# Nonhomogeneous Dirichlet Problems for Degenerate Parabolic-Hyperbolic Equations

Corrado Mascia [1], Alessio Porretta [2] and Andrea Terracina [3]

**Abstract**

The aim of the paper is to give a formulation for the initial boundary value problem of parabolic-hyperbolic type

$$\begin{cases} \partial_t u - \Delta b(u) + \operatorname{div} \Phi(u) = g, \\ u|_{t=0} = u_0, \qquad u|_{\partial\Omega\times(0,T)} = a_0. \end{cases}$$

in the case of nonhomogeneous boundary data $a_0$. Here $u = u(x,t) \in \mathbb{R}$, with $(x,t) \in Q = \Omega \times (0,T)$, where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary and $T > 0$. The function $b$ is assumed to be nondecreasing (allowing degeneration zones where $b$ is constant), $\Phi$ is Lipschitz continuous and $g \in L^\infty(\Omega \times (0,T))$. After introducing the definition of entropy solution to the above problem (in the spirit of Otto [13]), we prove results of uniqueness of the solution in the proposed setting. Moreover we prove that the entropy solution previously defined can be obtained as limit of solutions of regularized equations of nondegenerate parabolic type (specifically the diffusion function $b$ is approximated by functions $b^\varepsilon$ that are strictly increasing).

The approach proposed for the hyperbolic-parabolic problem can be used to prove similar results for the class of hyperbolic-elliptic boundary value problems of the form

$$u - \Delta b(u) + \operatorname{div} \Phi(u) = g \qquad\qquad u|_{\partial\Omega} = a_0.$$

again in the case of nonconstant boundary data $a_0$.

## 1 Introduction

In this paper we deal with the problem of giving a formulation for the initial boundary value problem

$$\partial_t u - \Delta b(u) + \operatorname{div} \Phi(u) = g, \tag{1.1}$$

$$u|_{t=0} = u_0, \qquad u|_{\partial\Omega\times(0,T)} = a_0. \tag{1.2}$$

Here $u = u(x,t) \in \mathbb{R}$, with $(x,t) \in Q = \Omega \times (0,T)$, where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary and $T > 0$. The function $\Phi$ is Lipschitz continuous and $f \in L^\infty(Q)$. The function $b$ is assumed to be continuous and nondecreasing. This monotonicity assumption allows some degeneration zone for the state variable $u$, so that the equation is of *hyperbolic-parabolic type*. Such kind of phenomena appear in many applicative models, see for instance [4, 5, 10].

[1]Dipartimento di Matematica, Università "La Sapienza", P.le Aldo Moro, 5, 00185, Roma, Italy
[2]Dipartimento di Matematica, Università "Tor Vergata", Via della Ricerca Scientifica, 00133, Roma, Italy
[3]Dipartimento di Matematica, Università "La Sapienza", P.le Aldo Moro, 5, 00185, Roma, Italy

The degeneration makes the problem much different from the nondegenerate parabolic one (in which the function $b'$ is strictly separated from 0) and from the hyperbolic one (in which $b' \equiv 0$) since the two behaviors (parabolic and hyperbolic) are mixed together in a way that depends on the solution itself. The analysis of this kind of equation should keep in mind both the features of the parabolic and of the hyperbolic equations and it has been considered in [4, 5, 3, 6]. Let us stress that the well-known porous media equation is contained in the class (1.1), in which case the degeneration set is restricted to a single point. In our analysis we also admit functions $b$ degenerating on intervals.

The main novelty of our work concerns with the boundary datum $a_0$ in (1.2). Indeed in all of the previous articles it has been considered the case of homogeneus boundary data, i.e. $a_0 \equiv 0$. When considering more general data, the formulation of the problem given in previous works have no straightforward extensions, and a new setting to problem (1.1-1.2) should be given. Our aim is to give a suitable definition of solution to (1.1-1.2) for smooth data $a_0$ guaranteeing uniqueness (through $L^1$ contractivity) and consistency with parabolic regularizations of the above equation in a sense that will be made more clear later on.

As we stressed before, problem (1.1-1.2), in general, is of hyperbolic-parabolic type. If $b' \geq \nu > 0$, the equation is non degenerate and the classical theory can be applied, obtaining existence and uniqueness of classical solutions. In this case the regularity of the solutions implies that the boundary conditions are assumed in the sense of traces. The opposite case is the completely degenerate one, that can be obtained from (1.1) by setting $b' \equiv 0$. In this situation the equation is of hyperbolic type and the problem is much more delicate, due to the possible presence of shocks (internal layers) in the solution. It is known the the correct setting for such kind of problems is based on the definition of entropy solution in the sense introduced by Kružkov in [11], i.e. weak solutions satisfying an additional assumption corresponding, poorly speaking, to entropy decreasing along discontinuity curves. When dealing with the boundary conditions (1.2) for first order equations, we also have to consider the possibility of boundary layers, i.e. the solution may not assume conditions at the boundary since the problem turns to be overdetermined. Hence these conditions should be interpreted as appropriate compatibility conditions. From many respects, boundary layers can be seen as discontinuity on the boundary and, as for the internal layers, the correct condition to be satisfied should be an entropy decreasing condition.

Many papers have been devoted to the hyperbolic problem

$$\begin{cases} \partial_t u + \text{div } \Phi(u) = g, \\ u|_{t=0} = u_0, \qquad u|_{\partial\Omega\times(0,T)} = a_0. \end{cases} \tag{1.3}$$

In all of these approaches, as it is natural, a function $u \in L^\infty(Q)$ is called entropy solution of the problem if it satisfies the entropy inequality for any positive test function that vanishes on the boundary, i.e.

$$\int_Q \{|u-k|\partial_t\xi + \text{sgn}\,(u-k)(\Phi(u) - \Phi(k))\nabla\xi + g\xi\}\ dxdt \geq 0 \qquad \forall \xi \in H_0^1(Q),$$

and, at the same time, it agrees an appropriate boundary condition. The main difference is in the formulation of this compatibility condition. In [2] it has been solved the problem of existence and uniqueness in the BV-setting. The regularity of the solution guarantees the existence of the trace of the solution $u$ on the boundary and the condition to be satisfied is imposed algebraically, i.e. at almost any point of the boundary the trace of the solution must belong to an appropriate set of admissible values, depending on the flux $\Phi$ and on the function $a_0$. This approach has been followed by many authors ([9], [15]) that have extended the previous results in different directions. In particular, in [9] it has been given an interesting interpretation of the boundary condition given in [2] for the one-dimensional case in terms of solutions of Riemann problems.

The main limit of the previous approach to problem (1.3) is the request of BV-regularity to the solution. In order to define a setting for more general data (namely, $L^\infty$ data) a new definition has been given by F. Otto [13] (see also [12]). This approach consists in asking the boundary condition to hold in integral form by introducing appropriate boundary entropy-entropy flux pairs. The same framework has been considered in [8], where it is stressed that to require the minimal regularity needed to give sense to the boundary trace of solutions it is useful to work in the setting of divergence-measure fields in the sense first introduced in [1]. Further generalizations of the work by F. Otto can be found in [16].

Let us go back to the hyperbolic-parabolic case. The main problem in dealing with problem (1.1)–(1.2) is that the region of parabolicity and of hyperbolicity are glued together in a way that depends on the solution itself, so that there is no hope to obtain a correct formulation by simply taking in account separately the parabolic and the hyperbolic features. Again, in the interior of domain $Q$, it is natural to give an entropy formulation: $b(u) \in L^2(0,T;H^1(\Omega))$ and

$$\int_Q \{|u-k|\partial_t\xi - \nabla|b(u)-b(k)|\nabla\xi + \operatorname{sgn}(u-k)(\Phi(u)-\Phi(k))\nabla\xi + g\xi\}\, dxdt \geq 0, \qquad (1.4)$$

for any $\xi \in H_0^1(Q)$ and the initial condition satisfied as $L^1-$limit as $t \to 0$. Since the function $b$ can degenerate, boundary layers may appear and the boundary condition, as in the first order equation, should be interpretated as compatibility condition. The approach in the same spirit of [2] would impose very strong regularity assumptions on the solution: both $u$ and $\nabla b(u)\cdot n$ (where $n$ is the outward normal vector to the boundary of $\Omega$) should have trace on $\partial\Omega$. A different approach seems more appropriate. In [6] an (entropy) solution to the problem with *homogeneous boundary datum* is defined by an integral formulation like (1.4) in $Q$, where test functions non zero on the boundary are also allowed provided $k$ is suitably chosen. Other results are contained in [3], for 1-dimensional case again with homogeneous boundary data, and in [4, 5] for some applicative models with boundary conditions different with respect to (1.2). In [6] existence and uniqueness of the solution is proved under suitable assumptions on initial data.

We propose here a definition of entropy solution to problem (1.1)–(1.2) that makes sense for nonconstant data $a_0$. We are not concerned here with the minimal regularity needed on $a_0$ in order that our notion of solution makes sense. In fact, we are mainly involved with the meaning that the boundary condition $u = a_0$ may have for such problem regardless of the possible technical extension to nonsmooth data (see also Remark 1.5).

For later use, we need to write the boundary condition in local coordinates near $\partial\Omega$. To this purpose, we will call $\{B_i\}$ a $C^2-$covering of $\partial\Omega$ if $B_i$ is a finite number of open sets such that $\partial\Omega \subset \cup\overline{B}_i$, and, in some local coordinates $x = (x', x_N)$, there exists a $C^2$ function $x_N = f_i(x')$ such that $B_i \cap \partial\Omega = \{x_N = f_i(x')\}$, $B_i \cap \Omega = \{x_N < f_i(x')\}$. Setting $\mathcal{B}$ the set of all possible $C^2-$coverings of $\partial\Omega$, for simplicity, in later use, we will drop the index $i$ and we will say that $B$ belongs to $\mathcal{B}$ to mean that $B$ belongs to a $C^2-$covering of $\partial\Omega$. We will also write $x = (x', x_N)$ for any $x \in B \cap \Omega$, where $(x', x_N)$ are the local coordinates induced by $B$, to which it also corresponds a $C^2$ function $x_N = f(x')$. We assume that $\partial\Omega$ admits at least one such covering (in other words, $\mathcal{B}$ is nonempty). Clearly this is true if $\Omega$ is a $C^2$ domain in $\mathbb{R}$.

In the method we propose the boundary conditions, written in integral form, are generally obtained as limit of volume integrals in $\Omega \times (0,T)$ which tend to concentrate on the boundary $\partial\Omega \times (0,T)$. To this purpose, we introduce the following definition.

**Definition 1.1** *We will call $\{\zeta_\delta\}$ a boundary layer sequence if $\zeta_\delta$ is a sequence of $C^2(\Omega) \cap C^0(\overline{\Omega})$ functions such that*

$$\lim_{\delta\to 0^+} \zeta_\delta = 1 \quad \textit{pointwise in } \Omega, \qquad 0 \le \zeta_\delta \le 1, \qquad \zeta_\delta = 0 \quad \textit{on } \partial\Omega.$$

The name of boundary layer sequence for $\zeta_\delta$ is obviously justified by the fact that the behavior of $\zeta_\delta$ as $\delta$ tends to zero gives an account of formation of boundary layers; indeed, we have that $-\nabla\zeta_\delta$ converges to the outward normal $\nu$ of the boundary and for any $\varphi \in (H^1(\Omega))^N$ we have:

$$\lim_{\delta\to 0} \int_\Omega \varphi \nabla\zeta_\delta \, dx = - \lim_{\delta\to 0} \int_\Omega \operatorname{div}(\varphi)\zeta_\delta \, dx = - \int_\Omega \operatorname{div}(\varphi) \, dx = - \int_{\partial\Omega} \varphi \cdot \nu \, d\sigma \, .$$

Note that our approach only formally differs from the one in [13] and in [8] where the boundary conditions are obtained as limit of integrals on suitable hypersurfaces which approximate the boundary from the interior domain. Our formulation could be equally translated in the framework used in these previous works.

Finally, through all the paper we use the following notation

$$\begin{aligned} F(u,v) &= \operatorname{sgn}(u-v)(\Phi(u) - \Phi(v)), \\ K_x(u,v) &= \nabla_x |b(u) - b(v)| - F(u,v), \\ H_x(u,v,w) &= K_x(u,v) + K_x(u,w) - K_x(w,v). \end{aligned} \tag{1.5}$$

Note that $K_x(u,v) = K_x(v,u)$ and $H_x(u,v,w) = H_x(u,w,v)$ for any $u, v, w$. Sometimes the index $x$ will be dropped in the above notation if clear from the context.

**Definition 1.2** *Let $g \in L^\infty(\Omega \times (0,T))$ and $a_0 \in L^\infty(\partial\Omega)$ be such that*

$$a_0 \in C^1(\partial\Omega)\,, \qquad \nabla b(a_0) \in L^2(\partial\Omega)\,. \tag{1.6}$$

*Given $T>0$, a function $u \in L^\infty(Q)$ is a* solution of the problem (1.1)–(1.2) *if*

*(i)* (regularity) *there holds:*

$$b(u) \in L^2(0,T;H^1(\Omega))\,,$$

*and, for every $B \in \mathcal{B}$, and any positive $\psi \in C_c^\infty(B)$ we have (here $x=(x',x_N)$, $x_N = f(x')$ on $B \cap \partial\Omega$ and $a_0 = a_0(x', f(x'))$):*

$$\Big(-|u-a_0|\,\psi, K_x(u,a_0)\psi\Big) \in \mathcal{DM}(Q)_2,$$

*where $\mathcal{DM}(Q)_2$ is the set of divergence-measure vector fields in $Q$ as defined in Section 2;*

*(ii)* (entropy condition in the interior of $Q$) *$u$ is an entropy solution of the equation with test functions zero on the boundary, i.e.*

$$\int_Q \{|u-k|\partial_t\xi - K_x(u,k)\nabla\xi + g\xi\}\, dxdt \geq 0, \tag{1.7}$$

*for any $\xi \in H_0^1(Q)$, $\xi \geq 0$;*

*(iii)* (initial condition) *the initial condition is assumed as limit in $L^1-$sense*

$$\lim_{t\to 0^+} \int_\Omega |u(x,t) - u_0(x)|\, dx = 0;$$

*(iv)* (boundary condition) *we have*

$$b(u) = b(a_0) \qquad \text{on } \partial\Omega \times (0,T) \text{ (in the sense of traces in } L^2(0,T;H^1(\Omega))\text{)},$$

*and for any boundary layer sequence $\zeta_\delta$, for every $B \in \mathcal{B}$, and any positive $\psi \in C_c^\infty(B)$ we have (here $x=(x',x_N)$, $x_N = f(x')$ on $B \cap \partial\Omega$ and $a_0 = a_0(x', f(x'))$):*

$$\liminf_{\delta\to 0} \int_Q H_x(u,k,a_0)\,\nabla\zeta_\delta\,\xi\,\psi\, dxdt \geq 0 \qquad \forall k \in \mathbb{R}, \tag{1.8}$$

*for any $\xi \in L^2(0,T;H^1(\Omega))$, $\xi \geq 0$.*

**Remark 1.3** In the hyperbolic case $b' \equiv 0$, the above boundary condition (1.8) reduces essentially to the Otto condition for conservation laws. Indeed, in this case, formula (1.8) becomes

$$\liminf_{\delta\to 0} \int_Q (F(u,k) + F(u,a_0) - F(a_0,k))\,\nabla\zeta_\delta\,\xi\, dx \leq 0 \qquad \forall k \in \mathbb{R},$$

where $F(u,v) = \operatorname{sgn}(u-v)(\Phi(u) - \Phi(v))$, that is equivalent to the boundary entropy-entropy flux condition proposed by Otto.

