_m \leq \lambda_p$.

5.3 Further tests

High-resolution methods such as PPM can be incorporated to the diffusive BGK models. A numerical example is depicted in Figure 6, for the same initial value problem in (5.1), where the discretization parameters are $\triangle x = 0.04$ and $\triangle t = 0.002$. The BGK model used here is the DRM1 (2.6), which is slightly more diffusive than the flux decomposition model, yet easier to implement.

The diffusive BGK model has also been successfully applied to high dimensions. For instance, Figure 9 shows the numerical solution at $t = 0.5$ for the 2-D Burgers type equation

$$u_t + (u^2)_x + (u^2)_y = 0.1(\nu(u)u_x)_x + 0.1(\nu(u)u_y)_y, \tag{5.4}$$

with strongly degenerate diffusion

$$\nu(u) = \begin{cases} 0, & |u| \leq 0.25, \\ 1, & |u| > 0.25, \end{cases}$$

and initial data

$$u(x, y, 0) = \begin{cases} -1, & (x - 0.5)^2 + (y - 0.5)^2 \leq 0.16, \\ 1, & (x + 0.5)^2 + (y + 0.5)^2 \leq 0.16, \\ 0, & \text{otherwise.} \end{cases}$$

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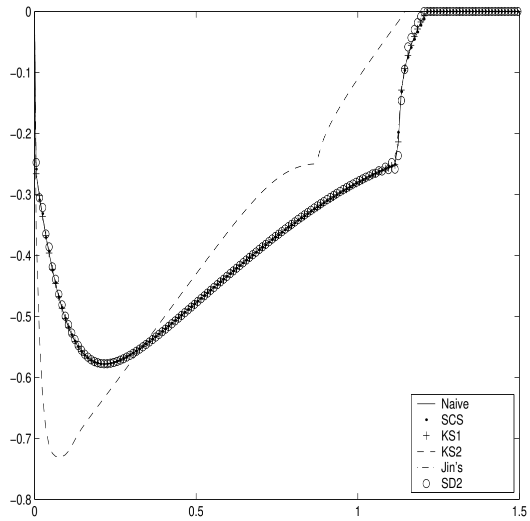

Figure 1: Comparison among different splitting methods.

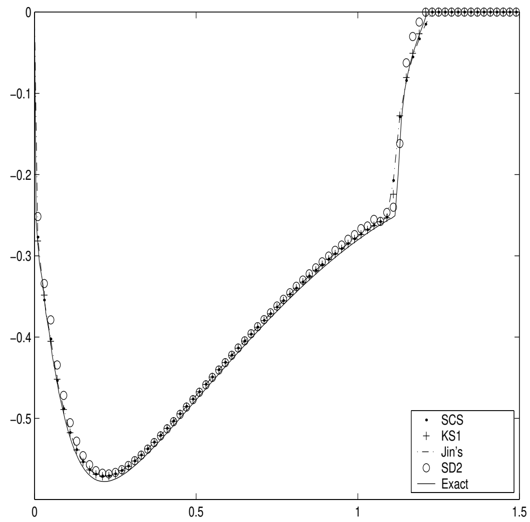

Figure 2: Comparison among different splitting methods, with $\triangle x = 0.02, \triangle t = 0.0005$.

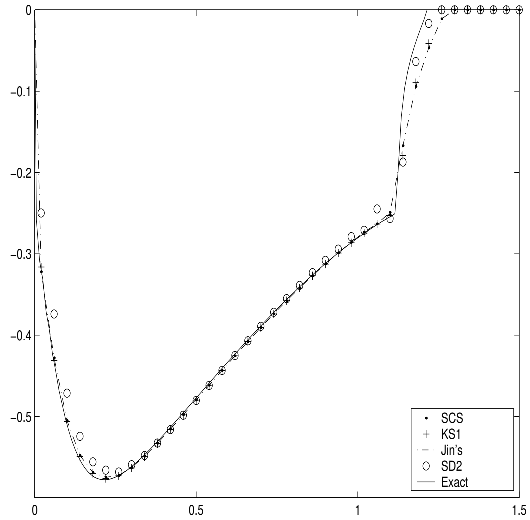

Figure 3: Comparison among different splitting methods, with $\triangle x = 0.04, \triangle t = 0.001$.

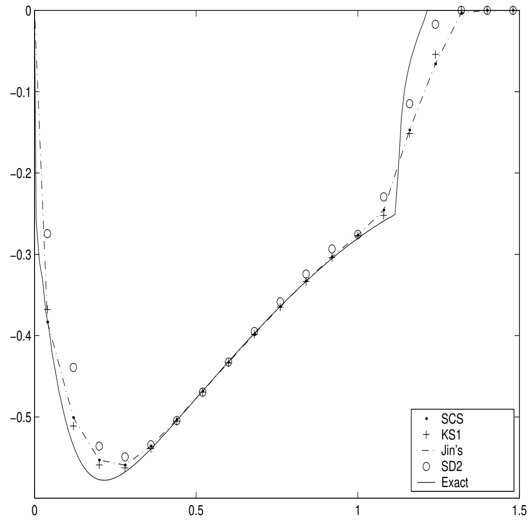

Figure 4: Comparison among different splitting methods, with $\triangle x = 0.08, \triangle t = 0.005$.

