Uniqueness and Stability of $\mathbf{L}^\infty$ Solutions for Temple Class Systems with Boundary and Properties of the Attainable Sets

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Abstract

We consider the initial-boundary value problem for a strictly hperbolic, genuinely non-linear, Temple class system of conservation laws

$$u_t + f(u)_x = 0 \qquad u \in I\!R^n, \tag{1}$$

on the domain $\Omega = \{(t,x) \in I\!R^2 : t \geq 0,\, x \geq 0\}$. For a class of initial data $\overline{u} \in \mathbf{L}^\infty(I\!R^+)$ and boundary data $\widetilde{u} \in \mathbf{L}^\infty(I\!R^+)$ with possibly unbounded variation, we construct a flow of solutions $(\overline{u}, \widetilde{u}) \to u(t) \doteq E_t(\overline{u}, \widetilde{u})$ that depend continuously, in the $\mathbf{L}^1$ distance, both on the initial data and on the boundary data. Moreover, we show that each trajectory $t \mapsto E_t(\overline{u}, \widetilde{u})$ provides the unique weak solution of the corresponding initial-boundary value problem that satisfies an entropy condition of Oleinik type.

Next, we study the initial-boundary value problem for (1) from the point of view of control theory taking the initial data $\overline{u}$ fixed and considering, in connection with a prescribed set $\mathcal{U}$ of boundary data regarded as admissible controls, the set of attainable profiles at a fixed time $T > 0$, and at a fixed point $\overline{x} > 0$:

$$\mathcal{A}(T, \mathcal{U}) \doteq \big\{E_T(\overline{u}, \widetilde{u})(\cdot) \;;\; \widetilde{u} \in \mathcal{U}\big\}, \qquad \mathcal{A}(\overline{x}, \mathcal{U}) \doteq \big\{E_{(\cdot)}(\overline{u}, \widetilde{u})(\overline{x}) \;;\; \widetilde{u} \in \mathcal{U}\big\}\,.$$

We establish closure and compactness of the sets $\mathcal{A}(T, \mathcal{U})$, $\mathcal{A}(\overline{x}, \mathcal{U})$ in the $\mathbf{L}^1{}_{loc}$ topology, for a class $\mathcal{U}$ of admissible controls satisfying convex constraints.

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1 Introduction

Consider the initial-boundary value problem for a nonlinear, strictly hyperbolic system of conservation laws in one space dimension

$$u_t + f(u)_x = 0, \tag{1.1}$$

$$u(0,x) = \overline{u}(x), \tag{1.2}$$

$$u(t,0) = \widetilde{u}(t), \tag{1.3}$$

on the domain $\Omega = \{(t,x) \in I\!R^2 \ ; \ t \geq 0, \, x \geq 0\}$. Here, $u = u(t,x) \in I\!R^n$ is the vector of the conserved quantities and the flux function $f : U \mapsto I\!R^n$ is a smooth vector field defined on some open set $U \subseteq I\!R^n$. We recall that, for problems of this type, classical solutions may develop discontinuities in finite time, no matter of the regularity of the initial and boundary data. Hence, it is natural to consider weak solutions in the sense of distributions. Moreover, in general, the Dirichlet condition (1.3) cannot be fulfilled pointwise a.e. (see [7, 15]), even when (1.1) is a linear system (cfr. [20]). For this reason, different weaker formulations of the boundary condition have been considered in the literature, both for characteristic boundaries (where the eigenvalues of the Jacobian matrix $Df(u)$ may coincide with the slopes of the boundary profile) and for non characteristic ones: see [1, 24] and references therein.

In this paper, the boundary $x = 0$ is possibly characteristic, but we assume that, for each i-th characteristic family, the i-th eigenvalue $\lambda_i(u)$ of $Df(u)$ is either always > 0 or else always ≤ 0, which implies that there is a fixed set of characteristic lines entering the interior of the domain Ω at any point of the boundary $x = 0$. Then, following [15, 18], we reformulate (1.3) in the weak form

$$f\big(u(t,0+)\big) \in f\big(\mathcal{V}(\widetilde{u}(t))\big), \qquad \text{for a.e.} \quad t > 0, \tag{1.4}$$

where $\mathcal{V}(\widetilde{u}(t)) \subset U$ is a time-dipendent set (the set of *admissible boundary values*) that is defined from the boundary data $\widetilde{u}$ using the notion of Riemann problem, while $f(u(t,0+))$ represents the weak trace of the flux $f(u(t,x))$ at $x = 0$. We are concerned with the well-posedness of (1.1)-(1.2),(1.4) within domains of $\mathbf{L}^\infty$ functions with possibly unbounded variations, having in mind to study the initial-boundary value problem (1.1)-(1.2),(1.4) from the point of view of control theory where it is natural to regard the boundary data as varying within a prescribed set of admissible $\mathbf{L}^\infty$ controls.

We recall that, for initial and boundary data with small total variation, the existence of global weak solutions of the corresponding mixed problem for (1.1), with various type of boundary conditions, was studied by T.P. Liu [22, 23], J. Goodman [17], B. Dubroca and G. Gallice [16], and M. Sablé-Tougeron [24], using the Glimm scheme, and by D. Amadori [1], developing a front tracking algorithm. More recently, the Lipschitz continuous dependence on the initial and boundary data of entropy admissible BV solutions was obtained in [2, 3], for systems of two equations, following the *semigroup approach* developed by A. Bressan and his collaborators to prove the well-posedness of the Cauchy problem for (1.1) (see [10]).

Notice that, while for scalar conservation laws the well-posedness theory for the mixed problem had been established within domains of $\mathbf{L}^\infty$ functions [20, 21, 25], in the case of systems the available stability results apply only to solutions with small total variation. In the present paper we extend these results to domains of $\mathbf{L}^\infty$ functions for a class of systems introduced by Temple [26, 25] in which rarefaction and Hugoniot curves coincide, under the assumption that all characteristic fields are genuinely nonlinear in the sense of Lax. Namely, for such systems, and for a domain $\mathcal{D}$ of $\mathbf{L}^\infty$ pairs of functions with possibly unbounded variation, we construct a continuous flow of solutions

$$(\,\overline{u},\,\widetilde{u}\,) \mapsto u(t,\,\cdot\,) \doteq E_t(\,\overline{u},\,\widetilde{u}\,)(\cdot), \qquad (\,\overline{u},\,\widetilde{u}\,) \in \mathcal{D} \tag{1.5}$$

of the mixed problem (1.1)-(1.2),(1.4), that, for every fixed $\delta > 0$, satisfy the stability estimate

$$\big\|E_t(\,\overline{u},\,\widetilde{u}\,) - E_t(\,\overline{v},\,\widetilde{v}\,)\big\|_{\mathbf{L}^1([\delta,+\infty[)} \leq L_t \cdot \Big\{\big\|\,\overline{u} - \overline{v}\,\big\|_{\mathbf{L}^1(\mathbb{R}^+)} + \big\|f(\widetilde{u}) - f(\widetilde{v})\big\|_{\mathbf{L}^1([0,\,t])}\Big\} \quad \forall\, t \geq \delta\,, \tag{1.6}$$

where the Lipschitz constant L_t takes the form $L_t = C(1 + \log(t/\delta))$, for some constant $C > 0$ depending on the system (1.1). Moreover, relying on a stability estimate of the same type, established for the map $(\,\overline{u},\,\widetilde{u}\,) \mapsto E_{(\cdot)}(\,\overline{u},\,\widetilde{u}\,)(x)$, $x > 0$, we prove that every solution $u(t,x) \doteq E_t(\,\overline{u},\,\widetilde{u}\,)(x)$ actually admits a strong $\mathbf{L}^1$ trace at the boundary $x = 0$. In the same spirit of [6, 13], the flow map E_t in (1.5) is constructed as the unique limit of a Cauchy sequence of flow maps E_t^ν whose trajectories are front tracking approximate solutions of (1.1) in the region Ω, that satisfy a stability estimate of the same type as (1.6) (with a Lipschitz constant independent on ν and on the total variation).

In order to obtain the well-posedness of the mixed problem (1.1)-(1.2),(1.4) with initial and boundary data $(\,\overline{u},\,\widetilde{u}\,) \in \mathcal{D}$, we next show that a distributional solution $u = u(t,x)$ of (1.1)-(1.2),(1.4) coincides with the corresponding trajectory of the flow map E_t if and only if, letting Ψ denote the weak trace of the flux $f(u)$ at $x = 0$, the following conditions hold:

(i) the map $(t,x) \to \big(u(t,\cdot),\, u(\cdot\,,x)\big)$ takes values within the domain $\mathcal{D}$;

(ii) u satisfies suitable Oleinik type conditions on the decay of positive waves in time and in space;

(iii) for every fixed $R > 0$, $T > 0$, the maps

$$t \mapsto \left\| u(t,\cdot\,) - \overline{u}\,\right\|^2_{\mathbf{L}^2([0,\,R])}, \qquad\qquad x \mapsto \left\| u(\cdot\,,x) - f^{-1}\circ\Psi\right\|^2_{\mathbf{L}^2([0,\,T])},$$

admit, respectively, at $t = 0$ and at $x = 0$, Lebesgue limit equal to 0.

Observing that Temple class systems are endowed with entropy-entropy flux pairs $(\eta, q) = (\eta(u), q(u))$ that have positive definite Hessian matrices $D^2\eta$, $D^2(q\circ f^{-1})$, one can recover the regularity conditions (iii) employing the corresponding distributional entropy inequality. We thus prove, in particular, that any weak solution of the mixed problem (1.1)-(1.2),(1.4) constructed by the Glimm scheme or by a front-tracking algorithm, which clearly satisfies the Oleinik type conditions (ii) and any entropy inequality, must coincide with the trajectory of the flow map E_t in (1.5).

The proof of the $\mathbf{L}^1$ stability estimate (1.6) is based on the same linearization technique developed in [11, 6, 13] of comparing two infinitesimally close solutions. The basic idea consists in "differentiating" a family of front tracking approximate solutions w.r.t. a parameter which determines the (space) locations of the jumps, and in providing a-priori bounds on the norm of the resulting "shift differential". In particular, given a piecewise constant solution $u = u(t, x)$ with an initial data $\overline{u}$ and a boundary data $\widetilde{u}$, we may consider a family of perturbed solutions $\theta \mapsto u^\theta(t,\cdot\,)$ obtained from u by shifting the time position t_0 of a single jump in $\widetilde{u}$ at constant rate $\widetilde{\xi}$ (or the space location x_0 of a single jump in $\overline{u}$ at constant rate ξ_0). Call $\widetilde{\sigma} \doteq f(\widetilde{u}(t_0+)) - f(\widetilde{u}(t_0-))$ the size of this jump in $f(\widetilde{u})$ (or let $\sigma \doteq \overline{u}(x_0+) - \overline{u}(x_0-)$ denote the size of the jump in $\overline{u}$). As long as the wave-front configuration of the perturbed and unperturbed solutions is the same, for every fixed $\delta > 0$, and any $t \geq \delta$, one can estimate the $\mathbf{L}^1$ distance between $u(t,\cdot\,)\restriction_{[\delta,+\infty[}$ and $u^\theta(t,\cdot\,)\restriction_{[\delta,+\infty[}$ by showing that, if the perturbed solution $u^\theta(t,\cdot\,)$ contains jumps of size $\sigma^\theta_1,\dots,\sigma^\theta_M$ located at $x^\theta_\alpha \geq \delta$, and shifted at shift rate (in space) $\xi^\theta_1,\dots,\xi^\theta_M$, then there holds

$$\sum_{\alpha=1}^{M} \left|\sigma^\theta_\alpha\,\xi^\theta_\alpha\right| \leq L_t\cdot\left|\widetilde{\sigma}\,\widetilde{\xi}\right| \tag{1.7}$$

for some constant $L_t = L_t(\delta) > 0$ depending on t and on the distance δ from the boundary $x = 0$. The key stability estimate (1.7) is obtained here as in [13] relying on two remarkable properties of genuinely nonlinear systems of Temple class:

a) By genuine nonlinearity and finite propagation speed, for every fixed $\delta > 0$, the total amount of waves in a solution $u(t,\cdot)$ to the mixed problem (1.1)-(1.2),(1.4), which can be influenced by shifting a single wave-front of the initial data $\overline{u}$ or of the boundary data $\widetilde{u}$, and are located at distance $\geq \delta$ from the boundary $x = 0$, remains uniformly bounded, for all $t \geq \delta$, by some constant depending on t and δ.

b) For solutions of Temple class systems, the support of perturbations satisfies a special localization property related with the representation formula for the solutions of the nonlinear equation $U_t + f(U_x) = 0$ in terms of envelopes of n families of hyperplanes [25].

Having in mind applications of Temple systems to problems of oil reservoir simulation, multicomponent chromatography, as well as in models for traffic flows, in the last part of the paper we focus our attention on the mixed problem (1.1)-(1.2),(1.4) from the point of view of control theory. Namely, following the same approach adopted by F. Ancona and A. Marson [4, 5] for scalar conservation laws, we fix an initial data $\overline{u} \in \mathbf{L}^\infty(I\!\!R^+)$ and, in connection with an assigned set $\mathcal{U} \subset \mathbf{L}^\infty(I\!\!R^+)$ of boundary data regarded as admissible controls, we consider the sets of attainable profiles at a fixed time T

$$\mathcal{A}(T, \mathcal{U}) \doteq \{E_T(\overline{u}, \widetilde{u})(\cdot) \ ; \ \widetilde{u} \in \mathcal{U}\},$$

and at a fixed point in space $\overline{x} > 0$

$$\mathcal{A}(\overline{x}, \mathcal{U}) \doteq \{E_{(\cdot)}(\overline{u}, \widetilde{u})(\overline{x}) \ ; \ \widetilde{u} \in \mathcal{U}\}.$$

Relying on the well-posedness theory provided by the previous results, we establish here, as in the scalar case [4, 5], the compactness of these sets in the $\mathbf{L}^1{}_{loc}$ topology for a class $\mathcal{U}$ of admissible boundary controls that satisfy convex constraints.

The paper is organized as follows. Section 2 contains the basic definitions and the statement of the main results. In Section 3 we provide an outline of the proof of Theorem 2.1 on the existence of a continuous flow of solutions depending continuously, in the $\mathbf{L}^1$ distance, on the initial and on the boundary data. The basic a priori estimates on shift differentials are contained in Sections 4-5, while the proof of Theorem 2.1 is given in Section 6, which contains also the proof of Theorem 2.2 and of the other results concerning uniqueness. The compactness property of the attainable sets stated in Theorem 2.3 is established in Section 7.

2 Preliminaries and statement of the main results

2.1 Formulation of the problem

Let $f : U \mapsto I\!\!R^n$ be the flux function of the strictly hyperbolic system (1.1) defined on a neighborhood of the origin $U \subseteq I\!\!R^n$, and denote by $\lambda_1(u) < \cdots < \lambda_n(u)$ the eigenvalues of the Jacobian matrix $Df(u)$. Choose right and left eigenvectors $r_i(u)$, $l_i(u)$, $i = 1, \ldots, n$, of $Df(u)$ normalized so that

$$\big|r_i(u)\big| = 1, \qquad\qquad l_i(u) \cdot r_j(u) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{2.1}$$

By possibly considering a sufficiently small restriction of the domain U, we may assume that a *uniform* strict hyperbolicity condition holds:

(SH1) For every $u, v \in U$, the characteristic speeds at these points satisfy

$$\lambda_j(u) > \lambda_i(v) \qquad\qquad \forall\ 1 \leq i < j \leq n\,. \tag{2.2}$$

Moreover, we fix a convex, compact set $K \subset U$ having the form

$$K = \Big\{ u \in U; \quad w_i(u) \in [a_i,\ b_i] \qquad i = 1, \ldots, n \Big\}, \tag{2.3}$$

and we assume that there is a fixed set of characteristic lines entering the interior of the domain Ω at every point of the boundary $x = 0$, i.e. that, for some index $p \in \{1, \ldots, n\}$, there holds

$$\lambda_{n-p}(u) \leq 0 < \lambda_{n-p+1}(u) \qquad\qquad \forall\ u \in K. \tag{2.4}$$

We also assume that each i-th characteristic field r_i is *genuinely nonlinear* in the sense of Lax, i.e. that, by choosing a suitable orientation of the eigenvectors $r_i(u)$, at every point $u \in U$ one has $D\lambda_i \cdot r_i(u) > 0$. We shall denote by $\widehat{\lambda}$ a uniform upper bound for all the characteristic speeds

$$|\lambda_i(u)| \leq \widehat{\lambda} \qquad\qquad \forall\ u \in U. \tag{2.5}$$

Moreover, the system (1.1) is of Temple class according with the following

Definition 2.1 *A system of conservation laws is of Temple class if there exists a system of coordinates $w = (w_1, \ldots, w_n)$ consisting of Riemann invariants, and such that the level sets $\big\{u \in U;\ w_i(u) = constant\big\}$ are hyperplanes (see [25]).*

It is not restrictive to assume that the Riemann coordinates are chosen so that $(w_1, \ldots, w_n)(0) = (0, \ldots, 0)$, and

$$\frac{\partial}{\partial w_i}\lambda_i(w) > 0 \qquad\qquad \forall\ w = w(u), \quad u \in U, \qquad i = 1, \ldots, n. \tag{2.6}$$

Throughout the paper, we will often write $w_i(t, x) \doteq w_i\big(u(t, x)\big)$ to denote the i-th Riemann coordinate of a solution $u = u(t, x)$ to (1.1). For a Temple class system, the integral curve of the vector field r_i through a point u_0 is the straight line described by the $n-1$ linear equations

$$w_j(u) = w_j(u_0) \qquad\qquad j \neq i. \tag{2.7}$$

In particular, shock and rarefaction curves coincide. Let $\sigma \mapsto R_i(u)[\sigma]$ denote the i-th rarefaction curve through $u \in U$.

Remark 2.1 Since the rarefaction curves are straight lines, the existence of Riemann coordinates implies

$$r_k\big(R_i(u)[\sigma]\big) \in \mathrm{span}\big\{r_k(u),\, r_i(u)\big\} \qquad \forall\, u \in U, \quad \forall\, \sigma, \quad \forall\, i,k = 1,\dots,n\,. \tag{2.8}$$

Relying on this property, one can easily verify that a strenghtned version of the strict hyperbolicity assumption on the linear independence of the eigenvectors $\{r_1(u),\dots,r_n(u)\}$, $u \in U$, holds, namely:

(SH2) Given any n-tuple of states $u^1,\dots,u^n \in U$, such that

$$u^{i+1} = R_i(u^i)[\sigma_i] \qquad\quad 1 \le i < n\,,$$

for some $\sigma_1,\dots,\sigma_n$, the eigenvectors $r_1(u^1),\dots,r_n(u^n)$ are linearly independent.

