Global $\mathbf{BV}$ entropy solutions and uniqueness for hyperbolic systems of balance laws [*]

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Abstract

We consider the Cauchy problem for $n \times n$ strictly hyperbolic systems of balance laws

$$\begin{cases} u_t + f(u)_x = g(x,u), \qquad x \in \mathbb{R}, t > 0 \\ u(0,.) = u_o \in \mathbf{L}^1 \cap \mathbf{BV}(\mathbb{R};\mathbb{R}^n), \\ |\lambda_i(u)| \geq c > 0 \text{ for all } i \in \{1,\dots,n\}, \\ \|g(.,u)\| + \|\nabla_u g(.,u)\| \leq \omega \in \mathbf{L}^1 \cap \mathbf{L}^\infty(\mathbb{R}), \end{cases}$$

each characteristic field being genuinely nonlinear or linearly degenerate. Assuming that $\|\omega\|_{\mathbf{L}^1(\mathbb{R})}$ and $\|u_o\|_{\mathbf{BV}(\mathbb{R})}$ are small enough, we prove the existence and uniqueness of global entropy solutions of bounded total variation as limits of special wave-front tracking approximations. Moreover, we give a characterization of the semigroup trajectories in terms of integral estimates.

Contents

[*]Work partially supported by TMR project HCL #ERBFMRXCT960033

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1 Introduction

We are interested in proposing an attractive unified approach to the existence and uniqueness theory for quasilinear strictly hyperbolic systems of balance laws. More precisely, we shall consider throughout this work the Cauchy problem in the large for the following $n \times n$ system of equations

$$u_t + f(u)_x = g(x,u), \qquad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

endowed with a (suitably small) initial data belonging to $\mathbf{L}^1 \cap \mathbf{BV}(\mathbb{R};\mathbb{R}^n)$, the space of integrable functions with bounded total variation (Tot.Var.) in the sense of [40] :

$$u(0,x) = u_o(x), \qquad x \in \mathbb{R}. \tag{1.2}$$

Here $u(t,x) \in \mathbb{R}^n$ is the unknowns vector, $f : \Omega \to \mathbb{R}^n$ denotes the fluxes, *i.e.* a smooth $\mathbf{C^2}$ function defined on Ω which is an open neighborhood of the origin in $\mathbb{R}^n$. The system (1.1) is supposed to be strictly hyperbolic, with each characteristic field either genuinely nonlinear or linearly degenerate in the sense of Lax, [32]. Concerning the source term g, we assume that it satisfies the following Caratheodory–type conditions:

(P_1) $g : \mathbb{R} \times \Omega \to \mathbb{R}^n$ is measurable with respect to (w.r.t.) x, for any $u \in \Omega$, and is $\mathbf{C^2}$ w.r.t. u, for any $x \in \mathbb{R}$;

(P_2) $\left\|g(x,\cdot)\right\|_{\mathbf{C^2}}$ is bounded over Ω, uniformly in $x \in \mathbb{R}$;

(P_3) there exists a function $\omega \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R})$ such that $|g(x,u)| \le \omega(x)$, $\left\|\nabla_u g(x,u)\right\| \le \omega(x)$ for all $(x,u) \in \mathbb{R} \times \Omega$.

Moreover, we assume that a *non-resonance* condition holds, that is, the characteristic speeds of the system (1.1) are bounded away from zero:

$$|\lambda_i(u)| \ge c > 0, \qquad \forall\, u \in \Omega,\ i \in \{1,\ldots,n\}. \tag{1.3}$$

An overview of the dramatic consequences of nonlinear resonance in the context of balance laws is to be found in *e.g.* [9, 27, 28, 34] and references therein. The main result of this paper is the following.

Theorem 1.1 *Assume (P_1)–(P_3) and (1.3). If the $\mathbf{L}^1$ norm of ω is sufficiently small, there exist a constant $L > 0$, a closed domain $\mathcal{D}$ of integrable functions with small total variation and a unique semigroup $P : [0,+\infty) \times \mathcal{D} \to \mathcal{D}$ satisfying*

i) $P_0 u = u$, $\quad P_{t+s}u = P_t P_s u$ *for all* $u, v \in \mathcal{D}$ *and* $t, s \ge 0$;

ii) $\|P_s u - P_t v\|_{\mathbf{L}^1(\mathbb{R})} \le L\Big(|s-t| + \|u-v\|_{\mathbf{L}^1(\mathbb{R})}\Big)$ *for all* $u, v \in \mathcal{D}$ *and* $t, s \ge 0$;

iii) for all $u_o \in \mathcal{D}$ the function $u(t,\cdot) = P_t u_o$ is a weak entropy solution of the Cauchy problem (1.1)–(1.2) and satisfies the integral estimates (5.19), (5.20).

Viceversa let $u : [0,T] \to \mathcal{D}$ be Lipschitz continuous as a map with values in $\mathbf{L}^1(\mathbb{R}, \mathbb{R}^n)$ and assume that $u(t,x)$ satisfies the integral conditions (5.19), (5.20). Then $u(t,\cdot)$ coincides with a trajectory of the semigroup P.

Our proofs rely on an extensive treatment of the sources as particular *nonconservative products*, [17, 33] (see also [11] for studies in a weaker functional framework). This formulation has been already used in several satisfactory numerical approaches, see [23, 24, 25] and justified rigorously in the scalar case, [22]. The present work can therefore be also seen as a further justification in the more difficult context of systems of balance laws.

Unfortunately, at the time being, only a few theoretical tools are available for the analysis of inhomogeneous systems. The oldest ones consist in some variations around the Glimm scheme, [20], but 2×2 systems can be also handled by means of Compensated Compactness, [39]. Concerning this last approach, we refer to *e.g.* [5, 9, 18, 29]. Since we aim to establish existence and continuous dependence of entropy solutions with respect to their initial data, we turn to the recent approach by Bressan and collaborators, see [7] and references therein, which relies on a wave-front tracking construction, [4, 13, 19, 37] whose **BV** norm is controlled by means of a Glimm functional. This technique has been mainly developed in the context of systems of homogeneous conservation laws, but some recent extensions to non-homogeneous problems are proposed in [3, 1, 2, 12] (see also [31]).

In these works, one handles the effects of the right-hand-side by means of a classical time-splitting technique; this is also of use in former references [10, 15, 16, 36] involving modified versions of the Glimm scheme. In this approach, one is led to assume that the inhomogeneous part dissipates strongly and deals therefore with *sink terms*. This does not apply to several physical models, including the case of geometrical source terms, see [9, 21, 34, 35].

An alternative has been proposed in [21, 35] for possibly accretive sources using another generalization of the Glimm scheme. Unfortunately, the proofs are intricate as they make use of a wave partitioning technique; our approach is simpler as we can get rid of it. The main idea is to localize the source term on certain lines in the x,t plane by means of a Dirac comb. Thus, it induces only stationary discontinuities (the "standing waves" in [28]) and most of the tools designed for homogeneous problems are still available for our analysis.

This paper is organized as follows. In Section 2 the h–Riemann solver is introduced and some of its properties are established. It will used as a building block for the construction of our approximate solutions to the nonhomogeneous problem. In Section 3 the basic existence theorems are proved: we recover as a particular case the existence result by T. P. Liu, [35]. We show also that our approximate solutions solve a family of nonconservative systems of balance laws. In Section 4 we construct a uniformly Lipschitz semigroup P whose trajectories are weak entropic solutions to the Cauchy problem (1.1)–(1.3) (see also [26]). Finally, in Section 5, we prove the uniqueness of the semigroup P through a characterization of its trajectories by means of integral estimates.

2 A self-similar non homogeneous Riemann solver

2.1 The steady-state equations

Consider the stationary equations associated to (1.1), namely the system of ordinary differential equations:

$$f(v(x))_x = g(x, v(x)). \tag{2.1}$$

The system (2.1) will play a central role in our construction. For any $x_o \in \mathbb{R}$, $v \in \Omega$, consider the initial data

$$v(x_o) = v. \tag{2.2}$$

For convenience, we introduce a suitable approximation of the solutions to (2.1), (2.2). Thanks to (1.3), the map $u \mapsto f(u)$ is invertible inside some neighborhood of the origin; in this neighborhood, for small $h > 0$, we can define

$$\Phi_h(x_o, u) \doteq f^{-1}\left[f(u) + \int_0^h g\,(x_o + s, u)\; ds\right]. \tag{2.3}$$

This map gives an approximation of the flow of (2.1) in the sense that

$$f\left(\Phi_h(x_o, u)\right) - f(u) = \int_0^h g\,(x_o + s, u)\; ds. \tag{2.4}$$

It is very easy to see that for fixed h, x_o the $\mathbf{C^2}$ norm of $u \mapsto \Phi_h(x_o, u)$ is bounded by a constant independent of h or x_o. Moreover the two maps $x_o \mapsto \Phi_h(x_o, u)$, $h \mapsto \Phi_h(x_o, u)$ are Lipschitz continuous. Throughout the paper we will use the Landau notation $\mathcal{O}(1)$ to indicate any function whose absolute value remains uniformly bounded, the bound depending only on f and on the uniform bound on $\left\|g(x, \cdot)\right\|_{\mathbf{C^2}}$.

The following lemma states some properties of $\Phi_h(x_o, u)$.

Lemma 2.1 *Let $x_o \in \mathbb{R}$ and u, u_1, u_2 be three states in a suitable neighborhood of the origin. For h suitably small, there holds*

$$\left|\Phi_h(x_o, u) - u\right| \;=\; \mathcal{O}(1) \int_0^h \omega(x_o + s)\; ds, \tag{2.5}$$

$$\left|\Phi_h(x_o, u_2) - \Phi_h(x_o, u_1) - (u_2 - u_1)\right| = \mathcal{O}(1)|u_2 - u_1| \int_0^h \omega(x_o + s) ds. \tag{2.6}$$

Proof. Let us introduce the function

$$b(y, u) \doteq f^{-1}\left[f(u) + y\right], \tag{2.7}$$

where $y, u \in \mathbb{R}^n$, y, u in a neighborhood of the origin. Observe that b is a $\mathbf{C^2}$ function, and that $b(0, u) = u$ for all $u \in \Omega$. With these notations, there holds

$$\Phi_h(x_o, u) \;=\; b\left(\int_0^h g\,(x_o + s, u)\; ds\;, u\right).$$

Let us start with (2.5). By the Lipschitz properties of b, one has

$$\begin{aligned}
\left|\Phi_h(x_o, u) - u\right| \;&=\; \left|b\left(\int_0^h g\,(x_o + s, u)\, ds\;, u\right) - b\,(0, u)\right| \\
&=\; \mathcal{O}(1)\left|\int_0^h g\,(x_o + s, u)\, ds\right| = \mathcal{O}(1)\int_0^h \omega(x_o + s)\; ds.
\end{aligned}$$

To prove (2.6), we first show that there holds

$$\left|b(y'', u_1 + y) - b(y', u_1) - y\right| = \mathcal{O}(1)\left(|y''||y| + |y'' - y'|\right) \tag{2.8}$$

for all u_1, y, y', y'' in a neighborhood of the origin. Indeed, by the triangle inequality,

$$\left|b(y'', u_1 + y) - b(y', u_1) - y\right| \leq \left|b(y'', u_1 + y) - b(y'', u_1) - y\right| + \left|b(y'', u_1) - b(y', u_1)\right|$$

Using again the Lipschitz continuity of b, the last term is bounded by $\mathcal{O}(1)|y''-y'|$. Concerning the first term, for convenience, we set

$$G(y'',y) \doteq b(y'',u_1+y) - b(y'',u_1) - y.$$

The function G is of class $\mathbf{C^2}$ and it satisfies $G(y'',0) \equiv G(0,y) \equiv 0$. Hence by standard arguments, [41], we deduce $G(y'',y) = \mathcal{O}(1)|y''||y|$. In conclusion, (2.8) holds true.

Finally, making the following substitutions in (2.8)

$$y = u_2 - u_1, \qquad y' = \int_0^h g\left(x_o+s,u_1\right)ds, \qquad y'' = \int_0^h g\left(x_o+s,u_2\right)ds,$$

one gets the desired inequality (2.6):

$$\begin{aligned}
&\left|\Phi_h(x_o,u_2) - \Phi_h(x_o,u_1) - u_2 + u_1\right| = \\
&\quad = \mathcal{O}(1)\left[|u_2-u_1|\int_0^h \omega(x_o+s)ds + \int_0^h \sup_{v\in\Omega}\left\|\nabla g(x_o+s,v)\right\|\cdot|u_2-u_1|ds\right] \\
&\quad = \mathcal{O}(1)|u_2-u_1|\int_0^h \omega(x_o+s)ds.
\end{aligned}$$

□

2.2 Introducing the h–Riemann solver

The next step is to introduce an approximate Riemann solver. Therefore, for any $x_o \in \mathbb{R}$ we consider the system (1.1), endowed with a Riemann initial datum:

$$u(0,x) = \begin{cases} u_\ell & \text{if } x < x_o \\ u_r & \text{if } x > x_o. \end{cases} \tag{2.9}$$

To this end, let us propose some notations. If the two states u_ℓ, u_r are sufficiently close, let Ψ be the unique entropic homogeneous Riemann solver which is the map such that

$$u_r = \Psi(\boldsymbol{\sigma})(u_\ell) = \psi_n(\sigma_n)\circ\ldots\circ\psi_1(\sigma_1)(u_\ell),$$

where $\boldsymbol{\sigma} = (\sigma_1,\ldots,\sigma_n)$ denotes the (signed) wave strengths vector in $\mathbb{R}^n$, [32]. Here ψ_j, $j=1,\ldots,n$ is the shock–rarefaction curve of the j^{th} family, parametrized as in [7] and related to the homogeneous system of conservation laws

$$u_t + f(u)_x = 0. \tag{2.10}$$

Observe that, due to (1.3), all the simple waves appearing in the solution of (2.10), (2.9) propagate with *non-zero* speed.

Let $p \in \{0,\ldots,n\}$ be the number of negative eigenvalues of the matrix $\nabla f(u)$, $u \in \Omega$ (note that p does not depend on u), then we set

$$\boldsymbol{\sigma}' \doteq (\sigma_1,\ldots,\sigma_p,0,\ldots,0), \quad \boldsymbol{\sigma}'' \doteq (0,\ldots,0,\sigma_{p+1},\ldots,\sigma_n), \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}' + \boldsymbol{\sigma}''. \tag{2.11}$$

To take into account the effects of the source term, we introduce a stationary discontinuity across the line $x = x_o$, that is, a wave whose speed is equal to 0; we call it a zero-wave. Now, given $h > 0$, we say that the particular Riemann solution:

$$u(t,x) = \begin{cases} u_\ell & \text{if } x < x_o \\ u_r & \text{if } x > x_o. \end{cases} \qquad \forall\, t \geq 0 \tag{2.12}$$

is admissible if and only if $u_r = \Phi_h(x_o, u_\ell)$, where Φ_h is the map defined at (2.3). Roughly speaking, we require u_ℓ, u_r to be (approximately) connected by a solution of the stationary equations (2.1).

In the same spirit, to define our general approximate Riemann solver for (1.1), we fix h suitably small and, having set $u_\ell = w_o$, $u_r = w_{n+1}$, we look for n intermediate states $w_1, \ldots, w_n$, such that

– if $i \neq p$, then the two states w_i, w_{i+1} are connected by a simple wave which is of the $(i+1)$ characteristic family if $i < p$, but belongs to the i^{th} family if $i > p$;

– if $i = p$, then $w_{p+1} = \Phi_h(x_o, w_p)$.