**Remark 1.4** It is interesting to note what happens when the boundary datum is assumed in the sense of traces (as in the case $b' > 0$). Then formula (1.8) (assuming the existence of normal derivatives along the boundary $\partial\Omega$) reads as

$$\int_0^T \int_{\partial\Omega} \frac{d}{d\mathbf{n}} (|b(u) - b(k)| + |b(u) - b(a_0)| - |b(a_0) - b(k)|)\xi \, d\sigma \le 0 \qquad (\mathbf{n} \text{ outer normal}, \xi \ge 0),$$

that is trivially satisfied since $|b(u) - b(k)| + |b(u) - b(a_0)| - |b(a_0) - b(k)|$ is a positive function that vanishes on the boundary.

**Remark 1.5** The boundary condition, as it is written in (1.8), points out the symmetry of the entropy inequality at the boundary which the solution $u$, the datum $a_0$ and the parameter $k$ have to satisfy. Apparently, this symmetric formulation requires a high regularity on $a_0$. In fact, it is not difficult to check (with easy choices of the sequence $\zeta_\delta$) that (1.8) may also be written as:

$$\begin{aligned} \liminf_{\delta \to 0} \frac{1}{\delta} \int_0^T \int_{x'} \int_{f(x')-\delta}^{f(x')} \left[ \frac{\partial}{\partial x_N} (|b(u) - b(k)| + |b(u) - b(a_0)|) - (F(u,k) + F(u,a_0)) \right] \cdot \\ \cdot (1 + |\nabla f(x')|^2) \, \xi \, \psi \, dx' \, dx_N \, dt \le \int_0^T \int_{\partial\Omega} F(a_0, k) \cdot \nu \, \xi \, \psi \, d\sigma \, dt \, , \end{aligned} \tag{1.9}$$

for any $k \in \mathbb{R}$, for any $\xi \in L^2(0,T; H^1(\Omega))$, $\xi \ge 0$. This shows the possible extension of our setting to data $a_0$ which are not smooth, since (1.9) is well defined even if $a_0$ is only bounded.

**Remark 1.6** Observe that if $b$ is Lipschitz continuous, assumption (1.6) only reduces to ask that $a_0$ is $C^1$.

In the above setting of entropy solutions, we are able to prove results on uniqueness and consistency with parabolic regularizations. Here are the statements of the Theorems. The proofs will be given in subsequent Sections.

**Theorem 1.7** *Assume $b \in C(\mathbb{R})$ be nondecreasing and $\Phi \in Lip\,(\mathbb{R}, \mathbb{R}^N)$. Let $a_0$ satisfy (1.6), $u_0, v_0 \in L^\infty(\Omega)$ and $T > 0$ be given. Let $u$, resp. $v$, be the solution of (1.1) with conditions $u|_{t=0} = u_0$ and $u|_{\partial\Omega\times(0,T)} = a_0$, resp. $v|_{t=0} = v_0$ and $v|_{\partial\Omega\times(0,T)} = a_0$. Then*

$$\int_\Omega |u(x,t) - v(x,t)| \, dx \le \int_\Omega |u(x,s) - v(x,s)| \, dx \qquad \forall s,t \quad \textit{with} \quad 0 \le s < t. \tag{1.10}$$

*In particular, there exists at most one solution of (1.1)–(1.2) in the sense of Definition 1.2.*

**Theorem 1.8** *Assume $b \in C(\mathbb{R})$ be nondecreasing and $\Phi \in Lip\,(\mathbb{R}, \mathbb{R}^N)$. Let $T > 0$, $u_0 \in L^\infty(\Omega)$ and let $a_0$ satisfy*

$$a_0 \in C^2(\partial\Omega)\,, \qquad \nabla b(a_0) \in L^2(\partial\Omega)\,, \qquad \Delta(b(a_0)) \in L^1(\partial\Omega)\,.$$

*Let $b^\varepsilon = b^\varepsilon(s)$ be such that $\frac{b^\varepsilon(s)-b^\varepsilon(t)}{s-t} \geq \nu_\varepsilon > 0$ for some constant $\nu_\varepsilon$, and assume that $b^\varepsilon$ converges to $b$ in $L^\infty_{loc}(\mathbb{R})$ as $\varepsilon \to 0^+$. Let $u_\varepsilon$ be the solution of the nondegenerate parabolic problem*

$$\begin{cases} \partial_t u_\varepsilon - \Delta b^\varepsilon(u_\varepsilon) + div\, \Phi(u_\varepsilon) = g, \\ u_\varepsilon|_{t=0} = u_0, \qquad u_\varepsilon|_{\partial\Omega\times(0,T)} = a_0. \end{cases}$$

*Assume that there exists $u \in L^\infty(Q)$ such that, as $\varepsilon \to 0$,*

$$\begin{aligned} &u_\varepsilon \to u \quad \text{strongly in } C([0,T]; L^1(\Omega)), \\ &b^\varepsilon(u_\varepsilon) \to b(u) \qquad \text{weakly in } L^2(0,T; H^1(\Omega)), \end{aligned}$$

*Then $u$ is an entropy solution in the sense of Definition 1.2 of the problem (1.1)–(1.2).*

As in the work of Carrillo (see [6]), the approach proposed for the hyperbolic-parabolic problem (1.1)-(1.2) can be used to prove similar results for the class of hyperbolic-elliptic equations

$$u - \Delta b(u) + \operatorname{div} \Phi(u) = g \qquad\qquad u|_{\partial\Omega} = a_0. \tag{1.11}$$

The last section of the present paper deals with the above problem, again aiming to give a suitable definition of solution to the problem in the case of nonconstant boundary data $a_0$ and guaranteeing uniqueness of the solution.

The structure of the paper is the following.

In Section 2 we prove the results on consistency of our definition with the nondegenerate parabolic case, i.e. if we solve an approximated version of the problem that is nondegenerate parabolic (so that the solution is smooth and the boundary conditions are assumed in classical sense) and then pass to the limit, the approximate sequence proves to converge to a solution of the limiting problem in the sense given by Definition 1.2. In this Section details on divergence-measure fields are given since this is the class to which the solution belongs.

Section 3 deals with the proof of uniqueness of solution, obtained by applying a Kružkov type technique.

Last Section concerns the problem (1.11). We give precise definition of solutions and we state the main results (that follow the same lines of the hyperbolic-parabolic case), without giving the proofs, but sketching the difference with the other case.

We also collect in the Appendix some results concerning a detailed study of a particular choice of boundary layer sequence $\zeta_\delta$ which are useful in our approach.

## 2 Divergence measure vector fields. Consistency of the formulation.

In this section we prove that the notion of entropy solution given above is consistent with the limit of parabolic approximations, in the sense that under appropriate regularity assumptions on the boundary datum $a_0$ every solution obtained through the vanishing viscosity method is an entropy solution. In order to deal with condition (i) of Definition 1.2 we need to introduce the notion of divergence measure vector fields.

We set

$$\mathcal{DM}(\Omega)_2 := \left\{ v \in (L^2(\Omega))^N \, : \exists C > 0 \, : \left| \int_\Omega v \nabla \varphi \, dx \right| \le C \, \|\varphi\|_{L^\infty(\Omega)} \quad \forall \varphi \in C_c^\infty(\Omega) \right\} ,$$

the space of $L^2(\Omega)$ vector fields whose divergence is a bounded Radon measure on $\Omega$. In fact, for every $v \in \mathcal{DM}(\Omega)_2$ we have that $\operatorname{div} v$ can be identified with a regular measure on the $\sigma$–algebra of the borelian subsets of $\Omega$. Then for any bounded continuous function $\varphi$ in $\Omega$ we will denote by $\int \varphi \operatorname{div} v$ the integral of $\varphi$ with respect to the measure $\operatorname{div} v$ which is well-defined. The properties of the space $\mathcal{DM}(\Omega)_2$ have been studied in [1] and in [8] mainly for $L^\infty(\Omega)$ vector fields $v$; although these two papers have different approaches, they both establish the existence of a trace for $v$ on $\partial\Omega$ in a weak sense and consequently an integration by parts formula. Let us sketch these arguments in the case of $L^2(\Omega)$ vector fields $v$; in the following we essentially follow the outlines of [1].

**Lemma 2.1** *Let $v \in \mathcal{DM}(\Omega)_2$. For every $u$ in $H^1(\Omega) \cap C(\Omega) \cap L^\infty(\Omega)$, let us set*

$$[v,u]_{\partial\Omega} := \int_\Omega u \, div v + \int_\Omega v \, \nabla u \, dx \, .$$

*Then there hold*

$$\begin{aligned} & [v,u]_{\partial\Omega} = \int_{\partial\Omega} u \, v \cdot \nu \, d\sigma \qquad \text{if } v \in (C^1(\bar\Omega))^N, \\ & [v,u]_{\partial\Omega} = 0 \quad \text{if } u \in H_0^1(\Omega), \\ & |[v,u]_{\partial\Omega}| \le C \, \|u\|_{H^1(\Omega)\cap L^\infty(\Omega)} \, . \end{aligned} \tag{2.1}$$

**Proof.** The first of (2.1) is trivially satisfied. To prove the second one, observe that if $u \in H_0^1(\Omega)$ is a bounded continuous function in $\Omega$ then $u$ can be approximated in $H_0^1(\Omega) \cap C_{\mathrm{loc}}(\Omega)$ with a sequence $u_j$ of $C_c^\infty(\Omega)$ functions with $L^\infty(\Omega)$ norm uniformly bounded by the $L^\infty(\Omega)$ norm of $u$. Thus $[v,u]_{\partial\Omega} = \lim_{j\to\infty} [v,u_j]_{\partial\Omega} = 0$. This implies that $[v,u]_{\partial\Omega}$ only depends on the trace of $u$ on $\partial\Omega$. Last inequality in (2.1) easily follows by definition of $[v,u]_{\partial\Omega}$ with $C = \|v\|_{L^2(\Omega)} + |\operatorname{div} v|(\Omega)$. ■

**Remark 2.2** As a consequence of (2.1) it is also possible to define a functional $\Lambda_v$ on $H^{\frac12}(\partial\Omega) \cap L^\infty(\partial\Omega)$ which can be regarded as the normal trace $v \cdot \nu$ at the boundary. Indeed, for every fixed $\xi \in H^{\frac12}(\partial\Omega) \cap L^\infty(\partial\Omega)$, $\Lambda$ is defined as $\Lambda(\xi) = [v, \bar\xi]_{\partial\Omega}$ for any $\bar\xi \in H^1(\Omega) \cap C(\Omega) \cap L^\infty(\Omega)$ such that $\bar\xi = \xi$ on $\partial\Omega$. Choosing $\bar\xi$ such that $\Delta\bar\xi = 0$ in $\Omega$ and $\bar\xi = \xi$ on $\partial\Omega$ implies that $\bar\xi \in H^1(\Omega) \cap C(\Omega) \cap L^\infty(\Omega)$ and $\|\bar\xi\|_{H^1(\Omega)\cap L^\infty(\Omega)} \le \|\xi\|_{H^{\frac12}(\partial\Omega)\cap L^\infty(\partial\Omega)}$. Then it follows that $|\Lambda(\xi)| \le C \|\xi\|_{H^{\frac12}(\partial\Omega)\cap L^\infty(\partial\Omega)}$. Denoting then the functional $\Lambda$ with $[v \cdot \nu]$, we have the integration by parts formula:

$$\langle [v \cdot \nu], \varphi \rangle = \int_\Omega \varphi \operatorname{div} v + \int_\Omega v \nabla \varphi \, dx \qquad \forall v \in \mathcal{DM}(\Omega)_2 \, , \quad \forall \varphi \in H^1(\Omega) \cap C(\Omega) \cap L^\infty(\Omega) \, , \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual application in $H^{\frac12}(\partial\Omega) \cap L^\infty(\partial\Omega)$.

We are not interested here in a detailed study of the properties of divergence measure vector fields. I n the following we will use Lemma 2.1 mainly to justify some limits of volume integrals in $\Omega$ which converge to surface integrals on $\partial\Omega$. This is done through the use of the above introduced boundary layer sequences defined in Definition 1.1.

