A Class of Approximate Riemann Solvers and Their Relation to Relaxation Schemes

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Abstract. We show that a simple relaxation scheme of the type proposed by Jin and Xin [Comm. Pure Appl. Math. 48(1995) pp. 235–276] can be reinterpreted as defining a particular approximate Riemann solver for the original system of m conservation laws. Based on this observation, a more general class of approximate Riemann solvers is proposed which allows as many as $2m$ waves in the resulting solution. These solvers are related to more general relaxation systems and connections with several other standard solvers are explored. The added flexibility of $2m$ waves may be advantageous in deriving new methods. Some potential applications are explored for problems with discontinuous flux functions or source terms.

1 Introduction

Consider the conservation law

$$u_t + f(u)_x = 0 \tag{1.1}$$

where $u \in \mathbb{R}^m$ and the flux function $f(u)$ may be nonlinear. We assume the system is hyperbolic, so the Jacobian $f'(u)$ is diagonalizable with real eigenvalues. Many finite-volume methods for this conservation law are based on solving Riemann problems at cell interfaces between the neighboring cell averages. The Riemann problem consists of the equation (1.1) together with piecewise constant data with a single discontinuity between two values U_l and U_r. Often some approximate Riemann solver is used, which consists of M_w waves $\mathcal{W}^p \in \mathbb{R}^m$ propagating at some speeds $s^p \in \mathbb{R}$ for $p = 1,\ 2,\ \ldots, M_w$. The vectors $\mathcal{W}^p$ represent the jump in u across each wave and must sum up to the total jump,

$$U_r - U_l = \sum_{p=1}^{M_w} \mathcal{W}^p. \tag{1.2}$$

For a linear system

$$u_t + Au_x = 0, \tag{1.3}$$

for example, the exact Riemann solution has this form, with $M_w = m$ and the s^p the eigenvalues of A. Each $\mathcal{W}^p$ is an associated eigenvector. The waves can be determined by first choosing some specific set of eigenvectors r^p and then solving the linear system

$$U_r - U_l = \sum_{p=1}^{m} \alpha^p r^p \tag{1.4}$$

for the scalar coefficients α^p and finally setting $\mathcal{W}^p = \alpha^p r^p$. We find that $\alpha = R^{-1}(U_r - U_l)$ where R is the matrix of right eigenvectors. In this case we also have

$$AU_r - AU_l = A(U_r - U_l) = \sum_{p=1}^{m} s^p \mathcal{W}^p. \tag{1.5}$$

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More generally, suppose for a nonlinear flux f that the decomposition (1.2) and choice of speeds s^p have the property that

$$f(U_r) - f(U_l) = \sum_{p=1}^{M_w} s^p \mathcal{W}^p, \tag{1.6}$$

which is a generalization of (1.5). Then an upwind algorithm and high-resolution variants can be defined in terms of these waves and speeds, for example using the wave-propagation approach of [36], which is reviewed in Section 8.

One important approximate Riemann solver of this form was proposed by Roe [45] and has been extensively used. The Jacobian matrix $f'(u)$ is evaluated at a special average of U_l and U_r, defining a matrix $\hat{A}$ with the property that

$$\hat{A}(U_r - U_l) = f(U_r) - f(U_l). \tag{1.7}$$

The eigenvalues $\hat{\lambda}^p$ of $\hat{A}$ are used as the wave speeds s^p and the eigenvectors of $\hat{A}$ are used to define the waves. Then (1.6) follows from (1.7). In this case again $M_w = m$. If the true Riemann solution contains a transonic rarefaction wave, however, then this approximate solution may need to be modified using an "entropy fix" as described in Section 5. One possible approach is to replace the corresponding wave by a pair of waves moving in opposite directions. In this case we could then view the approximate Riemann solution as consisting of $M_w = m + 1$ waves.

In this paper we explore a more general class of Riemann solvers in which $M_w = 2m$ waves are used, although some of these may coalesce so that the number can be reduced. Instead of the standard decomposition of the form (1.4) we perform a decomposition of the form

$$\begin{bmatrix} U_r - U_l \\ f(U_r) - f(U_l) \end{bmatrix} = \alpha^1 \begin{bmatrix} w^1 \\ \phi^1 \end{bmatrix} + \alpha^2 \begin{bmatrix} w^2 \\ \phi^2 \end{bmatrix} + \cdots + \alpha^{2m} \begin{bmatrix} w^{2m} \\ \phi^{2m} \end{bmatrix}, \tag{1.8}$$

where the vectors $w^p,\ \phi^p \in \mathbb{R}^m$ are first chosen in some manner (several possibilities are discussed below) along with speeds s^p for $p = 1,\ 2, \ldots,\ 2m$. The scalar coefficients $\alpha^1, \ldots,\ \alpha^{2m}$ are then determined by solving the $2m \times 2m$ linear system defined by (1.8). This is possible provided the $2m$ vectors appearing in (1.8) are linearly independent. The waves needed for updating cell averages are then

$$\mathcal{W}^p = \alpha^p w^p, \qquad p = 1,\ 2,\ \ldots,\ 2m, \tag{1.9}$$

with speeds s^p. We will refer to this as a *relaxation Riemann solver* for reasons which will be made clear in Section 2.

In general we will choose

$$\phi^p = s^p w^p, \tag{1.10}$$

although this is not entirely necessary (see Section 8). The rationale for this choice is that (1.6) will then automatically be satisfied by the resulting waves, no matter how the w^p and s^p are chosen. This is accomplished by introducing more degrees of freedom ($2m$ rather than m) and requiring that the jump in f is decomposed in a manner consistent with the jump in u. Note that (1.10) is reminiscent of the Rankine-Hugoniot jump conditions relating the jump in f across a shock to the jump in u.

One goal in this paper is to show that by considering a decomposition of the form (1.8) it is possible to gain a more unified view of several approximate Riemann solvers already in use, including the Roe solver, the Roe solver with an entropy fix, and the simple HLL and HLLE solvers. These solvers involve fewer than $2m$ waves in their natural implementation, but can be reinterpreted in the above form with some waves coalesced into fewer waves. See Sections 3 through 6.

We have also found that the relaxation scheme of Jin and Xin [26] (at least in the limit as the relaxation time vanishes) is closely related to an approximate Riemann solver of this type; see Section 2.

Indeed the decomposition (1.8) was first suggested to us by an attempt to generalize and improve the relaxation scheme. Another goal of this paper is to explain and explore this connection. The approximate Riemann solver interpretation may lead to new insights into relaxation schemes and how they might be improved. Conversely, methods based on standard approximate Riemann solvers such as Roe's can be reinterpreted as modified relaxation schemes. This may aid in the theoretical analysis of these methods (see Section 7).

The added flexibility of specifying $2m$ waves rather than m also allows some interesting new possibilities in deriving approximate Riemann solvers. Exploring some of these is the final goal of this paper. In Sections 9 and 10 we look at possible applications to conservation laws with discontinuous coefficients and with source terms.