Notice that the strict hyperbolicity condition **(SH2)** implies the invertibility of the map $f\,:\,U \mapsto f(U)$. Indeed, for any given pair of states $u, v \in U$, there will be some values $\sigma_1,\dots,\sigma_n$, so that, setting

$$z^1 \doteq u, \qquad\quad z^{i+1} \doteq z^i + \sigma_i\, r_i(z^i) \qquad 1 \le i \le n,$$

we can write

$$v - u = \sum_{i=1}^{n} \sigma_i\, r_i(z^i). \tag{2.9}$$

In turn, (2.9) yields

$$f(v) - f(u) = \sum_{i=1}^{n} \sigma_i\, \lambda_i(z^i,\, z^{i+1})\, r_i(z^i), \tag{2.10}$$

where $\lambda_i(z^i,\, z^{i+1})$ denotes the i-th eigenvalue of the averaged matrix

$$A(z^i,\, z^{i+1}) = \int_0^{\sigma_i} Df\big(z^i + \theta\lambda_i(z^i)\big)\; d\theta\,. \tag{2.11}$$

Observing that, by the genuine nonlinearity of the characteristic speeds and because of the assumption (2.4), one has

$$\sigma_i \neq 0 \quad\Longrightarrow\quad \lambda_i(z^i,\, z^{i+1}) \neq 0 \qquad\quad \forall\; i = 1,\dots,n\,,$$

using **(SH2)** one clearly deduces from (2.9)-(2.10) the injectivity of the flux function f.

We next introduce a definition of weak solution to (1.1)-(1.3) which includes an entropy admissibility condition of Oleinik type on the decay of positive waves. The boundary condition is formulated in terms of the weak trace of $f(u)$ at the the boundary $x = 0$, and is related to the notion of Riemann problem in the same spirit of [15]. To this purpose, letting $u(t,x) = W(\xi = x/t;\, u_L, u_R)$, $u_L, u_R \in U$, denote the self-similar solution of the Riemann problem for (1.1) with initial data

$$u(0,x) = \begin{cases} u_L & \text{if } x < 0, \\ u_R & \text{if } x > 0, \end{cases}$$

for any given boundary state $\widetilde{u} \in U$, we define the set of *admissible states at the boundary*

$$\mathcal{V}(\widetilde{u}) := \big\{ W(0+;\, \widetilde{u}, u_R)\ ;\ u_R \in U \big\}. \tag{2.12}$$

Definition 2.2 *A function $u : [0,T] \times \mathbb{R}^+ \mapsto U$ is an entropy weak solution of the initial-boundary value problem (1.1)-(1.3) on $\Omega_T \doteq [0,T] \times \mathbb{R}^+$, if it is continuous as a function from $]0,T]$ into ${\mathbf{L}^1}_{loc}$, and the following properties hold:*

(i) u is a distributional solution to the Cauchy problem (1.1)-(1.2) on Ω_T in the sense that, for every test function $\phi \in \mathcal{C}^1_c$ with compact support contained in the set $\{(t,x) \in \mathbb{R}^2;\ x > 0,\, t < T\}$, there holds

$$\int_0^T \int_0^{+\infty} \big(u(t,x) \cdot \phi_t(t,x) + f(u(t,x)) \cdot \phi_x(t,x) \big) dx\ dt + \int_0^{+\infty} \overline{u}(x) \cdot \phi(0,x) dx = 0.$$

(ii) the flux $f(u)$ admits a weak trace at the boundary $x = 0$, i.e. there exists a measurable function $\Psi : [0,T] \mapsto \mathbb{R}^n$ such that*

$$f(u(\cdot,x)) \ \underset{x \to 0^+}{\overset{*}{\rightharpoonup}}\ \Psi \qquad \text{in} \quad \mathbf{L}^\infty([0,T]), \tag{2.13}$$

and the boundary condition (1.3) is satisfied in the following sense

$$\Psi(t) \in f\big(\mathcal{V}(\widetilde{u}(t))\big) \qquad \text{for a.e.} \quad 0 \le t \le T. \tag{2.14}$$

(iii) *u satisfies the following entropy conditions on the decay of positive waves in time and in space. There exists some constant $C > 0$, depending only on the system (1.1), so that*

(a) For any $0 < t \leq T$, and for a.e. $0 < x < y$, there holds

$$w_i(t,y) - w_i(t,x) \leq C \cdot \frac{y-x}{t} \qquad \textit{if} \qquad i \in \{1,\dots,n-p\}, \tag{2.15}$$

$$w_i(t,y) - w_i(t,x) \leq C \cdot \left\{ \frac{y-x}{t} + \log\left(\frac{y}{x}\right) \right\} \qquad \textit{if} \qquad i \in \{n-p+1,\dots,n\}. \tag{2.16}$$

(b) For a.e. $x > 0$, and for a.e. $0 < \tau_1 < \tau_2 \leq T$, there holds

$$w_i(\tau_2,x) - w_i(\tau_1,x) \leq C \cdot \log\left(\frac{\tau_2}{\tau_1}\right) \qquad \textit{if} \qquad i \in \{1,\dots,n-p\}, \tag{2.17}$$

$$w_i(\tau_2,x) - w_i(\tau_1,x) \leq C \cdot \left\{ \frac{\tau_2-\tau_1}{x} + \log\left(\frac{\tau_2}{\tau_1}\right) \right\} \qquad \textit{if} \qquad i \in \{n-p+1,\dots,n\}. \tag{2.18}$$

Remark 2.2 The set of admissible flux values at the boundary $f\big(\mathcal{V}(\widetilde{u})\big)$ can be expressed in Riemann coordinates as

$$f\big(\mathcal{V}(\widetilde{u})\big) = \Big\{ f(u)\ ;\ w_j(u) = w_j(\widetilde{u}) \qquad \forall\, j = n-p+1,\dots,n \Big\}. \tag{2.19}$$

Hence, by the invertibiliy of the map $f : U \mapsto f(U)$, the above boundary condition (2.14) is equivalent to the set of equalities

$$w_j\big(f^{-1}(\Psi(t))\big) = w_j\big(\widetilde{u}(t)\big) \qquad \text{for a.e.} \quad 0 \leq t \leq T, \qquad j = n-p+1,\dots,n. \tag{2.20}$$

This means that the boundary condition (2.14) guarantees that, at almost every time $t \in [0, T]$, the solution to the Riemann problem for (1.1), having left and right initial states $u^L = \widetilde{u}(t)$, $u^R = f^{-1}(\Psi(t))$, contains only waves with non positive speeds and, in particular, its restriction to the region $[t, +\infty]\times]0, +\infty[$ takes costant value $f^{-1}(\Psi(t))$.

2.2 Stability and uniqueness of weak solutions

Due to the presence of the boundary data, the flow map $u(0,\cdot) \mapsto u(t,\cdot)$ induced by (1.1)-(1.3) is not time homogeneus. To recast the problem in a semigroup framework, it is thus convenient to incorporate the boundary data $\widetilde{u}$ in the domain of the semigroup. More precisely, in connection with a convex, compact set $K \subset U$ of the form (2.3), we consider the positively invariant domain of pairs of $\mathbf{L}^\infty$ functions, with possibly unbounded variations

$$\mathcal{D} \doteq \Big\{\mathbf{p} = (\,\overline{u},\, \widetilde{u}\,)\ ;\ \overline{u},\, \widetilde{u} \in \mathbf{L}^1(\mathbb{R}^+, K)\Big\}, \tag{2.21}$$

where $\mathbf{L}^1(\mathbb{R}^+, K)$ denotes the metric space of all $\mathbf{L}^1$ functions $u : \mathbb{R}^+ \mapsto K$, equipped with the usual $\mathbf{L}^1$ distance. Let $\mathcal{T}_t : \mathbf{L}^1(\mathbb{R}^+, K) \mapsto \mathbf{L}^1(\mathbb{R}^+, K)$ be the translation operator, i.e. $(\mathcal{T}_t\widetilde{u})(s) \doteq \widetilde{u}(t+s)$, and denote by $E : \mathbb{R}^+ \times \mathcal{D} \mapsto \mathbf{L}^1(\mathbb{R}^+, K)$ the evolution operator $E_t\mathbf{p} = u(t,\cdot)$, u being a solution to (1.1)-(1.3). With the above notations, we shall construct a semigroup S acting on $\mathcal{D}$, in the sense that

$$\begin{array}{ccccccc} S & : & \mathbb{R}^+ & \times & \mathcal{D} & \mapsto & \mathcal{D} \\ & & t & , & \mathbf{p} & \mapsto & S_t\mathbf{p}, \end{array} \tag{2.22}$$

where, if $\mathbf{p} = (\,\overline{u},\, \widetilde{u}\,)$, $S_t\mathbf{p} = \big(E_t\mathbf{p},\, \mathcal{T}_t\widetilde{u}\,\big)$.

Our main result is concerned with the existence of an $\mathbf{L}^1$ continuous semigroup of the form (2.22), generated by the system (1.1) on the domain $\mathcal{D}$.

Theorem 2.1 *Let (1.1) be a system of Temple class with all characteristic fields genuinely nonlinear, and assume that the strict hyperbolicity condition* (**SH1**) *holds. Then, there exist a continuous semigroup S of the form (2.22) and some constant $C > 0$, depending only on the system (1.1) and on the domain K, so that, for every fixed $\delta > 0$, and for all $(\,\overline{u},\, \widetilde{u}\,), (\,\overline{v},\, \widetilde{v}\,) \in \mathcal{D}$, letting $L_t \doteq L_t(\delta) = C(1 + \log(t/\delta))$, one has*

$$\big\|E_t(\,\overline{u},\, \widetilde{u}\,) - E_t(\,\overline{v},\, \widetilde{v}\,)\big\|_{\mathbf{L}^1([\delta,+\infty[)} \leq L_t \cdot \Big\{\big\|\,\overline{u} - \overline{v}\,\big\|_{\mathbf{L}^1(\mathbb{R}^+)} + \big\|f(\widetilde{u}) - f(\widetilde{v})\big\|_{\mathbf{L}^1([0,\,t])}\Big\} \tag{2.23}$$

for all $t \geq \delta$. Moreover, the map $(t,x) \mapsto E_t(\,\overline{u},\, \widetilde{u}\,)(x)$ yields an entropy weak solution (in the sense of Definition 2.2) to the initial-boundary value problem (1.1)-(1.3) on Ω, that admits a strong $\mathbf{L}^1$ trace at the boundary $x = 0$, i.e. there exists a measurable map $\psi : \mathbb{R}^+ \mapsto U$ such that

$$\lim_{x\to 0+} \int_0^\tau \big|E_t(\,\overline{u},\, \widetilde{u}\,)(x) - \psi(t)\big|\, dt = 0 \qquad \forall\, \tau \geq 0. \tag{2.24}$$

Remark 2.3 With the same arguments used in Section 6 to establish the existence of a strong $\mathbf{L}^1$ trace at the boundary $x = 0$ for the map $(t,x) \mapsto E_t(\,\overline{u},\, \widetilde{u}\,)(x)$ provided by Theorem 2.1, one can show the continuity w.r.t. the $\mathbf{L}^1{}_{loc}$ topology of $t \mapsto E_t(\,\overline{u},\, \widetilde{u}\,)$ at time $t = 0$. However, as in the case of the Cauchy problem [13], the map $t \mapsto E_t(\,\overline{u},\, \widetilde{u}\,)$ may not be Lipschitz continuous at $t = 0$ if the initial condition $\overline{u}$ has unbounded total variation. Moreover, the evolution operator $\mathbf{p} \mapsto E_t\mathbf{p}$ is not, in general, Lipschitz continuous w.r.t. the topology of $\mathbf{L}^1(\mathbb{R}^+, K)$ on the range $E_t(\mathcal{D})$.

Remark 2.4 The trajectories of the flow map E_t provided by Theorem 2.1 are obtained as limits of front tracking approximate solutions whose values are independent on the Riemann cooordinates of the boundary data that leave the domain Ω. Thus, the same property holds for the limit map E_t and hence, given any couple of initial data and boundary condition $(\overline{u}, \widetilde{u}) \in \mathcal{D}$, if we consider the auxiliary boundary condition $\widetilde{u}'$ defined in Riemann coordinates by

$$w_j(\widetilde{u}'(t)) \doteq \begin{cases} \overline{c}_j & \text{if} \quad j \leq n-p, \\ \\ w_j(\widetilde{u}(t)) & \text{if} \quad j > n-p, \end{cases} \qquad \forall\, t \geq 0\,, \tag{2.25}$$

for some constant values $\overline{c}_j \in [a_i, b_i]$, $j = 1, \ldots, n-p$, one has

$$E_t(\overline{u}, \widetilde{u}) = E_t(\overline{u}, \widetilde{u}') \qquad \forall\, t \geq 0\,. \tag{2.26}$$

Therefore, given any $(\overline{u}, \widetilde{u}), (\overline{v}, \widetilde{v}) \in \mathcal{D}$, by replacing $\widetilde{u}, \widetilde{v}$ in (2.23) with two auxiliary boundary data $\widetilde{u}', \widetilde{v}'$ having the property

$$w_j(\widetilde{u}'(t)) = w_j(\widetilde{v}'(t)) \qquad \forall\, t \geq 0, \qquad j = 1, \ldots, n-p,$$

we deduce for the flow map E_t the sharper estimate

$$\big\| E_t(\overline{u}, \widetilde{u}) - E_t(\overline{v}, \widetilde{v}) \big\|_{\mathbf{L}^1([\delta, +\infty[)} \leq L_t \cdot \left\{ \| \overline{u} - \overline{v} \|_{\mathbf{L}^1(\mathbb{R}^+)} + \sum_{j=n-p+1}^{n} \big\| w_j(\widetilde{u}) - w_j(\widetilde{v}) \big\|_{\mathbf{L}^1([0,t])} \right\}. \tag{2.27}$$

The next result states that every entropy weak solution to (1.1)-(1.3), satisfying some regularity conditions at time $t = 0$, and at the boundary $x = 0$, actually coincides with the corresponding trajectory $t \mapsto E_t(\overline{u}, \widetilde{u})$ of the flow map E_t constructed in Theorem 2.1 and hence, in particular, it admits a strong $\mathbf{L}^1$ trace at $x = 0$. As a consequence, we deduce the uniqueness (up to the domain) of a Lipschitz continuous map as E_t having the property that each trajectory provide an entropy weak solution to (1.1)-(1.3) that admits a strong $\mathbf{L}^1$ trace at the boundary $x = 0$, and at the initial time $t = 0$.

Theorem 2.2 *Let (1.1) be a system of Temple class satisfying the same assumptions as in Theorem 2.1. Let $u = u(t,x)$ be an entropy weak solution to the mixed problem (1.1)-(1.3) on the region $\Omega_T \doteq [0, T] \times \mathbb{R}^+$, in the sense of Definition 2.2. Assume that the following conditions hold.*

(i) The map $(t,x) \to \big(u(t,\cdot),\, u(\cdot,x)\big)$ takes values within the domain

$$\mathcal{D}_T \doteq \Big\{ \mathbf{p} = (\overline{u}, \widetilde{u})\ ;\ \overline{u} \in \mathbf{L}^1(\mathbb{R}^+, K),\, \widetilde{u} \in \mathbf{L}^1([0,T], K) \Big\}. \tag{2.28}$$

(ii) For every fixed $R > 0$, there holds

$$\lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau \int_0^R \left| u(t,x) - \overline{u}(x) \right|^2 dx\, dt = 0\,. \tag{2.29}$$

(iii) Letting Ψ denote the weak trace at the boundary $x = 0$ of the flux $f(u)$ according with Definition 2.2, one has

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon \int_0^T \left| u(t,x) - f^{-1}(\Psi(t)) \right|^2 dt\, dx = 0\,. \tag{2.30}$$

Then, u coincides with the corresponding trajectory of the flow map E_t, namely

$$u(t,\cdot) = E_t(\overline{u},\, \widetilde{u}\,)(\cdot) \qquad \forall\ 0 \leq t \leq T. \tag{2.31}$$

The following technical Lemmas show that, for Temple class systems, a convenient way to prove that the regularity conditions (2.29)-(2.30) are verified is to employ the distributional entropy inequality associated to a convex entropy $\eta = \eta(u)$ and an entropy flux $q = q(u)$ that have positive definite Hessian matrices $D^2\eta$, $D^2(q \circ f^{-1})$ (a similar result can be found in [14, Theorem 4.3.1]).

Lemma 2.1 *Let $u(t,x)$ be an entropy weak solution to the mixed problem (1.1)-(1.3) on the region $\Omega_T \doteq [0,\, T] \times I\!R^+$, in the sense of Definition 2.2. Assume that the essential range of the weak trace Ψ of the flux $f(u)$ at the boundary $x = 0$, is contained in a compact subset of $f(U)$, and that there exists an entropy-entropy flux pair $(\eta, q) = \big(\eta(u), q(u)\big)$ for (1.1) with the properties:*

a) $\eta = \eta(u)$ is a $\mathcal{C}^2$ map whose Hessian $D^2\eta(u)$ is positive definite for any $u \in U$;

b) $\xi = \xi(z) \doteq q \circ f^{-1}(z)$ is a $\mathcal{C}^2$ map whose Hessian $D^2\xi(z)$ is positive definite for any $z \in f(U)$;

c) for every nonnegative test function $\phi \in \mathcal{C}^1_c$, one has the entropy inequality

$$\begin{aligned}\int_0^T \int_0^{+\infty} & \Big\{ \eta(u(t,x)) \cdot \phi_t(t,x) + q(u(t,x)) \cdot \phi_x(t,x) \Big\}\ dx\ dt\ + \\ & + \int_0^{+\infty} \eta(\overline{u}(x)) \cdot \phi(0,x)\ dx + \int_0^T q \circ f^{-1}(\Psi(t)) \cdot \phi(t,0)\ dt \geq 0\,. \end{aligned} \tag{2.32}$$

Then, there hold the Lebesgue limits (2.29) for any $R > 0$, and (2.30).

Lemma 2.2 *Systems of Temple class admit entropy-entropy flux pairs $(\eta, q) = \big(\eta(u), q(u)\big)$ with the properties a)-b) stated in Lemma 2.1.*

2.3 Properties of the attainable sets

Following [4, 5], we turn now to study the mixed initial-boundary value problem (1.1)-(1.3) from the point of view of control theory, taking a fixed initial data $\overline{u} \in \mathbf{L}^1(I\!R^+, K)$ and considering, in connection with a prescribed set $\mathcal{U} \subseteq \mathbf{L}^1(I\!R^+, K)$ of boundary data regarded as admissible controls, the *attainable sets* for (1.1)-(1.3):

$$\mathcal{A}(T, \mathcal{U}) \doteq \Big\{ E_T(\overline{u}, \widetilde{u})(\cdot) \ ; \ \widetilde{u} \in \mathcal{U} \Big\}, \qquad \mathcal{A}(\overline{x}, \mathcal{U}) \doteq \Big\{ E_{(\cdot)}(\overline{u}, \widetilde{u})(\overline{x}) \ ; \ \widetilde{u} \in \mathcal{U} \Big\}, \tag{2.33}$$

i.e. the sets of all profiles that can be attained at a fixed time $T > 0$, or at a fixed point in space $\overline{x} > 0$, by entropy weak solutions of (1.1)-(1.3) with initial data $\overline{u}$, and boundary data $\widetilde{u}$ that vary in $\mathcal{U}$. Relying on the well-posedness theory provided by the above results, we establish here the compactness of $\mathcal{A}(T, \mathcal{U})$, $\mathcal{A}(\overline{x}, \mathcal{U})$ for a class $\mathcal{U}$ of admissible boundary controls that satisfy convex constraints.