**Lemma 2.3** *Let $v \in \mathcal{DM}(\Omega)_2$. Then for every $\xi \in H^1(\Omega) \cap C(\Omega) \cap L^\infty(\Omega)$ we have that*

$$\lim_{\delta\to 0} \int_\Omega v \nabla\zeta_\delta \xi \, dx = -[v, \xi]_{\partial\Omega} \,, \tag{2.3}$$

*for any boundary layer sequence $\zeta_\delta$. In particular if $\xi \in H^1_0(\Omega)$ we have that*

$$\lim_{\delta\to 0} \int_\Omega v \nabla\zeta_\delta \xi \, dx = 0 \,.$$

**Proof.** Since $\zeta_\delta = 0$ on $\partial\Omega$ we have, by Lemma 2.1:

$$\begin{aligned}
&[v, \xi]_{\partial\Omega} = [v, \xi(1-\zeta_\delta)]_{\partial\Omega} = \\
&\quad = \int_\Omega \xi(1-\zeta_\delta) \operatorname{div} v - \int_\Omega \xi \nabla\zeta_\delta \, v\varphi \, dx + \int_\Omega \nabla\xi \, v(1-\zeta_\delta) \, dx \,.
\end{aligned}$$

Since $1-\zeta_\delta$ converges to zero pointwise in $\Omega$ Lebesgue's theorem applied to the measure $\operatorname{div} v$ implies that:

$$\lim_{\delta\to 0} \int_\Omega \xi(1-\zeta_\delta) \operatorname{div} v = 0 \,.$$

Passing to the limit as $\delta$ tends to zero then implies (2.3). ∎

In order to deal with the degenerate parabolic problem (1.1)–(1.2) we will use the previous setting of divergence measure vector fields in a parabolic domain $Q = \Omega \times (0,T)$. In this case we set

$$\begin{aligned}
\mathcal{DM}(Q)_2 \, := \Big\{ &(w, v) \in L^2(Q) \times L^2(Q)^N \, : \\
&\exists C > 0 \, : \left| \int_Q w\varphi_t \, dxdt + \int_Q v \nabla\varphi \, dxdt \right| \le C \, \|\varphi\|_{L^\infty(Q)} \quad \forall \varphi \in C^\infty_c(Q) \Big\} \,.
\end{aligned}$$

Applying the result of Lemma 2.1 we can define the bilinear application at the boundary $[(w,v), \varphi]_{\partial Q}$ for any $\varphi \in H^1(Q) \cap L^\infty(Q) \cap C(Q)$ as follows:

$$\int_Q w\varphi_t \, dxdt + \int_Q v\nabla\varphi \, dxdt + \int_Q \varphi \operatorname{div}_{(t,x)}(w, v) = [(w, v), \varphi]_{\partial Q} \,. \tag{2.4}$$

Moreover $[(w,v), \varphi]_{\partial Q}$ satisfies:

$$[(w,v), \varphi]_{\partial Q} = 0 \quad \text{for any } \varphi \in H^1_0(Q), \qquad |[(w,v), \varphi]_{\partial Q}| \le C \|\varphi\|_{H^1(Q)\cap L^\infty(Q)} \,.$$

Let us observe that the boundary application $[(w,v), \varphi]_{\partial Q}$ can be furtherly detailed if $w$ has some continuity properties at $t = 0$ or $t = T$, as it happens in the context of parabolic equations. In fact, assume for instance that $w(t)$ converges towards a function $w_0$ in $L^1(\Omega)$ as $t$ tends to zero, that is:

$$\lim_{t\to 0}\int_\Omega |w(x,t)-w_0(x)|\,dx = 0\,.$$

Then

$$[(w,v),\varphi]_{\partial Q} = -\int_\Omega w_0(x)\psi(x,0)\,dx \tag{2.5}$$

for every $\psi \in H^1(Q)\cap L^\infty(Q)\cap C(Q)$ such that $\psi\in L^2(0,T;H_0^1(\Omega))$ and $\psi(T)=0$. Indeed let $\psi \in H^1(Q)\cap L^\infty(Q)\cap C(Q)$ such that $\psi\in L^2(0,T;H_0^1(\Omega))$ and $\psi(T)=0$. Let also $\theta_\varepsilon(t)=\max(\frac{t}{\varepsilon},1)$ for $t\in[0,T]$ and choose then $\varphi=\psi\theta_\varepsilon$ in (2.4). Since $\psi\theta_\varepsilon=0$ on $\partial Q$ we have:

$$\begin{aligned}\int_Q w\theta_\varepsilon\psi_t\,dxdt + \int_Q \theta_\varepsilon\nabla\psi\, v\,dxdt + \int_Q \psi\theta_\varepsilon \operatorname{div}_{(t,x)}(w,v)\,dxdt\\ = -\int_0^\varepsilon \frac{1}{\varepsilon}\int_\Omega w\psi\,dxdt = -\int_0^1\int_\Omega w(x,\varepsilon t)\psi(x,\varepsilon t)\,dxdt\,.\end{aligned}$$

Since $\theta_\varepsilon$ pointwise converges to 1 in $Q$ and using the convergence of $w(t)$ to $w_0$ in $L^1(\Omega)$ we get, as $\varepsilon$ tends to zero:

$$\int_Q w\psi_t\,dxdt + \int_Q \nabla\psi\, v\,dxdt + \int_Q \psi \operatorname{div}_{(t,x)}(w,v) = -\int_\Omega w_0\psi(0)\,dx \quad ,$$

for every $\psi \in H^1(Q)\cap L^\infty(Q)\cap C(Q)$ such that $\psi\in L^2(0,T;H_0^1(\Omega))$ and $\psi(T)=0$.

Clearly, if $w\in C([0,T];L^1(\Omega))$ the same arguments allow to define the normal trace on the lateral surface $\partial\Omega\times(0,T)$ by setting $[(w,v),\varphi]_{\partial\Omega\times(0,T)} = [(w,v),\varphi]_{\partial Q} + \int_\Omega w(0)\psi(0)\,dx - \int_\Omega w(T)\psi(T)\,dx$. Moreover, dropping $\psi(T)=0$, same arguments lead to

$$[(w,v),\varphi]_{\partial Q} = \int_\Omega w(x,T)\psi(T)\,dx - \int_\Omega w_0(x)\psi(x,0)\,dx, \tag{2.6}$$

for every $\psi \in H^1(Q)\cap L^\infty(Q)\cap C(Q)$ such that $\psi\in L^2(0,T;H_0^1(\Omega))$.

**Lemma 2.4** *Let $(w,v)\in\mathcal{DM}(Q)_2$, and assume that $w$ is continuous at $t=0$ with respect to the $L^1(\Omega)$ norm. Then for every $\xi\in H^1(Q)\cap L^\infty(Q)\cap C(Q)$*

$$\lim_{\delta\to 0}\int_Q v\nabla\zeta_\delta\xi\,dx = \int_\Omega w(x,T)\xi(T)\,dx - \int_\Omega w_0(x)\xi(x,0)\,dx - [(w,v),\xi]_{\partial Q}\,, \tag{2.7}$$

*for any boundary layer sequence $\zeta_\delta$. In particular if $\xi\in L^2(0,T;H_0^1(\Omega))$ we have*

$$\lim_{\delta\to 0}\int_Q v\nabla\zeta_\delta\xi\,dx = 0\,.$$

**Proof.** We take $\xi(1-\zeta_\delta)$ in (2.4), obtaining:

$$\int_Q \xi_t(1-\zeta_\delta)w\,dxdt - \int_Q \nabla\zeta_\delta v\,\xi\,dxdt + \int_Q (1-\zeta_\delta)v\nabla\xi\,dxdt$$
$$+\int_Q (1-\zeta_\delta)\xi\,\mathrm{div}_{(t,x)}(w,v) = [(w,v),\xi]_{\partial Q} - [(w,v),\zeta_\delta\xi]_{\partial Q}\,,$$

which yields by means of (2.5), since $\zeta_\delta\xi = 0$ on $(\partial\Omega\times(0,T))\cup(\Omega\times\{T\})$:

$$\int_Q \xi_t(1-\zeta_\delta)w\,dxdt - \int_Q \nabla\zeta_\delta v\,\xi\,dxdt + \int_Q (1-\zeta_\delta)v\nabla\xi\,dxdt$$
$$+\int_Q (1-\zeta_\delta)\xi\,\mathrm{div}_{(t,x)}(w,v) = [(w,v),\xi]_{\partial Q} + \int_\Omega w(0)\xi(0)\zeta_\delta\,dx\,.$$

Letting $\delta$ tend to zero and using Lebesgue's theorem with respect to the measure $\mathrm{div}_{(t,x)}(w,v)$ (recall that $(1-\zeta_\delta)$ pointwise converges to zero in $Q$) we obtain (2.7). Moreover if $\xi\in L^2(0,T;H^1_0(\Omega))$ and $\xi(T)=0$ thanks to (2.6) it follows from (2.7) that

$$\lim_{\delta\to 0}\int_Q v\nabla\zeta_\delta\xi\,dxdt = 0\,.$$

■

Next we consider a parabolic approximation of problem (1.1)–(1.2), obtained replacing the function $b$ with $b^\varepsilon$ such that $\frac{b^\varepsilon(s)-b^\varepsilon(t)}{s-t}\geq \nu_\varepsilon>0$ for some constant $\nu_\varepsilon$ (a possible choice is $b^\varepsilon = \varepsilon s + b(s)$. Let $u_\varepsilon$ be the solution of the parabolic problem:

$$\begin{cases} \partial_t u_\varepsilon - \Delta b^\varepsilon(u_\varepsilon) + \mathrm{div}\,\Phi(u_\varepsilon) = g, \\ u_\varepsilon|_{t=0} = u_0, \qquad u_\varepsilon|_{\partial\Omega\times(0,T)} = a_0. \end{cases} \tag{2.8}$$

Since $b^\varepsilon$ is strictly increasing we have that $u_\varepsilon$ is the unique solution of (2.8).

By a straightforward application of maximum principle and energy estimates, it can be proved that $u_\varepsilon$ is bounded in $L^\infty(Q)$ and $b(u_\varepsilon)$ is bounded in $L^2(0,T;H^1(\Omega))$, as stated by the following Lemma.

**Lemma 2.5** *Assume that $a_0\in C^1(\partial\Omega)$ and $b(a_0)\in L^2(0,T;H^{\frac12}(\partial\Omega))$, and let $g\in L^\infty(Q)$. Then there exists a constant $c$ which depends on $\|a_0\|_{L^\infty(\partial\Omega\times(0,T))}$, $\|b(a_0)\|_{L^2(0,T;H^{\frac12}(\partial\Omega))}$, $\|f\|_{L^\infty(Q)}$ but not on $\varepsilon$ such that:*

$$\|u_\varepsilon\|_{L^\infty(Q)} + \|b^\varepsilon(u_\varepsilon)\|_{L^2(0,T;H^1(\Omega))}\leq c \qquad \forall\varepsilon>0\,,$$

The estimates of Lemma 2.5 are not enough to deduce the almost everywhere convergence of $u_\varepsilon$ in $Q$. For this reason, in the sequel, we assume that there exists $u\in L^\infty(Q)$ for which

$$\begin{aligned} &u_\varepsilon\to u \quad \text{strongly in } C([0,T];L^1(\Omega)),\\ &b^\varepsilon(u_\varepsilon)\to b(u) \qquad \text{weakly in } L^2(0,T;H^1(\Omega)), \end{aligned} \tag{2.9}$$

Note that (2.9) implies that $b(u) \in b(a_0) + L^2(0,T;H_0^1(\Omega))$. We are going to prove that, under some regularity assumptions on $a_0$, if (2.9) holds true then $u$ is a solution of (1.1)–(1.2) in the sense of Definition 1.2. Let $a_0$ satisfy:

$$a_0 \in C^2(\partial\Omega)\,, \qquad \nabla b(a_0) \in L^2(\partial\Omega)\,, \qquad \Delta(b(a_0)) \in L^1(\partial\Omega)\,. \tag{2.10}$$

If $b$ is a $C^2$ function then (2.10) reduces to ask that $a_0 \in C^2(\partial\Omega)$; similarly the condition $\nabla b(a_0) \in L^2(\partial\Omega)$ is satisfied if $a_0 \in C^2(\partial\Omega)$ and $b$ is Lipschitz continuous.

**Proposition 2.6** *Assume that $a_0$ satisfies (2.10). Let $u_\varepsilon$ be the solution of (2.8) and let (2.9) hold true. Then $u$ satisfies (i) of Definition 1.2.*

**Proof.** Let us fix $\psi \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le \psi \le 1$, $\operatorname{supp}\psi \in B' \subset\subset B$, $B \cap \partial\Omega = \{x_N = f(x')\}$ and $B \cap \Omega = \{x_N < f(x')\}$. Here and after $a_0(x)$ stays for $a_0(x', f(x'))$.

Set $h_\sigma(t) = \frac{1}{\sigma}\max(-\sigma, \min(t,\sigma))$. Then $h_\sigma(t)$ converges to $\operatorname{sign}(t)$ where $\operatorname{sign}(0) = 0$ and $h'_\sigma(t) \ge 0$. From (2.8)

$$\begin{aligned}(u_\varepsilon - a_0)_t &- \Delta(b^\varepsilon(u_\varepsilon) - b^\varepsilon(a_0)) + \operatorname{div}(\Phi(u_\varepsilon) - \Phi(a_0)) \\ &= g - (a_0)_t + \Delta(b^\varepsilon(a_0)) - \operatorname{div}(\Phi(a_0)) \quad \text{in } B \times (0,T).\end{aligned} \tag{2.11}$$

Setting $G_\varepsilon = g - (a_0)_t + \Delta(b^\varepsilon(a_0)) - \operatorname{div}(\Phi(a_0))$, then, from (2.9) and from the assumptions on $a_0$, $G_\varepsilon$ strongly converges in $L^1(Q)$ to $g - u - (a_0)_t + \Delta(b(a_0)) - \operatorname{div}(\Phi(a_0))$ as $\varepsilon$ tends to zero. Multiplying (2.11) by $h_\sigma(b^\varepsilon(u_\varepsilon) - b^\varepsilon(a_0))\psi\xi$, where $\xi \in C^\infty(\bar{Q})$ and integrating by parts

$$\begin{aligned}\int_Q |\nabla(b^\varepsilon(u_\varepsilon) - b^\varepsilon(a_0))|^2\, h'_\sigma \psi\, \xi\, dxdt &= -\int_Q (u_\varepsilon - a_0)_t\, h_\sigma \psi \xi\, dxdt \\ -\int_Q \nabla(b^\varepsilon(u_\varepsilon) - b^\varepsilon(a_0))\nabla(\psi\,\xi) h_\sigma\, dxdt &+ \int_Q (\Phi(u_\varepsilon) - \Phi(a_0))\nabla(\psi\xi) h_\sigma\, dxdt \\ + \int_Q (\Phi(u_\varepsilon) - \Phi(a_0))\nabla(b^\varepsilon(u_\varepsilon) - b^\varepsilon(a_0)) h'_\sigma\, \psi\xi\, dxdt &+ \int_Q G_\varepsilon h_\sigma \psi \xi\, dxdt\,.\end{aligned} \tag{2.12}$$

where $h_\sigma$ and $h'_\sigma$ are calculated at $b^\varepsilon(u_\varepsilon) - b^\varepsilon(a_0)$.