2 Relaxation schemes

Relaxation schemes have recently been widely applied and studied, see for example [1], [9], [10], [18], [24], [28], [31], [38]. Another related class of numerical methods are the "kinetic schemes" based on the Boltzmann equation and relaxation towards equilibrium. See [2] for one discussion of these connections. Bouchut [6], [7] has recently presented an interpretation of kinetic schemes as approximate Riemann solvers for flux-vector splitting methods, though these take a rather different form from what we introduce here.

The original relaxation scheme of Jin and Xin [26] is based on replacing the conservation law (1.1) by a larger system of dimension $2m$,

$$\begin{aligned} u_t + v_x &= 0 \\ v_t + D^2 u_x &= \frac{1}{\tau}(f(u) - v), \end{aligned} \tag{2.1}$$

where $u,\ v \in \mathbb{R}^m$ and $D^2 \in \mathbb{R}^{m\times m}$ is a positive definite matrix. Jin and Xin choose it to be a diagonal matrix with positive diagonal elements and call it A, but we use A for linear systems such as (1.3) and call the matrix D^2 in (2.1) to avoid square roots in formulas below, and also because this form will be generalized below to a situation where D^2 is replaced by the product of two different matrices. We assume without loss of generality that D itself has positive eigenvalues $d^j > 0$ for $j = 1,\ 2,\ \ldots,\ m$. Then the matrix

$$\begin{bmatrix} 0 & I \\ D^2 & 0 \end{bmatrix} \tag{2.2}$$

appearing as the coefficient matrix on the left-hand side of the system (2.1) has $2m$ eigenvalues given by the pairs $\pm d^j$ for $j = 1,\ 2,\ \ldots,\ m$.

The original conservation law has been replaced by a linear hyperbolic system with a relaxation source term which rapidly drives $v \to f(u)$ when the relaxation time $\tau > 0$ is small. If we set $v \equiv f(u)$ then the first equation of (2.1) reduces to the original conservation law. In some cases it can be shown analytically that solutions to (2.1) approach solutions to the original problem as $\tau \to 0$. A general necessary condition for such convergence is that a *subcharacteristic condition* be satisfied. For (2.1), this requires that every eigenvalue λ of $f'(u)$ satisfies

$$-d_{\max} \le \lambda \le d_{\max} \tag{2.3}$$

where $d_{\max} = \max_j d^j$ is the spectral radius of D. This insures that the characteristic speeds of the hyperbolic part of (2.1) are at least as large as the characteristic speeds of the original problem. See [11], [21], [39], [41], [42], [49], [48] for some discussions of this condition and convergence properties.

The relaxation scheme we describe is not exactly the same as Jin and Xin's, but is the simplest variant of it. We use a fractional step method to advance the solution by one time step Δt:

1. Given U^n and V^n, apply some finite-volume method to update these over time Δt by solving the homogeneous linear hyperbolic system

$$\begin{bmatrix} u \\ v \end{bmatrix}_t + \begin{bmatrix} 0 & I \\ D^2 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_x = 0. \tag{2.4}$$

   Call the new values U^* and V^*.

2. Update U^*, V^* to U^{n+1}, V^{n+1} by solving the equations

$$\begin{aligned} u_t &= 0 \\ v_t &= \frac{1}{\tau}(f(u) - v) \end{aligned} \tag{2.5}$$

   over time Δt.

Clearly we have $U^{n+1} = U^*$. The variable v evolves according to a simple linear ODE since $f(u)$ is constant during this evolution. An implicit Runge-Kutta method was recommended in [26], or alternatively this can be solved exactly,

$$V^{n+1} = f(U^{n+1}) + e^{-\Delta t/\tau}(V^* - f(U^{n+1})). \tag{2.6}$$

We will only be concerned with the limiting case $\tau \to 0$ in which case we can simply set

$$V^{n+1} = f(U^{n+1}) = f(U^*). \tag{2.7}$$

(This is what is called the *relaxed scheme* in [26].) The relaxation scheme then consists of these two steps:

1. Solve a linear hyperbolic equation to update both U and V to get U^{n+1} and V^*,
2. Ignore V^* and set $V^{n+1} = f(U^{n+1})$.

In the standard implementation one works with a larger system of $2m$ equations, and updates both U and V using a hyperbolic method, even though the resulting V^* is then ignored. A slightly different and more efficient implementation makes it clear how this scheme is related to approximate Riemann solvers. Since we know $V^{n+1} = f(U^{n+1})$, we do not need to keep track of V separately and there is no need to update V^n to V^*. Instead we view the relaxation scheme as a way to update U^n to U^{n+1} and store only these m-vectors as in any other finite-volume method for the original conservation law. We do still work with the system (2.4) in the process of updating these vectors, but we now view this system only as a means for defining an approximate Riemann solver. Given states U_l and U_r we compute $V_l = f(U_l)$ and $V_r = f(U_r)$ and then solve the Riemann problem for (2.4) with data

$$\begin{bmatrix} U_l \\ f(U_l) \end{bmatrix}, \qquad \begin{bmatrix} U_r \\ f(U_r) \end{bmatrix}. \tag{2.8}$$

Solving this linear Riemann problem requires an eigendecomposition exactly of the form (1.8), where the vectors appearing in this decomposition are the $2m$ eigenvectors of the matrix (2.2). Once we have performed this decomposition, the waves $\mathcal{W}^p = \alpha^p w^p$ and speeds s^p (the corresponding eigenvalues of (2.2)) are used to update U^n to U^{n+1}. The relaxation process has disappeared altogether. Of course one could also store V^n and update it to V^* using the waves $\alpha^p \phi^p$, but this would be wasted effort since V^* is never used.

Let z^j be the eigenvector of the matrix D that corresponds to the eigenvalue d^j, for $j = 1,\ 2,\ \ldots,\ m$. Then for each j the matrix (2.2) has a pair of linearly independent eigenvectors

$$\begin{bmatrix} w^{2j-1} \\ \phi^{2j-1} \end{bmatrix} = \begin{bmatrix} z^j \\ -d^j z^j \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} w^{2j} \\ \phi^{2j} \end{bmatrix} = \begin{bmatrix} z^j \\ +d^j z^j \end{bmatrix}, \tag{2.9}$$

with eigenvalues $s^{2j-1} = -d^j$ and $s^{2j} = +d^j$ respectively. Note that these vectors satisfy the assumption (1.10).

In particular, for the case of a diagonal matrix $D = \text{diag}(d^1,\ d^2, \dots,\ d^m)$ as used in [26], we have $z^j = e^j$, the jth unit vector. The elements d^j of D must be chosen so that the subcharacteristic condition (2.3) is satisfied. For the Euler equations, Jin and Xin choose them in [26] as some approximations to the eigenvalues $u - c$, u, and $u + c$ of the Jacobian matrix (where u is the velocity and c the sound speed).