Theorem 2.3 *Let K be a set of the form (2.3), and $J \subseteq \{1, \ldots, n\}$ a set of indices such that $J \supseteq \{n-p, \ldots, n\}$. Define*

$$\mathcal{U} \doteq \Big\{ \widetilde{u} \in \mathbf{L}^1(I\!R^+, K) \ ; \ w_j(\widetilde{u}(t)) \in [c_j, \ d_j], \qquad \textit{for a.e. } t \geq 0, \qquad \forall\, j \in J \Big\}, \tag{2.34}$$

for some $-\infty < c_j \leq d_j < +\infty$, $j \in J$. Then $\mathcal{A}(T, \mathcal{U})$, $T > 0$, and $\mathcal{A}(\overline{x}, \mathcal{U})$, $\overline{x} > 0$, are compact subsets of $\mathbf{L}^1{}_{loc}(I\!R^+, K)$.

3 Outline of the proof of Theorem 2.1

We describe here the basic steps in the proof of Theorem 2.1. All the technical estimates involved in the proof will be then worked out in Sections 4-6. As in [13] we shall first construct a sequence of flow maps E^ν whose trajectories are front tracking approximate solutions [6, 9] of (1.1) in the region Ω, that depend $\mathbf{L}^1$ continuously on the initial and boundary data. Next, for any fixed $\mathcal{M} > 0$, we shall prove the convergence of such a sequence of flow maps to a continuous flow of solutions $\mathbf{p} \mapsto E_t\, \mathbf{p}$, defined on the domain

$$\mathcal{D}_{\mathcal{M}} \doteq \Big\{ \mathbf{p} \in \mathcal{D}; \ \text{Tot.Var.}\{\mathbf{p}\} \leq \mathcal{M} \Big\}, \tag{3.1}$$

where, if $\mathbf{p} = (\overline{u}, \widetilde{u})$,

$$\text{Tot.Var.}(\mathbf{p}) \doteq \text{Tot.Var.}(\overline{u}) + \text{Tot.Var.}(\widetilde{u}). \tag{3.2}$$

Finally, we will show that, for every fixed $\delta > 0$, and for any $t \geq \delta$, the map

$$\mathbf{p} \mapsto E_t \, \mathbf{p} \restriction_{[\delta,+\infty[}\,, \qquad \mathbf{p} \in \mathcal{D}_{\mathcal{M}} \tag{3.3}$$

is Lipschitz continuous with a Lipschitz constant depending on δ and t, but independent on the bound on the total variation $\mathcal{M}$.

We describe now a front-tracking algorithm which represents a natural extension of [12]. Fix an integer $\nu \geq 1$ and consider the discrete set of points in K whose coordinates are integer multiples of $2^{-\nu}$:

$$K^\nu \doteq \Big\{ u \in K\,;\ w_i(u) \in 2^{-\nu}\mathbb{Z}, \quad i = 1, \ldots, n \Big\}.$$

Moreover, consider the domain

$$\mathcal{D}^\nu \doteq \Big\{ \mathbf{p} = (u, u') : \mathbb{R}^+ \mapsto K^\nu \times K^\nu; \qquad u,\, u' \in \mathbf{L}^1, \quad u,\, u' \ \text{ are piecewise constant} \Big\}. \tag{3.4}$$

On $\mathcal{D}^\nu$ we now construct a flow map E^ν whose trajectories are front tracking approximate solutions of (1.1). To this end, we first describe how to solve a Riemann problem with left and right initial states $u^L, u^R \in K^\nu$. In Riemann coordinates, assume that

$$w(u^L) \doteq w^L = (w_1^L, \ldots, w_n^L) \qquad w(u^R) \doteq w^R = (w_1^R, \ldots, w_n^R).$$

Consider the intermediate states

$$z^0 = u^L, \qquad \ldots, \qquad z^i = u(w_1^R, \ldots, w_i^R, w_{i+1}^L, \ldots, w_n^L), \qquad \ldots, \qquad z^n = u^R. \tag{3.5}$$

If $w_i^R < w_i^L$, the solution will contain a single i-shock, connecting the states z^{i-1}, z^i, and travelling with Rankine-Hugoniot speed $\lambda_i(z^{i-1},\, z^i)$. Here and in the sequel, by $\lambda_i(u, u')$ we denote the i-th eigenvalue of the averaged matrix

$$A(u, u') \doteq \int_0^1 Df\big(\theta u + (1-\theta)u'\big)\, d\theta. \tag{3.6}$$

If $w_i^R > w_i^L$, the exact solution of the Riemann problem would contain a centered rarefaction wave. This is approximated by a rarefaction fan as follows. If $w_i^R = w_i^L + p_i\, 2^{-\nu}$ we insert the states

$$z^{i,\ell} = (w_1^R, \ldots, w_i^L + \ell\, 2^{-\nu}, w_{i+1}^L, \ldots, w_n^L) \qquad\qquad \ell = 0, \ldots, p_i, \tag{3.7}$$

so that $z^{i,0} = z^{i-1}$, $z^{i,p_i} = z^i$. Our front tracking solution will then contain p_i fronts of the i-th family, each connecting a couple of states $z^{i,\ell-1}$, $z^{i,\ell}$ and travelling with speed $\lambda_i\big(z^{i,\ell-1},\, z^{i,\ell}\big)$.

For a given pair of piecewise constant initial and boundary data $\mathbf{p} = (\overline{u}, \widetilde{u}) \in \mathcal{D}^\nu$, the approximate solution $u(t,\cdot) \doteq E_t^\nu \mathbf{p}$ is now constructed as follows. At time $t = 0$, for $x > 0$ we solve each of the Riemann problems determined by the jumps in $\overline{u}$ according to the above procedure, while at $x = 0$ we construct the solution to the Riemann problem with left and right initial states $u^L = \widetilde{u}(0+)$, $u^R = \overline{u}(0+)$ and take its restriction to the interior of the domain Ω. This yields a piecewise constant function with finitely many fronts, travelling with constant speeds. The solution is then prolonged up to the first time t_1 at which one of the following events takes place:

a) two or more discontinuities interact in the interior of Ω;
b) one or more discontinuities hit the boundary;
c) the boundary data $\widetilde{u}$ has a jump.

If the case a) occurs, we then solve the resulting Riemann problems applying again the above procedure, while in the other two cases b)-c) we construct the solution to the Riemann problem with left and right initial states $u^L = \widetilde{u}(t_1+)$, $u^R = u(t_1, 0+)$ and take its restriction to the interior of the domain Ω. This determines the solution $u(t,\cdot)$ untill the time $t_2 > t_1$ where one of the events a),b),c) again takes place, etc... Notice that at any time where case b) occurs but c) does not take place, no new wave is generated. Therefore, waves exiting the boundary are produced only by the jumps of the boundary data $\widetilde{u}$.

As in [6] and [13], one checks that these front tracking approximations are well defined for all times $t \geq 0$. Indeed, the following properties hold.

- For each $i = 1, \ldots, n$, the total variation of $u(t,\cdot)$, measured w.r.t. the Riemann coordinates, coincides with the total strength of waves in $w_i(t,\cdot)$ and is non-increasing in time.

- The number of wave-fronts in $u(t,\cdot)$ is non-increasing at each interaction. Hence, the total number of wave-fronts in $u(t,\cdot)$ remains finite.

It is now possible to define a ν-approximate semigroup $S^\nu : I\!R^+ \times \mathcal{D}^\nu \mapsto \mathcal{D}^\nu$ as in (2.22) by setting

$$S_t^\nu \mathbf{p} \doteq \left(E_t^\nu \mathbf{p},\, \mathcal{T}_t \widetilde{u}\right) \qquad \mathbf{p} = (\overline{u}, \widetilde{u}). \tag{3.8}$$

The uniqueness of the definition of the approximate solution $u(t,\cdot) = E_t^\nu \mathbf{p}$ guarantees that S_t^ν satisfy the standard semigroup properties, i.e.

$$S_0^\nu = \text{Identity}, \qquad S_t^\nu \circ S_s^\nu = S_{t+s}^\nu.$$

Each trajectory $t \mapsto E_t^\nu \mathbf{p}$ is a weak solution of (1.1) (because all fronts satisfy the Rankine-Hugoniot conditions), but may contain discontinuities that do not satisfy the usual Lax stability conditions (because of the presence of rarefaction fronts).

We next proceed towards an estimate of the Lipschitz constant for $\mathbf{p} \mapsto E_t^\nu \mathbf{p} \restriction_{[\delta,+\infty[}$, $\delta > 0$, following the same technique adopted in [13]. The basic idea to estimate the distance between two approximate solutions u, v, consists in constructing a continuous path of solutions u^θ connecting u, v, and then study how the length of the path $\theta \to u^\theta(t,\cdot)$ varies in time. In particular, given any two couples of initial and boundary data $\mathbf{p}_1 = (\overline{u}_1, \widetilde{u}_1)$, $\mathbf{p}_2 = (\overline{u}_2, \widetilde{u}_2)$ in $\mathcal{D}^\nu$, we introduce a suitable class of continuous paths (pseudopolygonals) that connect $\mathbf{fp}_1 \doteq (\overline{u}_1, f(\widetilde{u}_1))$ with $\mathbf{fp}_2 \doteq (\overline{u}_2, f(\widetilde{u}_2))$ by merely shifting the space and time positions of the jumps in $\overline{u}_1, \overline{u}_2$, and in $f(\widetilde{u}_1)$, $f(\widetilde{u}_2)$, respectively. More precisely, we consider a *pseudopolygonal* with values in

$$\mathcal{FD}^\nu \doteq \big\{\mathbf{fp} = (u, f(u')); \quad (u,u') \in \mathcal{D}^\nu\big\},$$

that is a finite concatenation of *elementary paths* $\gamma : \theta \mapsto \big(\overline{u}^\theta,\, f(\widetilde{u}^\theta)\big)$ of the form

$$\begin{aligned} &\overline{u}^\theta(x) = \textstyle\sum_{\alpha=1}^{N} \overline{\omega}_\alpha \cdot \chi_{]x^\theta_{\alpha-1},\, x^\theta_\alpha]}(x), \qquad && x^\theta_\alpha = x_\alpha + \xi_\alpha\theta, \quad x \geq 0, \\ & && && \theta \in [a,b], \\ &f(\widetilde{u}^\theta(t)) = \textstyle\sum_{\beta=1}^{\widetilde{N}} f(\widetilde{\omega}_\beta) \cdot \chi_{]t^\theta_{\beta-1},\, t^\theta_\beta]}(t), \qquad && t^\theta_\beta = t_\beta + \widetilde{\xi}_\beta\theta, \quad t \geq 0, \end{aligned} \tag{3.9}$$

with $x^\theta_{\alpha-1} < x^\theta_\alpha$, $t^\theta_{\alpha-1} < t^\theta_\alpha$, for all $\theta \in [a,\, b]$ and $\alpha = 1, \dots, N$, $\beta = 1, \dots, \widetilde{N}$. Here, χ_I is the characteristic function of the interval I, while $\overline{\omega}_\alpha$, $\widetilde{\omega}_\beta \in K^\nu$ are constant states and ξ_α, $\widetilde{\xi}_\beta$ are, respectively, the (space) shift rate of the jump in $\overline{u}^\theta$ at x_α, and the (time) shift rate of the jump in $f(\widetilde{u}^\theta)$ at t_β. A simple example of pseudopolygonal joining two couples of initial data and boundary flux $\mathbf{fp}_1 = (\overline{u}_1,\ f(\widetilde{u}_1))$, $\mathbf{fp}_2 = (\overline{u}_2,\ f(\widetilde{u}_2))$, is given by

$$\theta \mapsto \big(\overline{u}_1 \cdot \chi_{[0,\theta[} + \overline{u}_2 \cdot \chi_{]\theta,+\infty[}\,,\ \ f(\widetilde{u}_1) \cdot \chi_{[0,\theta[} + f(\widetilde{u}_2) \cdot \chi_{]\theta,+\infty[}\big).$$

The $\mathbf{L}^1$ length of an elementary path γ of the form (3.9), is then computed by

$$\begin{aligned} \|\gamma\|_{\mathbf{L}^1} &= \int_a^b \left\{ \sum_{\alpha=1}^{N} \big|\Delta\overline{u}^\theta(x_\alpha)\big| \left|\frac{\partial x^\theta_\alpha}{\partial\theta}\right| + \sum_{\beta=1}^{\widetilde{N}} \big|\widetilde{\Delta}\widetilde{u}^\theta(t_\beta)\big| \left|\frac{\partial t^\theta_\beta}{\partial\theta}\right| \right\} d\theta \\ &= \left\{ \sum_{\alpha=1}^{N} \big|\sigma_\alpha\big|\,|\xi_\alpha| + \sum_{\beta=1}^{\widetilde{N}} \big|\widetilde{\sigma}_\beta\big|\,|\widetilde{\xi}_\beta| \right\} (b-a), \end{aligned} \tag{3.10}$$

where

$$\sigma_\alpha \doteq \Delta\overline{u}^\theta(x_\alpha) = \overline{\omega}_{\alpha+1} - \overline{\omega}_\alpha, \qquad\qquad \widetilde{\sigma}_\beta \doteq \widetilde{\Delta}\widetilde{u}^\theta(t_\beta) = f(\widetilde{\omega}_{\beta+1}) - f(\widetilde{\omega}_\beta). \tag{3.11}$$

If we consider a pseudopolygonal $\gamma_0^\nu : \theta \mapsto \big(\overline{u}^\theta,\, f(\widetilde{u}^\theta)\big)$, $\theta \in [0,\, 1]$, with values in $\mathcal{FD}^\nu$, and let $u^\theta_\nu(t,\cdot) = E^\nu_t(\overline{u}^\theta,\, \widetilde{u}^\theta)$ be the corresponding solution, since the number of wave-fronts in these solutions is a-priori bounded and the locations of the interaction points in the t-x plane are determined by a linear system of equations, it follows that, at any time $t > 0$, the path

$$\gamma_t^\nu : \theta \mapsto \big(u_\nu^\theta(t,\cdot),\, f(\widetilde{u}^\theta)\big) \qquad \theta \in [0,\,1], \tag{3.12}$$

is still a pseudopolygonal with values in $\mathcal{FD}^\nu$. Moreover, there exist finitely many parameter values $0 = \theta_0 < \theta_1 < \ldots < \theta_m = 1$ such that the wave-front configuration of u_ν^θ remains the same as θ ranges on each of the open intervals $I_j \doteq]\theta_{j-1}, \theta_j[$. In this case, the length of the path γ_t^ν is measured by an expression of the form

$$\|\gamma_t^\nu\|_{\mathbf{L}^1} = \sum_{j=1}^m \int_{\theta_{j-1}}^{\theta_j} \sum_\alpha \big|\Delta u_\nu^\theta(t,\, x_\alpha^\theta)\big| \left|\frac{\partial x_\alpha^\theta}{\partial \theta}\right| d\theta + \sum_{j=1}^m \int_{\theta_{j-1}}^{\theta_j} \sum_\beta \big|\widetilde{\Delta}\widetilde{u}^\theta(t_\beta^\theta)\big| \left|\frac{\partial t_\beta^\theta}{\partial \theta}\right| d\theta\,. \tag{3.13}$$

Let $\pi_1(\,\overline{u},\, f(\widetilde{u})) = \overline{u}$, $\pi_2(\,\overline{u},\, f(\widetilde{u})) = f(\widetilde{u})$, denote the canonical projections for any couple $\mathbf{fp} = (\,\overline{u},\, f(\widetilde{u})) \in \mathcal{FD}^\nu$. In connection with any elementary path γ of the form (3.9), define the paths

$$\rho_{s_1,s_2}^i(\gamma) \;:\; \theta \longmapsto \pi_i(\gamma(\theta)) \restriction_{[s_1,\,s_2]} \qquad s_1,\, s_2 \geq 0\,, \tag{3.14}$$

and introduce the seminorms

$$\big\|\gamma\big\|_{\delta,t_1,t_2} \doteq \big\|\rho_{\delta,+\infty}^1(\gamma)\big\|_{\mathbf{L}^1} + \big\|\rho_{t_1,t_2}^2(\gamma)\big\|_{\mathbf{L}^1} \qquad \delta,\, t_i \geq 0. \tag{3.15}$$

Since the second term of the sum in (3.13) is constant in time, we have

$$\big\|\gamma_t^\nu\big\|_{\delta,0,t} = \big\|\rho_{\delta,+\infty}^1(\gamma_t^\nu)\big\|_{\mathbf{L}^1} + \big\|\rho_{0,t}^2(\gamma_0^\nu)\big\|_{\mathbf{L}^1} \qquad \forall\, t \geq 0\,. \tag{3.16}$$

Thus, to estimate the $\mathbf{L}^1$ distance between two approximate solutions $E_t^\nu\mathbf{p}_1$, $E_t^\nu\mathbf{p}_2$, one has to provide an a-priori bound on the integrand of the first term in (3.13). In particular, we shall first fix $\mathcal{M} > 0$, $\delta > 0$, and show that, for any $t \geq \delta$, there exists some constant $L_{\mathcal{M},t} = L_{\mathcal{M},t}(\delta) > 0$ such that there holds

$$\big\|\rho_{\delta,+\infty}^1(\gamma_t^\nu)\big\|_{\mathbf{L}^1} \leq L_{\mathcal{M},t} \cdot \big\|\gamma_0^\nu\big\|_{0,0,t} \qquad \forall\, t \geq \delta\,, \tag{3.17}$$

for every pseudopolygonal $\gamma_0^\nu : [0,\,1] \to \mathcal{FD}^\nu$ joining two couples of initial data and boundary flux in $\{\mathbf{fp};\;\; \mathbf{p} \in D_{\mathcal{M}} \cap \mathcal{D}^\nu\}$. Hence, defining the pseudometrics

$$d_{\delta,t_1,t_2}(\mathbf{p}_1,\mathbf{p}_2) \doteq \big\|\,\overline{u}_1 - \overline{u}_2\big\|_{\mathbf{L}^1([\delta,\,+\infty[)} + \big\|\,\widetilde{u}_1 - \widetilde{u}_2\,\big\|_{\mathbf{L}^1([t_1,\,t_2])} \qquad \mathbf{p}_i = (\,\overline{u}_i,\, \widetilde{u}_i\,), \quad \delta,\, t_i \geq 0, \tag{3.18}$$

and observing that the $\mathbf{L}^1$ lengths of the paths γ_0^ν, γ_t^ν, satisfy

$$\|\gamma_0^\nu\|_{0,0,t} \leq C_0 \cdot d_{0,0,t}(\mathbf{fp}_1,\, \mathbf{fp}_2), \tag{3.19}$$

$$\left\| E_t^\nu \mathbf{p}_1 - E_t^\nu \mathbf{p}_2 \right\|_{\mathbf{L}^1([\delta,+\infty[)} \le C_0 \cdot \|\gamma_t^\nu\|_{\delta,0,t}\,, \tag{3.20}$$

for some constant $C_0 > 0$ (depending only on the domain $\mathcal{D}$), we deduce from (3.16)-(3.17) a uniform Lipschitz estimate for the flow maps $\mathbf{p} \mapsto E_t^\nu\, \mathbf{p}\lceil_{[\delta,+\infty[}$ of the type

$$\left\| E_t^\nu \mathbf{p}_1 - E_t^\nu \mathbf{p}_2 \right\|_{\mathbf{L}^1([\delta,+\infty[)} \le L'_{\mathcal{M},t} \cdot d_{0,0,t}(\mathbf{fp}_1, \mathbf{fp}_2), \qquad \forall\, t \ge \delta\,, \tag{3.21}$$

for some other constant $L'_{\mathcal{M},t} = L'_{\mathcal{M},t}(\delta) > 0$ independent of ν, and for any $\mathbf{p}_1, \mathbf{p}_2 \in D_{\mathcal{M}} \cap \mathcal{D}^\nu$. As $\nu \to \infty$, the domain $D_{\mathcal{M}} \cap \mathcal{D}^\nu$ become dense in $\mathcal{D}_{\mathcal{M}}$. In the limit, a continuous flow map E_t is obtained, defined on the domain $\mathcal{D}_M$, and satisfying the estimate (2.23).