Since $h'_\sigma(t)t \le \chi_{\{0<|t|\le\sigma\}}$ and since for fixed $\varepsilon > 0$ the function $(b^\varepsilon)^{-1}$ is Lipschitz continuous (with Lipschitz constant $\le \frac{1}{\nu_\varepsilon}$) we have

$$|(\Phi(u_\varepsilon) - \Phi(a_0))\nabla(b^\varepsilon(u_\varepsilon) - b^\varepsilon(a_0))h'_\sigma| \le \frac{L}{\nu_\varepsilon}|\nabla(b^\varepsilon(u_\varepsilon) - b^\varepsilon(a_0))|\chi_{I_{\varepsilon,\sigma}}$$

where $L$ is the Lipschitz constant of $\Phi$ and $I_{\varepsilon,\sigma} = \{0 < |b^\varepsilon(u_\varepsilon) - b^\varepsilon(a_0)| \le \sigma\}$. Then

$$\begin{aligned}&\left|\int_Q (\Phi(u_\varepsilon) - \Phi(a_0))\nabla(b^\varepsilon(u_\varepsilon) - b^\varepsilon(a_0))h'_\sigma\, \psi\xi\, dxdt\right| \\ &\qquad \le \frac{L}{\nu_\varepsilon}\|\psi\|_{L^\infty(\Omega)}\|\xi\|_{L^\infty(Q)} \int\limits_{I_{\varepsilon,\sigma}} |\nabla(b^\varepsilon(u_\varepsilon) - b^\varepsilon(a_0))|\, dxdt\,.\end{aligned}$$

Since meas $I_{\varepsilon,\sigma}$ converges to zero as $\sigma \to 0$ (since it converges to the measure of the empty set), we deduce

$$\lim_{\sigma\to 0}\int_Q (\Phi(u_\varepsilon)-\Phi(a_0))\nabla(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))h'_\sigma\,\psi\xi\,dxdt = 0\,. \tag{2.13}$$

Setting $H_\sigma(t)=\int_0^t h_\sigma(s)\,ds$ and after integrating by parts, we also have

$$\begin{aligned}
-\int_Q (u_\varepsilon-a_0)_t\, h_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))\psi\xi\,dxdt &= -\int_Q (u_\varepsilon-a_0)_t\, h_\sigma(u_\varepsilon-a_0)\psi\xi\,dxdt\\
&\quad+\int_Q (u_\varepsilon-a_0)_t\,[h_\sigma(u_\varepsilon-a_0)-h_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))]\;\psi\xi\,dxdt\\
&=\int_\Omega H_\sigma(u(0)-a_0)\,\psi\xi(0)\,dx-\int_\Omega H_\sigma(u_\varepsilon(T)-a_0)\psi\xi(T)\,dx\\
+\int_Q H_\sigma(u_\varepsilon-a_0)\psi\xi_t\,dxdt &+\int_Q (u_\varepsilon-a_0)_t\,[h_\sigma(u_\varepsilon-a_0)-h_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))]\;\psi\xi\,dxdt\,.
\end{aligned}$$

Since $\operatorname{sign}(u_\varepsilon-a_0)=\operatorname{sign}(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))$ as $b^\varepsilon$ is increasing, last integral converges to zero as $\sigma$ tends to zero. Using that $H_\sigma(t)$ converges to $|t|$, we deduce

$$\begin{aligned}
\lim_{\sigma\to 0}-\int_Q (u_\varepsilon-a_0)_t\, h_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))\psi\xi\,dxdt &= \int_\Omega |u_0-a_0|\;\psi\xi(0)\,dx\\
-\int_\Omega |u_\varepsilon(T)-a_0|\psi\xi(T)\,dx &+ \int_Q |u_\varepsilon-a_0|\psi\xi_t\,dxdt\,.
\end{aligned}\tag{2.14}$$

By definition of $h_\sigma$ we also have, by Lebesgue's theorem

$$\begin{aligned}
&\lim_{\sigma\to 0}\int_Q \nabla(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))\nabla(\psi\,\xi)h_\sigma\,dxdt = \int_Q \nabla|b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0)|\nabla(\psi\,\xi)\,dxdt\,,\\
&\lim_{\sigma\to 0}\int_Q (\Phi(u_\varepsilon)-\Phi(a_0))\nabla(\psi\xi)h_\sigma\,dxdt = \int_Q F(u_\varepsilon,a_0)\nabla(\psi\xi)\,dxdt\,,\\
&\lim_{\sigma\to 0}\int_Q G_\varepsilon h_\sigma\psi\xi\,dxdt = \int_Q G_\varepsilon\operatorname{sign}(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))\psi\xi\,dxdt\,,
\end{aligned}\tag{2.15}$$

where $h_\sigma$ is calculated at $b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0)$ and we use the notation $F(u,v)$ introduced in (1.5). Thus we have proved that all the integrals in the right hand side of (2.12) have a limit as $\sigma$ tends to zero. In particular we also deduce that they are bounded uniformly with respect to $\sigma$. Choosing $\xi=1$ then there exists a constant $c_\varepsilon$, eventually depending on $\varepsilon$ but not on $\sigma$, such that:

$$\int_Q |\nabla(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))|^2\,h'_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))\psi\;dxdt \le c_\varepsilon\,.$$

The sequence $\{|\nabla(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))|^2\,h'_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))\psi\}_\sigma$ is thus bounded in $L^1(Q)$ (with respect to $\sigma$), hence it is also bounded in $C(\overline{Q})'$, the dual of the space $C(\overline{Q})$ of continuous functions on $\overline{Q}$. By compactness of the weak$-*$ topology of $C(\overline{Q})'$ we deduce that, up to subsequences, the sequence $\{|\nabla(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))|^2\, h'_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))\psi\}_\sigma$ converges towards an element $\lambda_\varepsilon$ of $C(\overline{Q})'$ in the weak$-*$ topology. Thus for any $\xi\in C^\infty(\overline{Q})$ we can pass to the limit in (2.12), obtaining

$$\begin{aligned}\langle\lambda_\varepsilon,\xi\rangle &= \int_\Omega |u_0-a_0|\psi\xi(0)\,dx-\int_\Omega |u_\varepsilon(T)-a_0|\psi\xi(T)\,dx\\ &+\int_Q |u_\varepsilon-a_0|\psi\xi_t\,dxdt-\int_Q \nabla|b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0)|\nabla(\psi\,\xi)\,dxdt\\ &+\int_Q F(u_\varepsilon,a_0)\nabla(\psi\xi)\,dxdt+\int_Q G_\varepsilon\,\mathrm{sign}(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))\psi\xi\,dxdt\,.\end{aligned}\tag{2.16}$$

On the other hand we have:

$$\begin{aligned}\frac{|\langle\lambda_\varepsilon,\xi\rangle|}{\|\xi\|_{L^\infty(Q)}} &= \lim_{\sigma\to0}\frac{1}{\|\xi\|_{L^\infty(Q)}}\left|\int_Q |\nabla(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))|^2\,h'_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))\psi\xi\,dxdt\right|\\ &\limsup_{\sigma\to0}\int_Q |\nabla(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))|^2\,h'_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))\psi\,dxdt\,,\end{aligned}$$

so that we get from (2.12), (2.13) and (2.15) (with $\xi=1$)

$$\begin{aligned}\frac{|\langle\lambda_\varepsilon,\xi\rangle|}{\|\xi\|_{L^\infty(Q)}} &\le \int_\Omega |u_0-a_0|\psi\,dx+\int_Q \nabla|b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0)|\nabla\psi\,dxdt\\ &+\int_Q F(u_\varepsilon,a_0)\nabla\psi\,dxdt+\int_Q G_\varepsilon\,\mathrm{sign}(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))\psi\,dxdt\,.\end{aligned}$$

Using the estimates on $u_\varepsilon$ in $L^\infty(Q)$ and on $b^\varepsilon(u_\varepsilon)$ in $L^2(0,T;H^1(\Omega))$, the fact that $G_\varepsilon$ strongly converges in $L^1(Q)$ and the regularity of $a_0$ we deduce that there exists a constant $c$ which does not depend on $\varepsilon$ such that:

$$\frac{|\langle\lambda_\varepsilon,\xi\rangle|}{\|\xi\|_{L^\infty(\Omega)}}\le c\qquad\forall\varepsilon>0\,.$$

Thus we have obtained that $\lambda_\varepsilon$ is bounded in $C(\overline{Q})'$, hence up to extracting a subsequence we have that $\lambda_\varepsilon$ converges, in the weak$-*$ topology, to a functional $\lambda$ in $C(\overline{Q})'$. Now we pass to the limit as $\varepsilon$ tends to zero in (2.16). To this aim, note that since $G_\varepsilon$ is strongly convergent in $L^1(Q)$ and $\mathrm{sign}(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))$ is uniformly bounded, we have that there exists a function $G\in L^1(Q)$ such that $G_\varepsilon\mathrm{sign}(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))$ weakly converges to $G$ in $L^1(Q)$. Moreover using (2.9) we get that $F(u_\varepsilon,a_0)$ strongly converges to $F(u,a_0)$ in $L^q(Q)$ for any $q<\infty$, $\nabla|b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0)|$ weakly converges to $\nabla|b(u)-b(a_0)|$ in $L^2(Q)^N$ and $|u_\varepsilon-a_0|$ strongly converges to $|u-a_0|$ in $C([0,T];L^1(\Omega))$. Passing to the limit as $\varepsilon$ tends to zero we finally obtain

$$\begin{aligned}\langle\lambda,\xi\rangle &= \int_\Omega |u_0-a_0|\,\xi(0)\psi\,dx-\int_\Omega |u(T)-a_0|\xi(T)\psi\,dx+\int_Q |u-a_0|\psi\xi_t\,dxdt\\ &-\int_Q \nabla|b(u)-b(a_0)|\nabla(\psi\,\xi)\,dxdt+\int_Q F(u,a_0)\nabla(\psi\xi)\,dxdt+\int_Q G\,\psi\xi\,dxdt\,,\end{aligned}\tag{2.17}$$

for every $\xi \in C^\infty(\overline{Q})$. In particular, since $G\psi$, $\nabla|b(u)-b(a_0)|\nabla\psi$, $F(u,a_0)\nabla\psi$ are all functions in $L^1(Q)$, and since $\lambda$ belongs to $C(\overline{Q})'$, we deduce that

$$\left|\int_Q |u-a_0|\psi\xi_t - \int_Q \psi\left(\nabla\left|b(u)-b(a_0)\right| - F(u,a_0)\right)\nabla\xi\, dxdt\right| \le C\,\|\xi\|_{L^\infty(Q)}\,,$$

for every $\xi \in C_c^\infty(Q)$. This implies that $(-|u-a_0|\psi\,,\, K_x(u,a_0)\psi) \in \mathcal{DM}(Q)_2$. ■

Let us see now the consistency of the boundary condition (iv) once the solutions $u_\varepsilon$ of (2.8) satisfy (2.9) and $a_0$ is smooth enough.

**Proposition 2.7** *Let $u_\varepsilon$ be the sequence of solutions of (2.8) and let (2.9) hold true. Assume that $a_0$ satisfies (2.10). Then $u$ satisfies (iv) of Definition 1.2.*

**Proof.** As before, let us fix $\psi \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le \psi \le 1$, with $\operatorname{supp}\psi \subset B' \subset\subset B$, $B\cap\partial\Omega = \{x_N = f(x')\}$ and $B\cap\Omega = \{x_N < f(x')\}$. Again, here and after, $a_0(x)$ stands for $a_0(x', f(x'))$.

Again, we use the function $h_\sigma(t) = \frac{1}{\sigma}\max(-\sigma, \min(t,\sigma))$, so that $H_\sigma(t) = \int_0^t h_\sigma(s)ds$ converges to $|t|$ as $\sigma \to 0$. Let $\xi \in H^1(Q)$ and $\varphi \in H_0^1(\Omega)$ be nonnegative functions. In the proof of Proposition 2.6, we already proved, using (2.11), (2.14) and (2.13), that

$$\begin{aligned}
&\lim_{\sigma\to 0}\int_Q \nabla(b^\varepsilon(u_\varepsilon) - b^\varepsilon(a_0))\nabla[h_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))\varphi\xi\psi]\,dxdt \\
&\qquad = \int_\Omega |u_0 - a_0|\psi\varphi\xi(0)\,dx - \int_\Omega |u_\varepsilon(T) - a_0|\psi\varphi\xi(T)\,dx \\
&\quad + \int_Q |u_\varepsilon - a_0|\psi\varphi\xi_t\,dxdt + \int_Q F(u_\varepsilon,a_0)\nabla(\varphi\xi\psi)\,dxdt + \int_Q \mu_\varepsilon\,\psi\xi\varphi\,dxdt\,,
\end{aligned} \tag{2.18}$$

where $\mu_\varepsilon = [g - (a_0)_t + \Delta(b^\varepsilon(a_0)) - \operatorname{div}(\Phi(a_0))]\operatorname{sign}(u_\varepsilon - a_0)$ is a sequence which is weakly convergent in $L^1(\Omega)$ thanks to the assumptions on $a_0$ and (2.9). Similarly, using the equation satisfied by $u_\varepsilon$ with test function $h_\sigma(b^\varepsilon(u_\varepsilon) - b^\varepsilon(k))\varphi\xi\psi$ which belongs to $H_0^1(\Omega)$, we have that for every $k > 0$:

$$\begin{aligned}
&\lim_{\sigma\to 0}\int_Q \nabla(b^\varepsilon(u_\varepsilon) - b^\varepsilon(k))\nabla\left[h_\sigma(b^\varepsilon(u_\varepsilon) - b^\varepsilon(k))\varphi\xi\psi\right]\,dx \\
&= \lim_{\sigma\to 0}\left(\int_\Omega \Psi_\sigma(u_\varepsilon,k)(0)\,\xi(0)\psi\varphi\,dx - \int_\Omega \Psi_\sigma(u_\varepsilon,k)(T)\xi(T)\psi\varphi\,dx + \int_Q \Psi_\sigma(u_\varepsilon,k)\xi_t\,\psi\varphi\,dxdt\right. \\
&+\left.\int_Q g\,h_\sigma(b^\varepsilon(u_\varepsilon) - b^\varepsilon(k))\varphi\xi\psi\,dxdt + \int_Q \Phi_\sigma(u_\varepsilon,k)\nabla(\varphi\xi\psi)\,dxdt\right)\,,
\end{aligned}$$

where $\Phi_\sigma(t,k) = \int_0^t \Phi'(s)h_\sigma(b^\varepsilon(s) - b^\varepsilon(k))\,ds$ (recall that $\Phi$ is Lipschitz continuous) and $\Psi_\sigma(t,k) = \int_0^t h_\sigma(b^\varepsilon(s) - b^\varepsilon(k))\,ds$. Since $b^\varepsilon(s)$ is increasing we have $\operatorname{sign}(b^\varepsilon(s) - b^\varepsilon(k)) =$

sign$(s-k)$, we get:

$$
\begin{aligned}
\lim_{\sigma\to 0}\int_Q \nabla(b^\varepsilon(u_\varepsilon)-b^\varepsilon(k))\nabla\left[h_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(k))\varphi\xi\psi\right]\,dxdt \\
=\int_\Omega |u_0-k|\psi\varphi\xi(0)\,dx-\int_\Omega |u_\varepsilon(T)-k|\psi\varphi\xi(T)\,dx \qquad (2.19)\\
+\int_Q |u_\varepsilon-k|\psi\varphi\xi_t\,dxdt+\int_Q g\,\mathrm{sign}(u_\varepsilon-k)\varphi\xi\psi\,dxdt+\int_Q F(u_\varepsilon,k)\nabla(\varphi\xi\psi)\,dxdt\,.
\end{aligned}
$$