Summarizing, we propose the following:

Definition 2.1 *Given $h > 0$ suitably small, $x_o \in \mathbb{R}$, we say that $u(t,x)$ is a h–Riemann solver for (1.1), (1.3), (2.9), if the following conditions hold*

(a) there exists two states u^-, u^+ which satisfy $u^+ = \Phi_h(x_o, u^-)$;

(b) $u(t,x)$ coincides, on the set $\{t \geq 0,\ x < x_o\}$, with the solution to the homogeneous Riemann Problem (2.10) with initial values u_ℓ, u^- and, on the set $\{t \geq 0,\ x > x_o\}$, with the solution to the homogeneous Riemann Problem with initial values u^+, u_r;

(c) the Riemann Problem between u_ℓ and u^- is solved only by waves with negative speed (i.e. of the families $1, \ldots, p$);

(d) the Riemann Problem between u^+ and u_r is solved only by waves with positive speed (i.e. of the families $p+1, \ldots, n$).

The next two lemmas establish existence and uniqueness for such h–Riemann solvers.

Lemma 2.2 *There exists $\delta_1,\ h_1 > 0$ such that the following holds: for any $x_o \in \mathbb{R}$, $h \in [0, h_1]$, u_ℓ, $u_r \in B(0, \delta_1)$, there exists a unique h–Riemann solver, in the sense of Def. 2.1.*

In a bit more general context, we have the following lemma.

Lemma 2.3 *For any $M > 0$ there exist $\delta_1',\ h_1' > 0$, depending only on M and the homogeneous system (2.10), such that the following holds. For all maps $\boldsymbol{\phi} \in \mathbf{C^2}(\mathbb{R}^n, \mathbb{R}^n)$ satisfying*

$$\|\boldsymbol{\phi}\|_{\mathbf{C^2}} \leq M, \qquad |\boldsymbol{\phi}(u) - u| \leq h_1', \qquad \|I - \nabla\boldsymbol{\phi}(u)\| \leq h_1'$$

and for all $u_\ell \in B(0, \delta_1')$, $u_r \in B(\boldsymbol{\phi}(0), \delta_1')$ there exist $n+1$ states $w_0, \ldots, w_{n+1}$ and n wave sizes $\sigma_1, \ldots, \sigma_n$, depending smoothly on u_ℓ, u_r, such that with previous notations, there holds:

i) $w_0 = u_\ell$, $w_{n+1} = u_r$;

ii) $w_i = \Psi_i(\sigma_i)(w_{i-1}), \qquad i = 1, \ldots, p$;

iii) $w_{p+1} = \boldsymbol{\phi}(w_p)$;

iv) $w_{i+1} = \Psi_i(\sigma_i)(w_i), \qquad i = p+1, \ldots, n$.

Proof. If $h_1' \in (0, M)$ is chosen sufficiently small, the following map (see (2.11), here $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$))

$$(\boldsymbol{\sigma}, u_r, u_\ell) \mapsto F(\boldsymbol{\sigma}, u_\ell, u_r) = \Psi(\boldsymbol{\sigma}'') \circ \boldsymbol{\phi} \circ \Psi(\boldsymbol{\sigma}')(u_\ell) - u_r \tag{2.13}$$

is well defined and of class $\mathbf{C^2}$ in a suitable neighborhood of the origin. Moreover its $\mathbf{C^2}$ norm has a bound depending only on M and the homogeneous system (2.10). Now we observe that for any $u_\ell \in \Omega$,

- $F(0,u_\ell,\boldsymbol{\phi}(u_\ell))\equiv 0$,
- $\nabla_{\boldsymbol{\sigma}} F(0,u_\ell,\boldsymbol{\phi}(u_\ell)) = \Big(\nabla\boldsymbol{\phi}(u_\ell)r_1(u_\ell),\ldots,\nabla\boldsymbol{\phi}(u_\ell)r_p(u_\ell), r_{p+1}\big(\boldsymbol{\phi}(u_\ell)\big),\ldots,r_n\big(\boldsymbol{\phi}(u_\ell)\big)\Big)$,

where $r_i(u)$ is the i–th right eigenvector of $\nabla f(u)$. It follows that, if $R(u_\ell)=(r_1,\ldots,r_n)(u_\ell)$, one gets:

$$\left\|\nabla_{\boldsymbol{\sigma}} F(0,u_\ell,\boldsymbol{\phi}(u_\ell)) - R(u_\ell)\right\| \le \mathcal{O}(1)\left[|\boldsymbol{\phi}(u_\ell)-u_\ell| + \|I-\nabla\boldsymbol{\phi}(u_\ell)\|\right] \le \mathcal{O}(1)h_1'.$$

Since $R(u_\ell)$ is invertible as soon as h_1' is sufficiently small (independently on $\boldsymbol{\phi}$), $\nabla_{\boldsymbol{\sigma}} F(0,0,\boldsymbol{\phi}(0))$ is an invertible matrix. By the implicit function theorem there exists $\delta_1'>0$ and a $\mathbf{C^2}$ function $\boldsymbol{\sigma}(u_\ell,u_r)$ defined on $B(0,\delta_1')\times B(\boldsymbol{\phi}(0),\delta_1')$, satisfying $F\left[\boldsymbol{\sigma}(u_\ell,u_r),u_\ell,u_r\right]\equiv 0$. Moreover δ_1' and the bound on the $\mathbf{C^2}$ norm of $\boldsymbol{\sigma}$ turn out to rely solely on M and the $\mathbf{C^2}$ norm of F which itself depends only on the homogeneous system through the function $f(u)$. □

Proof of Lemma 2.2. By Lemma 2.1 if $h_1>0$ is chosen sufficiently small then for any $h\in[0,h_1]$, $x_o\in\mathbb{R}$ the map $u\mapsto\Phi_h(x_o,u)$ meets the hypotheses of Lemma 2.3. Moreover taking h_1 eventually smaller we can obtain that there exists $\delta_1>0$ such that $B(0,\delta_1)\subset B(0,\delta_1')\cap B(\Phi_h(x_o,0),\delta_1')$, for any $h\in[0,h_1]$. □

In the sequel, we denote by E the implicit function given by Lemmas 2.3, 2.2:

$$\boldsymbol{\sigma}\doteq E(h,u_\ell,u_r;x_o),$$

which plays the role of a wave–size vector. We recall that E is a $\mathbf{C^2}$ function with respect to the variables u_ℓ,u_r and its $\mathbf{C^2}$ norm is bounded by a constant independent of h and x_o.

Remark 2.1 *It is clear that the aforementioned construction shares a lot of features with the solution of a Riemann problem involving a nonconservative product as proposed in e.g. [17, 33]. Indeed, the map Φ_h can be interpreted as a family of locally Lipschitzian paths giving a rigorous meaning to a product of the form $\Big[hg(x,u)\delta_{x_o}(x)\Big]_\Phi$ where $\delta_{x_o}(x)$ stands for the Dirac mass in $x=x_o$. This formulation has already been widely used for numerical purposes in e.g. [23, 24], exploiting the exact steady-state equations (2.1).*

We notice that, in contrast with the homogeneous case, the wave–size $\boldsymbol{\sigma}$ in the h–Riemann solver is not equivalent to the jump size $|u_\ell-u_r|$; an additional term appears coming from the "Dirac source term" (see the special case $u_\ell=u_r$).

Lemma 2.4 *Let δ_1, h_1 be the constants in Lemma 2.2. For $u_\ell,u_r\in B(0,\delta_1)$, $h\in[0,h_1]$, set $\boldsymbol{\sigma}=E[h,u_\ell,u_r;x_o]$. Then there holds:*

$$\begin{aligned} |u_\ell-u_r| &= \mathcal{O}(1)\left(|\boldsymbol{\sigma}| + \int_0^h \omega(x_o+s)\,ds\right)\\ |\boldsymbol{\sigma}| &= \mathcal{O}(1)\left(|u_\ell-u_r| + \int_0^h \omega(x_o+s)\,ds\right)\end{aligned} \tag{2.14}$$

Proof. Let $w_1,\ldots,w_{n+1}$ be the intermediate states of the h–Riemann solver. Then, using (2.5), one has

$$\begin{aligned} |u_\ell-u_r| &\le \sum_{i=1}^{n+1}|w_i-w_{i-1}| = \mathcal{O}(1)|\boldsymbol{\sigma}| + |w_{p+1}-w_p|\\ &= \mathcal{O}(1)|\boldsymbol{\sigma}| + \left|\Phi_h(x_o,w_p)-w_p\right|\\ &= \mathcal{O}(1)\left[|\boldsymbol{\sigma}| + \int_0^h \omega(x_o+s)\,ds\right].\end{aligned}$$

To prove the second estimate, we use the equality $E\left[h, u_\ell, \Phi_h(x_o, u_\ell); x_o\right] \equiv 0$ and the Lipschitz continuity of E to obtain

$$\begin{aligned}
|\boldsymbol{\sigma}| &= \Big| E\left[h, u_\ell, u_r; x_o\right] - E\left[h, u_\ell, \Phi_h(x_o, u_\ell); x_o\right] \Big| \\
&= \mathcal{O}(1) \left| u_r - \Phi_h(x_o, u_\ell) \right| \leq \mathcal{O}(1) \Big[|u_\ell - u_r| + \left| u_\ell - \Phi_h(x_o, u_\ell) \right| \Big] \\
&= \mathcal{O}(1) \left[|u_\ell - u_r| + \int_0^h \omega(x_o + s)\, ds \right].
\end{aligned}$$

□

3 Existence of a BV entropy solution

3.1 Some wave interaction estimates

The forthcoming Lemmas deal with the commutation defect between the wave curves of the homogeneous problem (2.10) and the steady zero-wave.

Lemma 3.1 *For any $u, y_1,\ y_2,\ \boldsymbol{\alpha} \in \mathbb{R}^n$ sufficiently small, there holds:*

$$\Big| y_2 + \Psi(\boldsymbol{\alpha})(u) - \Psi(\boldsymbol{\alpha})(u + y_1) \Big| = \mathcal{O}(1) \Big[|\boldsymbol{\alpha}||y_1| + |y_1 - y_2| \Big]. \tag{3.1}$$

Proof. First of all we observe that there holds

$$\Big| y_2 + \Psi(\boldsymbol{\alpha})(u) - \Psi(\boldsymbol{\alpha})(u + y_1) \Big| \quad \leq \quad |y_1 - y_2| + \underbrace{\Big| y_1 + \Psi(\boldsymbol{\alpha})(u) - \Psi(\boldsymbol{\alpha})(u + y_1) \Big|}_{|G(\boldsymbol{\alpha}, y_1)|}.$$

The function G is of class $\mathbf{C^2}$ and satisfies $G(\boldsymbol{\alpha}, 0) \equiv G(0, y_1) \equiv 0$, hence standard arguments show that $G(\boldsymbol{\alpha}, y_1) = \mathcal{O}(1)|\boldsymbol{\alpha}||y_1|$. □

Lemma 3.2 *For any $u, \boldsymbol{\alpha} \in \mathbb{R}^n$, $h > 0$ sufficiently small, and for any $x_o \in \mathbb{R}$ there holds:*

$$\left| \Phi_h\Big(x_o, \Psi(\boldsymbol{\alpha})(u)\Big) - \Psi(\boldsymbol{\alpha})\left(\Phi_h(x_o, u)\right) \right| = \mathcal{O}(1)|\boldsymbol{\alpha}| \cdot \int_{x_o}^{x_o + h} \omega(x)\, dx. \tag{3.2}$$

Proof. If we apply Lemma 3.1 with:

$$y_1 = \Phi_h\left(x_o, u\right) - u, \qquad y_2 = \Phi_h\left(x_o, \Psi(\boldsymbol{\alpha})(u)\right) - \Psi(\boldsymbol{\alpha})(u),$$

we get

$$\begin{aligned}
\Big| \Phi_h\left(x_o, \Psi(\boldsymbol{\alpha})(u)\right) - \Psi(\boldsymbol{\alpha})\left(\Phi_h(x_o, u)\right) \Big| \leq \mathcal{O}(1) \Big[&|\boldsymbol{\alpha}| \left| \Phi_h\left(x_o, u\right) - u \right| \\
&+ \Big| \Phi_h\left(x_o, \Psi(\boldsymbol{\alpha})(u)\right) - \Psi(\boldsymbol{\alpha})(u) - \Phi_h\left(x_o, u\right) + u \Big| \Big].
\end{aligned}$$

The first term in the right hand side of the previous inequality can be estimated directly by (2.5). In order to deal with the second one, applying (2.6) with $u_1 = u$ and $u_2 = \Psi(\boldsymbol{\alpha})(u)$, we derive

$$\begin{aligned}
&\Big| \Phi_h\left(x_o, \Psi(\boldsymbol{\alpha})(u)\right) - \Psi(\boldsymbol{\alpha})(u) - \Phi_h\left(x_o, u\right) + u \Big| \\
&\qquad \leq \mathcal{O}(1)|\Psi(\boldsymbol{\alpha})(u) - u| \cdot \int_0^h \omega(x_o + s) ds \leq \mathcal{O}(1)|\boldsymbol{\alpha}| \cdot \int_0^h \omega(x_o + s) ds.
\end{aligned}$$

□

Remark 3.1 *The commutation defect (3.2) has already been encountered in the special case of a scalar balance law. However, in this last situation, one can arrange a change of variables in order to redress the wave curves into straight lines in the states plane: see [22]. Annihilating this commutator permits in particular to establish, for the scalar balance law, stability of Godunov-type approximations for general* **BV** *data.*

The next Lemma is concerned with general wave interaction estimates for the h–Riemann solver and therefore lies at the heart of the matter.

Lemma 3.3 *Let δ_1, $h_1 > 0$ be the constants in Lemma 2.2 and let $x_o \in \mathbb{R}$, $h \in [0, h_1]$, $u_\ell, u_m, u_r \in B(0, \delta_1)$ satisfy (with the notations of (2.11)):*

- $u_m = \Psi(\boldsymbol{\alpha})(u_\ell)$
- $u_r = \Psi(\boldsymbol{\beta}'')\left(\Phi_h\left(x_o, \Psi(\boldsymbol{\beta}')(u_m)\right)\right)$,
- $u_r = \Psi(\boldsymbol{\sigma}'')\left(\Phi_h\left(x_o, \Psi(\boldsymbol{\sigma}')(u_\ell)\right)\right)$,

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\sigma} \in \mathbb{R}^n$. In other words, the Riemann problem (u_ℓ, u_m) is solved according to the homogeneous Riemann solver (2.10), while (u_m, u_r) and (u_ℓ, u_r) are solved at the point x_o with the h–Riemann solver. Then, there holds:

$$\sum_{i=1}^{n} |\sigma_i - \alpha_i - \beta_i| = \mathcal{O}(1) \left(\sum_{approaching} |\alpha_i \beta_j| + \int_{x_o}^{x_o+h} \omega(x)\, dx \cdot \sum_{i>p} |\alpha_i| \right), \tag{3.3}$$

where ω is the function in condition (P_3). Two waves α_i, β_j are said to be **approaching** *if either $i > j$ or else $j = i$ and at least one of them is a genuinely nonlinear shock. Analogously, if the Riemann problems (u_ℓ, u_m), (u_ℓ, u_r) are solved at x_o with the h–Riemann solver, while (u_m, u_r) is solved according to the homogeneous Riemann solver, then there holds:*

$$\sum_{i=1}^{n} |\sigma_i - \alpha_i - \beta_i| = \mathcal{O}(1) \left(\sum_{approaching} |\alpha_i \beta_j| + \int_{x_o}^{x_o+h} \omega(x)\, dx \cdot \sum_{i\le p} |\beta_i| \right). \tag{3.4}$$

Finally:

$$\Big| \left|\Phi_h(x_o, u_\ell) - \Phi_h(x_o, u_r)\right| - |u_\ell - u_r| \Big| \le \mathcal{O}(1)\, |u_\ell - u_r| \cdot \int_{x_o}^{x_o+h} \omega(x)\, dx. \tag{3.5}$$