To extend the flow map E_t to the whole domain $\mathcal{D}$ preserving the property (2.23), by similar arguments as above we will prove the estimate

$$\left\| E_t \mathbf{p}_1 - E_t \mathbf{p}_2 \right\|_{\mathbf{L}^1([\delta,+\infty[)} \le L''_t \cdot d_{0,0,t}(\mathbf{fp}_1, \mathbf{fp}_2), \qquad \forall\, t \ge \delta\,, \tag{3.22}$$

for some constant $L''_t = L''_t(\delta) > 0$ independent on the total variation, and for any $\mathbf{p}_1, \mathbf{p}_2 \in \mathcal{D}^\mu$, $\mu \ge 1$.

In the last part of the proof of Theorem 2.1, following the same technique adopted in the above analysis, we derive a stability estimate for the map $\mathbf{p} \mapsto f\big(E_{(\cdot)}\mathbf{p}(x)\big)$ of the same type as (3.22). Namely, we show that, for any $p_1, p_2 \in \mathcal{D}^\mu$, $\mu \ge 1$, there exists a countable set $\mathcal{N} = \mathcal{N}(\mathbf{p}_1, \mathbf{p}_2) \subset I\!R^+$ such that, for every $\tau_2 > \tau_1 > 0$, there holds

$$\left\| f\big(E_{(\cdot)}\mathbf{p}_1(x)\big) - f\big(E_{(\cdot)}\mathbf{p}_2(x)\big) \right\|_{\mathbf{L}^1([\tau_1,\,\tau_2])} \le L''' \cdot d_{0,0,\tau_2}(\mathbf{fp}_1, \mathbf{fp}_2) \qquad \forall\, x \in I\!R^+ \setminus \mathcal{N}\,, \tag{3.23}$$

where $L''' = L'''(\tau_1, \tau_2) > 0$ denotes some constant depending only on τ_1, τ_2, and on the domain $\mathcal{D}$. Relying on this property, and thanks to the invertibility of the flux f, we prove the existence of the strong $\mathbf{L}^1$ trace of $E_t\mathbf{p}(x)$ at $x = 0$, for any $\mathbf{p} \in \mathcal{D}$. Finally, we show that $E_t\mathbf{p}$ fulfills the boundary condition (2.20) and satisfies the Oleinik type estimates (2.15)-(2.18) on the decay of the positive waves.

4 Preliminary results

Fix $\nu \ge 1$, and consider a piecewise constant solution $u(t,\cdot) \doteq E_t^\nu(\overline{u}, \widetilde{u})$ of (1.1) in the region Ω, constructed by the front tracking algorithm described in Section 3, for some $(\overline{u}, \widetilde{u}) \in \mathcal{D}^\nu$. We may perturb this solution shifting the (space) locations x_α of the jumps in the initial data $\overline{u}$ at rates ξ_α, and the (time) locations t_α of the jumps in the boundary data $\widetilde{u}$, entering the interior of the domain Ω, at rates $\widetilde{\xi}_\alpha$ (fig. 1). This means that, if we let $x_\beta = x_\beta(t)$ denote the jumps in the unperturbed solution $u(t,\cdot)$, for θ suitably close

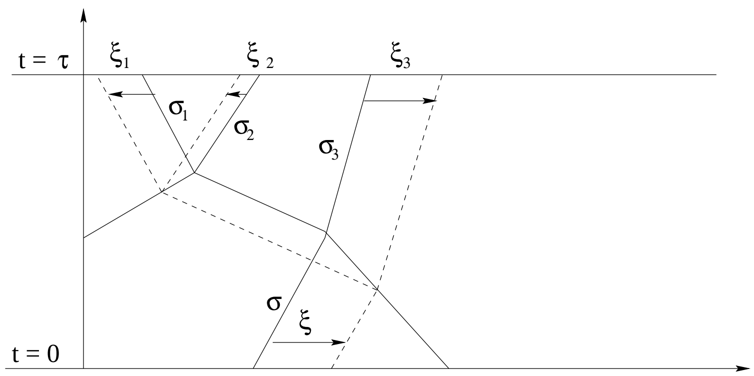

FIGURE 1

to zero the corresponding perturbed solution $u^\theta(t,\cdot)$ will be a function with jumps at the points $x^\theta_\beta = x_\beta + \theta\xi_\beta$. In the same way, $u^\theta(\cdot,x)$ will have jumps at times $t^\theta_\beta = t_\beta - \theta\widetilde{\xi}_\beta$ with $\widetilde{\xi}_\beta = \xi_\beta/\lambda_{k_\beta}$, where $t_\beta = t_\beta(x)$ denote the locations of all jumps in $u(\cdot,x)$ having nonzero slope λ_{k_β}. As long as the wave-front configuration of the functions u, u^θ is the same, the *space-shifts* $\xi_\beta(t)$ and the *time-shifts* $\widetilde{\xi}_\beta(x)$ are uniquely determined as linear functions of the initial time and space shifts ξ_α, $\widetilde{\xi}_\alpha$. In this section we collect some basic properties of these *shift differentials*, that can be obtained with entirely similar arguments as for the corresponding results in [13].

Remark 4.1 We denote by $\sigma_\alpha(t) = u(t,x_\alpha+) - u(t,x_\alpha-)$ the size of a jump in the solution u, occurring at $(t,x_\alpha(t))$, along the time direction, and by $\widetilde{\sigma}_\alpha(x) = f(u(t_\alpha+,x)) - f(u(t_\alpha-,x))$ the size of a jump in the flux $f(u)$ with nonzero speed, occurring at $(t,x_\alpha(t))$, along the space direction. Since approximate solutions are indeed weak solutions, by the Rankine-Hugoniot equations we have the identity

$$\widetilde{\xi}_\alpha\widetilde{\sigma}_\alpha = \frac{\xi_\alpha}{\lambda_{k_\alpha}}\lambda_{k_\alpha}\sigma_\alpha = \xi_\alpha\sigma_\alpha .$$

In the following we will use both notations, depending on convenience.

Lemma 4.1 *Consider a bounded, open region Γ in Ω. Call σ_α, $\alpha = 1,\dots,M$ the fronts entering Γ and let ξ_α be their space-shifts. Assume that the fronts leaving Γ, say σ_β, $\beta = 1,\dots,M'$, are linearly independent. Then the space-shifts ξ_β of the outgoing fronts are uniquely determined by the linear relation*

$$\sum_{\alpha=1}^{M}\xi_\alpha\sigma_\alpha = \sum_{\beta=1}^{M'}\xi_\beta\sigma_\beta\,. \tag{4.1}$$

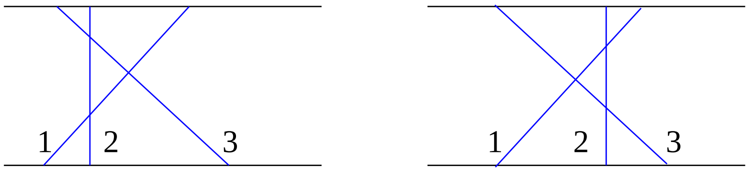

FIGURE 2

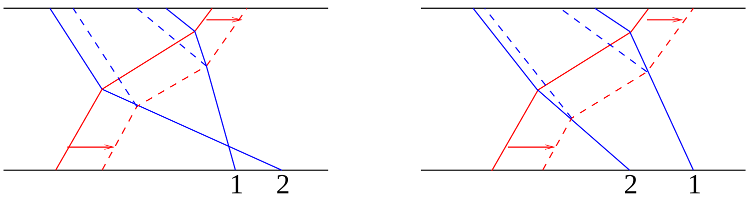

FIGURE 3

A proof of Lemma 4.1, based on an application of the divergence theorem, can be obtained by the same arguments in [13].

Remark 4.2 As observed in [13], according to Lemma 4.1 the shift rates of the outgoing fronts from a given region Γ depend only on the shift rates of the incoming ones, and not on the order in which these wave-fronts interact inside Γ. In particular, one can perform the following two operations, without changing the shift rates of the outgoing fronts:

(O1) Switch the order in which three fronts interact (fig. 2).

(O2) Invert the order of two fronts at $t = 0$ or at $x = 0$, provided that both fronts have zero shift rate (fig. 3).

This property will be repeatedly used in our future estimates. Indeed, in the computation of a shift rate, we can suitably alter the order of wave interactions and thus reduce the problem to a case where the wave-front configuration is particularly simple.

Lemma 4.2 *Assume that a front tracking solution $u(t,\cdot\,) = E_t^\nu(\,\overline{u},\,\widetilde{u}\,)$ contains two wave-fronts of the same characteristic family, say $t \mapsto x_{\alpha'}(t)$, $t \mapsto x_{\alpha''}(t)$, originating at distinct points $(\tau',\overline{x}')$, $(\tau'',\overline{x}'')$, $\tau' \geq \tau''$, $\overline{x}' \leq \overline{x}''$, of the boundary of Ω, and such that $x_{\alpha'}(t) \leq x_{\alpha''}(t)$, $t \in [\tau', T]$. Then, it is possible to assign space-shift rates ξ_α to all fronts in the initial data $\overline{u}$ and in the boundary data $\widetilde{u}$ so that the space-shift rate of the front at $x_{\alpha'}(\tau',\overline{x}')$ is $\xi_{\alpha'} = 1$, and in the corresponding perturbed solution u^θ, all fronts $x_\beta(t)$ outside the strip $\Gamma \doteq \big\{(t,x);\ \ t\in[\tau'',\tau'],\ 0\leq x\leq x_{\alpha''}(t)\big\} \cup \big\{(t,x);\ \ t\in[\tau',\ T],\ x_{\alpha'}(t)\leq x\leq x_{\alpha''}(t)\big\}$ have zero shift rate.*

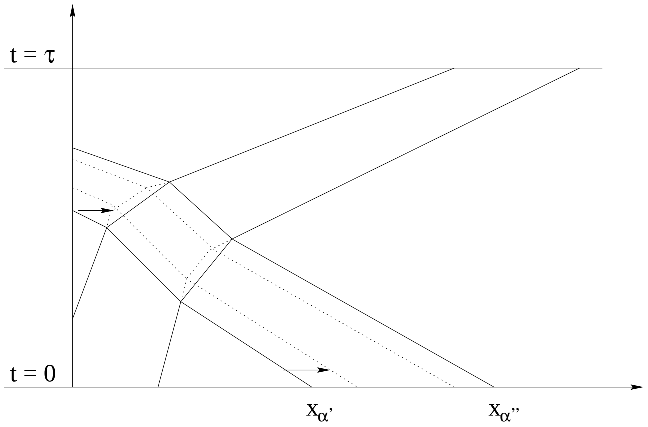

FIGURE 4

In other words, the perturbation of the initial data can be chosen so that one particular front shifts at unit rate, but the corresponding solution remains unaffected outside the region Γ (fig. 4). For a proof of the lemma, proceed as in [13].

Lemma 4.3 *Let u be a front-tracking solution of (1.1) in the region Ω, and consider two wave-fronts, say $t \mapsto x(t)$, $t \in [\tau_x, T]$, and $t \mapsto y(t)$, $t \in [\tau_y, T]$, originating at two points $(\tau_x, \overline{x})$, $(\tau_y, \overline{y})$ of the boundary of Ω. Then, there exists a second front tracking solution $\widehat{u}$ with two fronts $t \mapsto \widehat{x}(t)$, $t \in [\tau_x, T]$, $t \mapsto \widehat{y}(t)$, $t \in [\tau_y, T]$, having the following properties:*

(i) $\widehat{x}(\tau_x) = \overline{x}$, $\widehat{y}(\tau_y) = \overline{y}$, $\widehat{x}(T) = x(T)$, $\widehat{y}(T) = y(T)$.

(ii) $\widehat{u} = u$ *in a neighborhood of these points* $(\tau_x, \overline{x})$, $(\tau_y, \overline{y})$, $(T, x(T))$, $(T, y(T))$.

(iii) *Tot.Var.*$\big\{\big(\widehat{u}(0,\cdot),\, \widehat{u}(\cdot,0)\big)\big\} \le C_1$, *for some constant C_1 depending only on the system* (1.1) *and on the set K.*

PROOF. To fix the ideas, assume $\overline{x} = 0$, $\tau_x > 0$, $\overline{y} > 0$, $\tau_y = 0$. Call J_1, J_2, J_3 the three connected components of the set $\big(\{0\} \times I\!R^+ \cup [0,\ T] \times \{0\}\big) \setminus \{(0, \overline{y}),\, (\tau_x, 0)\}$ and, similarly, let J_1', J_2', J_3', J_4' denote the connected components of the set $\big([0,T] \times \{0\} \cup \{T\} \times I\!R^+\big) \setminus \{(\tau_x, 0),\, (T, x(T)),\, (T, y(T))\}$. Assume that u contains two wave-fronts $t \mapsto z'(t)$, $t \mapsto z''(t)$, of the same k-th family and with the same sign, starting at some points ζ_I', ζ_I'' within the same set J_i, and ending at some other points ζ_F', ζ_F'' that lie in the same set J_j'. Then, proceeding as in [13], we may apply Lemma 4.2 and obtain a second front tracking solution u^{θ_1} which has a smaller number of k-fronts than u, and coincides with u outside the region bounded by the fronts z', z''. We can repeat this construction as long as the resulting solution contains fronts of the same family and with the same sign, starting at points of the same set J_i and ending within the same set J'_j. In a finite number of steps, we then obtain a new solution $\widehat{u}$ which has the property that, for each $k = 1, \ldots, n$, and for any $i = 1, 2, 3,\ \ j = 1, 2, 3, 4$, there exists at most one point $\zeta^0 \in J_i$ where a positive k-wave originates, terminating within J'_j, and similarly for negative k-waves. This implies that the total variation of $(\widehat{u}(0, \cdot),\ \widehat{u}(\cdot, 0)\restriction_{[0,T]})$ is uniformly bounded by a constant C_1 depending only on n and on the diameter of the set K, which completes the proof of the Lemma since we may clearly modify $\widehat{u}(t, \cdot)$ for $t > T$ so that also $\text{Tot.Var.}\{\widehat{u}(\cdot, 0)\restriction_{[T,+\infty]}\} \le C_1$. □

Due to genuine nonlinearity, the amount of positive waves in $u(t, \cdot)$ contained in a bounded interval $[a,\ b]$, measured in Riemann coordinates, dacays in time. We have the following result:

Lemma 4.4 *Consider a front tracking solution* $u(t, \cdot) = E^\nu_t(\overline{u}, \widetilde{u})$, *with* $\overline{u}$, $\widetilde{u}$ *containing together at most* N *shock fronts of the* k*-th family. Then, there exists some constant* C_2 *depending only on the system (1.1) such that, for each* $\tau > 0$, *and for every bounded interval* $[a,\ b]$, $a > 0$, *one has:*

$$\textit{Tot.Var.}\{w_k(\tau, \cdot);\ [a,\ b]\} \le 2C_2 \cdot \frac{b-a}{\tau} + \|w_k\|_{L^\infty} + (N+1)2^{1-\nu}, \tag{4.2}$$

for $k = 1, \ldots, n-p$;

$$\textit{Tot.Var.}\{w_k(\tau, \cdot);\ [a,\ b]\} \le 2C_2 \cdot \left\{ \frac{b-a}{\tau} + \log\left(\frac{b}{a}\right) \right\} + \|w_k\|_{L^\infty} + (N+1)2^{1-\nu}, \tag{4.3}$$

for $k = n-p+1, \ldots, n$.

PROOF. We give the proof of the statement only for $k \in \{n-p+1, \ldots, n\}$, the other case being entirely similar. Relying on Lemma 4.3 and on the uniform strict hyperbolicity assumption **(SH1)**, with the same arguments used in the proof of Lemma 4 in [13] one can show that any two adiacent k-rarefaction fronts $x(t) \le y(t)$ of u, starting from the boundary $x = 0$, are separated at time $\tau > 0$ by a distance $\ge \kappa(\tau - t_0) \cdot 2^{-\nu}$, where $t_0 \ge 0$ is the beginning time of the rarefaction front $x(t)$, and $\kappa > 0$ denotes some constant depending only on the system. Hence, the distance between rarefaction fronts exiting the boundary grows at least linearly with the distance from the t axe. Therefore, the number of rarefaction fronts starting from the boundary and crossing any interval $[a, b]$, $a > 0$, is bounded by

$$1 + N + \frac{C_2}{2^{-\nu}} \log\left(\frac{b}{a}\right)$$

for some constant $C_2 > 0$ depending on the system (1.1). The positive variation of $w_k(\tau, \cdot)$

on $[a,b]$, i. e. the total amount of upward jumps, thus satisfies

$$\text{Pos.Var.}\big\{w_k(\tau,\cdot);\ [a,\ b]\big\} \leq (1+N)2^{-\nu} + C_2 \cdot \log\left(\frac{b}{a}\right). \tag{4.4}$$

On the other hand, the same decay estimates in [13, Lemma 5] hold for the waves starting from $t=0$:

$$\text{Pos.Var.}\big\{w_k(\tau,\cdot);\ [a,\ b]\big\} \leq (1+N)2^{-\nu} + C_2 \cdot \frac{b-a}{\tau}. \tag{4.5}$$

In turn, the total variation of $w_k(\tau,\cdot)$ on $[a,b]$ is bounded by $\|w_k\|_{\mathbf{L}^\infty}$ plus twice the positive variation of w_k. Hence (4.2)-(4.3) hold. □

5 Estimates on shift differentials

In this section we recover the key estimates on shift differentials of paths of approximate solutions. We will use the same technique developed in [13], and we refer to it for details.