Now observe that the quantity $\{H_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(k))+H_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))-H_\sigma(b^\varepsilon(a_0)-b^\varepsilon(k))\}$ is zero on $\partial\Omega\times(0,T)$ so that

$$
\begin{aligned}
&-\int_Q\{H_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(k))+H_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))-H_\sigma(b^\varepsilon(a_0)-b^\varepsilon(k))\}\xi\psi\Delta\varphi\,dxdt\\
&=\int_Q\nabla\varphi\nabla(\xi\psi)\{H_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(k))+H_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))-H_\sigma(b^\varepsilon(a_0)-b^\varepsilon(k))\}\,dxdt\\
&+\int_Q\nabla\varphi\nabla(b^\varepsilon(u_\varepsilon)-b^\varepsilon(k))h_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(k))\xi\,\psi\,dxdt\\
&+\int_Q\nabla\varphi\nabla(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))h_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))\xi\,\psi\,dxdt-\int_Q\nabla\varphi\nabla H_\sigma(b^\varepsilon(a_0)-b^\varepsilon(k))\,\xi\psi\,dxdt\,,
\end{aligned}
$$

and then

$$
\begin{aligned}
&-\int_Q\Big\{H_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(k))+H_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))-H_\sigma(b^\varepsilon(a_0)-b^\varepsilon(k))\Big\}\xi\psi\Delta\varphi\,dxdt\\
&=\int_Q\nabla\varphi\nabla(\xi\psi)\Big\{H_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(k))+H_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))-H_\sigma(b^\varepsilon(a_0)-b^\varepsilon(k))\Big\}\,dxdt\\
&+\int_Q\nabla(b^\varepsilon(u_\varepsilon)-b^\varepsilon(k))\nabla(h_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(k))\xi\,\psi\varphi\,dxdt\\
&-\int_Q\nabla(\xi\psi)\nabla(b^\varepsilon(u_\varepsilon)-b^\varepsilon(k))h_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(k))\varphi\,dxdt\\
&+\int_Q\nabla(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))\nabla(h_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))\xi\,\psi\varphi)\,dxdt\\
&-\int_Q\nabla(\xi\psi)\nabla(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))h_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))\varphi\,dxdt\\
&-\int_Q\nabla\varphi\nabla H_\sigma(b^\varepsilon(a_0)-b^\varepsilon(k))\,\xi\psi\,dxdt\,.\\
&-\int_Q\Big\{|\nabla b^\varepsilon(u_\varepsilon)|^2h'_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(k))+|\nabla(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))|^2h'_\sigma(b^\varepsilon(u_\varepsilon)-b^\varepsilon(a_0))\Big\}\xi\psi\varphi\,dxdt
\end{aligned}
$$

Now we let $\sigma$ tend to zero. The last integral is negative and can be desregarded; moreover we use (2.18) and (2.19) and that, for every $\eta\in L^2(0,T;H^1(\Omega))$, $H_\sigma(\eta)$ weakly converges to

$|\eta|$ in $L^2(0,T;H^1(\Omega))$ as $\sigma$ tends to zero. Thus

$$
\begin{aligned}
&- \int_Q \Big\{ |b^\varepsilon(u_\varepsilon) - b^\varepsilon(k)| + |b^\varepsilon(u_\varepsilon) - b^\varepsilon(a_0)| - |b^\varepsilon(a_0) - b^\varepsilon(k)| \Big\} \xi\psi\Delta\varphi \, dxdt \\
&\le \int_Q \nabla\varphi\nabla(\xi\psi) \Big\{ |b^\varepsilon(u_\varepsilon) - b^\varepsilon(k)| + |b^\varepsilon(u_\varepsilon) - b^\varepsilon(a_0)| - |b^\varepsilon(a_0) - b^\varepsilon(k)| \Big\} \, dxdt \\
&+ \int_\Omega \Big\{ \Big( |u_0 - k| + |u_0 - a_0| \Big) \xi(0) - \Big( |u_\varepsilon(T) - k| + |u_\varepsilon(T) - a_0| \Big) \xi(T) \Big\} \psi\varphi \, dxdt \\
&+ \int_Q \Big( |u_\varepsilon - k| + |u_\varepsilon - a_0| \Big) \psi\varphi\xi_t \, dxdt + \int_Q \Big\{ F(u_\varepsilon, k) + F(u_\varepsilon, a_0) \Big\} \nabla(\varphi\xi\psi) \, dxdt \\
&- \int_Q \nabla(\xi\psi) \Big\{ \nabla(|b^\varepsilon(u_\varepsilon) - b^\varepsilon(k)|) + \nabla(|b^\varepsilon(u_\varepsilon) - b^\varepsilon(a_0)|) \Big\} \varphi \, dxdt \\
&+ \int_\Omega \mu_\varepsilon \, \psi\xi\varphi \, dxdt + \int_Q g \, \mathrm{sign}(u_\varepsilon - k) \varphi\xi\psi \, dx - \int_Q \nabla\varphi\nabla(|b^\varepsilon(a_0) - b^\varepsilon(k)|) \, \xi\psi \, dxdt \, .
\end{aligned}
\tag{2.20}
$$

Let now $\zeta_\delta$ be a boundary layer sequence such that $-\Delta\zeta_\delta \ge 0$. For instance we can choose $\zeta_\delta$ solution of the Dirichlet problem (see Appendix)

$$
\begin{cases}
-\delta^2\Delta\zeta_\delta + \zeta_\delta = 1 & \text{in } \Omega, \\
\zeta_\delta = 0 & \text{on } \partial\Omega.
\end{cases}
$$

Choosing $\varphi = \zeta_\delta$ in (2.20) and using that the left hand side is nonnegative, thanks to the triangle inequality, we have (negleting the integrals on $t = T$)

$$
\begin{aligned}
0 &\le \int_Q \nabla\zeta_\delta\nabla(\xi\psi) \Big\{ |b^\varepsilon(u_\varepsilon) - b^\varepsilon(k)| + |b^\varepsilon(u_\varepsilon) - b^\varepsilon(a_0)| - |b^\varepsilon(a_0) - b^\varepsilon(k)| \Big\} \, dxdt \\
&+ \int_\Omega \Big( |u_0 - k| + |u_0 - a_0| \Big) \psi\zeta_\delta\xi(0) \, dxdt + \int_Q \Big\{ F(u_\varepsilon, k) + F(u_\varepsilon, a_0) \Big\} \nabla\zeta_\delta \, \xi\psi \, dxdt \\
&+ \int_Q \Big( |u_\varepsilon - k| + |u_\varepsilon - a_0| \Big) \psi\zeta_\delta\xi_t \, dxdt + \int_Q \Big\{ F(u_\varepsilon, k) + F(u_\varepsilon, a_0) \Big\} \nabla(\xi\psi)\zeta_\delta \, dxdt \\
&- \int_Q \nabla(\xi\psi) \Big\{ \nabla(|b^\varepsilon(u_\varepsilon) - b^\varepsilon(k)|) + \nabla(|b^\varepsilon(u_\varepsilon) - b^\varepsilon(a_0)|) \Big\} \zeta_\delta \, dxdt \\
&+ \int_Q \mu_\varepsilon \, \psi\xi\zeta_\delta \, dxdt + \int_Q g \, \mathrm{sign}(u_\varepsilon - k) \xi\psi\zeta_\delta \, dxdt - \int_Q \nabla\zeta_\delta\nabla(|b^\varepsilon(a_0) - b^\varepsilon(k)|) \, \xi\psi \, dxdt \, .
\end{aligned}
$$

When $\delta$ tends to zero, since $\zeta_\delta$ is a boundary layer sequence and $u_\varepsilon = a_0$ on $\partial\Omega \times (0,T)$, we get

$$
\begin{aligned}
0 &\le \int_\Omega \Big( |u_0 - k| + |u_0 - a_0| \Big) \psi\xi(0) \, dxdt - \int_0^T \int_{\partial\Omega} F(a_0, k) \cdot \nu \, \xi\psi \, dx'dt \\
&+ \int_Q \Big( |u_\varepsilon - k| + |u_\varepsilon - a_0| \Big) \psi\xi_t \, dxdt + \int_Q \Big\{ F(u_\varepsilon, k) + F(u_\varepsilon, a_0) \Big\} \nabla(\xi\psi) \, dxdt \\
&- \int_Q \nabla(\xi\psi) \Big\{ \nabla(|b^\varepsilon(u_\varepsilon) - b^\varepsilon(k)|) + \nabla(|b^\varepsilon(u_\varepsilon) - b^\varepsilon(a_0)|) \Big\} \, dxdt \\
&+ \int_Q \mu_\varepsilon \, \psi\xi \, dxdt + \int_Q g \, \mathrm{sign}(u_\varepsilon - k) \xi\psi \, dxdt - \lim_{\delta\to 0} \int_Q \nabla\zeta_\delta\nabla(|b^\varepsilon(a_0) - b^\varepsilon(k)|) \, \xi\psi \, dxdt \, .
\end{aligned}
$$

In fact the last limit exists since $a_0$ does not depend on the variable $x_N$; using the local behavior of $\zeta_\delta$ (see the Appendix) we have

$$\lim_{\delta\to 0}\int_Q \nabla\zeta_\delta \nabla(|b^\varepsilon(a_0)-b^\varepsilon(k)|)\,\xi\psi\,dxdt = \lim_{\delta\to 0}\int_Q \nabla_{x'}\zeta_\delta\cdot\nabla_{x'}(|b^\varepsilon(a_0)-b^\varepsilon(k)|)\,\xi\psi\,dx'dx_Ndt$$
$$= \lim_{\delta\to 0}\int_0^T\int_{\partial\Omega}\nabla f(x')\nabla(|b^\varepsilon(a_0)-b^\varepsilon(k)|)(x')\xi\psi\int_{-\infty}^{0}\frac{\exp(\frac{s}{\delta})}{\delta}\,ds\,dx'dt$$
$$= \int_0^T\int_{\partial\Omega}\nabla f\nabla(|b^\varepsilon(a_0)-b^\varepsilon(k)|)\xi\psi\,dx'dt\,.$$

Now we let $\varepsilon$ tend to zero, using that (2.9) holds true (hence in particular $F(u_\varepsilon,k)$ and $F(u_\varepsilon,a_0)$ converge to $F(u,k)$ and $F(u,a_0)$ respectively). Since $\mu_\varepsilon$ and $g\,\mathrm{sign}(u_\varepsilon-k)$ are weakly compact in $L^1(Q)$, we deduce that there exists a function $\theta(x)\in L^1(Q)$ such that:

$$\begin{aligned}0\le &\int_\Omega\Big(|u_0-k|+|u_0-a_0|\Big)\psi\xi(0)\,dxdt-\int_0^T\int_{\partial\Omega}F(a_0,k)\cdot\nu\,\xi\psi\,dx'dt\\
&+\int_Q\Big(|u-k|+|u-a_0|\Big)\psi\xi_t\,dxdt+\int_Q\Big\{F(u,k)+F(u,a_0)\Big\}\nabla(\xi\psi)\,dxdt\\
&-\int_Q\nabla(\xi\psi)\Big\{\nabla(|b(u)-b(k)|)+\nabla(|b(u)-b(a_0)|)\Big\}\,dxdt\\
&+\int_Q\theta\psi\xi\,dxdt-\int_0^T\int_{\partial\Omega}\nabla f\,\nabla(|b(a_0)-b(k)|)\,\xi\psi\,dx'dt\,.\end{aligned}\tag{2.21}$$

Note that (2.21) holds true for any $\xi\in H^1(Q)$, hence in particular we can replace $\xi$ with $\xi(1-\zeta_\delta)$, where $\zeta_\delta$ is a boundary layer sequence. This gives (recall that $\zeta_\delta=0$ on $\partial\Omega$):

$$\begin{aligned}&\int_Q\nabla\zeta_\delta\Big\{\Big(F(u,k)+F(u,a_0)\Big)-\Big(\nabla(|b(u)-b(k)|)+\nabla(|b(u)-b(a_0)|)\Big)\Big\}\xi\psi\,dxdt\\
&\le\int_\Omega\Big(|u_0-k|+|u_0-a_0|\Big)\psi(1-\zeta_\delta)\xi(0)\,dxdt+\int_Q\Big(|u-k|+|u-a_0|\Big)\psi(1-\zeta_\delta)\xi_t\,dxdt\\
&+\int_Q\Big\{\Big(F(u,k)+F(u,a_0)\Big)-\Big(\nabla(|b(u)-b(k)|)+\nabla(|b(u)-b(a_0)|)\Big)\Big\}\nabla(\xi\psi)(1-\zeta_\delta)\,dxdt\\
&+\int_Q\theta\psi\xi(1-\zeta_\delta)\,dxdt-\int_0^T\int_{\partial\Omega}\Big\{F(a_0,k)\cdot\nu+\nabla f\cdot\nabla(|b(a_0)-b(k)|)\Big\}\xi\psi\,dx'dt\,.\end{aligned}$$

Finally, it is enough to use that $1-\zeta_\delta$ almost everywhere converges to zero in $\Omega$ to obtain

$$\liminf_{\delta\to 0}\int_Q\nabla\zeta_\delta\Big\{\Big(\nabla(|b(u)-b(k)|)+\nabla(|b(u)-b(a_0)|)\Big)-\Big(F(u,k)+F(u,a_0)\Big)\Big\}\xi\psi\,dxdt$$
$$\ge\int_0^T\int_{\partial\Omega}\Big\{\nabla f\cdot\nabla(|b(a_0)-b(k)|)+F(a_0,k)\cdot\nu\Big\}\xi\psi\,dx'dt\,.$$

Recalling the notations in (1.5) and the limits made before on the last two terms this means

$$\liminf_{\delta\to 0}\int_Q H_x(u,k,a_0)\,\nabla\zeta_\delta\,\xi\,\psi\,dxdt \geq 0 \qquad \forall k\in\mathbb{R}, \quad \forall \xi\in L^2(0,T;H^1(\Omega)),$$

that is (iv) holds true. ■

**Proof of Theorem 1.8.** The proof is consequence of the previous two propositions. Indeed property (i) of Definition 1.2 follows from Proposition 2.6, (ii) from standard Kružkov entropy argument and convergence of $b^\varepsilon(u_\varepsilon)$ to $b(u)$, (iii) from convergence in $C([0,T];L^1(\Omega))$, and, finally, (iv) is guaranteed by Proposition 2.7. ■

# 3 Proof of the uniqueness result.

In this Section we prove $L^1$ contractivity and, therefore, uniqueness of entropy solutions.