Note that when $z^j = e^j$, the decomposition (1.8) splits into m decoupled 2×2 problems of the form

$$\begin{bmatrix} U_r^j - U_l^j \\ V_r^j - V_l^j \end{bmatrix} = \alpha^{2j-1} \begin{bmatrix} 1 \\ -d^j \end{bmatrix} + \alpha^{2j} \begin{bmatrix} 1 \\ d^j \end{bmatrix} \tag{2.10}$$

involving only the jth component of the vector U and the jth component of the flux vector $V = f(U)$. Solving this system gives

$$\begin{aligned} \alpha^{2j-1} &= \frac{1}{2d^j}(d^j(U_r^j - U_l^j) - (V_r^j - V_l^j)) \\ \alpha^{2j} &= \frac{1}{2d^j}(d^j(U_r^j - U_l^j) + (V_r^j - V_l^j)). \end{aligned} \tag{2.11}$$

When applied to the Euler equations, for example, the density and its flux are split into one pair of waves with speeds $\pm d^1$, the momentum is split into a second pair of waves with speeds $\pm d^2$, and the energy is split into a third pair of waves with speeds $\pm d^3$. The splitting of each component is done in such a way that conservation is maintained, which is guaranteed by the manner in which the flux differences are split along with the components of $U_r - U_l$ so that (1.6) is satisfied. In the next section we will show that this can be viewed as a generalization of the HLL Riemann solver.

3 Relation to the HLL solver

A simple approximate Riemann solver was discussed by Harten, Lax, and van Leer [20]. This HLL solver consists of approximating the Riemann solution by two waves (regardless of the dimension m of the system) with some speeds a_l and a_r chosen to approximate the minimum and maximum characteristic speeds of the system. It is often called the HLLE solver when the specific choice of a_l and a_r recommended by Einfeldt [14] is used. The wave strengths are

$$\mathcal{W}^1 = U_m - U_l, \qquad \mathcal{W}^2 = U_r - U_m, \tag{3.1}$$

where the middle state U_m is chosen to preserve conservation by requiring

$$(a_r - a_l)U_m = a_r U_r - a_l U_l - (f(U_r) - f(U_l)). \tag{3.2}$$

This yields

$$U_m = \frac{1}{a_r - a_l}\left(a_r U_r - a_l U_l - (f(U_r) - f(U_l))\right). \tag{3.3}$$

Suppose for the moment that $a_l = -a_r$ with $a_r > 0$. Then this is equivalent to the relaxation Riemann solver described above if we take D to be the diagonal matrix $D = a_r I$, so that

$$s^{2j-1} = -d^j = a_l \quad \text{and} \quad s^{2j} = d^j = a_r. \tag{3.4}$$

Then the jth system (2.10) splits the jth component of $U_r - U_l$ into 2 waves propagating at speeds a_l and a_r. After doing this for all m components we obtain $2m$ waves, each carrying a jump in only one component of u. But m of these waves travel at the same speed a_l and the other m at speed a_r and so we can lump these together into 2 waves, which then must be the HLL waves (3.1) since both approaches are conservative. We can verify directly that these are the same by using the solution (2.11). The left-going wave carries a jump α^{2j-1} in the jth component and so the intermediate state U_m has jth component

$$\begin{aligned} U_m^j &= U_l^j + \alpha^{2j-1} \\ &= \frac{1}{2d^j}(d^j(U_r^j + U_l^j) - (V_r^j - V_l^j)) \\ &= \frac{1}{2}(U_r^j + U_l^j) - \frac{1}{2d^j}(f(U_r^j) - f(U_l^j)). \end{aligned} \tag{3.5}$$

Since $d^j = a_r = -a_l$ we have $2d^j = a_r - a_l$ and this agrees with the jth component of (3.3).

We thus see that the relaxation scheme in the case $D = dI$ amounts to using the HLL Riemann solver with $a_l = -d$ and $a_r = d$. Let $\lambda^j(u)$ denote the jth eigenvalue of the Jacobian matrix $f'(u)$. If we choose

$$d = \max_{1\le j\le m} \left(\max(|\lambda^j(U_l)|,\ |\lambda^j(U_r)|)\right) \tag{3.6}$$

as an upper bound on the characteristic speeds (assuming the system is genuinely nonlinear) and then apply the first-order upwind method together with this approximate Riemann solver, the resulting method is simply Rusanov's method, as discussed in [46], for example. This method is also known as the Local Lax-Friedrichs (LLF) method. If we choose $d = \Delta x/\Delta t$, an upper bound on all possible wave speeds provided the CFL condition is satisfied for the grid being used, then this method reduces to the classical Lax-Friedrichs (LxF) method. We note in passing that the LxF and LLF methods can be extended to second-order accuracy to obtain the central schemes of Nessyahu & Tadmor [43] and Kurganov & Tadmor [30] respectively, and connections between these methods and relaxation schemes are briefly discussed in the introduction to [30].

If D is diagonal but the diagonal elements d^j are not equal (as in the choice of Jin and Xin [26]), we can view the relaxation scheme as a generalization of the HLL solver in which separate speeds $a_r^j = -a_l^j = d^j$ are chosen for each component of the vector u. It is not clear that this generalization will be an improvement, however, since in coupled systems of conservation laws we do not expect information in different components of u to propagate at different speeds. Rather, it is different eigencomponents of u (based on the eigenvectors of $f'(u)$) which propagate at different speeds. This suggests that a more substantial improvement might be made by replacing the unit vectors $z^j = e^j$ used in deriving HLL from (2.9) by some approximations to the eigenvectors of $f'(u)$. Generalizations of this form will be pursued in Section 6.

First we present a generalization of the relaxation scheme that agrees with the more general HLL method in the case when $a_l \neq -a_r$. Rather than using a matrix of the form (2.2), consider a relaxation system

$$\begin{bmatrix} u \\ v \end{bmatrix}_t + \begin{bmatrix} 0 & I \\ -D_l D_r & (D_l + D_r) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_x = \begin{bmatrix} 0 \\ (f(u) - v)/\tau \end{bmatrix}, \tag{3.7}$$

where $D_l = a_l I$ and $D_r = a_r I$. The coefficient matrix appearing in this system,

$$\begin{bmatrix} 0 & I \\ -D_l D_r & (D_l + D_r) \end{bmatrix}, \tag{3.8}$$

has eigenvector pairs

$$\begin{bmatrix} e^j \\ a_l e^j \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} e^j \\ a_r e^j \end{bmatrix}, \tag{3.9}$$

with eigenvalues a_l and a_r respectively. Note that if $a_l = -a_r$ then the matrix (3.8) reduces to (2.2). Using these vectors (3.9) in the decomposition (1.8) gives the HLL solver for arbitrary a_l and a_r.