Lemma 5.1 *Let $u(t,\cdot) = E^\nu_t(\overline{u}, \widetilde{u})$ be a front tracking solution, with $\overline{u}$, $\widetilde{u}$ containing together N shocks. Assume that the fronts of $\overline{u}$ located at x_β (respectively, the fronts of $f(\widetilde{u})$ starting at t_β) are shifted with shift rate ξ_β (respectively, $\widetilde{\xi}_\beta = \xi_\beta/\lambda_{k_\beta}$) and amplitude σ_β (respectively, $\widetilde{\sigma}_\beta$). Then there exists a constant C_3 depending only on the system (1.1) and on the domain K, such that for any $\delta > 0$, and for every $\tau \geq \delta$, calling $\xi_\alpha(\tau)$, $\sigma_\alpha(\tau)$ the shift rates and the amplitudes of the fronts in $u(\tau,\cdot)$, we have*

$$\sum_{x_\alpha(\tau)>\delta} \big|\xi_\alpha(\tau)\,\sigma_\alpha(\tau)\big| \leq C_3(1+N2^{-\nu})(1+\log(\tau/\delta)) \cdot \left(\sum_\beta \big|\xi_\beta\,\sigma_\beta\big| + \sum_\beta \big|\widetilde{\xi}_\beta\,\widetilde{\sigma}_\beta\big|\right). \tag{5.1}$$

Proof. Assume first that only one single front $\overline{\sigma}$ is shifted, starting at time $t=0$ in the positiont $x = \overline{x}$ (or leaving the boundary $x=0$ at time $t=\overline{t}$), say of the k-th family, with shift rate $\overline{\xi}$. Consider one particular front, say $x_{\alpha^*}(\cdot)$, of the j-th family, and call $\overline{y} \doteq x_{\alpha^*}(\tau)$ its terminal point at time τ. We claim that there are constants C_4, C_5, depending only on the system (1.1) and on the domain K, such that the following properties hold.

- **(P1)** If x_{α^*} is precisely the k-front starting at $\overline{x}$ ($\overline{t}$ respectively), then

$$\big|\xi_{\alpha^*}(\tau)\,\sigma_{\alpha^*}(\tau)\big| \leq C_5|\overline{\xi}\,\overline{\sigma}|. \tag{5.2}$$

- **(P2)** If x_{α^*} is a j-front, with $j \neq k$, and the backward j-characteristics ending at $\overline{y}$ include fronts starting from both sides of $\overline{x}$ ($\overline{t}$ respectively), then (5.2) again holds.

- **(P3)** If x_{α^*} is a j-front, and the j-fronts ending at $\overline{y}$ start all at the same side of $\overline{x}$

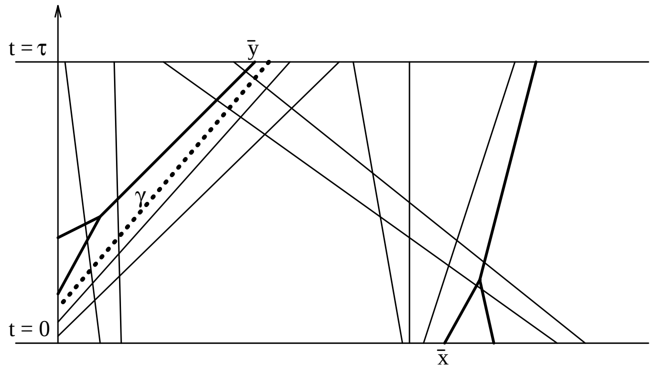

FIGURE 5

($\bar{t}$ respectively), one then has the sharper estimate

$$\left|\xi_{\alpha^*}(\tau)\right| \leq C_4 |\bar{\xi}\,\bar{\sigma}|. \tag{5.3}$$

Properties **(P1)**–**(P2)** can be established by the same arguments in [13], with minor changes. Hence, we limit ourselves here to give a proof of **(P3)**. Observe that, besides the fronts starting at $\overline{x}$ (or $\bar{t}$) and the ones ending at $\overline{y}$, one can single out four groups of waves:

(1) the waves starting on the left of $\overline{x}$ (respectively, after $\bar{t}$) and ending on the left of $\overline{y}$;

(2) the waves starting on the right of $\overline{x}$ (respectively, before $\bar{t}$) and ending on the right of $\overline{y}$;

(3) the waves starting on the right of $\overline{x}$ (respectively, before $\bar{t}$) and ending on the left of $\overline{y}$;

(4) the waves starting on the left of $\overline{x}$ (respectively, after $\bar{t}$) and ending on the right of $\overline{y}$.

According to Remark 4.2, in our computation of the shift rate $\xi_{\alpha^*}(\tau)$ of the front reaching $\overline{y}$, it is not restrictive to assume that the sets of waves in (1) and (2) are empty. Indeed, we can otherwise shift the locations of all these fronts of type (1) towards the left, until they all lie outside the domain influenced by the shift at $\overline{x}$. Similarly, fronts of type (2) can be shifted toward the right until they lie completely outside this domain of influence.

Let's consider the case (fig. 5) where no j-wave ending at $\overline{y}$ crosses the k-wave starting at $\overline{x}$ (or $\bar{t}$). Consider a curve γ running slightly to the right of the minimal backward j-front ending at $\overline{y}$. By Lemmas 4.1, 4.2, after performing the operations **(O1)**-**(O2)** a number of times, we can consider an equivalent configuration with the following properties:

No front crosses γ from left to right. There exists some index $\ell \leq j$ such that only fronts of families $i < \ell$ can cross γ from right to left, and we can assume that the waves of type (1) have zero shift rate at every time in the interval $[0,\ \tau]$.

Applying Remark 4.2 to the region on the right of γ we obtain

$$\sum_{\alpha\in C(\gamma)} \xi_\alpha\sigma_\alpha + \sum_{x_\alpha(\tau)>\gamma(\tau)} \xi_\alpha(\tau)\sigma_\alpha(\tau) = \bar{\xi}\bar{\sigma}. \tag{5.4}$$

Here the first summation extends to all fronts crossing the curve γ with non-zero shift rate. Call u^L and u^R the left and right states across the jump at $\bar{y}$. Observing that the two sums on the right hand side of (5.4) are contained in

$$\mathrm{span}\{r_1(u^R),\dots,r_{\ell-1}(u^R)\}, \qquad \mathrm{span}\{r_\ell(u^R),\dots,r_n(u^R)\}, \tag{5.5}$$

using the strict hyperbolicity condition **(SH2)**, we conclude

$$\left|\sum_{\alpha\in C(\gamma)} \xi_\alpha\sigma_\alpha\right| \le C_4\left|\bar{\xi}\,\bar{\sigma}\right|. \tag{5.6}$$

We now apply again Remark 4.2 to the region on the left of γ. Observing that the only incoming fronts which carry a nonzero shift rate are those crossing γ from right to left, and that the only outgoing shifted j-front is the one ending at $\bar{y}$, we obtain

$$\sum_{x_\alpha(\tau)<\bar{y}} \xi_\alpha\sigma_\alpha + \xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau) = \sum_{\alpha\in C(\gamma)} \xi_\alpha\sigma_\alpha. \tag{5.7}$$

Recalling the normalization at (2.1), we observe that (5.7) implies

$$\left|\xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau)\right| = l_j(u^L)\cdot \sum_{\alpha\in C(\gamma)} \xi_\alpha\sigma_\alpha. \tag{5.8}$$

On the other hand,

$$l_j(u^R)\cdot \sum_{\alpha\in C(\gamma)} \xi_\alpha\sigma_\alpha = 0. \tag{5.9}$$

Together, (5.8) and (5.9) imply

$$\left|\xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau)\right| \le C'\left|l_j(u^R)-l_j(u^L)\right|\left|\sum_{\alpha\in C(\gamma)} \xi_\alpha\sigma_\alpha\right| \le C_5\left|\sigma_{\alpha^*}(\tau)\right|\left|\bar{\xi}\,\bar{\sigma}\right|. \tag{5.10}$$

Hence, in this case, (5.3) holds.

We now prove **(P3)** in the case where $k>j$ and all j-waves running into $\bar{y}$ cross the k-wave starting from $\bar{t}$, as in figure 6. In this case, we construct a curve γ slightly to the left of the maximal backward j-front ending at $\bar{y}$. Observe that every wave-front crossing γ from left to right must be of a family $i>j$. Moreover, we can assume that no wave crosses γ from right to left. Applying Remark 4.2 to the region on the left of γ we obtain

$$\sum_{\alpha\in C(\gamma)} \xi_\alpha\sigma_\alpha + \sum_{x_\alpha(\tau)<\gamma(\tau)} \xi_\alpha(\tau)\sigma_\alpha(\tau) = \bar{\xi}\,\bar{\sigma}. \tag{5.11}$$

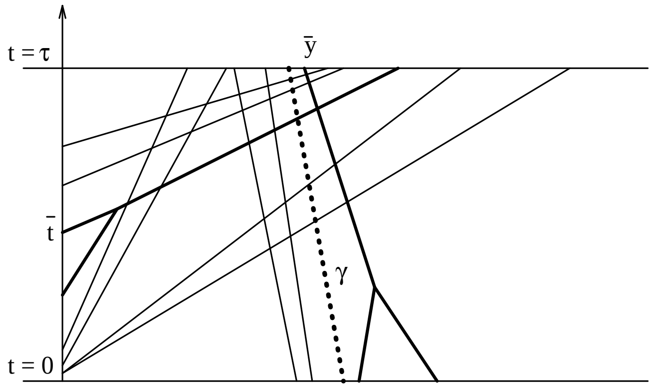

FIGURE 6

Since the waves crossing γ must belong to different families from the ones ending inside the interval $[0,\ \gamma(\tau)]$ (recall that interacting waves of the same family produce a single wave front), (5.11) implies

$$\left| \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha \right| \leq C_5 |\overline{\xi\sigma}|. \tag{5.12}$$

We now apply again Remark 4.2 to the region on the right of γ, observing that the set of outgoing fronts, crossing the line $t = \tau$, contains the j-front at $\overline{y}$ plus other fronts on the right of $\overline{y}$, of families $i > j$. This yields

$$\xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau) + \sum_{x_\alpha(\tau) > \overline{y}} \xi_\alpha(\tau)\sigma_\alpha(\tau) = \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha. \tag{5.13}$$

By (5.13),

$$\left|\xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau)\right| = l_j(u^R) \cdot \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha. \tag{5.14}$$

Observing that

$$l_j(u^L) \cdot \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha = 0,$$

we again have an estimate of the form (5.10), hence **(P3)** holds. The other cases are similar or easier.

We now complete the proof of Lemma 5.1. If we assume that only one single front is shifted leaving the boundary $x = 0$, say starting at time $t = \overline{t}$ (or starting at time $t = 0$ and located at $x = \overline{x}$), it follows that at a fixed time $\tau > 0$ the only fronts with nonzero shift rate can be the ones located inside the interval $[a_0,\ b_0] \doteq [0,\ \widehat{\lambda} \cdot (\tau - \overline{t})]$ (or inside the interval $[a_0,\ b_0] \doteq [\max\{0, \overline{x} - \widehat{\lambda} \cdot \tau\},\ \overline{x} + \widehat{\lambda} \cdot \tau])$, where $\widehat{\lambda}$ denotes the upper bound (2.5) for the absolute values of all characteristic speeds. Recalling the estimate (4.2)-(4.3) on the total variation, and using the properties **(P1)**–**(P3)**, we thus have

$$\begin{aligned}\sum_{x_\alpha(\tau)>\delta} \big|\xi_\alpha(\tau)\sigma_\alpha(\tau)\big| &\le nC_5|\overline{\xi}\overline{\sigma}| + C_4\big|\overline{\xi}\,\overline{\sigma}\big| \cdot \text{Tot.Var.}\{u(\tau)\,;\ \ [\delta,\ b_0]\} \\ &\le C_3(1+N2^{-\nu})(1+\log(\tau/\delta))\big|\overline{\xi}\,\overline{\sigma}\big|\,, \end{aligned} \tag{5.15}$$

for a suitable constant C_3, proving (5.1). Finally, we consider the case where all fronts in $\overline{u}$, $\tilde{u}$ are shifted. More precisely, let $\xi_\alpha(0)$ be the shift rate of the front located at $x_\alpha(0)$ ($t_\alpha(0)$ respectively), having amplitude $\sigma_\alpha(0)$. Call $\xi_\beta(\tau)$ the corresponding shift rate of the front of $u(\tau,\cdot)$ located at $x_\beta(\tau)$. Observing that the shift differential

$$\big(\xi_1(0),\dots\xi_M(0)\big) \mapsto \big(\xi_1(\tau),\dots\xi_{M'}(\tau)\big)$$

is a linear mapping, (5.1) easily follows from (5.15). □

In order to show that the trajectories of the flow map E_t that we shall construct in Section 6, provide solutions with a strong $\mathbf{L}^1$ trace at the boundary $x=0$, we will make use of the following estimates on shift-differentials of paths of approximate solutions along vertical segments of the domain Ω.

Lemma 5.2 *Let $u(t,x) = E^\nu_t(\overline{u},\,\widetilde{u}\,)(x)$ be a front tracking solution containing at most N shocks. Assume that the fronts of $\overline{u}$ located at x_β (respectively, the fronts of $f(\widetilde{u})$ entering the interior of the domain Ω, and starting at t_β) are shifted with shift rate ξ_β (respectively, $\widetilde{\xi}_\beta = \xi_\beta/\lambda_{k_\beta}$) and amplitude σ_β (respectively, $\widetilde{\sigma}_\beta$). Then, there exists some constant C_6 depending only on the system (1.1) and on the domain K such that, for any $t_2 > t_1 > 0$, and for every $x>0$, denoting with $\widetilde{\xi}_\alpha(x)$, $\widetilde{\sigma}_\alpha(x)$ the shift rates and the amplitudes of the fronts in $f(u(\cdot,x\,))$ having nonzero speed, and crossing the line $\{(t,x)\ ;\ t\ge 0\}$ at time $t_\alpha(x)$, there holds*

$$\sum_{t_\alpha(x)\in[t_1,\ t_2]} \big|\widetilde{\xi}_\alpha(x)\,\widetilde{\sigma}_\alpha(x)\big| \le C_6(1+N2^{-\nu})(1+\log(t_2/t_1))\cdot\Big(\sum_\beta \big|\xi_\beta\,\sigma_\beta\big| + \sum_\beta \big|\widetilde{\xi}_\beta\,\widetilde{\sigma}_\beta\big|\Big). \tag{5.16}$$

Proof. We give here only a sketch of the proof since it is quite similar to the one of Lemma 5.1. The estimates **(P1)**–**(P3)** can be recovered with minor modifications. As an example, we illustrate the case in which the particular front t_{α^*} crossing the segment $\{(t,x);\ t\in[t_1,\ t_2]\}$ that we are considering, starts at time $t=0$ and belongs to a family $j\in\{1,\dots.n-p\}$ (see fig. 7). Call $\overline{s} = t_{\alpha^*}(x)$ the crossing time of the segment $\{(t,x);\ t\in [t_1,\ t_2]\}$. Assume that only one single front $\widetilde{\sigma}(0)$, leaving the boundary $x=0$ at time $t=\overline{t}$, is shifted, say of family $k\in\{n-p+1,\dots,n\}$ (and no front in the initial data $\overline{u}$ is shifted).

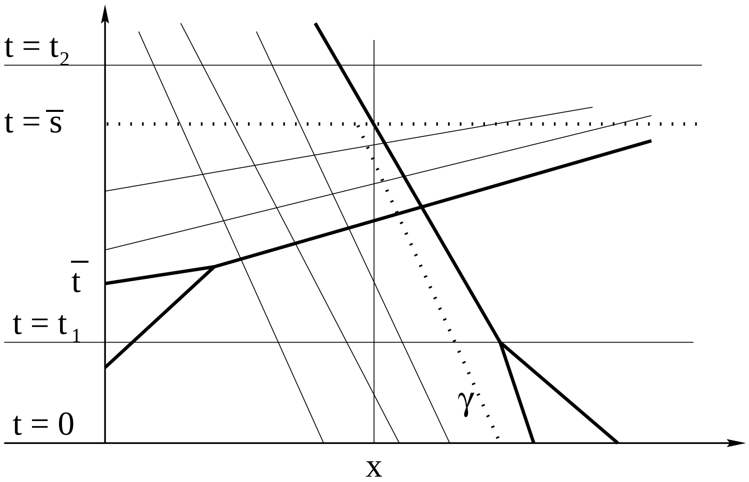

FIGURE 7

After performing the usual semplifications, we reduce to the situation illustrated in figure 7. Consider the straight line $t = \overline{s}$ and a curve γ running slightly to the left of the maximal backward j-front passing through $(\overline{s}, x)$. Applying the divergence theorem to the region on the left of γ we obtain

$$\widetilde{\xi}\,\widetilde{\sigma}(0) = \sum_{\alpha \in C(\gamma)} \xi_\alpha\,\sigma_\alpha + \sum_{x_\alpha(\overline{s}) < \gamma(\overline{s})} \xi_\alpha(\overline{s})\,\sigma_\alpha(\overline{s}). \tag{5.17}$$

By linear independence of the vectors on the left hand side of (5.17) we have

$$\Big| \sum_{\alpha \in C(\gamma)} \xi_\alpha\,\sigma_\alpha \Big| \le C \big| \widetilde{\xi}\,\widetilde{\sigma}(0) \big|. \tag{5.18}$$

Then we consider the region on the right of γ, where we compute

$$\sum_{\alpha \in C(\gamma)} \xi_\alpha\,\sigma_\alpha = \xi\,\sigma(x) + \sum_{x_\alpha(\overline{s}) > x} \xi_\alpha(\overline{s})\,\sigma_\alpha(\overline{s}). \tag{5.19}$$

From (5.19), by (2.1) and (5.18) we obtain

$$\begin{aligned} \big|\widetilde{\xi}\,\widetilde{\sigma}(\overline{s})\big| \; &= \; \big|\xi\,\sigma(x)\big| = \Big| l_j(u^R) \cdot \sum_{\alpha \in C(\gamma)} \xi_\alpha\,\sigma_\alpha \Big| \le C \big| l_j(u^R) - l_j(u^L) \big| \Big| \sum_{\alpha \in C(\gamma)} \xi_\alpha\,\sigma_\alpha \Big| \\ &\le C \big|\sigma(x)\big| \big|\widetilde{\xi}\,\widetilde{\sigma}(0)\big|, \end{aligned}$$

and we recover **(P3)**. Here u^L and u^R denote as usual the left and right states across the jump at $(\overline{s}, x)$.