First of all, let us recall the following result which is proved by J. Carrillo in [6].

**Lemma 3.1** *Let $\xi(x,t,y,s)$ be a nonnegative $C^\infty$ function such that:*

$$\begin{array}{lll}(x,t)\mapsto\xi(x,t,y,s) & \in C_c^\infty(Q) & \text{for every } (y,s)\in Q,\\ (y,s)\mapsto\xi(x,t,y,s) & \in C_c^\infty(Q) & \text{for every } (x,t)\in Q.\end{array}$$

*Let $u(x,t)$ and $v(y,s)$ satisfy (ii) of Definition 1.2. Then we have:*

$$\begin{gathered}-\iint\limits_{Q\times Q}|u-v|(\xi_t+\xi_s)+\iint\limits_{Q\times Q}\nabla_x|b(u)-b(v)|(\nabla_x\xi+\nabla_y\xi)-\iint\limits_{Q\times Q}F(u,v)\nabla_x\xi\\ +\iint\limits_{Q\times Q}\nabla_y|b(u)-b(v)|(\nabla_x\xi+\nabla_y\xi)-\iint\limits_{Q\times Q}F(u,v)\nabla_y\xi\leq 0.\end{gathered} \tag{3.1}$$

**Proof of Theorem 1.7.** Let us consider two entropy solutions $u$ and $v$, where $u=u(x,t)$, $v=v(y,s)$. If necessary, in the following we will write $Q_{(x,t)}$, resp. $Q_{(y,s)}$, to stress the domain of integration with respect to $(x,t)$, resp. $(y,s)$, although $Q\times Q$ will mainly denote the domain of integration with respect to the four variables.

For $i=1,2$, let $\psi_i\in C_c^\infty(\mathbb{R}^N)$, $0\leq\psi_i\leq 1$, with $\operatorname{supp}\psi_i\in B'\subset\subset B$, $B\cap\partial\Omega=\{x_N=f(x')\}$ and $B\cap\Omega=\{x_N<f(x')\}$; we also take $\psi_2=1$ on the support of $\psi_1$, so that $\psi_1(x)\psi_2(x)=\psi_1(x)$. We use $\psi_1$ as a function of $x$ and $\psi_2$ as a function of $y$, and we denote by $\psi(x,y)=\psi_1(x)\psi_2(y)$ (the rôle of $\psi_1$ and $\psi_2$ is to localize with respect to $x$ and $y$ in order to use the boundary condition $(iv)$ for the two solutions). Let moreover $\theta\in C_c^\infty(0,T)$, $\theta\geq 0$, then we choose in (3.1) $\xi(x,t,y,s)=\zeta_\delta(x)\zeta_\eta(y)\rho(x-y,t-s)\psi(x,y)\theta(t)$, where $\zeta_\delta$, $\zeta_\eta$ are boundary layer sequences, and $\rho$ is a sequence of mollifiers. Precisely, we take $\rho=\rho_{l,m,n}=\rho_l(t-s)\rho_m(x'-y')\rho_n(x_N-y_N)$, where $\rho_l$, $\rho_n$ are sequences of symmetric

mollifiers in $\mathbb{R}$ and $\rho_m$ is a sequence of symmetric mollifyers in $\mathbb{R}^{N-1}$, and $x = (x', x_N)$, $y = (y', y_N)$ are the local coordinates induced by $\psi$.

We denote $\nabla_{x+y} = \nabla_x + \nabla_y$ and $a_0(x) = a_0(x', f(x'))$, $a_0(y) = a_0(y', f(y'))$ and we use the notations in (1.5). Then we have

$$-\iint_{Q\times Q} |u-v|\, \zeta_\delta\, \zeta_\eta\, \rho\, \theta_t\, \psi + \iint_{Q\times Q} \nabla_x |b(u)-b(v)|\, \zeta_\delta\, \zeta_\eta\, \rho\, \theta\, \nabla_{x+y}\psi$$

$$+\iint_{Q\times Q} \nabla_y |b(u)-b(v)|\, \zeta_\delta\, \zeta_\eta\, \rho\, \theta\, \nabla_{x+y}\psi - \iint_{Q\times Q} F(u,v)\, \zeta_\delta\, \zeta_\eta\, \rho\, \theta\, \nabla_{x+y}\psi$$

$$+\iint_{Q\times Q} \Big\{ H_x(u,v,a_0(x)) \nabla_x \zeta_\delta\, \zeta_\eta + H_y(v,u,a_0(y)) \zeta_\delta\, \nabla_y \zeta_\eta \Big\} \rho\, \theta\, \psi$$

$$\le \iint_{Q\times Q} K_x(u,a_0(x)) \nabla_x \zeta_\delta\, \zeta_\eta\, \rho\, \theta\, \psi + \iint_{Q\times Q} K_y(v,a_0(y)) \zeta_\delta\, \nabla_y \zeta_\eta\, \rho\, \theta\, \psi$$

$$-\iint_{Q\times Q} \Big\{ \nabla_x |b(a_0(x)) - b(v)| \nabla_x \zeta_\delta\, \zeta_\eta + \nabla_y |b(a_0(y)) - b(u)| \zeta_\delta\, \nabla_y \zeta_\eta \Big\} \rho\, \theta\, \psi$$

$$+\iint_{Q\times Q} \Big\{ F(a_0(x),v) \nabla_x \zeta_\delta\, \zeta_\eta + F(a_0(y),u) \zeta_\delta\, \nabla_y \zeta_\eta \Big\} \rho\, \theta\, \psi$$

$$-\iint_{Q\times Q} \Big\{ \nabla_x |b(u)-b(v)| \zeta_\delta\, \nabla_y \zeta_\eta + \nabla_y |b(v)-b(u)| \nabla_x \zeta_\delta\, \zeta_\eta \Big\} \rho\, \theta\, \psi.$$

Since $\zeta_\delta$ and $\zeta_\eta$ are boundary layer sequences, we can use (1.8) both for $u$ (with $k = v(y,s)$, $\psi = \psi_1$) and for $v$ (with $k = u(x,t)$, $\psi = \psi_2$) obtaining:

$$-\iint_{Q\times Q} |u-v| \rho\, \theta_t\, \psi + \iint_{Q\times Q} \nabla_x |b(u)-b(v)|\, \rho\, \theta\, \nabla_{x+y}\psi + \iint_{Q\times Q} \nabla_y |b(u)-b(v)|\, \rho\, \theta\, \nabla_{x+y}\psi$$

$$-\iint_{Q\times Q} F(u,v)\, \rho\, \theta\, \nabla_{x+y}\psi \le \liminf_{\eta\to 0} \liminf_{\delta\to 0} \Big\{ I_1 + I_2 + I_3 + I_4 + I_5 \Big\},$$

where

$$I_1 = \iint_{Q\times Q} K_x(u,a_0(x)) \nabla_x \zeta_\delta\, \zeta_\eta\, \rho\, \theta\, \psi, \qquad I_2 = \iint_{Q\times Q} K_y(v,a_0(y)) \zeta_\delta\, \nabla_y \zeta_\eta\, \rho\, \theta\, \psi,$$

$$I_3 = \iint_{Q\times Q} \Big\{ F(a_0(x),v) \nabla_x \zeta_\delta\, \zeta_\eta + F(a_0(y),u) \zeta_\delta\, \nabla_y \zeta_\eta \Big\} \rho\, \theta\, \psi$$

$$I_4 = -\iint\limits_{Q\times Q} \Big\{ \nabla_x |b(u) - b(v)| + \nabla_y |b(a_0(y)) - b(u)| \Big\} \zeta_\delta \, \nabla_y \zeta_\eta \, \rho \, \theta \, \psi$$

$$I_5 = -\iint\limits_{Q\times Q} \Big\{ \nabla_y |b(v) - b(u)| + \nabla_x |b(a_0(x)) - b(v)| \Big\} \nabla_x \zeta_\delta \, \zeta_\eta \, \rho \, \theta \, \psi.$$

Next we calculate the limits of the five integrals $I_1, \dots, I_5$.

*Integrals $I_1$ and $I_2$.* Since $(-|u - a_0|\psi_1, K_x(u, a_0)\psi_1) \in \mathcal{DM}(Q)_2$, by Lemma 2.4 we have that there exists

$$< \mathcal{K}(u, a_0), \xi > := \lim_{\delta \to 0} \int_Q K_x(u, a_0) \nabla \zeta_\delta \psi_1 \xi \, dx dt$$

for every $\xi \in H^1(Q) \cap C(Q) \cap L^\infty(Q)$. Moreover

$$|< \mathcal{K}(u, a_0), \xi >| \le C \|\xi\|_{H^1(Q) \cap L^\infty(Q)} \, . \tag{3.2}$$

We apply this to the function $\xi = \theta(t) \int\limits_{Q_{(y,s)}} \zeta_\eta(y) \rho(x - y, t - s) \psi_2(y) \, dy ds$. Thus, since

$$\lim_{\eta \to 0} \int\limits_{Q_{(y,s)}} \zeta_\eta(y) \rho(x - y, t - s) \psi_2(y) = \int\limits_{Q_{(y,s)}} \rho(x - y, t - s) \psi_2(y) \quad \text{in } H^1 \cap L^\infty(Q_{(x,t)})$$

by (3.2), we get

$$\lim_{\eta \to 0} \lim_{\delta \to 0} I_1 =< \mathcal{K}\,(u, a_0), \int_{Q_{(y,s)}} \theta \, \rho(x - y, t - s) \psi_2 > .$$

Proceeding in the same way, we deduce

$$\lim_{\eta \to 0} \lim_{\delta \to 0} I_2 =< \mathcal{K}\,(v, a_0), \int_{Q_{(x,t)}} \theta \, \rho(x - y, t - s) \psi_1 > .$$

*Integral $I_3$.* By properties of $\zeta_\delta$, it follows

$$\lim_{\eta \to 0} \lim_{\delta \to 0} I_3 = - \iint\limits_{Q_{(x,t)} \times \Sigma_{(y,s)}} F(a_0(y), u(x)) \cdot \nu(y) \rho \, \theta \, \psi - \iint\limits_{Q_{(y,s)} \times \Sigma_{(x,t)}} F(a_0(x), v(y)) \cdot \nu(x) \rho \, \theta \, \psi,$$

where $\Sigma_{(x,t)} = Q_{(x,t)} \cap \partial\Omega_{(x,t)}$.

*Integrals $I_4$ and $I_5$.* By applying Green formulas and using $\zeta_\delta \Big|_{\partial\Omega} = 0$, we get

$$I_4 = \iint\limits_{Q\times Q} |b(u) - b(v)| \, \nabla_x \zeta_\delta \, \nabla_y \zeta_\eta \, \rho \, \theta \, \psi + \iint\limits_{Q\times Q} |b(u) - b(v)| \, \zeta_\delta \, \nabla_y \zeta_\eta \, \rho \, \theta \, \nabla_x \psi$$

$$+ \iint\limits_{Q\times Q} |b(u) - b(v)| \, \zeta_\delta \, \nabla_y \zeta_\eta \, \nabla_x \rho \, \theta \, \psi - \iint\limits_{Q\times Q} \nabla_y |b(a_0(y)) - b(u)| \, \zeta_\delta \, \nabla_y \zeta_\eta \, \rho \, \theta \, \psi$$

Since $a_0$ only depends on the boundary variable, we have $\frac{\partial a_0(y)}{\partial y_N} = 0$ and, since $\rho = \rho(x'-y', x_N - y_N)$, the last two integrals can be rewritten as

$$-\iint\limits_{Q\times Q} |b(u)-b(v)|\,\zeta_\delta\,\nabla_{y'}\zeta_\eta\,\nabla_{y'}\rho\,\theta\,\psi - \iint\limits_{Q\times Q} |b(u)-b(v)|\,\zeta_\delta\,\frac{\partial \zeta_\eta}{\partial y_N}\,\frac{\partial \rho}{\partial y_N}\,\theta\,\psi$$

$$-\iint\limits_{Q\times Q} \nabla_{y'}|b(a_0(y))-b(u)|\,\zeta_\delta\,\nabla_{y'}\zeta_\eta\,\rho\,\theta\,\psi = J_1+J_2+J_3.$$

Using the variable $\tilde{y}_N = y_N - f(y')$, and setting $z_\eta(y',\tilde{y}_N) = \zeta_\eta(y', \tilde{y}_N + f(y'))$, we have

$$J_1+J_2+J_3 = -\iint\limits_{Q\times Q} |b(u)-b(v)|\,\zeta_\delta\,\left(\nabla_{y'}z_\eta - \frac{\partial z_\eta}{\partial \tilde{y}_N}\nabla f\right)\left(\nabla_{y'}\rho - \frac{\partial \rho}{\partial \tilde{y}_N}\nabla f\right)\theta\,\psi$$

$$-\iint\limits_{Q\times Q} |b(u)-b(v)|\,\zeta_\delta\,\frac{\partial z_\eta}{\partial \tilde{y}_N}\,\frac{\partial \rho}{\partial \tilde{y}_N}\,\theta\,\psi - \iint\limits_{Q\times Q} \nabla_{y'}|b(a_0)-b(u)|\,\zeta_\delta\,\left(\nabla_{y'}z_\eta - \frac{\partial z_\eta}{\partial \tilde{y}_N}\nabla f\right)\rho\,\theta\,\psi.$$