Proof. Concerning (3.5), it is a direct consequence of (2.6). Concerning (3.3), having set

$$\tilde{u}_r = \Psi\left(\boldsymbol{\alpha}'' + \boldsymbol{\beta}''\right) \left(\Phi_h \left(x_o, \Psi\left(\boldsymbol{\alpha}' + \boldsymbol{\beta}'\right)(u_\ell) \right) \right),$$

the smoothness of E implies:

$$|\boldsymbol{\sigma} - (\boldsymbol{\alpha} + \boldsymbol{\beta})| = \left|E(h, u_\ell, u_r; x_o) - E(h, u_\ell, \tilde{u}_r; x_o)\right| = \mathcal{O}(1) |u_r - \tilde{u}_r|.$$

To proceed, we fix

$$u_1 = \Psi\left(\boldsymbol{\beta}''\right) \circ \Psi\left(\boldsymbol{\alpha}''\right) \left(\Phi_h \left(x_o, \Psi\left(\boldsymbol{\alpha}'\right)(u_\ell) \right) \right), \qquad u_2 = \Psi\left(\boldsymbol{\beta}''\right) \left(\Phi_h \left(x_o, \Psi\left(\boldsymbol{\alpha}\right)(u_\ell) \right) \right). \tag{3.6}$$

The proof of (3.3) is complete if we can show that there holds:

$$|u_r - \widetilde{u}_r + u_1 - u_2| = \mathcal{O}(1) \sum_{\text{approaching}} |\alpha_i \beta_j|, \tag{3.7}$$

$$|u_1 - u_2| = \mathcal{O}(1) \int_{x_o}^{x_o+h} \omega(x)\, dx \cdot \sum_{i>p} |\alpha_i|. \tag{3.8}$$

To prove (3.7), we observe that $u_r - \widetilde{u}_r + u_1 - u_2$ is a $\mathbf{C^2}$ function of $\boldsymbol{\alpha}, \boldsymbol{\beta}$, with Lipschitz continuous second derivatives. Moreover, assume first that there are no approaching waves; this implies

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_i, 0, \ldots, 0), \qquad \boldsymbol{\beta} = (0, \ldots, 0, \beta_j, \ldots, \beta_n), \qquad i \leq j.$$

The case $i = j$ corresponds to two rarefaction waves and therefore $\alpha_i \geq 0$ and $\beta_i \geq 0$. If $i \leq p$, then $\boldsymbol{\alpha}'' = 0$, $\Psi(\boldsymbol{\alpha}' + \boldsymbol{\beta}')(u) = \Psi(\boldsymbol{\beta}') \circ \Psi(\boldsymbol{\alpha}')(u)$ and

$$\begin{aligned}
u_r - \widetilde{u}_r &= \Psi(\boldsymbol{\beta}'') \left(\Phi_h(x_o, \Psi(\boldsymbol{\beta}') \circ \Psi(\boldsymbol{\alpha}')(u_\ell))\right) - \Psi(\boldsymbol{\beta}'') \left(\Phi_h(x_o, \Psi(\boldsymbol{\alpha}' + \boldsymbol{\beta}')(u_\ell))\right) \\
&= 0 \\
u_1 - u_2 &= \Psi(\boldsymbol{\beta}'') \left(\Phi_h(x_o, \Psi(\boldsymbol{\alpha}')(u_\ell))\right) - \Psi(\boldsymbol{\beta}'') \Big(\Phi_h(x_o, \Psi(\boldsymbol{\alpha}')(u_\ell))\Big) = 0.
\end{aligned}$$

On the other hand, if $i > p$ then it must be $\boldsymbol{\beta}' = 0$, $\Psi(\boldsymbol{\alpha}'' + \boldsymbol{\beta}'')(u) = \Psi(\boldsymbol{\beta}'') \circ \Psi(\boldsymbol{\alpha}'')(u)$ and

$$\begin{aligned}
u_r - \widetilde{u}_r &= \Psi(\boldsymbol{\beta}'') \left(\Phi_h(x_o, \Psi(\boldsymbol{\alpha})(u_\ell))\right) - \Psi(\boldsymbol{\alpha}'' + \boldsymbol{\beta}'') \left(\Phi_h(x_o, \Psi(\boldsymbol{\alpha}')(u_\ell))\right) \\
&= u_2 - u_1.
\end{aligned}$$

Hence (3.7) holds if no approaching waves are present. Now, (3.7) follows in the general case by means of classical considerations (see for instance [38, 14, 41]).

Concerning (3.8), we observe that

$$|u_1 - u_2| \leq \mathcal{O}(1) \Big| \Psi(\boldsymbol{\alpha}'') \left(\Phi_h(x_o, \Psi(\boldsymbol{\alpha}')(u_\ell))\right) - \Phi_h(x_o, \Psi(\boldsymbol{\alpha})(u_\ell)) \Big|.$$

Since $\Psi(\boldsymbol{\alpha})(u) = \Psi(\boldsymbol{\alpha}'') \circ \Psi(\boldsymbol{\alpha}')(u)$, we can apply Lemma 3.2 with $u = \Psi(\boldsymbol{\alpha}')(u_\ell)$ and then establish (3.8). The estimate (3.4) can be obtained using completely similar arguments. □

The estimates (3.3)–(3.5) suggest the local integrals of ω as a convenient way to measure the strength of the zero-waves associated to our treatment of the source term. More precisely, for a given space mesh h, we define the size of the zero-wave located at jh as

$$\sigma = \int_0^h \omega(jh + s)\, ds. \tag{3.9}$$

Observe that it does not depend on the left and right state of the discontinuity. For a wave front tracking approximate solution (which will be defined precisely in subsection 3.2), we let $\mathcal{A}$ stand for the usual set of pairs of approaching waves. According to (3.9), we can define $\tilde{\mathcal{A}}$ as an extended set of pairs of approaching waves for these approximate solutions of (1.1). More precisely, we shall say that for $x_\alpha < x_\beta$, the couple (α, β) belong to $\tilde{\mathcal{A}}$ in either one of the three following cases:

- (α, β) belongs to $\mathcal{A}$ as in the homogeneous case;
- if α is a zero-wave and β is a physical one, then $i_\beta \leq p$ and thus the wave in x_β has a negative speed;

– if β is a zero-wave and α is a physical one, then $i_\alpha > p$ and thus the wave in x_α has a positive speed.

In any case, two zero-waves are never approaching each other. This convention will be useful in the next section.

We close this section with some interaction type estimates needed for the forthcoming study of the uniqueness functional. To this end, for $u, v \in \Omega$, we introduce the notations

$$v = \mathbf{S}(\boldsymbol{\sigma})(u) = S_n(\sigma_n) \circ \ldots \circ S_1(\sigma_1)(u), \tag{3.10}$$

where S_j denote the shock curve of the j^{th} family, parametrized as in [7]. Observe that no zero-waves appear in this construction. By possibly taking a smaller δ_1 in Lemma 2.2, for any $u, v \in B(0, \delta_1)$ there is a unique way to connect u to v using the curves S_j, regardless to any entropy condition.

Lemma 3.4 *Let $u^\pm$, $v^\pm$, $\boldsymbol{\sigma}^\pm \in \mathbb{R}^n$ sufficiently small and $x_o \in \mathbb{R}$, if we define*

$$\begin{aligned} u^+ &= \Phi_h(x_o, u^-), \qquad & v^- &= \mathbf{S}(\boldsymbol{\sigma}^-)(u^-), \\ v^+ &= \Phi_h(x_o, v^-), \qquad & v^+ &= \mathbf{S}(\boldsymbol{\sigma}^+)(u^+), \end{aligned}$$

then there holds

$$\sum_{i=1}^n |\sigma_i^+ - \sigma_i^-| = \mathcal{O}(1) \sum_{i=1}^n |\sigma_i^-| \int_0^h \omega(x_o + s)\, ds. \tag{3.11}$$

Proof. With the same notation as in (3.10), we call $\tilde{E}$ the function $\tilde{E}(u, v) \doteq \boldsymbol{\sigma}$. We observe that:

$$\boldsymbol{\sigma} = \tilde{E}\left[\Phi_h(x_o, u^-), \mathbf{S}(\boldsymbol{\sigma})\left(\Phi_h(x_o, u^-)\right)\right].$$

Hence, from the Lipschitz continuity of $\tilde{E}$, we get

$$\begin{aligned} |\boldsymbol{\sigma}' - \boldsymbol{\sigma}| &= \left| \tilde{E}\left[\Phi_h(x_o, u^-), \Phi_h\left(x_o, \mathbf{S}(\boldsymbol{\sigma})(u^-)\right)\right] - \tilde{E}\left[\Phi_h(x_o, u^-), \mathbf{S}(\boldsymbol{\sigma})\left(\Phi_h(x_o, u^-)\right)\right] \right| \\ &= \mathcal{O}(1) \left| \Phi_h\left(x_o, \mathbf{S}(\boldsymbol{\sigma})(u^-)\right) - \mathbf{S}(\boldsymbol{\sigma})\left(\Phi_h(x_o, u^-)\right) \right|. \end{aligned}$$

This last quantity can be easily estimated using Lemma 3.2 that clearly holds with $\mathbf{S}$ in place of Ψ. □

3.2 The wave-front tracking approximate solutions

At this level, we are in position to construct a global **BV** solution to (1.1) by passing to the limit in a sequence of approximate solutions generated by a wave-front tracking technique, [4, 13, 19, 6, 37]. This approach has been extensively used by A. Bressan and collaborators: see [7] and the numerous quotations therein to which we refer for a general introduction. Ours is mainly inspired by his, the main difference coming from the use of the h–Riemann solver previously introduced.

The first step is thus to state a precise definition of our approximations.

Definition 3.1 *Given $\epsilon, h > 0$, we say that a continuous map*

$$u^{\epsilon,h} : [0, +\infty) \to \mathbf{L}^1{}_{loc}(\mathbb{R}, \mathbb{R}^n)$$

is an ϵ, h–approximate solution of (1.1)–(1.2) if the following holds:

– *As a function of two variables, $u^{\epsilon,h}$ is piecewise constant with discontinuities occurring along finitely many straight lines in the x,t plane. Only finitely many wave-front interactions occur, each involving exactly two wave-fronts, and jumps can be of four types: shocks (or contact discontinuities), rarefaction waves, non-physical waves and zero-waves: $\mathcal{J} = \mathcal{S} \cup \mathcal{R} \cup \mathcal{NP} \cup \mathcal{Z}$.*

– *Along each shock (or contact discontinuity) $x_\alpha = x_\alpha(t)$, $\alpha \in \mathcal{S}$, the values of $u^- = u^{\epsilon,h}(t, x_\alpha-)$ and $u^+ = u^{\epsilon,h}(t, x_\alpha+)$ are related by $u^+ = \psi_{k_\alpha}(\sigma_\alpha)(u^-)$ for some $k_\alpha \in \{1, ..., n\}$ and some wave-strength σ_α. If the ${k_\alpha}^{th}$ family is genuinely nonlinear, then the Lax entropy admissibility condition $\sigma_\alpha < 0$ also holds. Moreover, there holds*

$$|\dot{x}_\alpha - \lambda_{k_\alpha}(u^+, u^-)| \leq \epsilon$$

where $\lambda_{k_\alpha}(u^+, u^-)$ is the speed of the shock front (or contact discontinuity) prescribed by the classical Rankine-Hugoniot conditions.

– *Along each rarefaction front $x_\alpha = x_\alpha(t)$, $\alpha \in \mathcal{R}$, there holds $u^+ = \psi_{k_\alpha}(\sigma_\alpha)(u^-)$, $0 < \sigma_\alpha \leq \epsilon$ for some genuinely nonlinear family k_α. Moreover, we have: $|\dot{x}_\alpha - \lambda_{k_\alpha}(u^+)| \leq \epsilon$.*

– *All non-physical fronts $x = x_\alpha(t)$, $\alpha \in \mathcal{NP}$ travel at the same speed $\dot{x}_\alpha = \hat{\lambda} > \sup_{u,i} |\lambda_i(u)|$. Their total strength remains uniformly small, namely:*

$$\sum_{\alpha \in \mathcal{NP}} |u^{\epsilon,h}(t, x_\alpha+) - u^{\epsilon,h}(t, x_\alpha-)| \leq \epsilon, \qquad \forall\ t > 0.$$

– *The zero-waves are located at every point $x = jh$, $j \in (-\frac{1}{h\epsilon}, \frac{1}{h\epsilon}) \cap \mathbb{Z}$.*
Along a zero-wave located at $x_\alpha = j_\alpha h$, $\alpha \in \mathcal{Z}$, the values $u^- = u^{\epsilon,h}(t, x_\alpha-)$ and $u^+ = u^{\epsilon,h}(t, x_\alpha+)$ satisfy $u^+ = \Phi_h(x_\alpha, u^-)$ for all $t > 0$ except at the interaction points.

– *The total variation in space $\mathrm{Tot.Var.}\, u^{\epsilon,h}(t,\cdot)$ is uniformly bounded for all $t \geq 0$. The total variation in time $\mathrm{Tot.Var.} \left\{ u^{\epsilon,h}(\cdot, x); [0, +\infty) \right\}$ is uniformly bounded for $x \neq jh$, $j \in \mathbb{Z}$.*

Finally, we require that $\|u^{\epsilon,h}(0,.) - u_o\|_{\mathbf{L}^1(\mathbb{R})} \leq \epsilon$.

We aim at establishing the existence of ϵ, h–approximate solutions.

Theorem 3.1 *Under the assumptions (P_1)–(P_3) (1.3), there exists a constant $\delta_2 > 0$ such that the following holds. If $\|\omega\|_{\mathbf{L}^1(\mathbb{R})} < \delta_2$, then for any initial condition $u_o \in \mathbf{L}^1(\mathbb{R}, \mathbb{R}^n)$ satisfying*

$$\mathrm{Tot.Var.}\, u_o + \|\omega\|_{\mathbf{L}^1(\mathbb{R})} \leq \delta_2,$$

there exists an ϵ, h–approximate solution of problem (1.1)–(1.2) for $\epsilon > 0$ and $h > 0$ sufficiently small.

Proof.

Description of the wave front tracking algorithm For notational convenience, we shall drop hereafter the ϵ, h superscripts where there will be no ambiguity. Now, given any small value for these parameters, let us build up such an approximate solution.

• In order for our approximate solutions to be piecewise constant, we need to discretize the rarefactions; following [7], for a fixed small parameter δ, each rarefaction of size σ, when it is created for the first time, is divided into $m = \left[\frac{\sigma}{\delta}\right] + 1$ wave-fronts, each one with size $\sigma/m \leq \delta$.

• Given the initial data u_o, we can define a piecewise constant approximation $u(0,\cdot)$ satisfying the requirements of Def. 3.1; moreover, it is possible to guarantee that

$$\text{Tot.Var.}\, u(0,\cdot) \leq \text{Tot.Var.}\, u_o.$$

Then, $u(t,x)$ is constructed, for small t, by applying the h–Riemann solver at every point $x = jh$ with $j \in \left(-\frac{1}{h\epsilon}, \frac{1}{h\epsilon}\right) \cap \mathbb{Z}$ (even if in that point $u(0,\cdot)$ does not have a discontinuity), and by solving the remaining discontinuities in $u(0,\cdot)$ using a classical homogeneous Riemann solver for (2.10).

• At every interaction point, a new Riemann problem arises. Notice that because of their fixed speed, two non-physical fronts cannot interact with each other; neither can the zero-waves. Moreover, by a slight modification of the speed of some waves (only among shocks, contact discontinuities and rarefactions), it is possible to achieve the property that not more than two wave-fronts interact at a point.

After an interaction time, the number of wave-fronts may well increase. In order to prevent this number to become infinite in finite time, a particular treatment has been proposed for waves whose strength is below a certain threshold value ρ by means of a simplified Riemann solver [7]. We shall need a similar trick in our construction, in the same spirit as in the homogeneous case; this is also the purpose of the estimates (3.2), (3.5).

