A NOTE ON THE UNIQUENESS OF ENTROPY SOLUTIONS OF NONLINEAR DEGENERATE PARABOLIC EQUATIONS

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ABSTRACT. Following the lead of Carrillo [6], recently several authors have used Kružkov's device of "doubling the variables" to prove uniqueness results for entropy solutions of nonlinear degenerate parabolic equations. In all these results, the second order differential operator is not allowed to depend explicitly on the spatial variable, which certainly restricts the range of applications of entropy solution theory. The purpose of this paper is to extend a version of Carrillo's uniqueness result found in Karlsen and Risebro [14] to a class of degenerate parabolic equations with *spatially* dependent second order differential operator. The class is large enough to encompass several interesting nonlinear partial differential equations coming from the theory of porous media flow and the phenomenological theory of sedimentation-consolidation processes.

1. INTRODUCTION

In this paper, we are interested in entropy solutions of nonlinear degenerate parabolic initial value problems of the form

$$\begin{aligned} &\partial_t u + \mathrm{div}_x f(x,t,u) = \nabla_x \cdot \big(K(x,t)\nabla_x A(u)\big) + q(x,t,u), \quad (x,t) \in \Pi_T, \\ &u(x,0) = u_0(x), \quad x \in \mathbf{R}^d, \end{aligned} \tag{1.1}$$

where $\Pi_T = \mathbf{R}^d \times (0,T)$, $T > 0$ is fixed, $u = u(x,t)$ is the scalar unknown function that is sought, u_0 is the initial function, and $f = (f_1,\dots,f_d), K = \mathrm{diag}\{k_i \,|\, i = 1,\dots,d\}, A, q$ are given functions to be detailed in Section 2. For the moment, it suffices to say that $K \geq \underline{k} > 0$ and $A(\cdot)$ is nondecreasing with $A(0) = 0$, which implies that (1.1) is a (strongly) degenerate parabolic problem. For example, the hyperbolic conservation law ($A' \equiv 0$) is included in our setup. A special case of (1.1) that will be studied separately in this paper is

$$\begin{aligned} &u_t + \mathrm{div}_x\big(V(x,t)f(u)\big) = \nabla_x \cdot \big(K(x,t)\nabla_x A(u)\big) + q(x,t,u), \quad (x,t) \in \Pi_T, \\ &u(x,0) = u_0(x), \quad x \in \mathbf{R}^d, \end{aligned} \tag{1.2}$$

where $V = (V_1,\dots,V_d)$ is a vector field (not necessarily divergence free), f is a scalar function, and K, A, q are as before. Problems of the form (1.2) occur in several applications. Biased by our own interests, we mention here only flow in porous media (see, e.g., [7, 10]) and sedimentation-consolidation processes [4]. In porous media flow such as immiscible two-phase flow of water and oil in a reservoir, $V = V(x)$ is a driving velocity field coming from Darcy's law and $K = K(x)$ is the permeability tensor, which describes the flow properties of the porous medium (oil reservoir).

Since (1.1) is allowed to be degenerate, solutions are not necessarily smooth and weak solutions must be sought. Moreover, due to possible strong degeneracy, that is, $A'(\cdot)$ is allowed to be zero on an interval $[\alpha,\beta]$, weak solutions are not uniquely determined by their initial data and therefore we have to work in the L^1 framework of entropy solutions [22], i.e., weak solutions that satisfy a Kružkov-Vol'pert type entropy condition (see Section 2 for more details).

The purpose of this paper is to provide uniqueness and stability results for entropy solutions of (1.1) and (1.2) by adopting the "doubling of variables" strategy introduced in the important work by Carrillo [6], which in turn followed the route laid out in the pioneering work by Kružkov [15] on hyperbolic conservation laws. Various extensions of Carrillo's uniqueness result have appeared

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recently, see Rouvre and Gagneux [20], Bürger, Evje, and Karlsen [2], Karlsen and Risebro [14], Bürger and Karlsen [3], and Mascia, Porretta, and Terracina [17].

To mention only a few, other (less general) uniqueness results for entropy solutions of degenerate parabolic equations have been obtained by Vol'pert and Hudjaev [22], Wu and Yin [24], Bénilan and Touré [1], Escobedo, Vázquez, and Zuazua [9], and Volpert [23].