Without loss of generality we can choose $\zeta_\eta$ to be the boundary layer sequence considered in the Appendix. Then we know that $z_\eta$ behaves like $1-\exp\left(\tilde{y}_N/\eta\sqrt{1+|\nabla f|^2}\right)$. In particular, $\nabla_{y'}z_\eta$ weakly tends to 0 in $L^2(B)$ as $\eta \to 0$ and $\frac{\partial z_\eta}{\partial \tilde{y}_N}$ converges to $-\delta_{\{\tilde{y}_N=0\}}$. Thus we get, integrating by parts with respect to $y'$,

$$\lim_{\eta\to 0}\lim_{\delta\to 0} J_1+J_2+J_3 = -\iint\limits_{Q_{(x,t)}\times\Sigma_{(y,s)}} |b(u)-b(a_0(y))|\,\nabla f\,\nabla_{y'}\rho\,\theta\,\psi$$

$$-\iint\limits_{Q_{(x,t)}\times\Sigma_{(y,s)}} \nabla_{y'}|b(a_0(y))-b(u)|\,\nabla f\,\rho\,\theta\,\psi + \iint\limits_{Q_{(x,t)}\times\Sigma_{(y,s)}} |b(u)-b(a_0(y))|\,(1+|\nabla f|^2)\frac{\partial \rho}{\partial \tilde{y}_N}\theta\,\psi$$

$$= \iint\limits_{Q_{(x,t)}\times\Sigma_{(y,s)}} \Big\{|b(a_0(y))-b(u)|(\psi\Delta f + \nabla f\nabla_{y'}\psi)\,\rho\,\theta - |b(u)-b(a_0(y))|\,(1+|\nabla f|^2)\rho_l\,\rho_m\,\rho_n'\theta\,\psi\Big\}.$$

Finally, since the first integral in $I_4$ tends to 0 using $b(u)=b(a_0)$ in the sense of traces in $H^1$,

$$\lim_{\eta\to 0}\lim_{\delta\to 0} I_4 = -\iint\limits_{Q_{(x,t)}\times\Sigma_{(y,s)}} |b(u)-b(a_0(y))|\rho\,\theta\,\nu_y\,\nabla_x\psi$$

$$+\iint\limits_{Q_{(x,t)}\times\Sigma_{(y,s)}} \Big\{|b(a_0(y))-b(u)|\,(\psi\Delta f + \nabla f\nabla_{y'}\psi)\,\rho\,\theta - |b(u)-b(a_0(y))|(1+|\nabla f|^2)\rho_l\,\rho_m\,\rho_n'\theta\,\psi\Big\}.$$

Proceeding as for the integral $I_4$, we prove that

$$\lim_{\eta\to 0}\lim_{\delta\to 0} I_5 = -\iint\limits_{Q_{(y,s)}\times\Sigma_{(x,t)}} |b(a_0(x)) - b(v)|\rho\,\theta\,\nu_x\,\nabla_y\psi$$

$$+\iint\limits_{Q_{(y,s)}\times\Sigma_{(x,t)}} \Big\{|b(a_0(x)) - b(v)|\,(\psi\Delta f + \nabla f\nabla_{x'}\psi)\,\rho\,\theta + |b(v) - b(a_0(x'))|(1+|\nabla f|^2)\rho_l\,\rho_m\,\rho_n'\theta\,\psi\Big\}.$$

Summarizing we have obtained

$$-\int\!\!\int\limits_{Q\times Q} |u-v|\rho\,\psi\,\theta_t + \int\!\!\int\limits_{Q\times Q} (\nabla_x|b(u)-b(v)| + \nabla_y|b(u)-b(v)|)\,\rho\,\theta\,\nabla_{x+y}\psi$$

$$-\int\!\!\int\limits_{Q\times Q} F(u,v)\rho\,\theta\,\nabla_{x+y}\psi \le L_1 + L_2 + L_3 + L_4,$$

where

$$L_1 = < \mathcal{K}(u,a_0), \int_{Q_{(y,s)}} \theta\,\rho(x-y,t-s)\psi_2 > + < \mathcal{K}(v,a_0), \int_{Q_{(x,t)}} \theta\,\rho(x-y,t-s)\psi_1 >,$$

$$L_2 = -\iint\limits_{Q_{(x,t)}\times\Sigma_{(y,s)}} \big\{|b(u) - b(a_0(y'))|\,(1+|\nabla f|^2)\rho_n' + F(a_0(y'),u(x))\nu(y')\rho_n\big\}\,\rho_m\,\rho_l\,\theta\,\psi,$$

$$L_3 = \iint\limits_{Q_{(y,s)}\times\Sigma_{(x,t)}} \big\{|b(v) - b(a_0(x'))|\,(1+|\nabla f|^2)\rho_n' - F(a_0(x'),v(y))\nu(x')\rho_n\big\}\,\rho_m\,\rho_l\,\theta\,\psi.$$

$$L_4 = -\iint\limits_{Q_{(x,t)}\times\Sigma_{(y,s)}} |b(u) - b(a_0(y))|\,\rho\,\nu_y\,\theta\nabla_x\psi - \iint\limits_{Q_{(y,s)}\times\Sigma_{(x,t)}} |b(a_0(x)) - b(v)|\rho\,\nu_x\,\theta\nabla_y\psi$$

$$+\iint\limits_{Q_{(y,s)}\times\Sigma_{(x,t)}} |b(a_0(x)) - b(v)|(\psi\Delta f + \nabla f\nabla_{x'}\psi)\rho\theta + \iint\limits_{Q_{(x,t)}\times\Sigma_{(y,s)}} |b(a_0(y)) - b(u)|(\psi\Delta f + \nabla f\nabla_{y'}\psi)\rho\theta$$

Next we let $l\to\infty$, $m\to\infty$, and then finally $n\to\infty$.

*Term* $L_1$. There holds

$$\lim_{m\to+\infty}\lim_{l\to\infty}\int_{Q_{(y,s)}}\theta(t)\rho_m(x'-y'\rho_l(t-s)\rho_n(x_N-f(x')-\tilde{y}_N)\psi_2$$

$$=\lim_{m\to+\infty}\int\limits_{|y'|<1}\theta(t)\rho_m(x'-y')\int\limits_{-L}^{0}\rho_n(x_N-f(y')-\tilde{y}_N)\psi_2(y',f(y')+\tilde{y}_N)\,d\tilde{y}_N dy'$$

$$=\int\limits_{-L}^{0}\theta(t)\rho_n(x_N-f(x')-\tilde{y}_N)\psi_2(x',f(x')+\tilde{y}_N)\,d\tilde{y}_N$$

where the limits are taken in $H^1\cap L^\infty(Q_{(x,t)})$. Then, in virtue of (3.2), we obtain

$$\lim_{m\to+\infty}\lim_{l\to\infty}<\mathcal{K}(u,a_0),\theta\int_{Q_{(y,s)}}\rho_{l,m,n}(x-y,t-s)\,\psi_2>$$

$$=<\mathcal{K}(u,a_0),\theta\int\limits_{-L}^{0}\rho_n(x_N-f(x')-\tilde{y}_N)\psi_2(x',f(x')+\tilde{y}_N)\,d\tilde{y}_N>\ .$$

Since $<\mathcal{K}(u,a_0),\xi>$ only depends on the value of $\xi$ at the boundary $\partial\Omega\times(0,T)$ we get

$$\lim_{m\to+\infty}\lim_{l\to\infty}<\mathcal{K}(u,a_0),\int_{Q_{(y,s)}}\theta\rho_{l,m,n}(x-y,t-s)\,\psi_2>$$

$$=<\mathcal{K}(u,a_0),\theta\int\limits_{-L}^{0}\rho_n(-\tilde{y}_N)\psi_2(x',f(x')+\tilde{y}_N)\,d\tilde{y}_N>\ .$$

It is easy to see that

$$\lim_{n\to\infty}\int\limits_{-L}^{0}\rho_n(-\tilde{y}_N)\psi_2(x',f(x')+\tilde{y}_N)\,d\tilde{y}_N=\frac{1}{2}\psi_2(x',f(x'))\quad\text{in }H^1(\Omega)\cap L^\infty(\Omega)\,,$$

so that, using again (3.2), we get

$$\lim_{n\to+\infty}\lim_{m\to+\infty}\lim_{l\to\infty}<\mathcal{K}(u,a_0),\theta\int_{Q_{(y,s)}}\rho_{l,m,n}(x-y,t-s)\,\psi_2>=\frac{1}{2}<\mathcal{K}(u,a_0),\theta\psi_2>\ .$$

The other term can be treated in the same way, so that

$$\lim_{n\to+\infty}\lim_{m\to+\infty}\lim_{l\to\infty}L_1=\frac{1}{2}<\mathcal{K}(u,a_0),\theta\psi_2>+\frac{1}{2}<\mathcal{K}(v,a_0),\theta\psi_1>\ .$$

*Terms $L_2$ and $L_3$.* Passing to the limit $l$, $m \to +\infty$ in $L_2$ we get

$$-\int\limits_{Q_{(x,t)}} \left\{ |b(u) - b(a_0(x))| \, (1 + |\nabla f|^2)\rho_n' + F((a_0(x), u)\nu(x)\rho_n \right\} \theta \, \psi, \tag{3.3}$$

where $\rho_n$ and $\rho_n'$ are calculated at $x_N - f(x')$ and we have used

$$\iint\limits_{Q_{(x,t)}\times\Sigma_{(y,s)}} |F(a_0(x), u) - F(a_0(y), u)||\nu|\rho_{m,n}\theta\psi \le C \iint\limits_{Q_{(x,t)}\times\Sigma_{(y,s)}} |a_0(x) - a_0(y)||\nu|\rho_{m,n}\theta\psi,$$

and the fact that the right hand side tends to 0. Set $w_n = 2\int\limits_{x_N - f(x')}^{0} \rho_n(s)\, ds$. We can compute $\Delta w_n$ obtaining:

$$-\Delta w_n = 2(1 + |\nabla f|^2)\rho_n'(x_N - f(x')) - 2\Delta f \, \rho_n(x_N - f(x')) \,, \tag{3.4}$$

Note in particular that $w_n$ is a boundary layer sequence. Moreover we have that $\nabla w_n = -2\rho_n \, \nu$, so that, using also (3.4), (3.3) is equal to

$$\frac{1}{2} \int\limits_{Q_{(x,t)}} \{|b(u) - b(a_0)| \, \Delta w_n + F(a_0, u) \, \nabla w_n\} \, \theta \, \psi - \int\limits_{Q_{(x,t)}} |b(u) - b(a_0)| \Delta f \, \rho_n \, \theta \, \psi.$$

Integrating by parts, since $b(u) = b(a_0)$ in $\partial\Omega \times (0, T)$, we get

$$\begin{aligned} \lim_{m\to\infty} \lim_{l\to\infty} L_2 = & -\frac{1}{2} \int_{Q_{(x,t)}} K_x(u, a_0) \nabla w_n \psi_1 \, \theta\psi_2 \\ & - \frac{1}{2} \int_{Q(x,t)} |b(u) - b(a_0)| \nabla w_n \nabla\psi_1 \theta\psi_2 - \int\limits_{Q_{(x,t)}} |b(u) - b(a_0)| \Delta f \, \rho_n \, \theta \, \psi. \end{aligned}$$

Last two terms converge to zero as $n$ tends to infinity since $b(u) = b(a_0)$ on the boundary in the sense of traces in $H^1$. As for the other one, we can use that the limit exists thanks to (i) and Lemma 2.4. Hence

$$\lim_{n\to+\infty} \lim_{m\to\infty} \lim_{l\to\infty} L_2 = -\frac{1}{2} < \mathcal{K}\,(u, a_0), \theta\psi_2 > .$$

Term $L_3$ can be managed in the same way, giving

$$\lim_{n\to+\infty} \lim_{m\to+\infty} \lim_{l\to\infty} L_3 = -\frac{1}{2} < \mathcal{K}\,(v, a_0), \theta\psi_1 > .$$

*Term* $L_4$. Since $b(u)$ and $b(v)$ are both equals to $b(a_0)$ on $\partial\Omega \times (0,T)$ the term $L_4$ can be easily proved to converge to zero as $l, m, n \to +\infty$.

Finally, collecting together all the limits, we conclude (we also use $\psi_1\psi_2 = \psi_1$) that

$$-\int_Q |u-v|\,\psi_1\,\theta_t\,dxdt + \int_Q K(u,v)\,\nabla\psi_1\,dxdt \le 0,$$

for any $\psi_1 \in C_c^\infty(B' \subset\subset B)$. Hence, by using a partition of the unity and summing on $\psi_1$ we get :

$$-\int_Q |u-v|\,\theta_t\,dxdt \le 0\,,$$

for any $\theta \in C_c^\infty(0,T)$. Using the continuity of $u$ and $v$ at $t=0$ we get (1.10). ∎

# 4 Hyperbolic–Elliptic problems.

In this Section we point out that all the results obtained in this work can be extended to the hyperbolic-elliptic problem

$$\begin{cases} u - \Delta b(u) + \operatorname{div}\Phi(u) = f, \\ u|_{\partial\Omega} = a_0. \end{cases} \tag{4.1}$$

Following the evolution case we give the definition of entropy solution.