4 Relation to the Roe solver

Rather than using $w^{2j-1} = w^{2j} = e^j$ (for $j = 1, 2, \ldots, m$) in the decomposition (1.8), it is attractive to use approximations to the eigenvectors of the Jacobian matrix $f'(u)$ near U_l and U_r. One obvious choice is to use the eigenvectors $\hat{r}^j$ of the Roe matrix $\hat{A}$ satisfying (1.7). If we then choose two distinct speeds $s_l^j \neq s_r^j$ we will have two linearly independent vectors

$$\begin{bmatrix} w^{2j-1} \\ \phi^{2j-1} \end{bmatrix} = \begin{bmatrix} \hat{r}^j \\ s_l^j \hat{r}^j \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} w^{2j} \\ \phi^{2j} \end{bmatrix} = \begin{bmatrix} \hat{r}^j \\ s_r^j \hat{r}^j \end{bmatrix}. \tag{4.1}$$

Since the $\hat{r}^j$ for $j = 1, 2, \ldots, m$ are linearly independent it then follows that the full set of $2m$ vectors given by (4.1) for $j = 1, 2, \ldots, m$ will span $\mathbb{R}^{2m}$ and a decomposition of the form (1.8) can always be performed to define an approximate Riemann solution.

It turns out that we can also choose $s_l^j = s_r^j$ provided that we choose this value to be $\hat{\lambda}^j$, the jth eigenvalue of the matrix $\hat{A}$. In this case the two vectors in (4.1) are identical and there are only m distinct vectors in (1.8),

$$\begin{bmatrix} U_r - U_l \\ f(U_r) - f(U_l) \end{bmatrix} = \hat{\alpha}^1 \begin{bmatrix} \hat{r}^1 \\ \hat{\lambda}^1 \hat{r}^1 \end{bmatrix} + \cdots + \hat{\alpha}^m \begin{bmatrix} \hat{r}^m \\ \hat{\lambda}^m \hat{r}^m \end{bmatrix}. \tag{4.2}$$

This $2m \times m$ linear system has a unique solution $\hat{\alpha} \in \mathbb{R}^m$ in spite of the fact that it appears to be overdetermined. The particular vector on the left hand side of (4.2) lies in the span of these m vectors since $\hat{r}^j$ and $\hat{\lambda}^j$ come from the Roe matrix $\hat{A}$ satisfying (1.7). In fact, we can simply solve

$$U_r - U_l = \hat{\alpha}^1 \hat{r}^1 + \cdots + \hat{\alpha}^m \hat{r}^m \tag{4.3}$$

as one usually does with the Roe solver and then

$$f(U_r) - f(U_l) = \hat{\alpha}^1 \hat{\lambda}^1 \hat{r}^1 + \cdots + \hat{\alpha}^m \hat{\lambda}^m \hat{r}^m \tag{4.4}$$

will automatically be satisfied by (1.7), as is easily seen if we multiply (4.3) by $\hat{A}$.

5 Entropy fixes

A failing of the Roe solver is that it can lead to entropy-violating shocks in numerical solutions based on this decomposition. This typically happens if the true Riemann solution contains a transonic rarefaction wave in some family, say the kth family, with characteristic speeds that increase from negative to positive values through the rarefaction fan. This means that information in the kth family should travel partly to the left and partly to the right and affect cell averages on both sides. The Roe solver approximates every wave by a single discontinuity propagating at a speed given by an eigenvalue of $\hat{A}$, and in the transonic rarefaction case $\hat{\lambda}^k \approx 0$ typically and the proper spreading does not occur. An entropy fix is often used to address this problem. One possibility proposed by Harten and Hyman [19] (see also [35]) is to replace the single wave $\hat{\alpha}^k \hat{r}^k$ in this case by a pair of waves $\alpha_l^k \hat{r}^k$ and $\alpha_r^k \hat{r}^k$ propagating at speeds $s_l^k < 0 < s_r^k$ that are chosen to approximate the characteristic speeds at each edge of the rarefaction fan. The total wave strength should be the same, so we need

$$\alpha_l^k + \alpha_r^k = \hat{\alpha}^k, \tag{5.1}$$

and to maintain conservation we also require

$$\alpha_l^k s_l^k + \alpha_r^k s_r^k = \hat{\alpha}^k \hat{\lambda}^k. \tag{5.2}$$

This gives a linear system of two equations to solve for α_l^k and α_r^k, yielding

$$\begin{aligned} \alpha_l^k &= (\hat{\alpha}^k s_r^k - \hat{\lambda}^k)/(s_r^k - s_l^k) \\ \alpha_r^k &= (\hat{\lambda}^k - \hat{\alpha}^k s_l^k)/(s_r^k - s_l^k). \end{aligned} \tag{5.3}$$

Exactly this same method can be derived by using a relaxation Riemann solver of the form (1.8), which we now take to be of the special form

$$\left[\begin{array}{c} U_r - U_l \\ f(U_r) - f(U_l) \end{array}\right] = \alpha_l^1 \left[\begin{array}{c} \hat{r}^1 \\ s_l^1 \hat{r}^1 \end{array}\right] + \alpha_r^1 \left[\begin{array}{c} \hat{r}^1 \\ s_r^1 \hat{r}^1 \end{array}\right] + \cdots + \alpha_l^m \left[\begin{array}{c} \hat{r}^m \\ s_l^m \hat{r}^m \end{array}\right] + \alpha_r^m \left[\begin{array}{c} \hat{r}^m \\ s_r^m \hat{r}^m \end{array}\right]. \tag{5.4}$$

Here we are allowing each wave speed $\hat{\lambda}^j$ to be replaced by a pair of speeds s_l^j and s_r^j. If we take $s_l^j = s_r^j = \hat{\lambda}^j$ for every j then this reduces to the original Roe solver with each vector repeated twice. This system will have infinitely many solution since any α_l^j and α_r^j satisfying

$$\alpha_l^j + \alpha_r^j = \hat{\alpha}^j \tag{5.5}$$

provides a solution, where $\hat{\alpha}^j$ are the Roe coefficients in (4.2). For any such choice of α_l^j and α_r^j we essentially have the original Roe solver — we've simply replaced one wave by two waves propagating at the same speed and adding up to the original wave.

If the kth family has a transonic rarefaction, however, then we can choose $s_l^k < 0 < s_r^k$ (while still taking $s_l^j = s_r^j = \hat{\lambda}^j$ for $j \neq k$) and the decomposition (5.4) results in the Roe solver with the entropy fix described above. As in the discussion of the HLL method in Section 3, including Δf in the decomposition (5.4) ensures that conservation is maintained and leads to the same coefficients α_l^k and α_r^k as in (5.3).