Suppose that two wave-fronts of strengths σ, σ' interact at a given point (t,x). If $x \neq x_\alpha$, for any $\alpha \in \mathcal{Z}$, we use the classical accurate or simplified homogeneous Riemann solver as in [7]. Assume now that $x = x_\alpha = j_\alpha h$ with $\alpha \in \mathcal{Z}$, different situations can occur:

– if the wave approaching the zero wave is physical and there holds $|\sigma\sigma'| \geq \rho$ we use the (accurate) h–Riemann solver;

– if the wave incoming to the zero wave is physical and there holds $|\sigma\sigma'| < \rho$ we use a simplified one: Assume that the wave-front on the right is a zero–wave (the other case being similar). Let $u_\ell,\ u_m = \psi_j(\sigma)(u_\ell),\ u_r = \Phi_h(x, u_m)$ be the states before the interaction.
We define the auxiliary states

$$\tilde{u}_m = \Phi_h(x, u_\ell), \qquad \tilde{u}_r = \psi_j(\sigma)(\tilde{u}_m).$$

Then three fronts propagate after interaction: the zero-wave $(u_\ell, \tilde{u}_m)$, the physical front $(\tilde{u}_m, \tilde{u}_r)$ and the non-physical one $(\tilde{u}_r, u_r)$. Due to the commutation defect, we use the estimate (3.2) to ensure that the introduced error, i.e. the size of the generated non physical wave, is of second order.

– Suppose now that the wave on the left belongs to $\mathcal{NP}$. Again we use a simplified solver: let $u_\ell,\ u_m,\ u_r = \Phi_h(x, u_m)$ be the states before the interaction, and define the new state $\tilde{u}_\ell = \Phi_h(x, u_\ell)$. After the interaction time, only two fronts propagate: the zero–wave $(u_\ell, \tilde{u}_\ell)$ and the non-physical $(\tilde{u}_\ell, u_r)$. The estimate (3.5) ensures that the error we made is quadratic.

Stability of this algorithm We propose a slight modification (introducing the zero-waves) of the functionals introduced by J. Glimm [20]:

$$\begin{aligned} V(t) &= \sum_{\alpha \in \mathcal{J}} |\sigma_\alpha| = \sum_{\alpha \in \mathcal{S} \cup \mathcal{R} \cup \mathcal{NP} \cup \mathcal{Z}} |\sigma_\alpha| \\ Q(t) &= \sum_{\alpha, \beta \in \tilde{\mathcal{A}}} |\sigma_\alpha \sigma_\beta|, \end{aligned} \tag{3.12}$$

measuring respectively the total wave strengths and the interaction potential in $u(t,\cdot)$. Remember that if $\alpha \in \mathcal{Z}$ then the strength of the wave located in $x_\alpha = j_\alpha h$ is given by $\sigma_\alpha = \int_0^h \omega\,(j_\alpha h + s)\, ds$. Notice that there holds (see lemma 2.4) for some $C_1 > 1$

$$\frac{1}{C_1}\left[\|\omega\|_{\mathbf{L}^1(I)} + \text{Tot.Var.}\, u(t,\cdot)\right] \leq V(t) \leq C_1\left[\text{Tot.Var.}\, u(t,\cdot) + \|\omega\|_{\mathbf{L}^1(I)}\right] \tag{3.13}$$

where the interval I is defined by

$$I \doteq \bigcup_{\alpha\in\mathcal{Z}} \left[j_\alpha h, (j_\alpha+1)\,h\right] = \bigcup_{j\in\left(-\frac{1}{\epsilon h},\frac{1}{\epsilon h}\right)\cap\mathbb{Z}} \left[jh, (j+1)\,h\right].$$

Thus, according to the estimates (3.2)–(3.5) and the classical ones, for every time τ when any two wave fronts of strengths σ, σ' interact, we get:

$$V(\tau+) - V(\tau-) \leq \mathcal{O}(1)|\sigma\sigma'|, \tag{3.14}$$

$$Q(\tau+) - Q(\tau-) \leq |\sigma\sigma'|\left[\mathcal{O}(1)V(\tau-) - 1\right]. \tag{3.15}$$

Therefore if V remains sufficiently small (3.15) implies

$$Q(\tau+) - Q(\tau-) \leq -\frac{|\sigma\sigma'|}{2}. \tag{3.16}$$

By (3.14) and (3.16) we can choose a constant C_o large enough and $\delta_2' > 0$ so that (3.16) holds and moreover the quantity

$$\Upsilon(t) \doteq V(t) + C_o Q(t) \tag{3.17}$$

decreases at every interaction time τ provided that $V(\tau-) \leq \delta_2'$. Now we choose $\delta_2 \in (0, \delta_1)$ such that

$$\begin{aligned} C_1\delta_2 + C_oC_1^2\delta_2^2 &\leq \delta_2', \\ C_1\left[C_1\delta_2 + C_oC_1^2\delta_2^2\right] &\leq \delta_1. \end{aligned}$$

If the source satisfies $\|\omega\|_{\mathbf{L}^1(\mathbb{R})} < \delta_2$, we can choose a nontrivial initial data u_o satisfying

$$\text{Tot.Var.}\, u_o + \|\omega\|_{\mathbf{L}^1(\mathbb{R})} \leq \delta_2,$$

hence the approximate initial data also satisfies

$$\text{Tot.Var.}\, u(0,\cdot) + \|\omega\|_{\mathbf{L}^1(\mathbb{R})} \leq \delta_2.$$

Observing that $Q(t) \leq V(t)^2$ for any $t \geq 0$, by (3.13) we now have

$$V(t) \leq \Upsilon(t) \leq \Upsilon(0+) = V(0+) + C_oQ(0+) \leq C_1\delta_2 + C_oC_1^2\delta_2^2 \leq \delta_2', \tag{3.18}$$

$$\text{Tot.Var.}\left\{u(t,\cdot)\right\} + \|\omega\|_{\mathbf{L}^1(I)} \leq C_1V(t) \leq C_1\left[C_1\delta_2 + C_oC_1^2\delta_2\right] < \delta_1. \tag{3.19}$$

Indeed, if (3.18), (3.19) hold before an interaction time τ, then $V(\tau-) \leq \delta_2'$. Hence Υ decreases across the interaction and (3.18), (3.19) still hold after the time τ, and by induction, for all $t \geq 0$ for which $u(t,x)$ is defined.

Control of the wave fronts In order to guarantee that our approximate solution exists for any $t \geq 0$, we must establish that the total number of wave-fronts is finite. First of all, we can always choose a piecewise constant approximation $u^{\epsilon,h}(0,.)$ of u_o endowed with finitely many discontinuities. Then, observe that for $h, \epsilon > 0$, there is also finitely many zero-waves created at $t = 0+$. Therefore we only have to control the number of fronts generated by further interactions. At this point, we recall that the accurate Riemann solvers are used only for "strong enough" waves; when these waves interact the interaction potential $Q(t)$ decreases of a quantity proportional to ρ. Hence, being $Q(t)$ a non increasing function, these solvers are used at most $\mathcal{O}(1)\left[Q(0)/\rho\right]$ times. The simplified solvers do not increase the number of physical wave-fronts hence the total number of these fronts is finite. In turn, a new non–physical wave front is generated only when two physical ones interact. Clearly, any two physical fronts can interact only once. Hence the total number of non–physical wave fronts is also finite.

Final remarks In order to control the strength of both rarefaction and non-physical fronts, the same procedure as in [4, 7] can be followed. It can be shown that (possibly choosing a smaller δ_2) choosing appropriately the parameters δ and ρ, in the resulting wave front tracking pattern, the rarefaction waves have size less than ϵ and also the sum of all the sizes of the non–physical waves is less than ϵ. Indeed, our zero waves can be treated as a new family of physical waves that never suffer self–interactions and for which all the estimate for the interactions with waves of other families hold.
It remains only to show the uniform bound on the total variation $\mathrm{Tot.Var.}\left\{u(\cdot,x);[0,+\infty)\right\}$ for any $x \neq jh$, $j \in \mathbb{Z}$. This bound can be easily obtained by standard computations (see for instance [14]) using the fact that the line $\{(x,t) : t \geq 0\}$ is not characteristic and does not cross the zero waves. □

Remark 3.2 *The bounds on the total variation in each variable together with the ones on the velocity of the waves, $0 < c < |\lambda_i| < \hat{\lambda}$, imply the following uniform Lipschitz inequalities*

$$\int_{\mathbb{R}} \left|u(t',x) - u(t'',x)\right| dx \leq C'|t' - t''|, \qquad t', t'' \geq 0; \tag{3.20}$$

$$\int_0^T \left|u(t,x') - u(t,x'')\right| dt \leq C'|x' - x''|, \qquad x', x'' \in \left((j-1)h, jh\right), \ j \in \mathbb{Z} \tag{3.21}$$

where $C' > 0$ is a suitable constant.

3.3 Passing to the limits in ϵ and h

The following theorem is a first step towards establishing our algorithm's consistency with the continuous problem (1.1). Keeping $h > 0$ fixed, we are about to let first ϵ tend to zero. Hence we shall drop the superscript h for notational clarity.

Theorem 3.2 *Let u^ϵ be a family of ϵ, h–approximate solutions of (1.1)–(1.2). There exists a subsequence u^{ϵ_i} converging as $i \to +\infty$ in $\mathbf{L}^1_{loc}\left((0,+\infty)\times\mathbb{R}\right)$ to a function u which satisfies for any $\varphi \in \mathbf{C}^1_c\left((0,+\infty)\times\mathbb{R}\right)$:*

$$\int_0^\infty \int_{\mathbb{R}} \left[u\varphi_t + f(u)\varphi_x\right] dxdt + \int_0^\infty \sum_{j\in\mathbb{Z}} \varphi(t,jh) \left(\int_0^h g\left[jh+s, u(t,jh-)\right] ds\right) dt = 0. \tag{3.22}$$

Moreover $\mathrm{Tot.Var.}\, u(t,\cdot)$ *is uniformly bounded and u satisfies the Lipschitz property (3.20).*

Proof. Let $T, K > 0$ be two constants such that the support of φ is contained in the region $R = [0,T] \times [-K, K]$. In the sequel we will suppose ϵ small enough to satisfy $\frac{1}{\epsilon} > K$. Then we define the error

$$\mathcal{E}^\epsilon \doteq \mathcal{I}_1^\epsilon + \mathcal{I}_2^\epsilon \tag{3.23}$$

where

$$\mathcal{I}_1^\epsilon \doteq \int_0^\infty \int_{\mathbb{R}} \left[u^\epsilon \varphi_t + f(u^\epsilon)\varphi_x\right] dsdt, \tag{3.24}$$

$$\mathcal{I}_2^\epsilon \doteq \int_0^\infty \sum_{j\in\mathbb{Z}} \varphi(t, jh) \int_0^h g\left[jh + s, u^\epsilon(t, jh-)\right] dsdt. \tag{3.25}$$

The trajectories of wave fronts in u^ϵ subdivide R into finitely many polygons Γ_k in which u^ϵ is constant. Introducing the vector $\phi = (u^\epsilon\varphi, f(u^\epsilon)\varphi)$ we get

$$\mathcal{I}_1^\epsilon = \sum_k \int_{\Gamma_k} \text{div } \phi(t,x)\ dxdt = \sum_k \int_{\partial\Gamma_k} \phi \cdot d\vec{n},$$

where $\partial\Gamma_k$ is the oriented boundary and $\vec{n}$ denotes an outer normal vector. Hence if $x_1(t), \ldots, x_N(t)$ are the jumps in u^ϵ at the time t, by standard computations [7] we get

$$\mathcal{I}_1^\epsilon = \sum_k \int_{\partial\Gamma_k} \phi \cdot d\vec{n} = \int_0^T \sum_{\alpha\in\mathcal{J}} e_{\epsilon,\alpha}(t) \cdot \varphi(t, x_\alpha(t))\ dt$$

where

$$e_{\epsilon,\alpha}(t) = \dot{x}(t)\Delta u^\epsilon\left(t, x_\alpha(t)\right) - \Delta f\left(u^\epsilon\left(t, x_\alpha(t)\right)\right)$$

and $\Delta F(u^\epsilon(t, x_\alpha)) = F(u^\epsilon(t, x_\alpha+)) - F(u^\epsilon(t, x_\alpha-))$ for any function F.

Using the properties of the ϵ, h–approximate solutions, we have also:

$$\sum_{\alpha\in\mathcal{R}\cup\mathcal{S}\cup\mathcal{NP}} |e_{\epsilon,\alpha}(t)| = \mathcal{O}(1)\epsilon.$$

It remains only to see what happens when $\alpha \in \mathcal{Z}$. In this case $\dot{x}_\alpha(t) = 0$, therefore we have, denoting by $x_\alpha = j_\alpha h$ the location of the zero wave α and recalling (2.4),

$$\begin{aligned}
\mathcal{I}_1^\epsilon &= \mathcal{O}(1)\epsilon - \int_0^T \sum_{\alpha\in\mathcal{Z}} \Delta f\left(u^\epsilon\left(t, x_\alpha\right)\right) \cdot \varphi\left(t, x_\alpha\right) dt \\
&= \mathcal{O}(1)\epsilon - \int_0^T \sum_{\alpha\in\mathcal{Z}} \varphi\left(t, x_\alpha\right) \int_0^h g\left(x_\alpha + s, u^\epsilon(t, x_\alpha-)\right) dsdt \\
&= \mathcal{O}(1)\epsilon - \mathcal{I}_2^\epsilon.
\end{aligned} \tag{3.26}$$

Hence the error term (3.23) tends to zero as ϵ tends to zero. To conclude the proof we have to show that there exists a converging subsequence u^{ϵ_i} such that $\mathcal{I}_1^{\epsilon_i} + \mathcal{I}_2^{\epsilon_i}$ converges to the left hand side of (3.22). The convergence of $\mathcal{I}_1^{\epsilon_i}$ to the first term of the left hand side of (3.22) is ensured by the $\mathbf{L}^1_{loc}\left((0,+\infty)\times\mathbb{R}\right)$ compactness of u^ϵ coming from Helly's theorem.

Concerning $\mathcal{I}_2^{\epsilon_i}$, we need to show that $u^{\epsilon_i}(\cdot, jh-)$ converges to $u(\cdot, jh-)$ in $\mathbf{L}^1_{loc}\left((0,+\infty)\right)$. We begin taking a converging (in $\mathbf{L}^1_{loc}\left((0,+\infty)\times\mathbb{R}\right)$) subsequence u^{ϵ_i} to a function u. We can choose u in such a way that (eventually passing to a subsequence) there holds:

i) $u^{\epsilon_i}(t,\cdot)$ converges in $\mathbf{L}^1{}_{loc}(\mathbb{R})$ to $u(t,\cdot)$ for all $t \geq 0$;

ii) Tot.Var. $u(t,\cdot)$ is uniformly bounded for all $t \geq 0$;

iii) u^{ϵ_i} converges pointwise to u almost everywhere in $(0,+\infty)\times\mathbb{R}$.