Therefore, in the case where, at time $t = 0$ and at the boundary $x = 0$, only one single front is shifted, one derives (5.16) relying on the properties **(P1)**-**(P3)** and using similar estimates on the total variation as the ones in (4.2)-(4.3). Namely, there will be some positive constant depending only on the system (1.1), that we may call C_2 as the one in (4.2)-(4.3), such that, for every $b > a > 0$ and for any $x > 0$, there holds

$$\text{Tot.Var.}\big\{w_k(\cdot, x);\ [a,\ b]\big\} \le 2C_2 \cdot \log\left(\frac{b}{a}\right) + \|w_k\|_{L^\infty} + (N+1)2^{1-\nu}\,, \tag{5.20}$$

for $k = 1, \ldots, n-p$,

$$\text{Tot.Var.}\big\{w_k(\cdot, x);\ [a,\ b]\big\} \le 2C_2 \cdot \left\{\frac{b-a}{x} + \log\left(\frac{b}{a}\right)\right\} + \|w_k\|_{L^\infty} + (N+1)2^{1-\nu}\,, \tag{5.21}$$

for $k = n-p+1, \ldots, n$. The proof of the estimates (5.20)-(5.21) is enterely similar to the one of Lemma 4.4. Finally, the general case where all fronts in $\overline{u}$ and in $\widetilde{u}$ are shifted, is treated as in Lemma 5.1. □

6 Proof of Theorems 2.1-2.2

6.1 Existence of the semigroup on domains of BV functions

In order to construct the semigroup described in Theorem 2.1, we shall first define an $\mathbf{L}^1$ continuous flow map E_t on every domain

$$\mathcal{D}_{\mathcal{M}} \doteq \Big\{\mathbf{p} \in \mathcal{D};\quad \text{Tot.Var.}\{\mathbf{p}\} \le \mathcal{M}\Big\} \qquad \mathcal{M} > 0,$$

obtained as a limit of the approximate flow maps E^ν_t constructed in Section 3 on the domains $\mathcal{D}^\nu$. To this end, consider any two piecewise constant couples of initial and boundary data, say $\mathbf{p}_1,\ \mathbf{p}_2 \in \mathcal{D}_{\mathcal{M}} \cap \mathcal{D}^\nu$, and construct a pseudopolygonal path $\gamma^\nu_0 : \theta \mapsto \mathbf{fp}^\theta = (\overline{u}^\theta,\ f(\widetilde{u}^\theta))$ connecting $\mathbf{fp}_1$ with $\mathbf{fp}_2$ as described in Section 3. All functions $(\overline{u}^\theta, \widetilde{u}^\theta)$ lie in $\mathcal{D}_{\mathcal{M}} \cap \mathcal{D}^\nu$ and have a uniformly bounded number of shocks, say $\le N$. Call $u^\theta_\nu(t,\cdot) = E^\nu_t(\overline{u}^\theta, \widetilde{u}^\theta)$ the corresponding solution and consider the path $\gamma^\nu_t : \theta \mapsto \big(u^\theta_\nu(t,\cdot),\ f(\widetilde{u}^\theta)\big)$. Writing the length of this path in the form (3.13), and using Lemma 5.1, for any fixed $\delta > 0$, and for every $t \ge \delta$, we obtain the estimate

$$\big\|\rho^1_{\delta,+\infty}(\gamma^\nu_t)\big\|_{\mathbf{L}^1} = \sum_{j=1}^{m} \int_{\theta_{j-1}}^{\theta_j} \sum_{\{\alpha\ :\ x^\theta_\alpha > \delta\}} \big|\Delta u^\theta_\nu(t, x^\theta_\alpha)\big| \left|\frac{\partial x^\theta_\alpha(t)}{\partial\theta}\right| d\theta \ \le$$

$$
\begin{aligned}
&\leq \sum_{j=1}^{m} \int_{\theta_{j-1}}^{\theta_j} C_3(1+N2^{-\nu})(1+\log(t/\delta)) \cdot \\
&\quad \cdot \Bigg(\sum_{\beta} \big|\Delta u_\nu^\theta(0, x_\beta^\theta)\big| \bigg|\frac{\partial x_\beta^\theta(0)}{\partial\theta}\bigg| + \sum_{\{\beta' \,:\, t_{\beta'}^\theta < t\}} \big|\widetilde{\Delta} u_\nu^\theta(t_{\beta'}^\theta, 0)\big| \bigg|\frac{\partial t_{\beta'}^\theta(0)}{\partial\theta}\bigg| \Bigg)\, d\theta \\
&\leq C_3(1+N2^{-\nu})(1+\log(t/\delta)) \cdot \big\|\gamma_0^\nu\big\|_{0.0,t}\,, \qquad (6.1)
\end{aligned}
$$

where $\rho^1_{\delta,+\infty}$ and $\|\cdot\|_{0,0,t}$ denote the restriction map and the seminorm introduced at (3.14)-(3.15). Observing that any function in $\mathcal{D}_{\mathcal{M}} \cap \mathcal{D}^\nu$ has at most $2^\nu\mathcal{M}$ jumps, from (6.1) we derive (3.17) with $L_{\mathcal{M},t} = C_3(1+\mathcal{M})(1+\log(t/\delta))$, which, in turn, because of (3.19)-(3.20), clearly implies (3.21).

Once we have established the uniform Lipschitz continuity of the maps $\mathbf{p} \mapsto E_t^\nu\, \mathbf{p}\restriction_{[\delta,+\infty[}$, on the domains $\mathcal{D}_{\mathcal{M}} \cap \mathcal{D}^\nu$, since the union $\cup_{\nu\geq 1}\mathcal{D}_{\mathcal{M}} \cap \mathcal{D}^\nu$ is dense in $\mathcal{D}_{\mathcal{M}}$ we will define the map E_t on $\mathcal{D}_{\mathcal{M}}$ as the limit

$$
E_t(\mathbf{p}) \doteq \mathbf{L}^1 - \lim_{\nu\to\infty} E_t^\nu(\mathbf{p}^\nu) \qquad\qquad \mathbf{p}^\nu \in \mathcal{D}_{\mathcal{M}} \cap \mathcal{D}^\nu,\ \mathbf{p}^\nu \to \mathbf{p} \text{ in } \mathbf{L}^1\,. \qquad (6.2)
$$

In order to prove that the assignment (6.2) yields a well-defined map, since any sequence $E_t^\nu\mathbf{p}^\nu$ is uniformly bounded in $\mathbf{L}^\infty$, it is sufficient to show that, for every given $\mathbf{p} \in \mathcal{D}_{\mathcal{M}}$, and for any $\delta > 0$, if $\mathbf{p}^\nu \in \mathcal{D}_{\mathcal{M}} \cap \mathcal{D}^\nu$ is any sequence that converges to $\mathbf{p}$ in $\mathbf{L}^1$, then the sequence $E_t^\nu\mathbf{p}^\nu\restriction_{[\delta,+\infty[}$ is Cauchy in $\mathbf{L}^1$. Indeed, for any $\mu > \nu$, using (3.21) (possibly with a different constant $L'_{\mathcal{M},t}$), we obtain

$$
\begin{aligned}
\big\|E_t^\mu\mathbf{p}^\mu - E_t^\nu\mathbf{p}^\nu\big\|_{\mathbf{L}^1([\delta,+\infty[)} &\leq \big\|E_t^\mu\mathbf{p}^\mu - E_t^\mu\mathbf{p}^\nu\big\|_{\mathbf{L}^1([\delta,+\infty[)} + \big\|E_t^\mu\mathbf{p}^\nu - E_t^\nu\mathbf{p}^\nu\big\|_{\mathbf{L}^1([\delta,+\infty[)} \\
&\leq L'_{\mathcal{M},t} \cdot d_{0,0,t}(\mathbf{p}^\mu,\, \mathbf{p}^\nu) + d_{\delta,0,\infty}\big(S_t^\mu\mathbf{p}^\nu,\, S_t^\nu\mathbf{p}^\nu\big) \qquad (6.3)
\end{aligned}
$$

where $d_{\delta,0,\infty}$ denotes the pseudometric defined as in (3.18). To estimate the second term in (6.3), we shall use the same type of error estimate established in [10, Theorem 2.9] for the distance between a Lipschitz continuous map and the trajectory of a Lipschitz continuous semigroup which can be restated as follows.

Lemma 6.1 *Let $(B,\, d_B)$ be a metric space, let $d_{B'}$ be a pseudometric on B, and let $\mathcal{D}$ be a closed subset of B. Let $S : \mathcal{D} \times [0,\, T] \mapsto \mathcal{D}$ be a continuous semigroup and $\Gamma : [0,\, T] \mapsto \mathcal{D}$ a continuous map that satisfy*

$$
d_{B'}\big(S_t\mathbf{p}_1,\, S_s\mathbf{p}_2\big) \;\leq\; L \cdot \big\{d_B(\mathbf{p}_1,\, \mathbf{p}_2) + |t-s|\big\}, \qquad (6.4)
$$

$$
d_B\big(\Gamma(t),\, \Gamma(s)\big) \;\leq\; L \cdot |t-s|, \qquad (6.5)
$$

for some constant $L > 0$. Then, for any $\tau \in [0, T]$, one has the estimate

$$d_{B'}\big(\Gamma(\tau),\, S_\tau\Gamma(0)\big) \le L \cdot \int_0^\tau \left\{ \liminf_{h\to 0+} \frac{d_{B'}\big(\Gamma(t+h),\, S_h\Gamma(t)\big)}{h} \right\} dt. \tag{6.6}$$

Let B be the metric space $\mathbf{L}^1(I\!R^+, K) \times \mathbf{L}^1(I\!R^+, K)$ equipped with the usual $\mathbf{L}^1$ distance, and set

$$d_{B'} \doteq d_{\delta,0,\infty}, \qquad \mathcal{D} \doteq \mathcal{D}_{\mathcal{M}} \cap \mathcal{D}^\mu\,.$$

Observe that, if we let $S \doteq S^\mu$ be the approximate semigroup defined in (3.8), and $\Gamma : [0, T] \to \mathcal{D}_{\mathcal{M}} \cap \mathcal{D}^\nu \subset \mathcal{D}_{\mathcal{M}} \cap \mathcal{D}^\mu$ be the map $\Gamma(t) = S_t^\nu \mathbf{p}^\nu$, then the Lipschitz continuity (3.21) of $\mathbf{p} \mapsto E_t^\nu\, \mathbf{p} \restriction_{[\delta,+\infty[}$, together with the uniform bound on the total variation of $t \mapsto E_t^\nu \mathbf{p},\ \mathbf{p} \in \mathcal{D}_{\mathcal{M}} \cap \mathcal{D}^\nu,\ \ t \mapsto E_t^\mu \mathbf{p},\ \mathbf{p} \in \mathcal{D}_{\mathcal{M}} \cap \mathcal{D}^\mu$, clearly imply the estimates (6.4)-(6.5). Thus, we may apply Lemma 6.1 and, from (6.3), (6.6), we derive

$$\begin{aligned} \big\| E_t^\mu \mathbf{p}^\mu - E_t^\nu \mathbf{p}^\nu \big\|_{\mathbf{L}^1([\delta,+\infty[)} &\le L'_{\mathcal{M},t} \cdot d_{0,0,t}(\mathbf{p}^\mu,\, \mathbf{p}^\nu) + \\ &+ L \cdot \int_0^t \left\{ \liminf_{h\to 0+} \frac{d_{\delta,0,\infty}\big(S_{s+h}^\nu\, \mathbf{p}^\nu,\ S_h^\mu S_s^\nu \mathbf{p}^\nu\big)}{h} \right\} ds. \end{aligned} \tag{6.7}$$

With the same arguments in [6], letting $\mathbf{q} \doteq S_s^\nu \mathbf{p}^\nu$, we can now estimate the integrand in (6.7) by

$$\frac{1}{h}\, d_{\delta,0,\infty}\big(S_h^\nu\, \mathbf{q},\ S_h^\mu \mathbf{q}\big) = \frac{1}{h}\, \big\| E_h^\nu\, \mathbf{q} - E_h^\mu \mathbf{q} \big\|_{\mathbf{L}^1([\delta,+\infty[)} \le C_7 \cdot 2^{-\nu} \mathcal{M}, \tag{6.8}$$

for some constant $C_7 > 0$. Hence, (6.7) together with (6.8) yields

$$\big\| E_t^\mu \mathbf{p}^\mu - E_t^\nu \mathbf{p}^\nu \big\|_{\mathbf{L}^1([\delta,+\infty[)} \le L'_{\mathcal{M},t} \cdot d_{0,0,t}(\mathbf{p}^\mu,\, \mathbf{p}^\nu) + L C_7 \mathcal{M} \cdot 2^{-\nu} t, \tag{6.9}$$

which clearly shows that $E_t^\nu \mathbf{p}^\nu \restriction_{[\delta,+\infty[}$ is a Cauchy sequence in the $\mathbf{L}^1$ norm and that his limit does not depend on the choice of the sequence $\mathbf{p}^\nu$. Thus, the map in (6.2) is well-defined on every domain $\mathcal{D}_{\mathcal{M}}$.

6.2 Extension of the semigroup to domains of $\mathbf{L}^\infty$ functions

To ensure the existence of the map E_t on the whole domain $\mathcal{D}$ of functions of possibly unbounded variation, we will prove now the estimate (3.22) for some constant $L''_t > 0$ independent on the total variation. To this purpose, consider any two couples $\mathbf{p}_1,\ \mathbf{p}_2 \in \mathcal{D}^\mu$, and construct as above a pseudopolygonal path $\gamma_0 : \theta \mapsto \mathbf{fp}^\theta = (\overline{u}^\theta,\, f(\widetilde{u}^\theta))$ taking values in $\mathcal{FD}^\mu$, that connects $\mathbf{fp}_1$ with $\mathbf{fp}_2$ and has the following property. All functions $(\overline{u}^\theta,\, \widetilde{u}^\theta)$ have a uniformly bounded number of jumps and hence lie in some domain $\mathcal{D}_{\mathcal{M}},\ \mathcal{M} > 0$.

Then, calling $u_\nu^\theta(t,\cdot) = E_t^\nu(\,\overline{u}^\theta,\,\widetilde{u}^\theta\,)$ the corresponding ν-approximate solution, since by (6.2) we have

$$E_t(\,\overline{u}^\theta,\,\widetilde{u}^\theta\,) = \lim_{\nu\to\infty} E_t^\nu(\,\overline{u}^\theta,\,\widetilde{u}^\theta\,) = \lim_{\nu\to\infty} u_\nu^\theta(t,\cdot), \tag{6.10}$$

in order to establish (3.22) we will show that the length of the path $\gamma_t^\nu : \theta \mapsto \big(u_\nu^\theta(t,\cdot),\, f(\widetilde{u}^\theta)\big)$ remains a bounded multiple of the length of γ_0, independent on ν. Indeed, for any fixed $\delta > 0$, and for every $\nu \geq \mu$, letting N be a uniform bound on the number of shocks in $(\,\overline{u}^\theta,\,\widetilde{u}^\theta\,)$, and using Lemma 5.1, we obtain by the same arguments in (6.1) the estimate

$$\big\|\rho^1_{\delta,+\infty}\big(\gamma_t^\nu\big)\big\|_{\mathbf{L}^1} \leq C_3(1+N2^{-\nu})(1+\log(t/\delta))\cdot\|\gamma_0\|_{0,0,t} \qquad \forall\, t \geq \delta\,,$$

which, in turn, because of (3.16), (3.19)-(3.20), implies

$$\big\|E_t^\nu\mathbf{p}_1 - E_t^\nu\mathbf{p}_2\big\|_{\mathbf{L}^1([\delta,+\infty[)} \leq C_8(1+N2^{-\nu})(1+\log(t/\delta))\cdot d_{0,0,t}(\mathbf{fp}_1,\,\mathbf{fp}_2) \qquad \forall\, t \geq \delta\,, \tag{6.11}$$

for some other constant $C_8 > 0$. Letting $\nu \to \infty$ in (6.11), because of (6.10) we obtain (3.22) for all $\mathbf{p}_1,\ \mathbf{p}_2 \in \mathcal{D}^\mu$. Since the domains $\mathcal{D}^\mu,\ \mu \geq 1$, are dense in $\mathcal{D}$, relying on (3.22) we can now extend the map E_t by continuity to the whole domain $\mathcal{D}$ setting

$$E_t(\mathbf{p}) \doteq \mathbf{L}^1 - \lim_{\mu\to\infty} E_t(\mathbf{p}^\mu) \qquad\qquad \mathbf{p}^\mu \in \mathcal{D}^\mu,\ \mathbf{p}^\mu \to \mathbf{p} \text{ in } \mathbf{L}^1\,. \tag{6.12}$$

Clearly, the map in (6.12) preserves the property (3.22), proving (2.23). Moreover, any trajectory $t \mapsto E_t(\mathbf{p})$, being the limit of front tracking approximations, provides by standard arguments [9, 10] a weak solution to problem (1.1)-(1.2).

6.3 Stability estimates in space

Towards a proof of the existence of a strong $\mathbf{L}^1$ trace of $f(u(t,x)) \doteq f\big(E_t\mathbf{p}(x)\big)$ at the boundary $x = 0$, we shall first establish the stability estimate (3.23) for the map $\mathbf{p} \mapsto f\big(E_{(\cdot)}\mathbf{p}(x)\big)$. Fix $\tau_2 > \tau_1 > 0$, and observe that, because of (6.2), for every given $\mathbf{p} \in \mathcal{D}_\mathcal{M}$, the sequence $E^\nu_{(\cdot)}\mathbf{p}(\cdot)$ converges to $E_{(\cdot)}\mathbf{p}(\cdot)$ in $\mathbf{L}^1\big([0,\,\tau_2]\times I\!R^+;\,K\big)$. Hence, relying also on the continuity of the maps $x \mapsto E^\nu_{(\cdot)}\mathbf{p}(x),\ x \mapsto E_{(\cdot)}\mathbf{p}(x)$, we deduce that

$$f\big(E_{(\cdot)}(\mathbf{p})(x)\big)\restriction_{[\tau_1,\,\tau_2]} = \mathbf{L}^1 - \lim_{\nu\to\infty} f\big(E^\nu_{(\cdot)}(\mathbf{p})(x)\big)\restriction_{[\tau_1,\,\tau_2]} \qquad \forall\, x \geq 0. \tag{6.13}$$