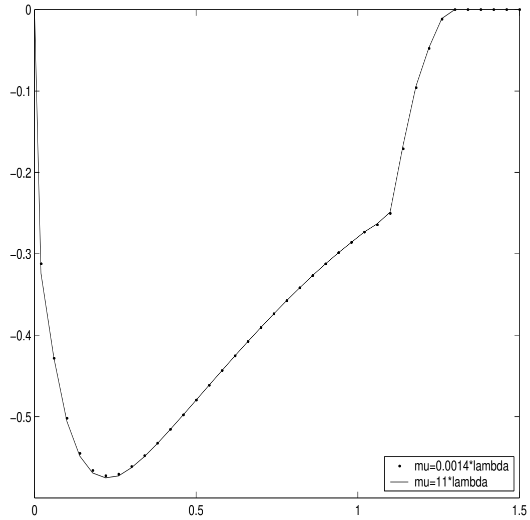

Figure 5: Effect of different μ, with $\triangle x = 0.04, \triangle t = 0.001$.

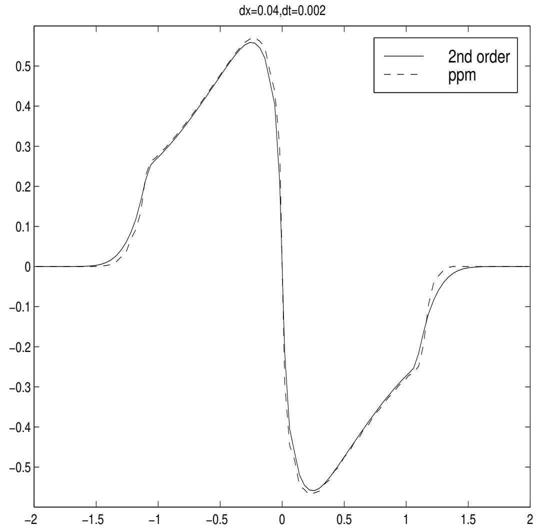

Figure 6: Comparison between second-order scheme and PPM.

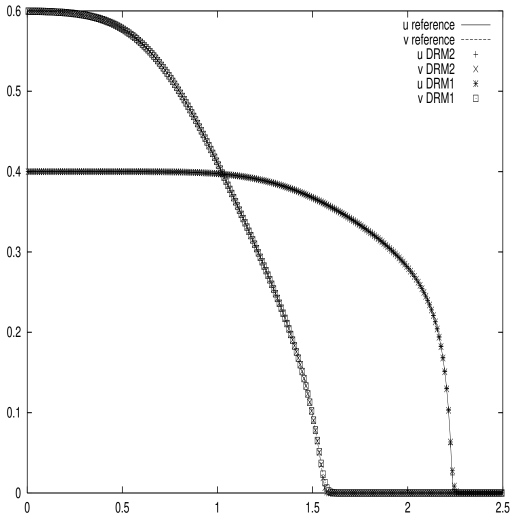

Figure 7: Approximation to a 2x2 system: $\Delta x = 0.01$.

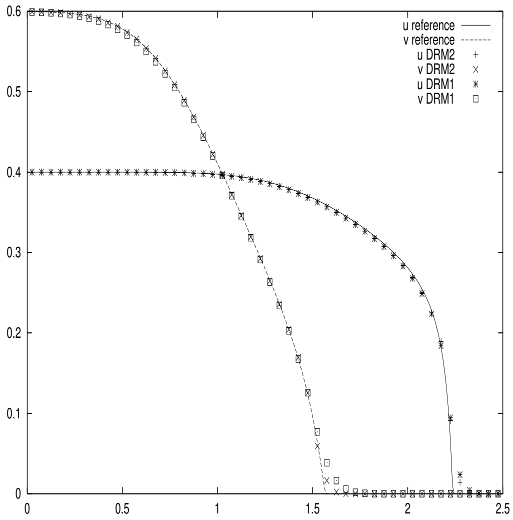

Figure 8: Approximation to a 2x2 system: $\Delta x = 0.05$.

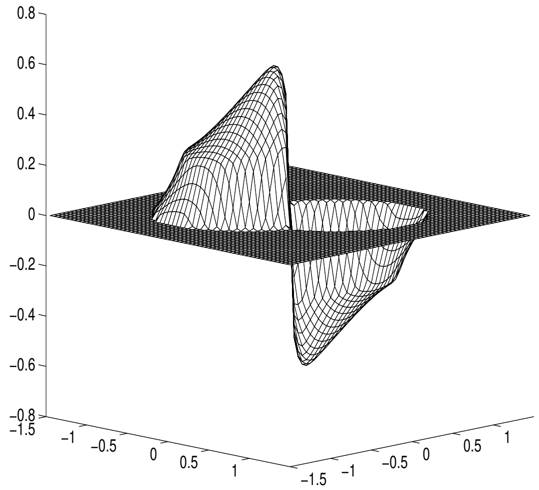

Figure 9: Approximation to 2-D Burgers type equation.