Property ii) ensures that the second term of the left hand side of (3.22) is meaningful. Property iii) ensures that there exists a set $\mathcal{N}\subset\mathbb{R}$ with Lebesgue measure zero such that for all $x\notin\mathcal{N}$ the sequence $u^{\epsilon_i}(\cdot,x)$ converges almost everywhere in $(0,+\infty)$ to $u(\cdot,x)$ and this implies the $\mathbf{L}^1_{loc}\left((0,+\infty)\right)$ convergence by the Lebesgue theorem. Therefore for all $x',x''\in\left((j-1)h,jh\right)\setminus\mathcal{N}$ we can pass to the limit in (3.21) and obtain that for such x',x'' u satisfies that inequality. Moreover taking a sequence $x''_k\in\left((j-1)h,jh\right)\setminus\mathcal{N}$ converging to jh we can again pass to the limit in (3.21) and obtain this inequality with $jh-$ in place of x''. Since φ has support in R, the summations in (3.25) and (3.22) actually extend from $-[K/h]$ to $[K/h]$ where the brackets $[\cdot]$ denote the integer part of a number. Finally we take $x'_j\in\left((j-1)h,jh\right)\setminus\mathcal{N}$ and, using the smoothness of g with respect to u, we compute

$$\begin{aligned}&\left|\mathcal{I}_2^{\epsilon_i}-\int_0^\infty\sum_{j\in\mathbb{Z}}\varphi(t,jh)\int_0^h g\left[jh+s,u(t,jh-)\right]dsdt\right|\\ &\leq\mathcal{O}(1)\sum_{j=-[K/h]}^{[K/h]}\int_0^T\int_0^h\left|u(t,jh-)-u^{\epsilon_i}(t,jh-)\right|dsdt\\ &\leq\mathcal{O}(1)\sum_{j=-[K/h]}^{[K/h]}\left\{|x'_j-jh|+\int_0^T\int_0^h\left|u(t,x'_j)-u^{\epsilon_i}(t,x'_j)\right|dsdt\right\}\end{aligned}\tag{3.27}$$

If in this last estimate we take the "lim sup" as $i\to+\infty$, we finally get

$$\limsup_{i\to+\infty}\left|\mathcal{I}_2^{\epsilon_i}-\int_0^\infty\sum_{j\in\mathbb{Z}}\varphi(t,jh)\int_0^h g\left[jh+s,u(t,jh-)\right]dsdt\right|\leq\mathcal{O}(1)\sum_{j=-[\frac{K}{h}]}^{[\frac{K}{h}]}|x'_j-jh|.$$

This proves the theorem since $x'_j\in\left((j-1)h,jh)\right)\setminus\mathcal{N}$ is arbitrary. □

Now we are in position to prove an existence theorem for the Cauchy problem (1.1), (1.2). We can apply again Helly's compactness theorem to get a subsequence u^{h_i} converging to some function u in $\mathbf{L}^1{}_{loc}$ whose total variation in space is uniformly bounded for all $t\geq 0$; moreover, working as in [2], Prop. 5.1, one can prove that $u^{h_i}(t,\cdot)$ converges in $\mathbf{L}^1$ to $u(t,\cdot)$, for all $t\geq 0$.

Theorem 3.3 *Let u^{h_i} be a subsequence of solutions of equation (3.22) with uniformly bounded total variation converging as $i\to+\infty$ in $\mathbf{L}^1$ to some function u. Then u is a weak solution to the Cauchy problem (1.1)–(1.2).*

Proof. We pick up a test function $\varphi\in\mathbf{C}^1_c((0,+\infty)\times\mathbb{R})$. The first term of equation (3.22) obviously converges to

$$\int_0^\infty\int_{\mathbb{R}}\left[u\varphi_t+f(u)\varphi_x\right]dxdt.$$

Concerning the second one we observe that since φ has bounded support, the interval of integration in t can be reduced to $[0,T]$ for some $T>0$ and the summation extends from $-[K/h_i]$ to $[K/h_i]$ for some constant $K>0$ independent on h_i. Therefore φ is Lipschitz continuous and we get

$$\begin{aligned}&\int_0^T\sum_{j=-[K/h_i]}^{[K/h_i]}\varphi(t,jh_i)\int_0^{h_i}g\left[jh_i+s,u^{h_i}(t,jh_i-)\right]dsdt=\\ &=\mathcal{O}(1)\left[\frac{K}{h_i}\right]h_i^2+\int_0^T\sum_{j=-[K/h_i]}^{[K/h_i]}\int_0^{h_i}\varphi(t,jh_i+s)g\left[jh_i+s,u^{h_i}(t,jh_i-)\right]dsdt.\end{aligned}$$

Because of the smoothness of g with respect to the u variable, using the equality

$$\begin{aligned} g\left[jh_i+s,u^{h_i}(t,jh_i-)\right] &= \mathcal{O}(1)\,\text{Tot.Var.}\left\{u^{h_i}(t,\cdot);[(j-1)h_i,(j+1)h_i]\right\} \\ &\quad +g\left[jh_i+s,u^{h_i}(t,jh_i+s)\right], \end{aligned}$$

we get that the second term of (3.22) becomes

$$\begin{aligned} &\mathcal{O}(1)h_i+\mathcal{O}(1)\int_0^T\sum_{j=-[K/h_i]}^{[K/h_i]}\int_0^{h_i}\text{Tot.Var.}\left\{u^{h_i}(t,\cdot),[(j-1)h_i,(j+1)h_i]\right\}dsdt \\ &+\int_0^T\sum_{j=-[K/h_i]}^{[K/h_i]}\int_0^{h_i}\varphi(t,jh_i+s)\cdot g\left[jh_i+s,u^{h_i}(t,jh_i+s)\right]dsdt \\ &=\mathcal{O}(1)h_i+\mathcal{O}(1)h_i\int_0^T\text{Tot.Var.}\,u^{h_i}(t,\cdot)dt+\int_0^T\int_{-K}^K g\left[x,u^{h_i}(t,x)\right]\varphi(t,x)dxdt \\ &=\mathcal{O}(1)h_i+\int_0^\infty\int_{\mathbb{R}} g\left[x,u^{h_i}(t,x)\right]\varphi(t,x)dxdt. \end{aligned}$$

In the limit $i\to+\infty$, this boils down to the correct contribution for the source term. □

Consequently, we recover in particular the following existence theorem first proved by T. P. Liu in [35].

Theorem 3.4 *Under the assumptions (P_1)–(P_3) and (1.3), there exists $\delta_2>0$ such that if $\|\omega\|_{\mathbf{L}^1(\mathbb{R})}<\delta_2$, then for any $u_o\in\mathbf{L}^1\cap\mathbf{BV}(\mathbb{R};\mathbb{R}^n)$ satisfying*

$$\text{Tot.Var.}\,u_o+\|\omega\|_{\mathbf{L}^1(\mathbb{R})}\le\delta_2,$$

there exists a global weak solution $u:[0,+\infty)\to\mathbf{L}^1\cap\mathbf{BV}(\mathbb{R};\mathbb{R}^n)$ to the Cauchy problem (1.1)–(1.2). Moreover, for any possible convex entropy-entropy flux pair ϕ,ψ verifying $\nabla\psi=\nabla\phi\cdot\nabla f$ there also holds in the sense of distributions:

$$\phi(u)_t+\psi(u)_x\le\nabla\phi(u)\cdot g(x,u). \tag{3.28}$$

Of course, in the situation where the system (1.1) is endowed with a convex entropy pair ϕ,ψ, one establishes (3.28) using completely similar arguments.

4 Lipschitz semigroups of BV entropy solutions

4.1 Some positively invariant domains

The main existence theorem proved in the last section applies to initial data with sufficiently small total variation. Here we construct closed domains $\mathcal{D}_h(\delta)\subset\mathbf{L}^1(\mathbb{R},\mathbb{R}^n)$ for $h,\delta>0$ with the property that, for every initial data $u_o\in\mathcal{D}_h(\delta)$, the front tracking algorithm yields a solution $u(t,x)$ of the Cauchy problem (3.22) satisfying $u(t,\cdot)\in\mathcal{D}_h(\delta)$, for any $t\ge0$.

Let $u:\mathbb{R}\to\mathbb{R}^n$ be a piecewise constant function with small total variation. For all $t>0$ small enough (before any interaction occur) and $x\in\mathbb{R}$, let $u(t,x)$ be the function obtained solving all the h–Riemann problems on the lattice $\{jh\}_{j\in\mathbb{Z}}$ and all the homogeneous Riemann problems for the remaining discontinuities of u. We can define for $u(t,x)$, $t>0$, the quantities in (3.12) with the difference that here there are no non physical waves and, moreover, since we are solving all the countable h–Riemann solvers, the sums do not have necessarily a finite number of terms. We call these two quantities respectively the *Total strength of waves* and the *Interaction potential* of u and we denote them by $\boldsymbol{V}_h(u)$ and $\boldsymbol{Q}_h(u)$.

Observe the slight differences between this last definition and (3.12).

– The quantities V and Q in (3.12) refer only to a solution constructed by the wave front tracking algorithm and are defined only where no interaction takes place. On the other hand, $\boldsymbol{V}_h$ and $\boldsymbol{Q}_h$ refer to any piecewise constant function u.

– In the case of a wave front tracking ϵ, h–approximate solution, we have

$$\left|V\left(u(t)\right) - \boldsymbol{V}_h\left(u(t)\right)\right| = \mathcal{O}(1)\left[\epsilon + \int_{\mathbb{R}\setminus(-1/\epsilon,1/\epsilon)} \omega(x)dx\right] = \mathcal{O}(1)r(\epsilon), \tag{4.1}$$

where $r(\epsilon)$ is a function that tends to zero as ϵ tends to zero. In (4.1) the integral term is due to the fact that in $\boldsymbol{V}_h$, waves appear from all the countable h–Riemann solvers whereas in V, there are zero waves only in the interval $(-1/\epsilon, 1/\epsilon)$. The other term of the order ϵ comes from the different way in which the non physical waves are treated in V and $\boldsymbol{V}_h$ respectively (see [7] for details). Analogously, there holds

$$\left|Q\left(u(t)\right) - \boldsymbol{Q}_h\left(u(t)\right)\right| = \mathcal{O}(1)r(\epsilon), \tag{4.2}$$

together with

$$V\left(u(0+)\right) \leq \boldsymbol{V}_h\left(u(0)\right), \qquad Q\left(u(0+)\right) \leq \boldsymbol{Q}_h\left(u(0)\right), \tag{4.3}$$

since for t small enough, no interaction has yet occurred and therefore no non physical waves appeared.

We also observe that Lemma 2.4 implies for any piecewise constant function u:

$$\frac{1}{C_1}\left[\|\omega\|_{\mathbf{L}^1(\mathbb{R})} + \text{Tot.Var.}\, u\right] \leq \boldsymbol{V}_h(u) \leq C_1\left[\text{Tot.Var.}\, u + \|\omega\|_{\mathbf{L}^1(\mathbb{R})}\right]. \tag{4.4}$$

for some constant $C_1 > 1$.

Now, for any piecewise constant function with small total variation, we define the functional

$$\boldsymbol{\Upsilon}_h(u) = \boldsymbol{V}_h(u) + C_o\boldsymbol{Q}_h(u). \tag{4.5}$$

Finally, the domains

$$\mathcal{D}_h(\delta) = \text{cl}\left\{u \in \mathbf{L}^1\left(\mathbb{R}, \mathbb{R}^n\right) : \ u \text{ piecewise constant }, \boldsymbol{\Upsilon}_h(u) < \delta\right\} \tag{4.6}$$

where "cl" stands for the closure in $\mathbf{L}^1$.

Thanks to (4.1), (4.2) and recalling that $\Upsilon(t) = V(t) + C_oQ(t)$ is non increasing along ϵ, h–approximate solutions, we obtain following [7] that the domains $\mathcal{D}_h(\delta)$ are positively invariant with respect to the flow generated by front tracking solutions of (3.22).

To ensure that the domains $\mathcal{D}_h(\delta)$ are not empty, for some $\delta > 0$ and all $h > 0$ small enough, one can proceed the following way. Let δ_2 be the constant in Theorem 3.1; picking up a $\delta \in \left(0, \delta_2/C_1\right)$, assume that ω satisfies $\|\omega\|_{\mathbf{L}^1} + C_oC_1^2\|\omega\|_{\mathbf{L}^1}^2 < \delta$.

This condition on the source garantees that $\mathcal{D}_h(\delta) \neq \emptyset$, and the fact that $\delta < \frac{\delta_2}{C_1}$ ensures that for any $u \in \mathcal{D}_h(\delta)$, ϵ, h–approximate solutions can be constructed. Observe that since the constant C_1 in (4.4) does not depends on h, the intersection $\bigcap_{h>0} \mathcal{D}_h(\delta)$ is not empty and contains all integrable functions with sufficiently small total variation.

4.2 An extended "almost-decreasing" functional

The existence result of Theorem 3.4 comes from a compactness argument and does not provide any information on the eventual uniqueness of the weak solutions one gets in the limit $\epsilon, h \to 0$. In the scalar case, the theory of Kružkov, [30], ensures both uniqueness and $\mathbf{L}^1_{loc}$ continuous dependence as soon as (3.28) holds for any convex ϕ.

Recently, a somewhat simplified framework has been proposed in order to establish uniqueness and continuous dependence for entropy solutions to systems of conservation laws, see [8], by means of a functional uniformly equivalent to the $\mathbf{L}^1(\mathbb{R})$ distance. We are about to follow a similar approach to establish our next result. Since we keep h fixed, we drop it from the superscripts.

Let u, v be two ϵ, h–approximate solutions of (1.1) in the sense of Def. 3.1 with two possibly different initial data u_o, v_o; for any $x \in \mathbb{R}$, $t \geq 0$, we consider the scalar functions q_i, $i = 1, ..., n$ implicitly defined by:

$$v(t,x) = \mathbf{S}(\mathbf{q})(u(t,x)) = S_n(q_n) \circ S_{n-1}(q_{n-1}) \circ ... \circ S_1(q_1)(u(t,x)), \tag{4.7}$$

where S_i, $i = 1, ..., n$ are the shock curves associated to (2.10) regardless to the entropy condition. Inside a compact neighborhood of the origin, we have for some $C_2 > 1$:

$$\frac{1}{C_2}|u - v| \leq \sum_{i=1}^{n} |q_i| \leq C_2|u - v|.$$

Therefore, we define the following functional still equivalent to the $\mathbf{L}^1(\mathbb{R})$ distance:

$$\Xi(u,v)(t) \doteq \sum_{i=1}^{n} \int_{\mathbb{R}} |q_i(t,x)| X_i(t,x)dx. \tag{4.8}$$

The weights X_i, $i = 1, ..., n$ are given by:

$$X_i(t,x) = 1 + \kappa_1 B_i(t,x) + \kappa_2[Q(u(t,\cdot)) + Q(v(t,\cdot))] \geq 1,$$

where κ_1, κ_2 are (big) constants to be adjusted later. The differences with the homogeneous case go like this:

(i) the "extended" interaction potentials (3.12), taking the zero-waves into account, are to be considered;

(ii) the zero-waves appear also as follows, according to the notations of [7, 8]:

$$B_i(t,x) = A_i(t,x) + \begin{cases} \displaystyle\sum_{x_\alpha < x,\ \alpha \in \mathcal{Z}} |\sigma_\alpha| & i \leq p \\ \displaystyle\sum_{x_\alpha > x,\ \alpha \in \mathcal{Z}} |\sigma_\alpha| & i > p. \end{cases} \tag{4.9}$$

Observe that the parameter $h > 0$ is the same for both approximate solutions. Since $B_i(t,x) \leq V(u(t,.)) + V(v(t,.))$, we can follow [7, 8] and claim that, as soon as the constants κ_1, κ_2 are fixed, the small parameter δ in the domains $\mathcal{D}_h(\delta)$ can be reduced in order to have $1 \leq X_i(t,x) \leq 2$.