6 Generalized Roe solvers

We could go further and allow $s_l^j \neq s_r^j$ in each family, while still using the eigenvectors $\hat{r}^j$ of the Roe matrix and a decomposition of the form (5.4). One possible choice might be

$$s_l^j = \lambda^j(U_l), \qquad s_r^j = \lambda^j(U_r), \tag{6.1}$$

for $j = 1, 2, \ldots, m$. This choice would automatically give spreading across any rarefaction wave, including transonic ones. On the other hand if $\lambda^j(U_l) > \lambda^j(U_r)$ then the jth wave in the true Riemann solution is presumably a shock, but we would be approximating it by two waves. We can still solve the system (5.4). The state that arises in the approximate solution between these two waves can be viewed as an approximation to the value that would be found by averaging an overturned compression wave as in Brenier's transport-collapse method [8] or the large time step method of [34]. This also has similarities to the method developed by Engquist & Osher [15] for scalar problems and Osher & Solomon [44] for systems, often called the Osher solver in general. In this approach only the integral curves of the eigenvectors are used to compute an approximate Riemann solution, so that rarefaction waves and overturned compression waves are used in every family. Hence (5.4) with the choice (6.1) might be viewed as an approximation to the Osher solver based on Roe averages. Perhaps a closer connection can be made with a different choice of eigenvectors and speeds in (5.4).

Note that if s_l^j and s_r^j have the same sign then the generalization proposed in this section does not really change the contribution from the jth family to the numerical solution, at least not at the level of a first-order upwind method based on these waves. This is because the two waves in this family affect only one of the neighboring cell averages and might as well be lumped into a single wave. We can combine them as

$$\alpha_l^j \left[\begin{array}{c} \hat{r}^j \\ s_l^j \hat{r}^j \end{array}\right] + \alpha_r^j \left[\begin{array}{c} \hat{r}^j \\ s_r^j \hat{r}^j \end{array}\right] = \beta^j \left[\begin{array}{c} \hat{r}^j \\ s^j \hat{r}^j \end{array}\right], \tag{6.2}$$

for some choice of β^j and s^j, where s^j should then be used as the speed of this lumped wave. We can easily solve for the required values:

$$\begin{aligned} \beta^j &= \alpha_l^j + \alpha_r^j \\ s^j &= \frac{\alpha_l^j s_l^j + \alpha_r^j s_r^j}{\alpha_l^j + \alpha_r^j}. \end{aligned} \tag{6.3}$$

On the other hand, we know there is a unique decomposition of $U_r - U_l$ into the eigenvectors $\hat{r}^j$ with the coefficients $\hat{\alpha}^j$, and from this we can deduce that in fact $\beta^j = \hat{\alpha}^j$ and also that $s^j = \hat{\lambda}^j$, the corresponding Roe velocity.

It is only in the transonic case that something different is obtained by the more general choice (6.1). For a transonic rarefaction this gives a standard entropy fix, as already discussed. For a transonic shock this would introduce additional dissipation. This may also be desirable in some cases, since the lack of dissipation in shocks for which $\hat{\lambda}^k \approx 0$ is also known to cause numerical difficulties, such as nonphysical oscillations near slowly moving shocks. The addition of more dissipation in this case is one approach to improving solutions in this case. See for example [3], [13], [25], [27].

A further generalization of this solver is obtained by using vectors $\hat{r}^j$ in (5.4) that are not the eigenvectors of the Roe matrix. This may be useful for problems where a Roe average satisfying (1.7) is not available, and instead one wishes to use a simpler average such as $\hat{A} = f'\left(\frac{1}{2}(U_l + U_r)\right)$. By taking the $\hat{r}^j$ in (5.4) to be the eigenvectors of this matrix and choosing some reasonable values for s_l^j and s_r^j, for example (6.1), it is possible to obtain consistent decompositions of $U_r - U_l$ and $f(U_r) - f(U_l)$ in terms of these $2m$ waves. Moreover, we can merge each pair of waves into a single wave using (6.2) with β^j and s^j defined by (6.3) if desired (typically in all but the transonic rarefaction case). We then have a decomposition into m waves,

$$U_r - U_l = \sum_{j=1}^{m} \beta^j \hat{r}^j \tag{6.4}$$

and set of speeds s^j for which

$$f(U_r) - f(U_l) = \sum_{j=1}^{m} s^j \beta^j \hat{r}^j \tag{6.5}$$

holds. This mimics an important property of the Roe solver that is useful in wave-propagation implementations (see Section 8).

One possible application of this idea would be to use the Roe eigenvectors coming from a simpler but related system of equations as an approximation. This could be useful for problems where a Roe matrix cannot be found directly. A similar idea has been proposed by by Coquel and Perthame [12] and implemented by In [22] for one particular system. They use the classical Roe solver for the polytropic Euler equations in order to solve real-gas problems with more complicated equations of state. An additional energy variable is added to the system and relaxation in the energy equations is used to couple the two. In the limit of zero relaxation time this can be viewed as defining a new approximate Riemann solver for the real-gas problem. This method is not directly in the form of the relaxation Riemann solver (5.4), however.

7 The Roe solver as a relaxation scheme

The Roe solver and the generalization presented in the previous section can be viewed in the context of relaxation schemes using the connection introduced in Section 2. This may be useful in analyzing the Roe scheme. The decomposition (5.4) arises naturally in the process of solving the $2m \times 2m$ linear system appearing in a relaxation scheme based on the relaxation system (3.7). However, the matrices D_l and D_r are no longer diagonal, but instead are given by

$$D_l = \hat{R} S_l \hat{R}^{-1}, \qquad D_r = \hat{R} S_r \hat{R}^{-1}, \tag{7.1}$$

where $\hat{R}$ is the matrix of Roe eigenvectors, $S_l = \mathrm{diag}(s_l^1, \ldots, s_l^m)$ and $S_r = \mathrm{diag}(s_r^1, \ldots, s_r^m)$. If the speeds are all distinct, $s_l^j \neq s_r^j$, then the coefficient matrix (3.8) is diagonalizable and the vectors appearing in (5.4) are the eigenvectors. For the original Roe scheme, on the other hand, $s_l^j = s_r^j = \hat{\lambda}^j$

for all j. In this case $D_l = D_r = \hat{A}$ and this matrix reduces to

$$\begin{bmatrix} 0 & I \\ -\hat{A}^2 & 2\hat{A} \end{bmatrix}. \tag{7.2}$$

This matrix is defective: each eigenvalue $\hat{\lambda}^j$ has algebraic multiplicity 2 but geometric multiplicity 1 and there are only m distinct vectors in (5.4) as already discussed.