We refer to Carrillo [5], Otto [19], and Cockburn and Gripenberg [8], Evje, Karlsen, and Risebro [11], Ohlberger [18], Eymard, Gallouet, and Herbin [12], Eymard, Gallouet, Herbin, and Michel [13] for various applications of Kružkov's "doubling device" in the context of second order strongly degenerate parabolic equations.

For a partial historical overview of the L^1 theory of parabolic equations as well as corresponding numerical theory, we refer to [10].

A typical feature of all the works [6, 20, 2, 14, 3, 17] that employ the "doubling device" to prove uniqueness results for degenerate parabolic equations seems to be that the second order differential operator is not allowed to depend explicitly on spatial variable x (i.e., $K = I$), which certainly restricts the range of applications.

In this paper, we will show that by properly adopting the "doubling device" one can indeed prove uniqueness of entropy solutions for degenerate parabolic equations with x - dependent second order differential operator, at least when the equations take a particular form as in (1.1) with the diagonal matrix K bounded away from zero. Furthermore, in the $L^\infty(0,T;BV(\mathbf{R}^d))$ class of entropy solutions, we will prove an L^1 stability result for (1.2) which is independent of $\mathrm{div}_x V$. Our results generalize those obtained by Karlsen and Risebro [14], who dealt with the case $K \equiv 1$.

It is worth while mentioning that we do not know how to treat the case where K is zero on sets of non-zero Lebesgue measure. Nevertheless, our results are general enough to include the nonlinear partial differential equations coming from the theory of porous media flow [7, 10] and the phenomenological theory of sedimentation-consolidation processes [4].

By following the approach in [17], it is possible to extend the results obtained in this paper to problems with Dirichlet boundary conditions. In future work, we will present explicit continuous dependence estimates as well convergence results for numerical methods for (1.1) and (1.2).

The rest of this paper is organized as follows: In Section 2, we state the definition of an entropy solution as well as our main results. Section 3 is devoted to preliminaries and the identification of a certain entropy dissipation term that is needed in the uniqueness proof. Finally, Sections 4 and 5 are devoted to the proofs of our main results.

2. Statement of Results

We start by stating sufficient conditions on the "data" f, K, A, q for Theorems 2.1 and 2.2 below to hold. Concerning f, we assume that

$$f(\cdot,\cdot,u) \in L^1(0,T;W^{1,1}_{\mathrm{loc}}(\mathbf{R}^d))\ \forall u; \quad f(\cdot,\cdot,0), f_x(\cdot,\cdot,0) \in L^1(\Pi_T)\bigcap L^\infty(\Pi_T); \tag{2.1}$$

$$f(x,t,\cdot), f_x(x,t,\cdot) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{R}) \text{ uniformly in } x,t, \tag{2.2}$$

where f_x denotes the function obtained by taking the divergence of the flux f with respect to the x - variable. With the phrase "uniformly in x, t" in (2.2), we mean

$$|f(x,t,v) - f(x,t,u)|,\ |f_x(x,t,v) - f_x(x,t,u)| \le C|v-u|, \quad \forall x,t,v,u,$$

for some constant C that is independent of x, t, v, u. Note that the assumptions on f, f_x imply that $f(\cdot,\cdot,u), f_x(\cdot,\cdot,u) \in L^\infty(\Pi_T)$ for all u.

We need an additional condition to prove uniqueness of the entropy solution. Here we shall assume that

$$\big|F_i(x,t,v,u) - F_i(y,s,v,u)\big| \le \gamma\,|x-y|\,|v-u|, \qquad \forall x,y,t,v,u, \qquad i = 1,\dots,d, \tag{2.3}$$

for some constant $\gamma > 0$ (independent of x, t, v, u), where

$$F_i(x,t,v,u) := \mathrm{sign}\,(v-u)\,\big[f_i(x,t,v) - f_i(x,t,u)\big]. \tag{2.4}$$

Although we will not bother to do so here, we mention that condition (2.3) can be replaced by a less restrictive "one-sided" Lipschitz condition, see [14] for details.