Lemma 4.1 *For suitable constants $C_3, \kappa_1, \kappa_2, \delta_3 > 0$, the following holds.*
Let u, v be two ϵ, h–approximate solutions of (1.1) in the sense of Def. 3.1 with parameters respectively equal to (ϵ, h) and (ϵ', h), $0 < \epsilon' \leq \epsilon$. Assume furthermore

$$\Upsilon\left(u(0,\cdot)\right) < \delta_3, \qquad \Upsilon\left(v(0,\cdot)\right) < \delta_3 \quad \forall t \geq 0, \tag{4.10}$$

then the functional Ξ *in (4.8) satisfies*

$$\Xi\left(u(t,\cdot),v(t,\cdot)\right)-\Xi\left(u(s,\cdot),v(s,\cdot)\right)\leq C_3\tilde{r}(\epsilon)(t-s),\qquad \textit{for all}\quad 0\leq s\leq t, \tag{4.11}$$

where the function $\tilde{r}(\epsilon)$ *tends to zero as* ϵ *tends to zero.*

Proof. The constants ϵ,ϵ',h are now fixed. We denote by $\mathcal{R}=\mathcal{R}(u)\cup\mathcal{R}(v)$ the set of the rarefactions for both u and v, by $\mathcal{S}=\mathcal{S}(u)\cup\mathcal{S}(v)$ the set of the shocks (or contact discontinuity) for both u and v, and by $\mathcal{Z}=\mathcal{Z}(v)\supset\mathcal{Z}(u)$ the set of zero waves. Observe that since h is fixed and equal for both u and v, their zero waves coincide in the spatial region $(-1/\epsilon,1/\epsilon)$. Finally we denote by $\mathcal{J}=\mathcal{R}\cup\mathcal{S}\cup\mathcal{NP}\cup\mathcal{Z}$ the set of all the jumps.
Notice that for a given $t\geq 0$, $x\mapsto q_i(t,x)$ and $x\mapsto X_i(t,x)$ are constant on the intervals whose endpoints are discontinuities for u or v, hence we have

$$\Xi(u,v)(t)=\sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{n}|q_i(t,x_\alpha+)|X_i(t,x_\alpha+)\Big[x_{\alpha+1}(t)-x_\alpha(t)\Big].$$

The functional $t\mapsto\Xi(u,v)(t)$ is continuous in $\mathbf{L}^1$ except at the times at which there is an interaction of waves in u or v. At each of these times, it is possible to check that Ξ decreases, by the very definition of the weights X_i and a suitable choice of the constants κ_1, κ_2.

Outside interaction instants, we can differentiate w.r.t. time and obtain with unambiguous notations:

$$\frac{d}{dt}\Xi(u,v)(t)=\sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{n}E_{i,\alpha}$$

where:

$$E_{i,\alpha}=|q_i^{\alpha+}|X_i^{\alpha+}(\lambda_i^{\alpha+}-\dot{x}_\alpha)-|q_i^{\alpha-}|X_i^{\alpha-}(\lambda_i^{\alpha-}-\dot{x}_\alpha)$$

$$\begin{tikzcd}
u^- \arrow[r, "\Phi_h(x,.)"] \arrow[d, "q^-"'] & u^+ \arrow[d, "q^+"] \\
v^- \arrow[r, "\Phi_h(x,.)"] & v^+
\end{tikzcd}$$

Figure 1:

and λ_i is the speed of the (possibly non-entropic) shock connecting the two states separated by q_i in the decomposition (4.7). We need to establish the bounds

$$\sum_{i=1}^{n}E_{i,\alpha}\quad\leq\quad\mathcal{O}(1)|\sigma_\alpha|\qquad\quad \alpha\in\mathcal{NP}, \tag{4.12}$$

$$\sum_{i=1}^{n}E_{i,\alpha}\quad\leq\quad\mathcal{O}(1)\epsilon|\sigma_\alpha|\qquad\quad \alpha\in\mathcal{R}\cup\mathcal{S}\cup\mathcal{Z}(u), \tag{4.13}$$

$$\sum_{i=1}^{n}E_{i,\alpha}\quad\leq\quad\mathcal{O}(1)|\sigma_\alpha|\qquad\quad \alpha\in\mathcal{Z}(v)\backslash\mathcal{Z}(u). \tag{4.14}$$

that together give

$$\begin{aligned}\frac{d}{dt}\Xi(u,v)(t) &= \mathcal{O}(1)\sum_{\alpha\in\mathcal{NP}}|\sigma_\alpha| + \mathcal{O}(1)\epsilon\sum_{\alpha\in\mathcal{R}\cup\mathcal{S}\cup\mathcal{Z}(u)}|\sigma_\alpha| + \mathcal{O}(1)\sum_{\alpha\in\mathcal{Z}(v)\setminus\mathcal{Z}(u)}|\sigma_\alpha| \\ &= \mathcal{O}(1)\left[\epsilon\left(1+V\left(u(t,\cdot)\right)+V\left(v(t,\cdot)\right)\right)+\int_{\mathbb{R}\setminus\left(-\frac{1}{\epsilon}+h,\frac{1}{\epsilon}\right)}\omega(x)dx\right] = \mathcal{O}(1)\tilde{r}(\epsilon).\end{aligned}$$

Take now $\delta_3 \in (0,\delta_2)$ small enough; in order to derive (4.12), one can proceed in a similar way to [7, 8]. The same approach yields (4.13) in the case of $\alpha \in \mathcal{R}\cup\mathcal{S}$.
We are just left with the case $\alpha \in \mathcal{Z}$. Let $\alpha \in \mathcal{Z}(u)$, this means in particular

$$\dot{x}_\alpha = 0, \qquad \sigma_\alpha = \int_0^h \omega(j_\alpha h + s)ds \geq 0.$$

By Def. 2.1, we have for all positive times except interaction instants (see Figure 1):

$$u(t,x_\alpha+) = \Phi_h(x_\alpha, u(t,x_\alpha-)) \qquad \text{and} \qquad v(t,x_\alpha+) = \Phi_h(x_\alpha, v(t,x_\alpha-)).$$

Let us rewrite $E_{i,\alpha}$ as follows

$$E_{i,\alpha} = \left(X_i^{\alpha+} - X_i^{\alpha-}\right)|q_i^{\alpha-}|\lambda_i^{\alpha-} + X_i^{\alpha+}\left(|q_i^{\alpha+}|\lambda_i^{\alpha+} - |q_i^{\alpha-}|\lambda_i^{\alpha-}\right). \tag{4.15}$$

By Lipschitz continuity and by Lemma 2.1 we get

$$\begin{aligned}\left|\lambda_i^{\alpha+} - \lambda_i^{\alpha-}\right| &= \mathcal{O}(1)\left[\left|u\left(t,x_\alpha+\right) - u\left(t,x_\alpha-\right)\right| + \left|v\left(t,x_\alpha+\right) - v\left(t,x_\alpha-\right)\right|\right] \\ &= \mathcal{O}(1)\int_0^h \omega(j_\alpha h + s)ds = \mathcal{O}(1)\sigma_\alpha,\end{aligned} \tag{4.16}$$

moreover we use Lemma 3.4 to get

$$q_i^{\alpha+} - q_i^{\alpha-} = \mathcal{O}(1)\sigma_\alpha\sum_{k=1}^n |q_k^{\alpha-}|. \tag{4.17}$$

Hence by (4.16), (4.17) we obtain

$$X_i^{\alpha+}\left(|q_i^{\alpha+}|\lambda_i^{\alpha+} - |q_i^{\alpha-}|\lambda_i^{\alpha-}\right) = \mathcal{O}(1)\sigma_\alpha\sum_{k=1}^n |q_k^{\alpha-}|. \tag{4.18}$$

Let now $\mathcal{I}$ and $\mathcal{I}'$ two sets of indexes defined by

$$\begin{aligned}\mathcal{I} &= \left\{i \in \{1,\ldots,n\} :\ q_i^{\alpha+}q_i^{\alpha-} > 0\right\}, \\ \mathcal{I}' &= \left\{i \in \{1,\ldots,n\} :\ q_i^{\alpha+}q_i^{\alpha-} \leq 0\right\}.\end{aligned}$$

For $i \in \mathcal{I}$, the weight $A_i(t,x)$ as defined in [7] does not change across the zero wave located in x_α. Hence the only variation in $B_i(t,x)$ comes from to the term related to the zero waves (see (4.9)) and the non resonance assumption implies

$$\begin{cases} i \in \{1,\ldots,p\}\cap\mathcal{I} & \Rightarrow\ \lambda_i < 0 \text{ and } X_i^{\alpha+} - X_i^{\alpha-} = \kappa_1\sigma_\alpha \\ i \in \{p+1,\ldots,n\}\cap\mathcal{I} & \Rightarrow\ \lambda_i > 0 \text{ and } X_i^{\alpha+} - X_i^{\alpha-} = -\kappa_1\sigma_\alpha \end{cases}$$

which in turn gives

$$\left(X_i^{\alpha+} - X_i^{\alpha-}\right)\lambda_i^{\alpha-} = -\kappa_1\sigma_\alpha|\lambda_i^{\alpha-}| \leq -\kappa_1\sigma_\alpha c \qquad \forall i \in \mathcal{I}. \tag{4.19}$$

Suppose now $i \in \mathcal{I}'$. We can compute

$$\left|q_i^{\alpha+}\right| + \left|q_i^{\alpha-}\right| = \left|q_i^{\alpha+} - q_i^{\alpha-}\right| = \mathcal{O}(1)\sigma_\alpha \sum_{k=1}^n |q_k^{\alpha-}|. \tag{4.20}$$

Therefore we have

$$\sum_{k=1}^n |q_k^{\alpha-}| = \sum_{k\in\mathcal{I}} |q_k^{\alpha-}| + \sum_{k\in\mathcal{I}'} |q_k^{\alpha-}| = \sum_{k\in\mathcal{I}} |q_k^{\alpha-}| + \mathcal{O}(1)\sigma_\alpha \sum_{k=1}^n |q_k^{\alpha-}|$$

and hence

$$\big(1 - \mathcal{O}(1)\sigma_\alpha\big)\sum_{k=1}^n |q_k^{\alpha-}| = \sum_{i\in\mathcal{I}} |q_k^{\alpha-}|. \tag{4.21}$$

This last relation show that, possibly choosing a smaller δ_3 (to make σ_α sufficiently small), the quantity $\sum_{k=1}^n |q_k^{\alpha-}|$ is controlled by the quantity $\sum_{i\in\mathcal{I}} |q_k^{\alpha-}|$.
Finally putting together (4.15), (4.18), (4.19), (4.20), we obtain

$$\begin{aligned}
\sum_{i=1}^n E_{i,\alpha} &= \sum_{i\in\mathcal{I}}\left(X_i^{\alpha+} - X_i^{\alpha-}\right)|q_i^{\alpha-}|\lambda_i^{\alpha-} + \sum_{i\in\mathcal{I}'}\left(X_i^{\alpha+} - X_i^{\alpha-}\right)|q_i^{\alpha-}|\lambda_i^{\alpha-} + \mathcal{O}(1)\mathcal{O}(1)\sigma_\alpha \sum_{k=1}^n |q_k^{\alpha-}| \\
&\leq -\kappa_1\sigma_\alpha c \sum_{i\in\mathcal{I}} |q_i^{\alpha-}| + \mathcal{O}(1)\sigma_\alpha \sum_{i\in\mathcal{I}} |q_i^{\alpha-}| \\
&= \big(-\kappa_1 c + \mathcal{O}(1)\big)\ \sigma_\alpha \sum_{i\in\mathcal{I}} |q_i^{\alpha-}| \leq 0
\end{aligned}$$

by choosing κ_1 large enough. Hence (4.13) is proved.
Concerning (4.14) let $\alpha \in \mathcal{Z}(v)\backslash\mathcal{Z}(u)$. By Def. 2.1, we have for all positive times except interactions.

$$u(t, x_\alpha+) = u(t, x_\alpha-) \qquad \text{and} \qquad v(t, x_\alpha+) = \Phi_h(x_\alpha, v(t, x_\alpha-)).$$

By Lipschitz continuity and by Lemma 2.1 we get

$$\left|\lambda_i^{\alpha+} - \lambda_i^{\alpha-}\right| = \mathcal{O}(1)\left|v\left(t, x_\alpha+\right) - v\left(t, x_\alpha-\right)\right| = \mathcal{O}(1)\sigma_\alpha, \tag{4.22}$$

moreover we use again Lemma 3.4 to derive

$$q_i^{\alpha+} - q_i^{\alpha-} = \mathcal{O}(1)\sigma_\alpha. \tag{4.23}$$

Hence by (4.22), (4.23) we obtain

$$X_i^{\alpha+}\left(|q_i^{\alpha+}|\lambda_i^{\alpha+} - |q_i^{\alpha-}|\lambda_i^{\alpha-}\right) = \mathcal{O}(1)\sigma_\alpha. \tag{4.24}$$

Let now $\mathcal{I}, \mathcal{I}'$ the set of indexes previously defined. We can compute

$$\begin{aligned}
&\left|X_i^{\alpha+} - X_i^{\alpha-}\right| = \kappa_1\sigma_\alpha && \text{if} \quad i \in \mathcal{I}, \\
&\left|q_i^{\alpha+}\right| + \left|q_i^{\alpha-}\right| = \left|q_i^{\alpha+} - q_i^{\alpha-}\right| = \mathcal{O}(1)\sigma_\alpha && \text{if} \quad i \in \mathcal{I}'.
\end{aligned} \tag{4.25}$$

Hence (4.14) is proved putting (4.25), (4.24) in (4.15) and we are done. □

4.3 The emerging semigroups P^h and P

Since (4.11) hold true for fixed $h > 0$ small enough, we can define a family of semigroups P^h describing the weak solutions of the system (3.22) obtained in the limit $\epsilon \to 0$ of our wave front tracking algorithm. This is the purpose of the next Theorem.

Theorem 4.1 *There exists $\delta > 0$ such that if $\|\omega\|_{\mathbf{L}^1(\mathbb{R})}$ is sufficiently small, then for any (small) $h > 0$ there exist a non empty closed domain $\mathcal{D}_h(\delta)$ defined by (4.6) and a unique uniformly Lipschitz semigroup $P^h : [0, +\infty) \times \mathcal{D}_h(\delta) \to \mathcal{D}_h(\delta)$ whose trajectories $u(t, .) = P^h_t u_o$ solve (3.22) and are obtained as limit of any sequence of $\epsilon, h-$approximate solutions as ϵ tends to zero with fixed h. In particular the semigroups P^h satisfy for any $u_o, v_o \in \mathcal{D}_h(\delta)$, $t, s \geq 0$*

$$P^h_0 u_o = u_o, \qquad P^h_t P^h_s u_o = P^h_{s+t} u_o, \tag{4.26}$$

$$\left\| P^h_t u_o - P^h_s v_o \right\|_{\mathbf{L}^1(\mathbb{R})} \leq L \left[\|u_o - v_o\|_{\mathbf{L}^1(\mathbb{R})} + |t - s| \right] \tag{4.27}$$

for some $L > 0$, independent on h.

Proof. It is a direct consequence of lemma 4.1 following the standard arguments of [7, 8] for fixed h. □

So far we have kept h fixed. To obtain the solutions of the original system (1.1) we have to let h tend to zero. Unfortunately we are not able to compare two approximations with different grid size h. Consequently, in this section, we only prove the existence of a Lipschitz semigroup P whose trajectories are solution of (1.1). This matches the main result proved in [26].

Uniqueness will be studied in section 5 through a characterization of the trajectories of P.

Theorem 4.2 *If $\|\omega\|_{\mathbf{L}^1(\mathbb{R})}$ is sufficiently small, there exist a constant $L > 0$, a non empty closed domain $\mathcal{D}$ of integrable functions with small total variation and a semigroup $P : [0, +\infty) \times \mathcal{D} \to \mathcal{D}$ with the following properties*

i) $P_0 u = u, \quad \forall u \in \mathcal{D}; \qquad P_{t+s} u = P_t P_s u, \quad \forall u \in \mathcal{D},\ t, s \geq 0.$

ii) $\|P_s u - P_t v\|_{\mathbf{L}^1(\mathbb{R})} \leq L \Big(|s - t| + \|u - v\|_{\mathbf{L}^1(\mathbb{R})} \Big), \quad \forall u \in \mathcal{D},\ t, s \geq 0.$

iii) for all $u_o \in \mathcal{D}$, the function $u(t, \cdot) = P_t u_o$ is a weak entropy solution of system (1.1).

iv) for some $\delta > 0$ and all $h > 0$ small enough $\mathcal{D} \subset \mathcal{D}_h(\delta)$.

v) There exists a sequence of semigroups P^{h_i} such that $P^{h_i}_t u$ converges in $\mathbf{L}^1$ to $P_t u$ as $i \to +\infty$ for any $u \in \mathcal{D}$.