Therefore we may proceed as in the proof of (3.22) to establish the estimate (3.23). Given any pair of couples $\mathbf{p}_1,\ \mathbf{p}_2 \in \mathcal{D}^\mu$, we construct a pseudopolygonal path $\gamma_0 : \theta \mapsto \mathbf{fp}^\theta = (\,\overline{u}^\theta,\, f(\widetilde{u}^\theta))$ taking values in $\mathcal{FD}^\mu$, and connecting $\mathbf{fp}_1$ with $\mathbf{fp}_2$, so that all functions $(\,\overline{u}^\theta,\,\widetilde{u}^\theta)$ have a uniformly bounded number of shocks $\leq N$, and lie in some domain $\mathcal{D}_\mathcal{M},\ \ \mathcal{M} > 0$. Then, for every $\nu \geq \mu$, calling $u_\nu^\theta(t,\cdot) \doteq E_t^\nu(\,\overline{u}^\theta,\,\widetilde{u}^\theta\,)(\cdot)$ the corresponding ν-approximate solution, we consider the pseudopolygonal path

$$\gamma_x^\nu : \theta \mapsto \big(\overline{u}^\theta,\, f\big(u_\nu^\theta(\cdot, x)\big)\big) \tag{6.14}$$

with values in $\mathcal{FD}^\nu$. Let $\rho^i_{\tau_1,\tau_2}$ and $\|\cdot\|_{0,0,\tau_2}$ denote the restriction map and the seminorm defined as in (3.14)-(3.15), and let $\mathcal{N} = \mathcal{N}(\mathbf{p}_1, \mathbf{p}_2)$ be the countable set of all points $x > 0$ such that $E^\nu_{(\cdot)}\mathbf{p}_1(x)$ or $E^\nu_{(\cdot)}\mathbf{p}_2(x)$ contains some wave-front travelling with zero speed. Then, using Lemma 5.2, we compute as in (6.1)

$$\begin{aligned}
\big\|\rho^2_{\tau_1,\tau_2}(\gamma_x^\nu)\big\|_{\mathbf{L}^1} &= \sum_{j=1}^m \int_{\theta_{j-1}}^{\theta_j} \sum_{\{\alpha\,:\, t^\theta_\alpha \in [\tau_1,\tau_2]\}} \big|\widetilde{\Delta} u_\nu^\theta(t^\theta_\alpha, x)\big| \left|\frac{\partial t^\theta_\alpha(x)}{\partial\theta}\right| d\theta \\
&\le \sum_{j=1}^m \int_{\theta_{j-1}}^{\theta_j} C_6(1+N2^{-\nu})(1+\log(\tau_2/\tau_1)) \cdot \\
&\quad \cdot \left(\sum_\beta \big|\Delta u_\nu^\theta(0, x^\theta_\beta)\big| \left|\frac{\partial x^\theta_\beta(0)}{\partial\theta}\right| + \sum_{\{\beta'\,:\, t^\theta_{\beta'}<\tau_2\}} \big|\widetilde{\Delta} u_\nu^\theta(t^\theta_{\beta'}, 0)\big| \left|\frac{\partial t^\theta_{\beta'}(0)}{\partial\theta}\right|\right) d\theta \\
&\le C_6(1+N2^{-\nu})(1+\log(\tau_2/\tau_1)) \cdot \|\gamma_0\|_{0,0.\tau_2}\,,
\end{aligned} \tag{6.15}$$

for every $x \in I\!R^+ \setminus \mathcal{N}$. Observe that the $\mathbf{L}^1$ length of the path γ_x^ν satisfies

$$\|\gamma_x^\nu\|_{0,\tau_1,\tau_2} = \big\|\rho^1_{0,+\infty}(\gamma_0^\nu)\big\|_{\mathbf{L}^1} + \big\|\rho^2_{\tau_1,\tau_2}(\gamma_x^\nu)\big\|_{\mathbf{L}^1}\,, \tag{6.16}$$

$$\big\|f\big(E^\nu_{(\cdot)}\mathbf{p}_1(x)\big) - f\big(E^\nu_{(\cdot)}\mathbf{p}_2(x)\big)\big\|_{\mathbf{L}^1([\tau_1,\,\tau_1])} \le C_9 \cdot \|\gamma_x^\nu\|_{0,\tau_1,\tau_2}\,, \tag{6.17}$$

for every $x \in I\!R^+ \setminus \mathcal{N}$, and for some constant $C_9 > 0$. Hence, recalling also (3.19), we deduce from (6.15) the estimate

$$\big\|f\big(E^\nu_{(\cdot)}\mathbf{p}_1(x)\big) - f\big(E^\nu_{(\cdot)}\mathbf{p}_2(x)\big)\big\|_{\mathbf{L}^1([\tau_1,\,\tau_1])} \le C_{10}(1+N2^{-\nu})(1+\log(\tau_2/\tau_1)) \cdot d_{0,0,\tau_2}(\mathbf{fp}_1, \mathbf{fp}_2) \tag{6.18}$$

for some other constant $C_{10} > 0$. Letting $\nu \to \infty$ in (6.18), thanks to (6.13) we obtain (3.23) with $L''' = C_{10}(1+\log(\tau_2/\tau_1))$. By continuity, and relying on the density of the domains $\mathcal{D}^\mu$, $\mu \ge 1$ in $\mathcal{D}$, we then extend the estimate (3.23) to any pair $\mathbf{p}_1$, $\mathbf{p}_2$ in $\mathcal{D}$.

6.4 Oleinik type estimates

Concerning the entropy admissibility conditions (2.15)-(2.16) on the decay of the positive waves, consider a couple of initial data and boundary condition $(\overline{u}, \widetilde{u}) \in \mathcal{D}$, and fix any interval $[a, b]$. Thanks to (3.21)-(3.22), we can now approximate the weak solution constructed as above $u(t,\cdot) = E_t(\overline{u}, \widetilde{u})$ with a sequence of front tracking solutions $u^\nu(t,\cdot) = E^\nu_t(\overline{u}^\nu, \widetilde{u}^\nu)$, choosing initial and boundary data $(\overline{u}^\nu, \widetilde{u}^\nu) \in \mathcal{D}^\nu$ having a number of shocks $N_\nu \le \nu$.

By (4.4)-(4.5), the total number of positive wave-fronts in $u^\nu(\tau,\cdot) = E^\nu_\tau(\overline{u}^\nu, \widetilde{u}^\nu)$ on $[a,b]$ satisfies

$$\text{Pos.Var.}\{w^\nu_k(\tau,\cdot);\ [a,\ b]\} \le C_2 \cdot \frac{b-a}{\tau} + (N_\nu + 1)2^{1-\nu}\,, \tag{6.19}$$

for $k = 1, \dots, n-p$,

$$\text{Pos.Var.}\{w^\nu_k(\tau,\cdot);\ [a,\ b]\} \le C_2 \cdot \left\{\frac{b-a}{\tau} + \log\left(\frac{b}{a}\right)\right\} + (N_\nu + 1)2^{1-\nu}\,, \tag{6.20}$$

for $k = n-p+1, \dots, n$, where $w^\nu_k \doteq w_k(u^\nu)$ denotes as usual the k-th Riemann coordinate of u^ν. Letting $\nu \to \infty$ in (6.19)-(6.20), by the lower semicontinuity of the total variation we obtain (2.15)-(2.16). The estimates (2.17)-(2.18) on the decay of the positive variation of $w_k(\cdot\,,x)$ can be established in entirely similar way relying on the corresponding estimates for $w^\nu_k(\cdot,x)$ which, in turn, are obtained with the same type of arguments used to prove the ones in (5.20)-(5.21).

6.5 Boundary conditions

Let $u(t,x) \doteq E_t\mathbf{p}(x)$, $\mathbf{p} = (\overline{u}, \widetilde{u}) \in D$, be the weak solution defined at (6.12), and consider a sequence $\mathbf{p}^\nu = (\overline{u}^\nu, \widetilde{u}^\nu) \in \mathcal{D}^\nu$ converging to $\mathbf{p}$ in $\mathbf{L}^1$ as $\nu \to \infty$. Call $u^\nu(t,x) \doteq E_t\mathbf{p}^\nu(x)$ the corresponding solution. Since every $\mathbf{p}^\nu$ is piecewise constant and lies in some domain $D_{\mathcal{M}_\nu}$, one can easily verify that any function $u^\nu(t,x)$, $\nu \ge 1$, has bounded total variation and pointwise satisfies the boundary condition (2.20)

$$\lim_{x\to 0+} w_j\big(u^\nu(t,x)\big) = w_j\big(\widetilde{u}^\nu(t)\big) \qquad \text{for a.e.} \quad t \ge 0, \quad j = n-p+1, \dots, n. \tag{6.21}$$

Now, fix $\tau_2 > \tau_1 > 0$. By (3.23) and because of the invertibility property of the flux function f, there will be a countable set $\mathcal{N}$, and some constant $C_{11} = C_{11}(\tau_1,\tau_2) > 0$ (depending only on τ_1, τ_2) such that

$$\big\|w_j\big(u^\nu(\cdot,x)\big) - w_j\big(u(\cdot,x)\big)\big\|_{\mathbf{L}^1([\tau_1,\ \tau_2])} \le C_{11}\cdot d_{0,0,\tau_2}(\mathbf{p}^\nu, \mathbf{p}) \qquad \forall\, x \in \mathbb{R}^+ \setminus \mathcal{N}, \quad \nu \ge 1\,. \tag{6.22}$$

Then, (6.21), (6.22) together imply that, for any $j = n-p+1, \dots, n$, the functions $w_j\big(u(\cdot,x)\big)$, $w_j\big(f(u(\cdot,x))\big)$ have a strong limit as $x \to 0$ and

$$\lim_{x\to 0+} \int_{\tau_1}^{\tau_2} \big|w_j\big(u(t,x)\big) - w_j\big(\widetilde{u}(t)\big)\big|dt = 0\,, \tag{6.23}$$

thus showing that $u(t,x)$ fulfills the boundary condition (2.20). On the other hand, because of the Oleinik type conditions (2.15) on the decay of the positive waves, also $w_j\big(u(\cdot,x)\big)$, $j = 1, \dots, n-p$, have a strong limit as $x \to 0$, which completes the proof of the existence of the strong $\mathbf{L}^1$ trace of $u(t,x)$ at $x = 0$, and hence concludes the proof of Theorem 2.1.

6.6 Uniqueness

The proof of Theorem 2.2 is given in two steps.

STEP 1. Let u be an entropy weak solution to (1.1)-(1.3) on the region on $\Omega_T \doteq [0, T] \times I\!R^+$ according with Definition 2.2, and assume that conditions (i)-(iii) stated in Theorem 2.2 hold. Let $\widehat{\lambda}$ be the upper bound (2.5) for the absolute value of all characteristic speeds, and fix any $R > \widehat{\lambda} \cdot T$. We will show that there exists some constant $C_{12} = C_{12}(R) > 0$ such that, for every $0 < \delta < (R - \widehat{\lambda} \cdot T)/2$, letting $J_{\delta,R}(t) \doteq [2\delta,\, R - \widehat{\lambda} \cdot t]$, one has

$$\int_{J_{\delta,R}(t)} \big|u(t,x) - E_t(\overline{u},\, \widetilde{u}\,)(x)\big| dx \leq C_{12} \cdot (1 + \log(t/\delta)) \cdot \int_0^t \big|u(\sigma,\delta) - E_\sigma(\overline{u},\, \widetilde{u}\,)(\delta)\big| d\sigma \tag{6.24}$$

for any $t \in [\delta, T]$.

Towards this goal, observe first that, because of the entropy conditions (2.15)-(2.16) on the decay of the positive waves, for every fixed $s \in [\delta, T]$ the restrictions of $u(\sigma, \cdot)$, $s \leq \sigma \leq T$, to the interval $J_{\delta,R}(\sigma)$ have uniformly bounded total variation. Thus, the same type of uniqueness results in [12] yield

$$u(t,\cdot) = E_{t-s}\big(u(s,\,\delta + \cdot\,),\; u(s + \cdot\,,\,\delta)\big)(-\delta + \cdot\,) \qquad \text{restricted} \quad \text{to} \quad J_{\delta,R}(t), \tag{6.25}$$

for every $\delta \leq s \leq t \leq T$. Moreover, by the definition of $J_{\delta,R}(t)$, the domain of dependence of a solution to (1.1)-(1.3) along the segment $\{(t,x)\ ;\ x \in J_{\delta,R}(t)\}$ is contained in the set $\{(s,x) \in I\!R^2\ ;\ 0 \leq s \leq t,\ \ 0 \leq x \leq R\}$. Hence, we can restate the Lipschitz estimate (2.23) provided by Theorem 2.1 as

$$\begin{aligned}
&\int_\delta^{R-\widehat{\lambda}\cdot t-\delta} \Big|E_{t-s}\big(u(s,\,\delta+\cdot\,),\; u(s+\cdot\,,\,\delta)\big)(x) - E_{t-s}\big(E_s(\overline{u},\,\widetilde{u}\,)(\delta+\cdot\,),\; E_{(s+\cdot)}(\overline{u},\,\widetilde{u}\,)(\delta)\big)(x)\Big| dx \\
&\qquad \leq C(1+\log(t/\delta)) \cdot \left\{ \int_\delta^R \big|u(s,\,x) - E_s(\overline{u},\,\widetilde{u}\,)(x)\big| dx + \int_s^t \big|u(\sigma,\,\delta) - E_\sigma(\overline{u},\,\widetilde{u}\,)(\delta)\big| d\sigma \right\},
\end{aligned} \tag{6.26}$$

for every $\delta \leq s \leq t \leq T$. Therefore, relying on (6.25)-(6.26), and using the Hölder inequality, we derive

$$\begin{aligned}
&\left(\int_{J_{\delta,R}(t)} \big|u(t,x) - E_t(\overline{u},\,\widetilde{u}\,)(x)\big| dx \right)^2 \leq \\
&\leq C_{13}\big(1+\log(t/\delta)\big)^2 \cdot \left\{ \int_\delta^R \big|u(s,\,x) - E_s(\overline{u},\,\widetilde{u}\,)(x)\big|^2 dx + \left(\int_s^t \big|u(\sigma,\,\delta) - E_\sigma(\overline{u},\,\widetilde{u}\,)(\delta)\big| d\sigma \right)^2 \right\},
\end{aligned} \tag{6.27}$$

for any $\delta \le s \le t \le T$, and for some constant $C_{13} > 0$. Now fix $\tau \in]\delta, t[$, $\delta < t \le T$. Since the left hand side of (6.27) is independent on s, multiplying the right hand side of (6.27) by $1/\tau$, and integrating w.r.t. $s \in [\delta, \tau]$, we obtain

$$\left(\int_{J_{\delta,R}(t)} \big|u(t,x) - E_t(\overline{u},\, \widetilde{u}\,)(x)\big| dx \right)^2 \le C_{13}\big(1+\log(t/\delta)\big)^2 \cdot \Big\{ I_1(\tau) + I_2(\tau) + I_3 \Big\}\,, \tag{6.28}$$

with

$$I_1(\tau) \doteq \frac{1}{\tau} \int_0^\tau \int_0^R \big|u(s,x) - \overline{u}(x)\big|^2 dx ds\,, \tag{6.29}$$

$$I_2(\tau) \doteq \frac{1}{\tau} \int_0^\tau \int_0^R \big|E_s(\overline{u},\, \widetilde{u}\,)(x) - \overline{u}(x)\big|^2 dx ds\,, \tag{6.30}$$

$$I_3 \doteq \left(\int_0^t \big|u(\sigma,\delta) - E_\sigma(\overline{u},\, \widetilde{u}\,)(\delta)\big| d\sigma \right)^2. \tag{6.31}$$

Observing that the Lebesgue limit (2.29) together with the continuity of $t \mapsto E_t(\overline{u},\, \widetilde{u}\,)(\cdot)$ at $t = 0$ (cfr. Remark 2.3) imply

$$\lim_{\tau \to 0+} I_1(\tau) = \lim_{\tau \to 0+} I_2(\tau) = 0\,, \tag{6.32}$$

from (6.28), letting $\tau \to 0+$, we recover (6.24).

Step 2. We will show now that there exists some constant $C_{14} > 0$ such that, for every $0 < \delta < (R - \widehat{\lambda} \cdot T)/2$, and for any $t \in [\delta, T]$, one has

$$\int_0^t \big|u(\sigma,\delta) - E_\sigma(\overline{u},\, \widetilde{u}\,)(\delta)\big| d\sigma \le C_{14} \cdot \delta\big(1 + \log(t/\delta)\big)\,. \tag{6.33}$$

To this end, observe as in the previous step that, because of the entropy conditions (2.17)-(2.18) on the decay of the positive waves, for every fixed $y \in]0, \delta]$ the restrictions of $u(\cdot, z)$, $y \le z \le \delta$, to the interval $[\delta, T]$ have uniformly bounded total variation. Thus, the same type of uniqueness results in [12] yield

$$u(\delta + \cdot,\, \delta) = E_{(\cdot)}\big(u(\delta,\, y + \cdot),\; u(\delta + \cdot\,,\, y)\big)(\delta - y) \qquad \text{restricted to} \quad [\delta,\, T - \delta], \tag{6.34}$$

for every $0 < y \le \delta$. Moreover, since the choice of $\delta < (R - \widehat{\lambda} \cdot T)/2$ implies

$$\delta + \widehat{\lambda} \cdot (t - s) \le R - \widehat{\lambda} \cdot s \qquad \forall\ 0 \le s \le t \le T,$$

it follows that, for any fixed $0 < t \le T$, the domain of dependence of a solution to (1.1)-(1.3) along the vertical segment $\{(s,\delta)\ ;\ 2\delta \leq s \leq t\}$ is contained in the set $\{(s,x) \in I\!R^2\ ;\ 0 \leq s \leq t,\ \ 0 \leq x \leq R - \widehat{\lambda} \cdot \delta\}$. Hence, recalling Remark 2.4, we can restate the Lipschitz estimate (3.23) (cfr. also (6.18)) in the form

$$\begin{aligned}
&\int_{\delta}^{t-\delta} \big|E_\sigma\big(u(\delta,\, y+\cdot\,),\ u(\delta+\cdot\,,\, y)\big)(\delta-y) - \\
&\qquad\qquad -E_\sigma\big(E_\delta(\overline{u},\, \widetilde{u}\,)(y+\cdot\,),\ E_{(\delta+\cdot\,)}(\overline{u},\, \widetilde{u}\,)(y)\big)(\delta-y)\big|d\sigma \ \leq \\
&\leq C_{10}(1+\log(t/\delta)) \cdot \bigg\{ \int_y^{R-\widehat{\lambda}\cdot\delta} \big|u(\delta,\, x) - E_\delta(\overline{u},\, \widetilde{u}\,)(x)\big|dx + \\
&\qquad\qquad + \sum_{j=n-p+1}^{n} \int_\delta^t \big|w_j\big(u(\sigma,\, y)\big) - w_j\big(E_\sigma(\overline{u},\, \widetilde{u}\,)(y)\big)\big|d\sigma \bigg\},
\end{aligned} \tag{6.35}$$

for a.e. $0 < y \leq \delta$, and any $\delta \leq t \leq T$. Thus, relying on (6.34)-(6.35), and using again the Hölder inequality, we derive

$$\begin{aligned}
&\bigg(\int_{2\delta}^{t} \big|u(\sigma,\delta) - E_\sigma(\overline{u},\, \widetilde{u}\,)(\delta)\big|d\sigma\bigg)^2 \leq \\
&\quad \leq C_{15}\big(1+\log(t/\delta)\big)^2 \cdot \bigg\{ \bigg(\int_0^{R-\widehat{\lambda}\cdot\delta} \big|u(\delta,\, x) - E_\delta(\overline{u},\, \widetilde{u}\,)(x)\big|dx\bigg)^2 + \\
&\qquad\qquad + \sum_{j=n-p+1}^{n} \int_\delta^t \big|w_j\big(u(\sigma,\, y)\big) - w_j\big(E_\sigma(\overline{u},\, \widetilde{u}\,)(y)\big)\big|^2 d\sigma \bigg\},
\end{aligned} \tag{6.36}$$

for a.e. $0 < y \leq \delta,\ \ t \in [2\delta, T]$, and for some constant $C_{15} > 0$. Now fix $\xi \in]0,\, \delta[$. Since the left hand side of (6.36) is independent on y, multiplying the right hand side of (6.36) by $1/\xi$, and integrating w.r.t. $y \in [0,\, \xi]$, we obtain

$$\bigg(\int_{2\delta}^{t} \big|u(\sigma,\delta) - E_\sigma(\overline{u},\, \widetilde{u}\,)(\delta)\big|d\sigma\bigg)^2 \leq C_{15}\big(1+\log(t/\delta)\big)^2 \cdot \Big\{ I_4 + \sum_{j=n-p+1}^{n} \big(I_{5,j}(\xi) + I_{6,j}(\xi)\big) \Big\}, \tag{6.37}$$

with

$$I_4 \doteq \bigg(\int_0^{R-\widehat{\lambda}\cdot\delta} \big|u(\delta,x) - E_\delta(\overline{u},\, \widetilde{u}\,)(x)\big|dx\bigg)^2 \tag{6.38}$$

$$I_{5,j}(\xi) \doteq \frac{1}{\xi}\int_0^\xi \int_0^t \big|w_j\big(u(\sigma,y)\big) - w_j\big(\widetilde{u}(\sigma)\big)\big|^2 d\sigma dy\,, \tag{6.39}$$

$$I_{6,j}(\xi) \doteq \frac{1}{\xi}\int_0^\xi \int_0^t \big|w_j\big(E_\sigma(\overline{u},\, \widetilde{u}\,)(y) - w_j\big(\widetilde{u}(\sigma)\big)\big|^2 d\sigma dy\,. \tag{6.40}$$