**Definition 4.1** *Let $g \in L^\infty(\Omega)$ and $a_0 \in L^\infty(\partial\Omega)$ such that*

$$a_0 \in C^1(\partial\Omega)\,, \qquad \nabla b(a_0) \in L^2(\partial\Omega)\,. \tag{4.2}$$

*A function $u \in L^\infty(\Omega)$ is a* solution of the problem (4.1) *if*

*(i) $b(u) \in b(a_0) + H_0^1(\Omega)$ and, for every $B \in \mathcal{B}$, and any positive $\psi \in C_c^\infty(B)$*

$$\psi\,[\nabla_x|b(u)-b(a_0)| - sgn\,(u-a_0)(\Phi(u)-\Phi(a_0))] \in \mathcal{DM}(\Omega),$$

*(here $x = (x', x_N)$, $x_N = f(x')$ on $B \cap \partial\Omega$ and $a_0 = a_0(x', f(x'))$);*

*(ii) $u$ is an entropy solution of the equation with test function in $H_0^1(\Omega)$, i.e.*

$$\int_\Omega \{|u-k|\xi - \nabla|b(u)-b(k)|\nabla\xi + sgn\,(u-k)(\Phi(u)-\Phi(k))\nabla\xi + f\xi\}\;dxdt \ge 0,$$

*for any $\xi \in H_0^1(\Omega)$, $\xi \ge 0$ and for any $k \in \mathbb{R}$;*

*(iii) for any boundary layer sequence $\zeta_\delta$, for every $B \in \mathcal{B}$, and any positive $\psi \in C_c^\infty(B)$*

$$\liminf_{\delta\to 0} \int_\Omega H_x(u(x), k, a_0(x))\,\nabla\zeta_\delta\,\xi\,\psi\,dx \ge 0 \qquad \forall k \in \mathbb{R},$$

*for any $\xi \in H^1(\Omega)$ (here $x = (x', x_N)$, $x_N = f(x')$ on $B \cap \partial\Omega$ and $a_0 = a_0(x', f(x'))$).*

Also in this case we analyse an approximation of the problem (4.1) considering non degenerate funtions $b_\varepsilon(s)$ such that $\frac{b^\varepsilon(s)-b^\varepsilon(t)}{s-t} \geq \nu_\varepsilon > 0$ for some constant $\nu_\varepsilon$, uniformly converging to $b$ on compact sets. More precisely we study the problem

$$\begin{cases} u_\varepsilon - \Delta b^\varepsilon(u_\varepsilon) + \operatorname{div}(\Phi(u_\varepsilon)) = f \quad \text{in } \Omega, \\ u_\varepsilon|_{\partial\Omega} = a_0. \end{cases} \tag{4.3}$$

The following a priori estimates hold

**Lemma 4.2** *Assume that $a_0 \in C^1(\partial\Omega)$ and $b(a_0) \in H^{\frac{1}{2}}(\partial\Omega)$, and let $g \in L^\infty(\Omega)$. Then there exists a constant $c$ which depends on $\|a_0\|_{L^\infty(\partial\Omega)}$, $\|b(a_0)\|_{H^{\frac{1}{2}}(\partial\Omega)}$, $\|g\|_{L^\infty(\Omega)}$ but not on $\varepsilon$ such that:*

$$\|u_\varepsilon\|_{L^\infty(\Omega)} + \|b^\varepsilon(u_\varepsilon)\|_{H^1(\Omega)} \leq c \qquad \forall \varepsilon > 0.$$

As in the parabolic case to obtain the consistency of our definition with the limit of the elliptic problem 4.3, we assume

$$a_0 \in C^2(\partial\Omega)\,, \qquad \nabla b(a_0) \in L^2(\partial\Omega)\,, \qquad \Delta(b(a_0)) \in L^1(\partial\Omega)\,. \tag{4.4}$$

We assume, up to subsequences, that there exists $u \in L^\infty(\Omega)$ such that:

$$\begin{aligned} &u_\varepsilon \to u \quad \text{strongly in } L^1(\Omega), \\ &b^\varepsilon(u_\varepsilon) \to b(u) \qquad \text{weakly in } H^1(\Omega), \end{aligned} \tag{4.5}$$

The following Proposition can be proved following the proofs of the Propositions 2.6–2.7.

**Proposition 4.3** *Suppose that $a_0$ verifies (4.4) and let $u_\varepsilon$ be the solution of (4.3). Then, if $u_\varepsilon, u$ satisfy (4.5), properties (i) and (iii) of Definition 4.1 hold.*

To prove uniqueness of the entropy solution we need the following lemma which is proved in [6].

**Lemma 4.4** *Let $\xi(x,y)$ be a nonnegative $C^\infty$ function such that:*

$$\begin{aligned} x \mapsto \xi(x,y) \quad &\in C_c^\infty(\Omega) \quad \text{for every } y \in \Omega, \\ y \mapsto \xi(x,y) \quad &\in C_c^\infty(\Omega) \quad \text{for every } x \in \Omega. \end{aligned}$$

*Let $u(x)$ and $v(y)$ satisfy (ii) of Definition 4.1. Then we have:*

$$\begin{aligned} &\iint\limits_{\Omega\times\Omega} |u-v|\xi + \iint\limits_{\Omega\times\Omega} \nabla_x|b(u)-b(v)|(\nabla_x\xi + \nabla_y\xi) - \iint\limits_{\Omega\times\Omega} F(u,v)\nabla_x\xi + \\ &\quad + \iint\limits_{\Omega\times\Omega} \nabla_y|b(u)-b(v)|(\nabla_x\xi + \nabla_y\xi) - \iint\limits_{\Omega\times\Omega} F(u,v)\nabla_y\xi \leq 0. \end{aligned} \tag{4.6}$$

**Theorem 4.5** *Suppose that $a_0$ is $C^1(\partial\Omega)$ and $\nabla b(a_0) \in L^2(\partial\Omega)$. Then there exists at most one entropy solution of problem (4.1).*

The proof of this result is essentially similar to the one of Theorem 1.7. In this case we choose in (4.6) test functions $\xi(x,y) = \zeta_\delta(x)\zeta_\eta(y)\rho_m(x'-y')\rho_n(x_N - y_N)\psi_1(x)\psi_2(y)$, where $\zeta_\delta, \zeta_\eta\, \rho_m, \rho_n, \psi_1, \psi_2$ are choosen as in the proof of Theorem 1.7. Letting $\delta$ tends to 0, $\eta$ to 0, $m$ to infinity and finally $n$ to infinity we obtain

$$\int_\Omega |u-v|\, dx \leq 0,$$

that yields uniqueness.

# 5 Appendix

Let $\Omega$ be an open subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and let $\xi_\delta = \xi_\delta(x)$ be the unique solution of

$$\begin{cases} -\delta^2\Delta\xi_\delta + \xi_\delta = 1 & x \in \Omega, \\ \xi_\delta = 0 & x \in \partial\Omega. \end{cases} \tag{5.1}$$

Note that by maximum principle $0 \leq \zeta_\delta \leq 1$, hence $-\Delta\zeta_\delta \geq 0$. Moreover it is easy to see that $\zeta_\delta \to 1$ in $H^1_{\text{loc}}$ as $\delta \to 0$. Our aim is to estimate the local behavior of $\xi_\delta$ near the boundary of the domain $\Omega$. After an appropriate changes of frame, we can assume that the domain $\Omega$ is locally represented by

$$\{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \,:\, x' \in Q,\, 0 < x_N < f(x')\},$$

for some $Q \subset \mathbb{R}^{N-1}$ and $f\,:\,Q \to \mathbb{R}$ smooth.

Let us introduce the change of coordinates $(x', x_N) \to (y', y_N) := (x', f(x') - x_N)$ and set $\eta_\delta(y', y_N) := \xi_\delta(x', x_N)$. The equation for $\eta_\delta$ is

$$-\delta^2\left(\Delta\eta_\delta + |\nabla f|^2\frac{\partial^2\eta_\delta}{\partial y_N^2} + 2\sum_{i=1}^{N-1}\frac{\partial f}{\partial x_i}\frac{\partial^2\eta_\delta}{\partial y_i\partial y_N} + \sum_{i=1}^{N-1}\frac{\partial^2 f}{\partial x_i^2}\frac{\partial\eta_\delta}{\partial y_N}\right) + \eta_\delta = 1. \tag{5.2}$$

This equation can be rewritten in divergence form as follows

$$-\delta^2\sum_{h=1}^N\frac{\partial}{\partial y_h}\left(\sum_{k=1}^N a_{hk}\frac{\partial\eta_\delta}{\partial y_k}\right) + \eta_\delta = 1, \tag{5.3}$$

where

$$A = (a_{hk}) = \begin{pmatrix} I_{N-1} & \nabla f^T \\ \nabla f & 1 + |\nabla f|^2 \end{pmatrix}$$

with $\nabla f = (\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \quad \cdots \quad \frac{\partial f}{\partial x_{N-1}})$. Then there exists a constant $\alpha$ such that $(Av, v) \geq \alpha|v|^2$ for any $v \in \mathbb{R}^N$. Heuristically, in order to determine a first order approximation of $\eta_\delta$, let us set $w_\delta = \eta(y', \delta y_N)$. Then the equation solved by $w_\delta$ is

$$-(1+|\nabla f|^2)\frac{\partial^2 w_\delta}{\partial y_N^2} - \delta^2 \sum_{i=1}^{N-1} \frac{\partial^2 w_\delta}{\partial y_i^2} - 2\delta \sum_{i=1}^{N-1} \frac{\partial f}{\partial x_i}\frac{\partial^2 w_\delta}{\partial y_i \partial y_N} - \delta \sum_{i=1}^{N-1} \frac{\partial^2 f}{\partial x_i^2}\frac{\partial w_\delta}{\partial y_N} + w_\delta = 1.$$

Let $w$ be the solution of the equation obtained formally by the previous one by setting $\delta = 0$

$$-(1+|\nabla f|^2)\frac{\partial^2 w}{\partial y_N^2} + w = 1, \qquad \text{with} \quad w(y', 0) = 0 \quad w(y', +\infty) = 1. \tag{5.4}$$

Therefore

$$w(y', \delta^{-1} y_N) = 1 - \exp\left(-\frac{y_N}{\delta\sqrt{1+|\nabla f(y')|^2}}\right).$$

Our aim is to estimate the $H^1_{\text{loc}}-$norm of $v_\delta(y', y_N) := \eta_\delta(y', y_N) - w(y', \delta^{-1}y_N)$.

Note that $|v_\delta| \leq 1$. We consider the test function $v_\delta\zeta^2$ in the distributional formulation for the equation of $v_\delta$. Here $\zeta$ belongs to $C_c^\infty(\mathbf{R}^N)$ with $\text{supp}(\zeta) \subset Q \times (-L, L)$ for some $L > 0$, so that $v_\delta\zeta^2$ is zero on the boundary of $Q \times (0, L)$.

Denoting all the positive constants with $C$, we get, using the ellipticity of $A$,

$$\begin{aligned}
C\left\{\delta^2 \int_Q\int_0^L |\nabla v_\delta|^2\zeta^2 + \int_Q\int_0^L v_\delta^2\zeta^2\right\} &\leq -2\delta^2 \int_Q\int_0^L v_\delta\,\zeta\, A(y')\,\nabla v_\delta \cdot \nabla\zeta + \\
+\delta^2 \int_Q\int_0^L \left(\sum_{i=1}^{N-1} \frac{\partial w(y', \delta^{-1}y_N)}{\partial y_i}\frac{\partial v_\delta}{\partial y_i}\zeta^2 + 2\sum_{i=1}^{N-1} \frac{\partial w(y', \delta^{-1}y_N)}{\partial y_i}\frac{\partial \zeta}{\partial y_i} v_\delta\,\zeta\right) \\
-\delta^2 \int_Q\int_0^L \left(2\sum_{i=1}^{N-1} \frac{\partial f}{\partial x_i}\frac{\partial^2 w(y', \delta^{-1}y_N)}{\partial y_i \partial y_N} + \sum_{i=1}^{N-1} \frac{\partial^2 f}{\partial x_i^2}\frac{\partial w(y', \delta^{-1}y_N)}{\partial y_N}\right) v_\delta\zeta^2.
\end{aligned} \tag{5.5}$$

By applying Young's inequality and dividing by $\delta^2$, we obtain

$$\begin{aligned}
\int_Q\int_0^L |\nabla v_\delta|^2\zeta^2 + \frac{1}{\delta^2}\int_Q\int_0^L v_\delta^2\zeta^2 &\leq C\int_Q\int_0^L v_\delta^2|\nabla\zeta|^2 \\
+C\int_Q\int_0^L \left(\sum_{i=1}^{N-1} \left|\frac{\partial w(y', \delta^{-1}y_N)}{\partial y_i}\right|^2 \zeta^2 + \sum_{i=1}^{N-1} \frac{\partial w(y', \delta^{-1}y_N)}{\partial y_i}\frac{\partial \zeta}{\partial y_i} v_\delta\,\zeta\right) \\
-C\int_Q\int_0^L \left(2\sum_{i=1}^{N-1} \frac{\partial f}{\partial x_i}\frac{\partial^2 w(y', \delta^{-1}y_N)}{\partial y_i \partial y_N} + \sum_{i=1}^{N-1} \frac{\partial^2 f}{\partial x_i^2}\frac{\partial w(y', \delta^{-1}y_N)}{\partial y_N}\right) v_\delta\zeta^2.
\end{aligned}$$

By explicit calculation of the derivatives of $w(y', \delta^{-1}y_N)$ and since $f$ belongs to $C^2(Q)$ last terms can be estimated as follows:

$$C\int_Q\int_0^L \left(\sum_{i=1}^{N-1}\left|\frac{\partial w(y',\delta^{-1}y_N)}{\partial y_i}\right|^2\zeta^2 + \sum_{i=1}^{N-1}\frac{\partial w(y',\delta^{-1}y_N)}{\partial y_i}\frac{\partial \zeta}{\partial y_i}v_\delta\,\zeta\right)$$

$$-C\int_Q\int_0^L\left(2\sum_{i=1}^{N-1}\frac{\partial f}{\partial x_i}\frac{\partial^2 w(y',\delta^{-1}y_N)}{\partial y_i\partial y_N}+\sum_{i=1}^{N-1}\frac{\partial^2 f}{\partial x_i^2}\frac{\partial w(y',\delta^{-1}y_N)}{\partial y_N}\right)v_\delta\zeta^2$$

$$\le C\int_Q\int_0^L \exp\left(-\frac{y_N}{\delta\sqrt{1+|\nabla f(y')|^2}}\right)\left[\frac{1}{\delta}+\frac{y_N}{\delta^2}+\frac{y_N}{\delta}+\frac{y_N^2}{\delta^2}\right]dy'\,dy_N$$

$$\le C\int_Q\int_0^\infty \exp(-t)[1+t+\delta t+\delta t^2]\,dy'\,dt\,,$$

so that

$$\int\!\!\int|\nabla v_\delta|^2\zeta^2+\frac{1}{\delta^2}\int\!\!\int v_\delta^2\zeta^2\le C.$$

This implies that $v_\delta\zeta$ is bounded in $H_0^1$ and since $v_\delta$ converges to zero almost everywhere in $\Omega$ we conclude that $v_\delta\,\zeta$ weakly converges to zero in $H_0^1$.

In fact, it easily follows from (5.5) that the convergence of $v_\delta\,\zeta$ in $H^1(\Omega)$ is strong. Therefore, we can conclude the following result.

**Proposition 5.1** *Let $\xi_\delta$ be the solution of (5.1). Then for any system of coordinates $(x', x_N)$ locally representing $\partial\Omega$ we have:*

$$\xi_\delta(x',x_N)=1-\exp\left(-\frac{f(x')-x_N}{\delta\sqrt{1+|\nabla f(x')|^2}}\right)+\varepsilon_\delta\,,$$

*where $\varepsilon_\delta$ strongly converges to zero in $H^1(B)$, $B$ being the local portion of $\Omega$ with $|x'|\le 1$, $x_N\le f(x')$.*

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