Proof. Referring to the estimate (4.4), we observe that if δ', δ'' satisfy

$$C_1^2 \delta' + C_o C_1^4 \left(\delta'\right)^2 \leq \delta'' \leq \frac{\delta_3}{C_1}, \tag{4.28}$$

then for u piecewise constant with $\boldsymbol{\Upsilon}_{h'}(u) < \delta'$, one has

$$\begin{aligned} \boldsymbol{\Upsilon}_{h''}(u) &\leq \boldsymbol{V}_{h''}(u) + C_o \boldsymbol{V}_{h''}(u)^2 \leq C_1 \left(\text{Tot.Var.}\, u + \|\omega\|_{\mathbf{L}^1}\right) + C_o \left[C_1 \left(\text{Tot.Var.}\, u + \|\omega\|_{\mathbf{L}^1}\right)\right]^2 \\ &\leq C_1^2 \boldsymbol{V}_{h'}(u) + C_o C_1^4 \boldsymbol{V}_{h'}(u)^2 < C_1^2 \delta' + C_o C_1^4 \left(\delta'\right)^2 \leq \delta''. \end{aligned}$$

Hence $\mathcal{D}_{h'}(\delta') \subset \mathcal{D}_{h''}(\delta'')$ for any $h', h'' > 0$. Now, let us pick up $\delta'' = \frac{\delta_3}{C_1}$ and $\delta' > 0$ such that (4.28) holds, and assume that ω satisfies $C_1^2 \|\omega\|_{\mathbf{L}^1} + C_o C_1^4 \|\omega\|_{\mathbf{L}^1}^2 < \delta'$ to ensure that the domains are not empty. Consequently, there holds

$$\mathcal{D}_{h'}(\delta') \subset \mathcal{D}_{h''}(\delta''), \qquad \forall h', h'' > 0.$$

Hence we have $\mathcal{D}_0 \subset \mathcal{D}_0' \subset \mathcal{D}_1 \subset \mathcal{D}_1'$, where the sets $\mathcal{D}_0,\ \mathcal{D}_0',\ \mathcal{D}_1,\ \mathcal{D}_1'$ are defined by

$$\mathcal{D}_0 = \bigcap_{h>0} \mathcal{D}_h(\delta'), \quad \mathcal{D}_0' = \bigcap_{h>0} \mathrm{cl}\left\{ \bigcup_{k<h} \mathcal{D}_k(\delta') \right\}, \quad \mathcal{D}_1 = \bigcap_{h>0} \mathcal{D}_h(\delta''), \quad \mathcal{D}_1' = \bigcap_{h>0} \mathrm{cl}\left\{ \bigcup_{k<h} \mathcal{D}_k(\delta'') \right\}.$$

Observe that $\mathcal{D}_0 \subset \mathcal{D}_1$ are common domains of definition for the semigroups P^h. Let $\widetilde{B}$ be a countable dense subset of $\mathcal{D}_1$: by a diagonal argument one can prove that there exists a sequence of semigroups P^{h_i} such that for any $u \in \widetilde{B}$ and $t \geq 0$ the sequence $u_i(t,\cdot) = P_t^{h_i} u$ converges in $\mathbf{L}^1{}_{loc}$ to a function $u(t,\cdot) \doteq P_t u$. The regularity (4.27) implies that $P_t^{h_i} u$ converges to a function $P_t u$ for any $u \in \mathcal{D}_1$ and $t \geq 0$. The map P is Lipschitz continuous inside $[0,+\infty) \times \mathcal{D}_1$ and satisfies

$$P_t u \ \in\ \mathcal{D}_1', \quad \text{if } u \in \mathcal{D}_1; \tag{4.29}$$

$$P_t u \ \in\ \mathcal{D}_0', \quad \text{if } u \in \mathcal{D}_0. \tag{4.30}$$

Now we define the domain $\mathcal{D}$ with $\mathcal{D}_0 \subset \mathcal{D} \subset \mathcal{D}_0'$ in the following way

$$\mathcal{D} = \mathrm{cl}\left\{ v \in \mathcal{D}_0' :\ \exists u \in \mathcal{D}_0,\ t \geq 0 \text{ such that } v = P_t u \right\}. \tag{4.31}$$

Since $\mathcal{D} \subset \mathcal{D}_1$, properties *ii), v)* and *iv)* with $\delta = \delta''$ hold. Property *iii)* is a direct consequence of *v)* and Theorem 3.3. By (4.30) and *ii)* we can pass to the limit in (4.26) and obtain that *i)* holds when $u \in \mathcal{D}_0$. By (4.31), if we take $u \in \mathcal{D}$ then there exists two sequences $u_k \in \mathcal{D}_0$, $t_k \geq 0$ such that $P_{t_k} u_k$ converges to u. But for any $t \geq 0$ we have $P_t P_{t_k} u_k = P_{t+t_k} u_k$, hence the Lipschitz continuity of P_t implies that $P_{t+t_k} u_k$ converges to $P_t u$ which therefore belongs to $\mathcal{D}$. So the domain $\mathcal{D}$ turns out to be invariant with respect to P. Hence we can again pass to the limit in (4.26) and obtain that *i)* holds for any $u \in \mathcal{D}$. □

5 Uniqueness of BV entropy solutions

5.1 Some preliminary results

In order to establish our main characterization result, we shall make use of the following technical lemmas whose proofs can be found in [7].

Lemma 5.1 *Let (a,b) a (possibly unbounded) open interval, and let $\hat{\lambda}$ be an upper bound for all wave speeds. If $\bar{u},\ \bar{v} \in \mathcal{D}_h(\delta)$ then for all $t \geq 0$ and $h > 0$, one has*

$$\int_{a+\hat{\lambda}t}^{b-\hat{\lambda}t} \left| \left(P_t^h \bar{u}\right)(x) - \left(P_t^h \bar{v}\right)(x) \right| dx \leq L \int_a^b \left| \bar{u}(x) - \bar{v}(x) \right| dx. \tag{5.1}$$

Lemma 5.2 *Given any interval $I_0 = [a,b]$, define the interval of determinacy*

$$I_t = [a + \hat{\lambda}t, b - \hat{\lambda}t], \qquad t < \frac{b-a}{2\hat{\lambda}}. \tag{5.2}$$

For every Lipschitz continuous map $w : [0,T] \mapsto \mathcal{D}_h(\delta)$ and $h > 0$ there holds

$$\left\| w(t) - P_t^h w(0) \right\|_{\mathbf{L}^1(I_t)} \leq L \int_0^t \left\{ \liminf_{\eta \to 0} \frac{\left\| w(s+\eta) - P_\eta^h w(s) \right\|_{\mathbf{L}^1(I_{s+\eta})}}{\eta} \right\} ds. \tag{5.3}$$

Remark 5.1 *Lemmas 5.1, 5.2 hold also substituting P^h with the operator P, in this case we have obviously to substitute the domains $\mathcal{D}_h(\delta)$ with the domain $\mathcal{D}$ of Theorem 4.2.*

Let now u_ℓ, u_r two nearby states and $\lambda < \hat{\lambda}$; we consider the function

$$v(t,x) = \begin{cases} u_\ell & \text{if} \quad x < \lambda t + x_o \\ u_r & \text{if} \quad x \geq \lambda t + x_o. \end{cases} \tag{5.4}$$

Lemma 5.3 *Call $w(t,x)$ the self-similar solution given by the standard homogeneous Riemann Solver with the Riemann data (2.9).*

(i) In the general case, one has

$$\frac{1}{t} \int_{-\infty}^{+\infty} \left| v(t,x) - w(t,x) \right| dx = \mathcal{O}(1) |u_\ell - u_r| ; \tag{5.5}$$

(ii) Assuming the additional relations $u_r = R_i(\sigma)(u_\ell)$ and $\lambda = \lambda_i(u_r)$ for some $\sigma > 0$, $i = 1, \ldots, n$ one has the sharper estimates

$$\frac{1}{t} \int_{-\infty}^{+\infty} \left| v(t,x) - w(t,x) \right| dx = \mathcal{O}(1) \sigma^2 ; \tag{5.6}$$

(iii) Let $u^* \in \Omega$ and call $\lambda_1^* < \ldots < \lambda_n^*$ the eigenvalues of the matrix $A^* = \nabla f(u^*)$. If for some i there holds $A^*(u_r - u_\ell) = \lambda_i^*(u_r - u_\ell)$ and $\lambda = \lambda_i^*$ in (5.4) then one has

$$\frac{1}{t} \int_{-\infty}^{+\infty} \left| v(t,x) - w(t,x) \right| dx = \mathcal{O}(1) |u_\ell - u_r| \left(|u_\ell - u^*| + |u^* - u_r| \right) ; \tag{5.7}$$

We now prove the next result which is directly concerned with our h-Riemann solver.

Lemma 5.4 *Call $w(t,x)$ the self-similar solution given by the h–Riemann Solver in x_o with the Riemann data (2.9).*

(i) In the general case one has

$$\frac{1}{t} \int_{-\infty}^{+\infty} \left| v(t,x) - w(t,x) \right| dx = \mathcal{O}(1) \left[|u_\ell - u_r| + h \right] ; \tag{5.8}$$

(ii) Assuming the additional relation $u_r = u_\ell + \left[\nabla f\right]^{-1}(u^*) \int_0^h g(x_o + s, u^*) ds$ with $\lambda = 0$ in (5.4) one has the sharper estimate

$$\frac{1}{t} \int_{-\infty}^{+\infty} \left| v(t,x) - w(t,x) \right| dx = \mathcal{O}(1) h \left[h + |u_\ell - u^*| \right] . \tag{5.9}$$

Proof. Estimate *(i)* is a direct consequence of Lemma 2.4.
Let us prove now *(ii)*. Since $\lambda = 0$ we derive

$$\begin{aligned}
\frac{1}{t}\int_{-\infty}^{+\infty} \left|v(t,x) - w(t,x)\right| dx &= \frac{1}{t}\int_{-\hat{\lambda}t}^{0} \left|u_\ell - w(t,x)\right| dx + \frac{1}{t}\int_{0}^{\hat{\lambda}t} \left|u_r - w(t,x)\right| dx \\
&= \mathcal{O}(1)\left[\sum_{\iota=1}^{p} |\sigma_\iota| + \sum_{\iota=p+1}^{n} |\sigma_\iota|\right] = \mathcal{O}(1)|\boldsymbol{\sigma}|.
\end{aligned}$$

And now, we compute

$$\begin{aligned}
|\boldsymbol{\sigma}| &= \left|E\left[h, u_\ell, u_r; x_o\right] - E\left[h, u_\ell, \Phi_h(x_o, u_\ell); x_o\right]\right| \\
&= \mathcal{O}(1)\left|u_r - \Phi_h(x_o, u_\ell)\right|.
\end{aligned}$$

To estimate this last term, we recall the definition of b (2.7) and write for some $y_1,\ y_2$:

$$\begin{aligned}
\left|u_\ell + \left[\nabla f\right]^{-1}(u^*)y_1 - b(y_2, u_\ell)\right| &\leq \mathcal{O}(1)|y_1| \cdot |u^* - u_\ell| \\
&\quad + \mathcal{O}(1)|y_1 - y_2| + \left|u_\ell + \left[\nabla f\right]^{-1}(u_\ell)y_2 - b\left(y_2, u_\ell\right)\right|.
\end{aligned}$$

The function $z(y_2) = u_\ell + \left[\nabla f\right]^{-1}(u_\ell)y_2 - b\left(y_2, u_\ell\right)$ satisfies $z(0) = D_{y_2}z(0) = 0$, hence we have the estimate

$$\left|u_\ell + \left[\nabla f\right]^{-1}(u^*)y_1 - b(y_2, u_\ell)\right| \leq \mathcal{O}(1)\left[|y_1| \cdot |u^* - u_\ell| + |y_1 - y_2| + |y_2|^2\right].$$

If in this last expression we substitute

$$y_1 = \int_0^h g(x_o + s, u^*)ds, \qquad y_2 = \int_0^h g(x_o + s, u_\ell)ds$$

then, we get

$$\left|u_r - \Phi_h(x_o, u_\ell)\right| = O(1)h\left[h + |u_\ell - u^*|\right]$$

which proves (5.9). □

5.2 Characterization of the trajectories of P

In this section we are about to give necessary and sufficient conditions for a function $u(t,\cdot) \in \mathcal{D}$ to coincide with a semigroup's trajectory, proving therefore the uniqueness of the semigroup P and the convergence of all the sequence of the semigroups P^h towards P as $h \to 0$.

We begin by introducing some notations: given a function $u = u(t,x)$ and a point (τ, ξ), we denote by $U^\sharp_{(u;\tau,\xi)}$ the solution of the homogeneous Riemann Problem (2.9) with data

$$u_\ell = \lim_{x \to \xi-} u(\tau, \xi), \qquad u_r = \lim_{x \to \xi+} u(\tau, \xi), \qquad x_o = \xi. \tag{5.10}$$

Moreover we define $U^\flat_{(u;\tau,\xi)}$ as the solution of the linear hyperbolic Cauchy problem with constant coefficients

$$w_t + \widetilde{A}w_x = \widetilde{g}(x), \qquad w(\tau, x) = u(\tau, x), \tag{5.11}$$

with $\widetilde{A} = \nabla f\left(u(\tau,\xi)\right)$, $\widetilde{g}(x) = g\left(x, u(\tau,\xi)\right)$.

A crucial difference between the solution of (5.11) and $P_t^h u$ lies in the treatment of the source term. On the one hand, it is rendered through static waves induced by a Dirac comb. On the other, it is processed in a very usual way. For this reason, it is convenient to approximate the solution of the linear system (5.11) by the solutions of another linear system for which the source is still localized on a lattice.

Define $u^* \doteq u(\tau, \xi)$ and let $\lambda_i = \lambda_i(u^*)$, $r_i = r_i(u^*)$, $l_i = l_i(u^*)$ be respectively the i^{th} eigenvalue, the i^{th} right/left eigenvectors of the matrix $\widetilde{A}$. Thanks to the non–resonance assumption, the solution of linear system (5.11) can be written down explicitly as

$$w(t,x) = \sum_{i=1}^{n} \left\{ \Big\langle l_i, u\left(\tau, x - \lambda_i(t-\tau)\right) \Big\rangle + \frac{1}{\lambda_i} \int_{x-\lambda_i(t-\tau)}^{x} \langle l_i, \widetilde{g}(x') \rangle \, dx' \right\} r_i. \tag{5.12}$$

Let now $\overline{v}$ be a piecewise constant function and define

$$w^h(t,x) = \sum_{i=1}^{n} \left\{ \Big\langle l_i, \overline{v}\left(x - \lambda_i(t-\tau)\right) \Big\rangle + \frac{1}{\lambda_i} \Big\langle l_i, G^h(t,x) \Big\rangle \right\} r_i, \tag{5.13}$$

where the function $G^h(t,x) = \displaystyle\sum_{i=1}^{n} G_i^h(t,x) r_i$ is defined by

$$G_i^h(t,x) = \begin{cases} \displaystyle\sum_{j:\, jh \in \left(x-\lambda_i(t-\tau), x\right)} \int_0^h \langle l_i, \widetilde{g}\,(jh+s) \rangle \, ds & \text{if } \lambda_i > 0 \\ \displaystyle -\sum_{j:\, jh \in \left(x, x-\lambda_i(t-\tau)\right)} \int_0^h \langle l_i, \widetilde{g}\,(jh+s) \rangle \, ds & \text{if } \lambda_i < 0. \end{cases} \tag{5.14}$$

Observe that w^h solves the following Cauchy problem

$$(w^h)_t + \widetilde{A}.(w^h)_x = \sum_{j \in \mathbb{Z}} \delta(x - jh) \int_0^h \widetilde{g}(jh+s)\, ds, \qquad w^h(\tau, x) = \overline{v}(x).$$

It is not difficult to establish that

$$\left| G_i^h(t,x) - \int_{x-\lambda_i(t-\tau)}^{x} \langle l_i, \widetilde{g}(x') \rangle \, dx \right| = \mathcal{O}(1)h. \tag{5.15}$$

Hence, for any $a, b \in \mathbb{R}$ with $a < b$, we have the error estimate

$$\int_{a+\hat{\lambda}(t-\tau)}^{b-\hat{\lambda}(t-\tau)} \left| w(t,x) - w^h(t,x) \right| dx \le \mathcal{O}(1) \left[\int_a^b \left| u(\tau,x) - \overline{v}(x) \right| dx + (b-a)h \right]. \tag{5.16}$$

From (5.13), (5.14), it is easy to see that $w^h(t,x)$ is piecewise constant with discontinuities occurring along finitely many lines on compact sets in the (t,x) plane for $t \ge \tau$. Only finitely many wave front interactions occur in a compact set, and jumps can be of two types: contact discontinuities or zero waves. The zero waves are located at the points jh, $j \in \mathbb{Z}$ and satisfy

$$w^h(t, jh+) - w^h(t, jh-) = \left[\nabla f\right]^{-1}(u^*) \int_{jh}^{(j+1)h} \widetilde{g}\,(jh+s)\, ds. \tag{5.17}$$

Conversely a contact discontinuity of the i^{th} family located at the point $x_\alpha(t)$ satisfies $\dot{x}_\alpha(t) = \lambda_i(u^*)$ and

$$w^h(t, x_\alpha(t)+) - w^h(t, x_\alpha(t)-) = \sigma r_i(u^*) \tag{5.18}$$

for some $\sigma \in \mathbb{R}$.