Normally a relaxation system of the form (2.1) or (3.7) yields a solution $u(x,t)$ which may converge to the solution of the original conservation law as $\tau \to 0$, but will not agree with this solution for $\tau > 0$. Instead, it approximates the solution to a viscous conservation law of the form

$$u_t + f(u)_x = \tau(B(u)u_x)_x + \mathcal{O}(\tau^2), \tag{7.3}$$

where the viscosity matrix $B(u)$ can be determined by a Chapman-Enskog expansion [10], [39]. The structure of this viscosity matrix can play a role in determining whether the correct entropy-satisfying solution is obtained in the limit $\tau \to 0$, see for example [6], [7], [17], [40]. For a relaxation system with the coefficient matrix (7.2), we find that

$$B(u) = -\hat{A}^2 + 2\hat{A}f'(u) - (f'(u))^2. \tag{7.4}$$

Note that if $U_l \approx U_r$ then $\hat{A} \approx f'(u)$ and the viscosity matrix vanishes in the case of equality. Moreover, even when there is a large jump between U_l and U_r it is possible that this relaxation system will reproduce an exact weak solution to the original conservation law even when $\tau > 0$, as if there were no viscosity. This happens in the special case when we consider a Riemann problem between states U_l and U_r that satisfy the Rankine-Hugoniot jump condition for some scalar value s,

$$f(U_r) - f(U_l) = s(U_r - U_l). \tag{7.5}$$

Suppose we solve the relaxation system

$$\begin{bmatrix} u \\ v \end{bmatrix}_t + \begin{bmatrix} 0 & I \\ -\hat{A}^2 & 2\hat{A} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_x = \begin{bmatrix} 0 \\ (f(u) - v)/\tau \end{bmatrix} \tag{7.6}$$

with this Riemann initial data U_l, U_r and $V_l = f(U_l)$, $V_r = f(U_r)$ and with the matrix $\hat{A}$ chosen to be the Roe matrix for this data (and frozen at this value even if u and v evolve). Then the property (1.7) of the Roe matrix implies that $U_r - U_l$ is an eigenvector of $\hat{A}$, proportional to $\hat{r}^k$ for some k, and that $s = \hat{\lambda}^k$ is the corresponding eigenvalue. It follows that

$$\begin{aligned} u(x,t) &= u(x - \hat{\lambda}^k t, 0) \\ v(x,t) &= v(x - \hat{\lambda}^k t, 0) = f(u(x,t)) \end{aligned} \tag{7.7}$$

is the solution to the relaxation system (7.6) for any value of τ. The jump discontinuity simply propagates with speed $\hat{\lambda}^k$ and since $v \equiv f(u)$, the source term vanishes. This is a weak solution of the original conservation law in this case, though it may not satisfy the entropy condition. If the discontinuity should spread into a rarefaction wave, the relaxation system will instead produce the expansion shock.

Of course if this relaxation system is now used numerically as part of a relaxation scheme, then numerical viscosity may be added when the linear system is solved numerically. But in the case $\hat{\lambda}^k = 0$ it is possible that no smearing is introduced, as for example in Roe's first-order method which produces entropy-violating solutions in the transonic case. It is well known that this is caused by a lack of numerical viscosity, which has been extensively analyzed by other means, but it is interesting to observe that this phenomenon is connected with a relaxation system which itself lacks viscosity and produces entropy-violating weak solutions even in the case when the data is not transonic. It is also interesting to note that the viscosity matrix (7.4) generally fails to be positive definite. In fact if $\hat{A}$ and $f'(u)$ commute then $B(u) = -(\hat{A} - f'(u))^2$ is negative definite.

Note that adding an entropy fix to Roe's method, as described in Sections 5 and 6, changes the relaxation system to one of the more general form (3.7). The entropy-violating weak solution is no longer an exact solution, as we have explicitly added spreading of this wave.

8 Wave-propagation algorithms

A relaxation Riemann solver of the general type we have discussed could be used in conjunction with any numerical method that is based on approximate Riemann solvers. One simple finite-volume method that gives high-resolution results and directly uses a wave decomposition of the form (1.2) is the wave-propagation method described in [36] and implemented in the CLAWPACK software [33]. This method uses an updating formula of the form

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}(\mathcal{A}^+\Delta U_{i-1/2} + \mathcal{A}^-\Delta U_{i+1/2}) - \frac{\Delta t}{\Delta x}(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}), \tag{8.1}$$

where $\mathcal{A}^\pm\Delta U_{i-1/2}$ are the left-going and right-going *fluctuations* resulting from the Riemann solution at the grid interface $x_{i-1/2}$ and $\tilde{F}_{i-1/2}$ are *correction fluxes* yielding high resolution. For the Riemann problem at $x_{i-1/2}$ the data is $U_l = U_{i-1}^n$ and $U_r = U_i^n$ and we denote the resulting waves and speeds by $\mathcal{W}_{i-1/2}$ and $s_{i-1/2}$. The simplest method of this form based on (1.2) has $\tilde{F}_{i-1/2} = 0$ and

$$\mathcal{A}^\pm\Delta U_{i-1/2} = \sum_{p=1}^{M_w}(s_{i-1/2}^p)^\pm \mathcal{W}_{i-1/2}^p, \tag{8.2}$$

where M_w is the number of waves produced by the Riemann solver, $s^- = \min(s, 0)$ and $s^+ = \max(s, 0)$. This is the first-order upwind method (Godunov's method) based on the approximate Riemann solution. In order to be conservative we require that

$$\mathcal{A}^-\Delta U_{i-1/2} + \mathcal{A}^+\Delta U_{i-1/2} = f(U_i) - f(U_{i-1}). \tag{8.3}$$

If a relaxation Riemann solver of the form (1.8) is used, then this will hold provided (1.10) is satisfied. Alternatively, with an arbitrary choice of $w_{i-1/2}^p$ and $\phi_{i-1/2}^p$ in (1.8) we can obtain a conservative method by setting

$$\mathcal{A}^\pm\Delta U = \sum_{p=1}^{M_w}\frac{1}{2}(1 \pm \operatorname{sgn}(s_{i-1/2}^p))\phi_{i-1/2}^p. \tag{8.4}$$

The high-resolution correction fluxes $\tilde{F}_{i-1/2}$ are defined in general by

$$\tilde{F}_{i-1/2} = \frac{1}{2}\sum_{p=1}^{M_w}|s_{i-1/2}^p|\left(1 - \frac{\Delta t}{\Delta x}|s_{i-1/2}^p|\right)\tilde{\mathcal{W}}_{i-1/2}^p, \tag{8.5}$$

where $\tilde{\mathcal{W}}_{i-1/2}^p$ is a limited version of the wave $\mathcal{W}_{i-1/2}^p$, obtained by comparing $\mathcal{W}_{i-1/2}^p$ with the pth wave from the adjacent Riemann problem at either $x_{i-3/2}$ (if $s_{i-1/2}^p > 0$) or at $x_{i+1/2}$ (if $s_{i-1/2}^p < 0$). Any standard limiter can be applied. See [36] for more details.