Observe that, by Remark 2.2, the Lebesgue limit (2.30) together with the existence of a strong $\mathbf{L}^1$ trace (2.24) of $E_t(\overline{u},\widetilde{u})(x)$ at $x=0$, imply

$$\lim_{\xi\to 0+} I_{5,j}(\xi) = \lim_{\xi\to 0+} I_{6,j}(\xi) = 0 \qquad j = n-p+1,\dots,n\,. \tag{6.41}$$

Hence, letting $\xi \to 0+$ in (6.37), we obtain

$$\int_{2\delta}^{t} \big|u(\sigma,\delta) - E_\sigma(\overline{u},\widetilde{u})(\delta)\big|d\sigma \le C_{15}\big(1+\log(t/\delta)\big)\cdot \int_0^{R-\widehat{\lambda}\cdot\delta} \big|u(\delta,x) - E_\delta(\overline{u},\widetilde{u})(x)\big|d\sigma \tag{6.42}$$

for any $t\in[2\delta,T]$. Then, if we set $M \doteq \sup_{u\in K}|u|$, using (6.42) and the estimate (6.24) with $t=\delta$, we derive

$$\begin{aligned}\int_0^t \big|u(\sigma,\delta) - E_\sigma(\overline{u},\widetilde{u})(\delta)\big|d\sigma &\le \\ &\le 4M\delta + C_{15}\big(1+\log(t/\delta)\big)\cdot\left\{4M\delta + \int_{2\delta}^{R-\widehat{\lambda}\cdot\delta}\big|u(\delta,x) - E_\delta(\overline{u},\widetilde{u})(x)\big|d\sigma\right\}\\ &\le 4M\delta + C_{15}\big(1+\log(t/\delta)\big)\cdot\big\{4M\delta + 2C_{12}(1+\log 2)M\delta\big\}\,,\end{aligned} \tag{6.43}$$

which yields (6.33) for every $t\in[\delta,T]$. Hence, observing that (6.24), (6.33) together imply

$$\int_{J_{\delta,R}(t)} \big|u(t,x) - E_t(\overline{u},\widetilde{u})(x)\big|dx \le C_{12}\,C_{14}\cdot\delta(1+\log(t/\delta))^2\,, \tag{6.44}$$

and then letting $\delta\to 0+$ in (6.44), we obtain

$$\int_0^{R-\widehat{\lambda}\cdot t} \big|u(t,x) - E_t(\overline{u},\widetilde{u})(x)\big|dx = 0 \qquad \forall\, t\in[0,T]\,. \tag{6.45}$$

Since $R > \widehat{\lambda}\cdot T$ was arbitrary, this concludes the proof of Theorem 2.2.

Proof of Lemma 2.1. A derivation of condition (2.29) under the assumption that the system (1.1) is endowed with an entropy-entropy flux pair (η,q) having the properties a), c), can be found in [14, Theorem 4.3.1]. Hence, we will show here only how to obtain condition (2.30) whenever the entropy-entropy flux pair (η,q) enjoys the properties b), c).

Fix $\tau>0$ and $0<\varepsilon<\min\{1,\tau\}$. Let $\delta,\gamma:\mathbb{R}\mapsto[0,1]$, be $\mathcal{C}^\infty$ functions such that

$$\delta(z) = 0 \quad \text{if} \quad z\notin[-1,1], \qquad \int_{-\infty}^0 \delta(z)\,dz = \int_0^\infty \delta(z)\,dz = 1, \tag{6.46}$$

$$\gamma(z) = \begin{cases} 1 & \text{if} \quad z \in [0,\, 2+\tau], \\ 0 & \text{if} \quad z \notin [-1,\, 3+\tau], \end{cases} \tag{6.47}$$

and define

$$\delta_\varepsilon(z) = \frac{1}{\varepsilon}\cdot\delta\left(\frac{z}{\varepsilon}\right), \qquad \alpha_\varepsilon(z) = \frac{\gamma(z)}{2}\cdot\int_z^\infty \delta_\varepsilon(s-\tau)\; ds, \qquad \beta_\varepsilon(z) = \gamma(z)\cdot\int_z^\infty \delta_\varepsilon(s)\; ds\,. \tag{6.48}$$

Observe that, letting $\phi^\varepsilon(t,x) \doteq \alpha_\varepsilon(t)\cdot\beta_\varepsilon(x)$, one has $\phi^\varepsilon(t,0) = \alpha_\varepsilon(t) = 1$, for any $t \in [0,\, \tau-\varepsilon]$, and

$$\phi^\varepsilon_t(t,x) = -\delta_\varepsilon(t-\tau)\cdot\beta_\varepsilon(x), \qquad \phi^\varepsilon_x(t,x) = -\delta_\varepsilon(x)\cdot\alpha_\varepsilon(t) \qquad \forall\; t,\, x \geq 0\,. \tag{6.49}$$

Then, writing (2.32) for the test function ϕ^ε, we find

$$-\int_0^\varepsilon \delta_\varepsilon(x)\cdot\left(\int_0^T q\big(u(t,x)\big)\; dt\right)\, dx + \int_0^T q\circ f^{-1}\big(\Psi(t)\big)\; dt \geq C_{16}\cdot\varepsilon\,, \tag{6.50}$$

where $C_{16} > 0$ denotes some constant independent on ε. Therefore, setting $\xi(z) \doteq q\circ f^{-1}(z)$, and combining (6.50) with (2.13), we obtain

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon\int_0^T \Big\{\xi\big(f(u(t,x))\big) - \xi\big(\Psi(t)\big) - D\xi\big(\Psi(t)\big)\big[f(u(t,x)) - \Psi(t)\big]\Big\}\; dtdx \leq 0\,. \tag{6.51}$$

Since, by assumption, the Hessian $D^2\xi(z)$ is positive definite, uniformly on compact sets, and the essential range of the weak trace Ψ is contained in a compact subset of $f(U)$, from (6.51) we derive

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon\int_0^T \big|f\big(u(t,x)\big) - \Psi(t)\big|^2 dtdx = 0\,, \tag{6.52}$$

which, in turn, by the regularity properties of the flux f, yields (2.30). □

Proof of Lemma 2.2. Recalling that Temple class systems are endowed with convex entropies constructed in terms of the Riemann invariants, following [25, Chapter 13] we consider, for every j-th Riemann invariant, a smooth function $(\sigma,u) \mapsto e^j(\sigma,u)$, with the properties:

i) for every $\sigma \in \mathbb{R}$, $u \mapsto e^j(\sigma,u)$ is an affine map such that

$$\Pi_j^\sigma \doteq \big\{u \in U\;;\; w_j(u) = \sigma\big\} = \big\{u \in U\;;\; e^j(\sigma,u) = 0\big\}\,;$$

ii) $w_j(u) \geq \sigma$ if and only if $e^j(\sigma,u) \geq 0$;

iii) $h^j(u) \doteq -\dfrac{\partial}{\partial\sigma}\, e^j(w_j(u),u) > 0$ for any $u \in U$.

Then, choosing n continuous, integrable maps $P^j(\sigma)$, $j = 1, \dots, n$, with positive real values, we define the entropy

$$\eta(u) \doteq \sum_{j=1}^{n} \int_{-\infty}^{w_j(u)} e^j(\sigma, u)\, P^j(\sigma)\; d\sigma\,, \tag{6.53}$$

with associated entropy flux

$$q(u) \doteq \sum_{j=1}^{n} \int_{-\infty}^{w_j(u)} e^j(\sigma, f(u))\, P^j(\sigma)\; d\sigma\,. \tag{6.54}$$

A straight computation for the Hessians of η and $\xi \doteq q \circ f^{-1}$ yields

$$D^2\eta(u)\big[h,\, h\big] = \sum_{j=1}^{n} h^j(u)\, P^j(w_j(u)) \big(Dw_j(u) \cdot h\big)^2, \tag{6.55}$$

$$D^2\xi(f(u))\big[h,\, h\big] = \sum_{j=1}^{n} h^j(u)\, P^j\big(w_j(u)\big) \Big(Dw_j(u) \cdot (Df(u))^{-1} \cdot h\Big)^2. \tag{6.56}$$

Observing that, by the properties of the Riemann invariants, the vectors $Dw_1(u), \dots$ $\dots, Dw_n(u)$, are linearly independent, and since by construction $P^j(\sigma) > 0$, $h^j(u) > 0$, from (6.55)-(6.56) we deduce that both the Hessians $D^2\eta(u)$, $D^2\xi(z)$, are positive definite for any $u \in U$, $z \in f(U)$. □

7 Proof of Theorem 2.3

We give here only the proof of the compactness of the attainable sets $\mathcal{A}(T, \mathcal{U})$, $T > 0$, in connection with the sets of admissible boundary controls $\mathcal{U}$ defined in (2.34), the procedure to establish the compactness of $\mathcal{A}(\,\overline{x}, \mathcal{U})$, $\overline{x} > 0$, being entirely similar.

Fix $T > 0$. Given $\overline{u} \in \mathbf{L}^1(I\!R^+, K)$, consider a sequence of boundary data $\{\widetilde{u}^\nu\}_{\nu \geq 0} \subset \mathcal{U}$, and let $u^\nu(t, x) \doteq E_t(\overline{u},\, \widetilde{u}^\nu\,)(x)$ be the corresponding solutions. Observe that all solutions $u^\nu(t, x)$, $\nu \geq 0$, are uniformly bounded since they take values in the compact set K. Moreover, thanks to the Oleinik type estimates (2.15)-(2.16) on the time-decay of the positive waves, for every fixed $0 < a < b$, and $0 < \tau \leq T$, there exist constants $C' = C'(a, b, \tau) > 0$, $C'' = C''(a, b, \tau) > 0$, such that

$$\text{Tot.Var.}\big\{u^\nu(t, \cdot)\;;\; [a,\, b]\big\} \leq C' \qquad \forall\, t \in [\tau,\, T], \qquad \forall\, \nu \geq 0\,, \tag{7.1}$$

$$\int_a^b |u^\nu(t, x) - u^\nu(s, x)|\; dx \leq C''|t - s| \qquad \forall\, t, s \in [\tau,\, T], \qquad \forall\, \nu \geq 0\,. \tag{7.2}$$

Hence, applying Helly's Theorem, we deduce that there exists a subsequence $\{u^{\nu_j}\}_{j \geq 0}$ so that $\{u^{\nu_j}(t, \cdot)\restriction_{[a,\, b]}\}_{j \geq 0}$ converges in $\mathbf{L}^1([a, b],\, K)$ to some function $u_{a,b,\tau}(t, \cdot)$, for any $t \in [\tau, T]$.

Therefore, by considering sequences of real number $a_k \to 0+$, $b_k \to \infty$, $\tau_k \to 0+$, and by using a diagonal procedure, we construct a subsequence $\{u^{\nu'}(t,\cdot)\}_{\nu'\geq 0}$ that converges in $\mathbf{L}^1{}_{loc}(I\!R^+, K)$ to some function $u(t,\cdot)$, for any $t \in [0,T]$. We claim that there exists a boundary data $\widetilde{u} \in \mathcal{U}$ such that

$$u(t,\cdot) = E_t(\overline{u}, \widetilde{u}) \qquad \forall\, t \in [0,T]\,. \tag{7.3}$$

Notice that, by construction, the map $(t,x) \to \big(u(t,\cdot),\, u(\cdot,x)\big)$ takes values within the domain $\mathcal{D}_T$ defined in (2.28). Moreover, the estimate (7.2) implies the continuity of $u : [0,T] \times I\!R^+ \mapsto U$ as a function from $]0,T]$ into $\mathbf{L}^1{}_{loc}(I\!R^+)$. Hence, thanks to Theorem 2.2 and Lemma 2.1, Lemma 2.2, in order to prove the claim it will be sufficient to show that :

1. There exists a boundary data $\widetilde{u} \in \mathcal{U}$ so that $u(t,x)$ is an entropy weak solution to (1.1)-(1.3) on the region $[0,T] \times I\!R^+$, in the sense of Definition 2.2, and the essential range of the weak trace of $f(u)$ at $x = 0$ is contained in a compact subset of $f(U)$.
2. Given an entropy-entropy flux pair (η, q) with the properties a)-b) stated in Lemma 2.1, u satisfies the corresponding distributional entropy inequality (2.32).

Towards a proof of **1.** observe that, because of (2.24), all fluxes $f(u^\nu)$, $\nu \geq 0$, admit a strong $\mathbf{L}^1$ trace Ψ^ν at $x = 0$, whose essential range is contained in the compact set $f(K)$. Hence, the sequence $\{\Psi^\nu\}_{\nu\geq 0}$ is weak* relatively compact in $\mathbf{L}^\infty(I\!R^+)$ and, by possibly taking a subsequence, we have

$$\Psi^\nu \overset{*}{\rightharpoonup} \Psi \qquad \text{in} \quad \mathbf{L}^\infty(I\!R^+), \tag{7.4}$$

for some function $\Psi \in \mathbf{L}^\infty(I\!R^+, I\!R^n)$. Moreover, by Theorem 2.1, every u^ν is a distributional solution of the corresponding initial-boundary value problem on $[0,T] \times I\!R^+$, i.e. there holds

$$\int_0^T \int_0^{+\infty} \Big\{ u^\nu(t,x) \cdot \phi_t(t,x) + f(u^\nu(t,x)) \cdot \phi_x(t,x) \Big\}\, dx\; dt\; + \\ + \int_0^{+\infty} \overline{u}(x) \cdot \phi(0,x)\; dx + \int_0^T \Psi^\nu(t) \cdot \phi(t,0)\; dt = 0\,. \tag{7.5}$$

for any test function $\phi \in \mathcal{C}^1_c$ with compact support contained in the set $]-\infty, T\,[\times I\!R$. Therefore, passing to the limit as $\nu \to \infty$ in (7.5), we get

$$\int_0^T \int_0^{+\infty} \Big\{ u(t,x) \cdot \phi_t(t,x) + f(u(t,x)) \cdot \phi_x(t,x) \Big\}\, dx\; dt\; + \\ + \int_0^{+\infty} \overline{u}(x) \cdot \phi(0,x)\; dx + \int_0^T \Psi(t) \cdot \phi(t,0)\; dt = 0\,. \tag{7.6}$$

By considering, in particular, test functions $\phi \in \mathcal{C}_c^1$ with compact support contained in the set $]-\infty, T\,[\times]0, \infty[$, from (7.6) we deduce that $u(t,x)$ is a distributional solution of the Cauchy problem (1.1)-(1.2) on the region $[0,T] \times I\!R^+$, as required by Definition 2.2-(i). On the other hand, given any $\mathcal{C}^1$ function $\alpha = \alpha(t)$, writing (7.6) for test functions $\phi^\varepsilon(t,x) = \alpha(t) \cdot \beta^\varepsilon(t,x)$, $\beta^\varepsilon \in \mathcal{C}_c^1$, supported on the semi-strips $]0,T[\times]-\infty, \varepsilon[$, $\varepsilon > 0$, shrinking to the region $]0,T[\times]-\infty, 0]$ around the boundary $]0,T[\times\{0\}$, and such that $\phi^\varepsilon(t,0) = \alpha(t)$, we obtain

$$\lim_{x \to 0^+} \int_0^T f(u(t,x)) \cdot \alpha(t)\ dt = \int_0^T \Psi(t) \cdot \alpha(t)\ dt\,, \tag{7.7}$$

thus proving (2.13).

Observe now that, by Remark 2.2, the definition (2.34) of the set $\mathcal{U}$ of admissible boundary data implies

$$w_j\big(f^{-1}(\Psi^\nu(t))\big) \in [c_j,\, d_j] \qquad \text{for a.e.} \quad t \geq 0, \qquad j = n-p+1, \dots, n. \tag{7.8}$$

Hence, for every flux trace Ψ^ν, $\nu \geq 0$, one has

$$\Psi^\nu(t) \in \mathcal{G} \doteq \Big\{ f(u)\ ;\ w_j(u) \in [c_j,\, d_j] \qquad \forall\, j = n-p+1, \dots, n \Big\} \qquad \text{for a.e.} \quad t \geq 0. \tag{7.9}$$

Since, by the properties of the Riemann invariants, the set $\mathcal{G}$ is closed and convex, it follows that the weak limit of Ψ^ν satisfies

$$\Psi(t) \in \mathcal{G} \qquad \text{for a.e.} \quad t \geq 0. \tag{7.10}$$

Therefore, if we consider the boundary data $\widetilde{u}$ defined in Riemann coordinates by

$$w_j(\widetilde{u}(t)) \doteq \begin{cases} \overline{\gamma}_j & \text{if} \quad j \leq n-p, \\ \\ w_j\big(f^{-1}(\Psi(t))\big) & \text{if} \quad j > n-p, \end{cases} \qquad \forall\, t \geq 0\,, \tag{7.11}$$

for some constant values

$$\overline{\gamma}_j \in \begin{cases} [c_j,\, d_j] \ \text{if} \quad j \in J, \\ \\ [a_j,\, b_j] \ \text{otherwise}, \end{cases} \qquad j \leq n-p\,, \tag{7.12}$$

(J denoting the set of indices in the definition (2.34) of $\mathcal{U}$, and a_j, b_j, being the constants in the definition (2.3) of the set K) we clearly have $\widetilde{u} \in \mathcal{U}$, and, by Remark 2.2, $u(t,x)$ satisfies the boundary condition (2.14) of Definition 2.2-(ii). To complete the proof of **1.** observe that the Oleinik type conditions of Definition 2.2-(iii) can be recovered by the lower semicontinuity of the total variation, since the map $u(t,x)$ is obtained as the $\mathbf{L}^1{}_{loc}$ limit of a sequence of maps satisfing (2.15)-(2.18).

Finally, regarding **2.** observe that every solution $t \mapsto u^\nu(t,\cdot)$, being a trajectory of the flow map E_t, is obtained as the limit of front-tracking approximations, and hence, by standard arguments satisfies the entropy inequality associated to any entropy-entropy flux pair. Clearly, this property is preserved by the $\mathbf{L}^1{}_{loc}$ limit u of the sequence u^ν. This concludes the proof of Theorem 2.3.

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