Now, we can state the following important result.

Theorem 5.1 *Let $P : \mathcal{D} \times [0, +\infty) \to \mathcal{D}$ be the semigroup of Theorem 4.2 and let $\hat{\lambda}$ be an upper bound for all wave speeds. Then every trajectory $u(t, \cdot) = P_t u_0$, $u_0 \in \mathcal{D}$, satisfies the following conditions at every $\tau \geq 0$.*

(i) For every ξ, one has

$$\lim_{\theta \to 0} \frac{1}{\theta} \int_{\xi - \theta\hat{\lambda}}^{\xi + \theta\hat{\lambda}} \left| u(\tau + \theta, x) - U^{\sharp}_{(u;\tau,\xi)}(\theta, x) \right| dx = 0. \tag{5.19}$$

(ii) There exists a constant C such that, for every $a < \xi < b$ and $0 < \theta < \frac{b-a}{2\hat{\lambda}}$, one has

$$\frac{1}{\theta} \int_{a+\theta\hat{\lambda}}^{b-\theta\hat{\lambda}} \left| u(\tau + \theta, x) - U^{\flat}_{(u;\tau,\xi)}(\tau + \theta, x) \right| dx \leq C \Big[\text{Tot.Var.} \left\{ u(\tau); (a, b) \right\} + (b - a) \Big]^2. \tag{5.20}$$

Viceversa let $u : [0, T] \to \mathcal{D}$ be Lipschitz continuous as a map with values in $\mathbf{L}^1(\mathbb{R}, \mathbb{R}^n)$ and assume that the conditions (i), (ii) hold at almost every time τ. Then $u(t, \cdot)$ coincides with a trajectory of the semigroup P.

Proof.

Part 1: Necessity Given a semigroup trajectory $u(t, \cdot) = P_t \bar{u}$, $\bar{u} \in \mathcal{D}$ we now show that the conditions *(i)*, *(ii)* hold for every $\tau \geq 0$.

To prove *(i)*, let $h,\ \theta,\ \varepsilon > 0$ be given and define the open intervals

$$J_t = J_t^- \cup J_t^o \cup J_t^+$$

with

$$\begin{aligned} J_t^- &= \Big(\xi - (2\theta - t + \tau)\,\hat{\lambda}, \xi - (t - \tau)\hat{\lambda} \Big)\,; \\ J_t^o &= \Big[\xi - (t - \tau)\hat{\lambda}, \xi + (t - \tau)\hat{\lambda} \Big]\,; \\ J_t^+ &= \Big(\xi + (t - \tau)\hat{\lambda}, \xi + (2\theta - t + \tau)\,\hat{\lambda} \Big)\,. \end{aligned}$$

Now, choose a piecewise constant function $\bar{v}$ such that

$$\bar{v}(\xi) = u(\tau, \xi), \qquad \bar{v}(\xi\pm) = u(\tau, \xi\pm), \tag{5.21}$$

$$\int_{J_\tau} \left| \bar{v}(x) - u(\tau, x) \right| \, dx \leq \varepsilon, \tag{5.22}$$

$$\text{Tot.Var.} \left\{ \bar{v}; J_\tau \right\} \leq \text{Tot.Var.} \left\{ u(\tau); J_\tau \right\}. \tag{5.23}$$

Define the function $U^{\sharp,\varepsilon}_{(u;\tau,\xi)}(\theta, x)$ as the piecewise constant function obtained from $U^{\sharp}_{(u;\tau,\xi)}(\theta, x)$ dividing the centered rarefaction waves in equal parts and replacing them by rarefaction fans containing wave fronts whose strength is less than ε. Observe that there holds:

$$\frac{1}{t} \int_{-\infty}^{+\infty} \left| U^{\sharp,\varepsilon}_{(u;\tau,\xi)}(\theta, x) - U^{\sharp}_{(u;\tau,\xi)}(\theta, x) \right| \, dx = \mathcal{O}(1)\varepsilon. \tag{5.24}$$

Finally for $t \geq \tau$, we define the function

$$v^\varepsilon(t,x) = \begin{cases} \bar{v}\left(x + (t-\tau)\hat{\lambda}\right) & \text{if} \quad x \in J_t^- \\ U^{\sharp,\varepsilon}_{(u;\tau,\xi)}(t-\tau,x) & \text{if} \quad x \in J_t^o \\ \bar{v}\left(x - (t-\tau)\hat{\lambda}\right) & \text{if} \quad x \in J_t^+. \end{cases} \tag{5.25}$$

Observe that v^ε provides an approximate solution of the homogeneous system (2.10) in the region were $|x-\xi| \leq \hat{\lambda}(t-\tau)$. Moreover, for every t, the fronts of v^ε contained in the interval J_t^+ coincide with the fronts of $\bar{v}$ located inside $\left(\xi, \xi + 2(\theta - t + \tau)\hat{\lambda}\right)$ and hence

$$\text{Tot.Var.}\left\{v^\varepsilon(t); J_t^+\right\} \leq \text{Tot.Var.}\left\{\bar{v}; J_\tau^+\right\}. \tag{5.26}$$

Applying estimate (5.3) to the function v^ε we obtain

$$\int_{J_{\tau+\theta}} \left| v^\varepsilon(\tau+\theta,x) - \left(P_\theta^h \bar{v}\right)(x)\right| dx \leq L \int_\tau^{\tau+\theta} \liminf_{\eta\to 0} \frac{\left\| v^\varepsilon(t+\eta) - P_\eta^h v^\varepsilon(t)\right\|_{\mathbf{L}^1\left(J_{t+\eta}\right)}}{\eta} dt. \tag{5.27}$$

The discontinuities of v^ε do not cross the Dirac comb for almost all times $t \in (\tau, \tau+\theta)$. Therefore we compute for such a time t:

$$\begin{aligned} &\frac{1}{\eta}\int_{J_{t+\eta}} \left| v^\varepsilon(t+\eta,x) - \left(P_\eta^h v^\varepsilon(t)\right)(x)\right| dx \qquad (5.28) \\ &\qquad = \frac{1}{\eta}\left\{\int_{J_{t+\eta}^-} + \int_{J_{t+\eta}^o} + \int_{J_{t+\eta}^+}\right\} \left| v^\varepsilon(t+\eta,x) - \left(P_\eta^h v^\varepsilon(t)\right)(x)\right| dx. \end{aligned}$$

Using (5.8) with $u_\ell = u_r$ and (5.6), we get for η small

$$\frac{1}{\eta}\int_{J_{t+\eta}^o} \left| v^\varepsilon(t+\eta,x) - \left(P_\eta^h v^\varepsilon(t)\right)(x)\right| dx = \mathcal{O}(1)\left[2\hat{\lambda}(t-\tau) + h\right] + \mathcal{O}(1)\varepsilon. \tag{5.29}$$

Moreover, following [7] and using (5.5) and (5.8) with $u_\ell = u_r$ we get

$$\begin{aligned} \frac{1}{\eta}\int_{J_{t+\eta}^+} \left| v^\varepsilon(t+\eta,x) - \left(P_\eta^h v^\varepsilon(t)\right)(x)\right| dx \;&=\; \mathcal{O}(1)\left[2\hat{\lambda}(\tau+\theta-t) + h\right] \qquad (5.30) \\ &\quad + \mathcal{O}(1)\,\text{Tot.Var.}\left\{v^\varepsilon(t); J_t^+\right\}. \end{aligned}$$

A similar estimate holds for the interval $J_{t+\eta}^-$. Putting together (5.28), (5.29), (5.30), (5.26) one has

$$\frac{1}{\eta}\int_{J_{t+\eta}} \left| v^\varepsilon(t+\eta,x) - \left(P_\eta^h v^\varepsilon(t)\right)(x)\right| dx = \mathcal{O}(1)\left[h + \theta + \varepsilon + \text{Tot.Var.}\left\{\bar{v}; J_\tau^- \cup J_\tau^+\right\}\right].$$

Hence by (5.27), (5.21), (5.23), we have

$$\int_{J_{\tau+\theta}} \left| v^\varepsilon(\tau+\theta,x) - \left(P_\theta^h \bar{v}\right)(x)\right| dx = \mathcal{O}(1)\theta\left[h + \theta + \varepsilon + \text{Tot.Var.}\left\{u(\tau); J_\tau^- \cup J_\tau^+\right\}\right]. \tag{5.31}$$

Finally we take the sequence P^{h_i} converging to P. Using (5.1) and (5.22) we have

$$\left\| P_\theta^{h_i} u(\tau) - P_\theta^{h_i}\bar{v}\right\|_{\mathbf{L}^1\left(J_{\tau+\theta}\right)} = \mathcal{O}(1)\left\| u(\tau) - \bar{v}\right\|_{\mathbf{L}^1(J_\tau)} = \mathcal{O}(1)\varepsilon$$

and so by (5.24), (5.31), since $U^\sharp_{(u;\tau,\xi)}(\theta,x) = v^\varepsilon(\tau+\theta,x)$ we derive:

$$\frac{1}{\theta}\int_{\xi-\theta\hat\lambda}^{\xi+\theta\hat\lambda}\left|u(\tau+\theta,x) - U^\sharp_{(u;\tau,\xi)}(\theta,x)\right| dx = \mathcal{O}(1)\frac{\left\|P_\theta u(\tau) - P_\theta^{h_i}u(\tau)\right\|_{\mathbf{L}^1(\mathbb{R})}}{\theta}$$
$$+\mathcal{O}(1)\left[\frac{\varepsilon}{\theta} + h_i + \theta + \varepsilon + \text{Tot.Var.}\left\{u(\tau); J_\tau^- \cup J_\tau^+\right\}\right].$$

The left hand side of the previous estimate does not depend on ε and h_i, hence there holds

$$\frac{1}{\theta}\int_{\xi-\theta\hat\lambda}^{\xi+\theta\hat\lambda}\left|u(\tau+\theta,x) - U^\sharp_{(u;\tau,\xi)}(\theta,x)\right| dx = O(1)\left[\theta + \text{Tot.Var.}\left\{u(\tau); J_\tau^- \cup J_\tau^+\right\}\right].$$

Taking the limit as $\theta \to 0$ in the previous estimate yields (5.19).

To prove *(ii)* let $\theta > 0$ and a point (τ,ξ) be given together with an open interval (a,b) containing ξ. Fix $\varepsilon > 0$ and choose a piecewise function $\bar v \in \mathcal{D}$ satisfying (5.21) together with

$$\int_a^b \left|\bar v(x) - u(\tau,x)\right|\, dx \le \varepsilon, \tag{5.32}$$

$$\text{Tot.Var.}\left\{\bar v; (a,b)\right\} \le \text{Tot.Var.}\left\{u(\tau); (a,b)\right\} \tag{5.33}$$

Let now w^h be defined by (5.13). From (5.16), (5.32) we have the estimate

$$\int_{a+\theta\hat\lambda}^{b-\theta\hat\lambda}\left|U^\flat_{(u;\tau,\xi)}(\tau+\theta,x) - w^h(\tau+\theta,x)\right| dx \le \mathcal{O}(1)\left[\varepsilon + h(b-a)\right]. \tag{5.34}$$

Using (5.2), (5.3) we get

$$\int_{a+\theta\hat\lambda}^{b-\theta\hat\lambda}\left|w^h(\tau+\theta,x) - \left(P_\theta^h w^h(\tau)\right)(x)\right| dx \le L\int_\tau^{\tau+\theta}\liminf_{\eta\to0}\frac{\left\|w^h(t+\eta) - P_\eta^h w^h(t)\right\|_{\mathbf{L}^1(I_{t-\tau+\eta})}}{\eta}dt. \tag{5.35}$$

Let $t \in (\tau,\tau+\theta)$ be a time for which there is no interaction in w^h; in particular, discontinuities which travel with a non-zero velocity do not cross the Dirac comb (this happens for almost all t). We observe that by the explicit formula (5.13), there holds

$$\text{Tot.Var.}\left\{w^h; I_{t-\tau}\right\} = \mathcal{O}(1)\left[\text{Tot.Var.}\left\{\bar v; (a,b)\right\} + (b-a)\right] \tag{5.36}$$

$$\left|w^h(t,x) - \bar v(\xi)\right| = \mathcal{O}(1)\left[\text{Tot.Var.}\left\{\bar v; (a,b)\right\} + h + t - \tau\right] \tag{5.37}$$

From (5.7), (5.18), (5.9), (5.17), (5.36), (5.37), we obtain

$$\frac{1}{\eta}\int_{I_{t-\tau+\eta}}\left|w^h(t+\eta,x) - \left(P_\eta^h w^h(t)\right)(x)\right| dx = \mathcal{O}(1)\left[(b-a)+h\right]\left(\text{Tot.Var.}\left\{\bar v; (a,b)\right\} + h + t - \tau\right)$$

$$+\left[(b-a) + \text{Tot.Var.}\left\{\bar v; (a,b)\right\}\right]\cdot\left(\text{Tot.Var.}\left\{\bar v; (a,b)\right\} + h + t - \tau\right). \tag{5.38}$$

Let now P^{h_i} be the subsequence converging to P. Since $w^h(\tau) = \bar{v}$ using (5.34), (5.35), (5.38), (5.33), we get

$$\frac{1}{\theta}\int_{a+\theta\hat{\lambda}}^{b-\theta\hat{\lambda}} \left|u(\tau+\theta,x) - U^{\flat}_{(u;\tau,\xi)}(\tau+\theta,x)\right| dx = \mathcal{O}(1)\frac{1}{\theta}\Big[\varepsilon + h_i(b-a) + \|u(\tau)-\bar{v}\|_{\mathbf{L}^1(I_0)} + \|P_\theta u(\tau) - P_\theta^{h_i}u(\tau)\|_{\mathbf{L}^1(\mathbb{R})}\Big] + \mathcal{O}(1)\Big[\mathrm{Tot.Var.}\,\big\{u(\tau);(a,b)\big\} + (b-a) + h_i\Big]\cdot\Big[\mathrm{Tot.Var.}\,\big\{u(\tau);(a,b)\big\} + h_i + \theta\Big].$$

So for $\varepsilon,\ h_i \to 0$ we obtain the desired inequality since $\theta \le (b-a)/(2\hat{\lambda})$.

Part 2: Sufficiency By Remark 5.1 we can apply (5.3) to P and hence the proof for the homogeneous case presented in [7] can be followed exactly for our case, hence it will be not repeated here. □

Proof of Theorem 1.1 It is now a direct consequence of Theorem 4.2 and Theorem 5.1. □

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