If $M_w > m$ then it may be possible and more efficient to combine some waves together and use a smaller set of waves in these correction terms. For example, if the Roe solver with an entropy fix is used then it is common to apply the entropy fix only in computing $\mathcal{A}^\pm\Delta U_{i-1/2}$ and then use only the original m waves resulting from the Roe solver to define the correction fluxes. This is typically necessary when limiters are used since the neighboring Riemann problems are generally not transonic and provide only m waves for comparison.

We also note that if $s_{i-1/2}^p = 0$ for any wave then it makes no contribution to $\mathcal{A}^\pm\Delta U_{i-1/2}$ in (8.2) or to $\tilde{F}_{i-1/2}$ in (8.5). In the applications discussed below in Sections 9 and 10, for example, there will be $2m$ waves produced by the relaxation solver, but only m of these will have nonzero speeds.

9 Discontinuous flux functions

As one example of how a relaxation Riemann solver with $2m$ waves might prove useful, consider a conservation law with a spatially varying flux function $f(u,x)$. One way to solve this problem numerically is to use a finite-volume method with the flux function discretized so that the ith grid cell has a flux function $f_i(u)$ associated with it. At a cell interface we must then solve a Riemann problem with data U_l, U_r and two different flux functions $f_l(u)$ and $f_r(u)$. When f_l and f_r are nonlinear, determining the exact Riemann solution for this situation may be nontrivial, *e.g.*, [16], [29], [32], [47]. We are currently investigating the possibility of using a Riemann solver of the form (1.8) for such problems and here only report some preliminary observations.

One natural way to use (1.8) might be to compute two sets of eigenvectors and eigenvalues using the two Jacobian matrices $f_l'(U_l)$ and $f_r'(U_r)$. Call these λ_l^j, r_l^j and λ_r^j, r_r^j. These could be used to define $2m$ vectors for use in (1.8). This does not seem to be a good idea in general, however. Often both λ_l^j and λ_r^j will have the same sign, indicating what direction the jth wave is propagating. Suppose these are both positive, for example, indicating that this wave is propagating into the cell on the right, where the flux function is $f_r(u)$. Then the eigenvector r_r^j may be a useful component in the decomposition, but r_l^j may be completely irrelevant.

Instead, it is useful to observe that the Riemann solution must typically involve a stationary discontinuity in u (moving at speed 0) at the interface, between two values U_l^* and U_r^* related via

$$f_l(U_l^*) = f_r(U_r^*). \tag{9.1}$$

This is required because the flux must be continuous at the interface. This suggests that the Riemann solver should include m waves that allow jumps in each of the m components of u and combine to give no jump in f. If we also have some m vectors r^j and speeds s^j (for $j = 1,\ 2,\ \ldots,\ m$) that represent the propagating waves we expect the Riemann solution to contain, then we can look for a decomposition of the form

$$\begin{aligned} \begin{bmatrix} U_r - U_l \\ f_r(U_r) - f_l(U_l) \end{bmatrix} &= \alpha^1 \begin{bmatrix} r^1 \\ s^1 r^1 \end{bmatrix} + \cdots + \alpha^m \begin{bmatrix} r^m \\ s^m r^m \end{bmatrix} \\ &\quad + \alpha^{m+1} \begin{bmatrix} e^1 \\ 0 \end{bmatrix} + \cdots + \alpha^{2m} \begin{bmatrix} e^m \\ 0 \end{bmatrix}. \end{aligned} \tag{9.2}$$

The hard part in general may be to determine a suitable choice for r^j and s^j.

We illustrate this for one simple example, the variable-coefficient advection equation

$$u_t + (a(x)u)_x = 0, \tag{9.3}$$

where $a(x) > 0$ everywhere. The Riemann problem with data U_l, U_r and speeds a_l, a_r has flux functions $f_l(u) = a_l u$ and $f_r(u) = a_r u$. Physically this might model the density of items traveling on a system of conveyer belts, at the junction between two belts moving at different speeds. The exact solution of this Riemann problem is

$$u(x,t) = \begin{cases} U_l & \text{if } x/t < 0 \\ U^* & \text{if } 0 < x/t < a_r \\ U_r & \text{if } x/t > a_r \end{cases} \tag{9.4}$$

where

$$U^* = \frac{a_l U_l}{U_r}, \tag{9.5}$$

as determined by the requirement that $a_l U_l = a_r U^*$ so that the flux is continuous.

Applying the decomposition (9.2) with $m = 1$ to this simple problem yields the correct Riemann solution if we take $r^1 = 1$ and $s^1 = a_r$. We have

$$\begin{bmatrix} U_r - U_l \\ a_r U_r - a_l U_l \end{bmatrix} = \alpha^1 \begin{bmatrix} 1 \\ a_r \end{bmatrix} + \alpha^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{9.6}$$

Solving for α^1, α^2 yields

$$\begin{aligned}\alpha^1 &= U_r - a_l U_l/a_r = U_r - U^*,\\ \alpha^2 &= a_l U_l/a_r - U_r = U^* - U_l,\end{aligned} \tag{9.7}$$

where U^* is given by (9.5). These waves propagate with speeds $s^1 = a_r$ and $s^2 = 0$ and so the exact solution (9.4) is achieved.

Another standard way to approach this variable-flux problem is to view $a(x)$ as a second variable in a system of two equations

$$\begin{bmatrix} u \\ a \end{bmatrix}_t + \begin{bmatrix} au \\ 0 \end{bmatrix}_x = 0. \tag{9.8}$$

This is now a nonlinear system of two conservation laws in which both fields are linearly degenerate (but there is no longer a spatially-varying coefficient). The flux Jacobian is

$$\begin{bmatrix} a & u \\ 0 & 0 \end{bmatrix}, \tag{9.9}$$

with eigenvalues $\lambda^1 = 0$, $\lambda^2 = a$, and eigenvectors

$$r^1 = \begin{bmatrix} u \\ -a \end{bmatrix}, \qquad r^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{9.10}$$

Solving the Riemann problem for this system again gives the solution (9.4).

Notice that attempting to use a Roe solver for the system (9.8) (as might be desired for more complicated problems of this type) would be less successful than the simple Riemann solver proposed in (9.6). The Roe matrix for the system (9.8) is given by

$$\hat{A} = \begin{bmatrix} \hat{a} & \hat{u} \\ 0 & 0 \end{bmatrix}, \tag{9.11}$$

where $\hat{a} = \frac{1}{2}(a_l + a_r)$ and $\hat{u} = \frac{1}{2}(U_l + U_r)$, so the Roe solver uses the decomposition

$$\begin{bmatrix} U_r - U_l \\ a_r U_r - a_l U_l \end{bmatrix} = \hat{\alpha}^1 \begin{bmatrix} \hat{u} \\ -\hat{a} \end{bmatrix} + \hat{\alpha}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{9.12}$$

This does not give the exact Riemann solution since the propagation speed $\hat{a}$ of the moving wave is not the correct speed a_r unless $a_l = a_r$.

For the simple scalar linear problem (9.3) there are many ways to determine the exact solution, as we have just illustrated. For more general nonlinear systems with discontinuous fluxes some sort of approximate Riemann solver must be used. We hope that the relaxation Riemann solvers might provide a better starting point than augmented systems of the form (9.8).

There is a simplification of the relaxation Riemann solver (9.2) that arises naturally in an implementation based on the wave-propagation method of Section 8 and which gives another interpretation of this Riemann solver. Since the waves numbered $m+1$ through $2m$ are viewed as being stationary at the interface we have $s^{m+1} = \cdots = s^{2m} = 0$ and these waves do not contribute to the fluctuations (8.2) or to the correction fluxes (8.5). Hence we only need to determine the coefficients $\alpha^1, \ \ldots, \ \alpha^m$. These can be determined by considering only the lower part of the system in (9.2), *i.e.*,

$$f_r(U_r) - f_l(U_l) = \alpha^1 s^1 r^1 + \cdots + \alpha^m s^m r^m, \tag{9.13}$$

which is a linear system of m equations for the m unknowns $\alpha^1, \ \ldots, \ \alpha^m$. Note that this is similar to the standard Riemann solver (1.4), but we decompose the jump in f into eigencomponents rather than the jump in u. This makes sense since there is no jump in f across the stationary interface and so the full jump $f_r(U_r) - f_l(U_l)$ can be split into pieces corresponding to propagating waves, whereas u has an unknown jump across the interface. This is consistent with standard Riemann solvers in the case of a single flux function, at least in some cases. For a linear constant coefficient system with $f(u) = Au$, or for Roe's method based on a matrix $\hat{A}$ satisfying (1.7), performing the decomposition (9.13) would result in exactly the same coefficients α^p as performing the decomposition (1.4).

Approximate Riemann solvers based on splitting the jump in f have recently been studied numerically for various applications in work with Bale and Rossmanith [4]. This work started directly from (9.13) and we only recently realized the connection with relaxation schemes. Preliminary results indicate that it may be a useful approach for many problems, including the implementation of finite-volume methods for conservation laws on curved manifolds, which leads to spatially-varying flux functions due to the metric terms. This approach is also being used in a wave-propagation algorithm for the Einstein equations in numerical relativity work by Bardeen and Buchman [5].

10 Source terms

Now consider a conservation law

$$u_t + f(u)_x = \psi \tag{10.1}$$

with a source term ψ. One common approach to solving this equation is to use a fractional step method, alternating between solving the homogeneous conservation law and the ODE $u_t = \psi$. This leads to inaccuracy in some cases, for example if the solution is nearly in steady state with $f(u)_x \approx \psi$ and we wish to study the propagation of small disturbances on this background state (see the discussion in [37], for example).

Another approach is to somehow incorporate ψ into the solution of Riemann problems. One way to do this is to discretize the source terms as a sum of delta function singularities with strength proportional to Δx at the cell interfaces, so that the effect of the source is concentrated at these points. This approach is taken by Jenny and Müller [23] in their *Rankine-Hugoniot Riemann solver*, for example. In this case we must solve a more general Riemann problem of the form

$$u_t + f(u)_x = \Psi\delta(x) \tag{10.2}$$

where $\Psi = \Delta x\, \psi_{i-1/2}$ is the strength of the delta function at this interface and the data U_l, U_r comes from the cells to the left and right. The solution to this Riemann problem consists of propagating waves satisfying the usual Rankine-Hugoniot jump conditions away from $x/t = 0$ (where the source term vanishes) along with jumps in u across $x/t = 0$ that satisfy

$$f(U_r^*) - f(U_l^*) = \Psi. \tag{10.3}$$

This is similar to (9.1) but now the flux is not continuous at the interface because of the singular source. This suggests that we use a Riemann solver analogous to (9.2) but with the source term included,

$$\begin{aligned}\begin{bmatrix} U_r - U_l \\ f(U_r) - f(U_l) - \Psi \end{bmatrix} &= \alpha^1 \begin{bmatrix} r^1 \\ s^1 r^1 \end{bmatrix} + \cdots + \alpha^m \begin{bmatrix} r^m \\ s^m r^m \end{bmatrix} \\ &\quad + \alpha^{m+1} \begin{bmatrix} e^1 \\ 0 \end{bmatrix} + \cdots + \alpha^{2m} \begin{bmatrix} e^m \\ 0 \end{bmatrix}.\end{aligned} \tag{10.4}$$

As in the wave-propagation implementation of methods based on (9.1), we only need $\alpha^1, \ldots, \alpha^m$ and these can be obtained by solving the smaller $m \times m$ system

$$f(U_r) - f(U_l) - \Psi = \alpha^1 s^1 r^1 + \cdots + \alpha^m s^m r^m, \tag{10.5}$$

and then using

$$\mathcal{W}^p = \alpha^p r^p \tag{10.6}$$

as the pth wave in the algorithm described in Section 8.

Note that a numerical steady state will be maintained by this method. Suppose that the cell averages U_i^n and source terms $\psi_{i-1/2}^n$ satisfy

$$\frac{f(U_i^n) - f(U_{i-1}^n)}{\Delta x} = \psi_{i-1/2}^n \tag{10.7}$$

at time t_n. Then the left-hand side of (10.5) will be zero and hence $\alpha^p = 0$ for $p = 1, 2, \ldots, m$. All waves $\mathcal{W}^p$ arising from each modified Riemann problem will then have zero strength and a wave-propagation algorithm will reduce to giving $U_i^{n+1} = U_i^n$.

If the solution does not satisfy (10.7) exactly but is close to a steady state, then it is the deviation from steady state that is used to define the waves in the approximate Riemann solution. This is similar in spirit to the quasi-steady wave-propagation algorithm proposed in [37]. In that algorithm the delta function singularities were placed at cell centers rather than cell interfaces, however, and a new set of Riemann problems at these points was introduced to cancel out the source terms. An algorithm based on (10.5) is easier to implement than the one proposed in [37], and preliminary results indicate that it may also be more robust when the solution deviates further from steady state.

11 Conclusions

We have explored the connection between a simple relaxation scheme of the type proposed by Jin and Xin and a class of approximate Riemann solvers for the original conservation law. For a system of m conservation laws this solver uses $2m$ waves and is based on splitting up both the jump in u and the jump in the flux simultaneously. This insures that conservation is maintained, and in the simplest case can be directly related to the HLL Riemann solver. Approximate Riemann solvers based on characteristic decompositions, such as Roe's, can be related to more general relaxation systems. It may be possible to exploit this connection to improve our understanding of both types of methods.

The added flexibility of these more general approximate Riemann solvers may be useful in some applications. We briefly discussed some possibilities for systems where a Roe matrix is unavailable, for problems with discontinuous flux functions, and for problems with source terms. Some specific applications are now being explored and will be reported on elsewhere.

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