Concerning K, we assume that it is a $d\times d$ diagonal matrix, that is,

$$K(x,t)=\operatorname{diag}\{k_i(x,t)\,|\,i=1,\dots,d\},$$

and for k_i, $i=1,\dots,d$, we need to assume the following two conditions:

$$k_i\in L^\infty(\Pi_T)\bigcap L^2(0,T;H^1(\mathbf{R}^d))\bigcap L^2(0,T;\operatorname{Lip}(\mathbf{R}^d)) \tag{2.5}$$

and, for some constant $\underline{k}>0$,

$$k_i(x,t)\geq \underline{k},\qquad \forall (x,t)\in\Pi_T. \tag{2.6}$$

Concerning A, we assume that

$$A\in \operatorname{Lip}_{\mathrm{loc}}(\mathbf{R}) \text{ and } A(\cdot) \text{ is nondecreasing with } A(0)=0. \tag{2.7}$$

Concerning q, we assume that

$$q(\cdot,\cdot,0)\in L^1(\Pi_T)\bigcap L^\infty(\Pi_T);\quad q(x,t,\cdot)\in\operatorname{Lip}_{\mathrm{loc}}(\mathbf{R}) \text{ uniformly in } x,t. \tag{2.8}$$

With the phrase "uniformly in x,t" in (2.8), we mean

$$|q(x,t,v)-q(x,t,u)|\leq C|v-u|,\quad \forall x,t,v,u,$$

for some constant C that is independent of x,t,v,u. Note that the assumptions on q imply that $q(\cdot,\cdot,u)\in L^\infty(\Pi_T)$ for all u.

We shall use the following definition of an entropy solution of (1.1):

Definition 2.1 (Entropy Solution)**.** *A function $u=u(x,t)$ is an entropy solution of* (1.1) *if:*

(**D**.1) $u\in L^1(\Pi_T)\bigcap L^\infty(\Pi_T)\bigcap C(0,T;L^1(\mathbf{R}^d))$.

(**D**.2) $\sqrt{K}\nabla_x A(u)\in L^2(\Pi_T)$.

(**D**.3) *For all $c\in\mathbf{R}$ and all non-negative test functions in $C_0^\infty(\Pi_T)$, the following entropy inequality holds:*

$$\begin{aligned}\iint\limits_{\Pi_T}\Big(&|u-c|\partial_t\phi+\operatorname{sign}(u-c)\Big[f(x,t,u)-f(x,t,c)-K(x,t)\nabla_x A(u)\Big]\cdot\nabla_x\phi\\ &-\operatorname{sign}(u-c)\big(\operatorname{div}_x f(x,t,c)-q(x,t,u)\big)\phi\Big)\,dt\,dx\geq 0.\end{aligned} \tag{2.9}$$

(**D**.4) *Essentially as $t\downarrow 0$, $\|u(\cdot,t)-u_0\|_{L^1(\mathbf{R}^d)}\to 0$.*

Note that $\sqrt{K}$ is defined as $\operatorname{diag}\{\sqrt{k_i}\,|\,i=1,\dots,d\}$.

We will prove the following theorem:

Theorem 2.1 (Uniqueness)**.** *Assume that* (2.1)-(2.3) *and* (2.5)-(2.8) *hold. Let v,u be two entropy solutions of* (1.1) *with initial data $u_0\in L^1(\mathbf{R}^d)\bigcap L^\infty(\mathbf{R}^d)$. Then $v=u$ a.e. in Π_T. In other words, there exists at most one entropy solution of* (1.1).

This theorem is a generalization of Theorem 1.1 in Karlsen and Risebro [14].

Our second result concerns (1.2) and states that in the $L^\infty(0,T;BV(\mathbf{R}^d))$ class of entropy solutions, an L^1 contraction principle holds if

$$f\in\operatorname{Lip}_{\mathrm{loc}}(\mathbf{R});\quad f(0)=0;\quad V\in L^1(0,T;W^{1,1}_{\mathrm{loc}}(\mathbf{R}^d))\bigcap C(\Pi_T);\quad V,\operatorname{div}_x V\in L^\infty(\Pi_T). \tag{2.10}$$

More precisely, we prove the following theorem:

Theorem 2.2 (L^1 Stability)**.** *Assume that* (2.5)-(2.8) *and* (2.10) *hold. Let $v,u\in L^\infty(0,T;BV(\mathbf{R}^d))$ be entropy solutions of* (1.2) *with initial data $v_0,u_0\in L^1(\mathbf{R}^d)\bigcap L^\infty(\mathbf{R}^d)\bigcap BV(\mathbf{R}^d)$, respectively. Then there exists a constant C, not depending on $\operatorname{div}_x V$, such that for a.e. $t\in(0,T)$,*

$$\|v(\cdot,t)-u(\cdot,t)\|_{L^1(\mathbf{R}^d)}\leq\exp(Ct)\|v_0-u_0\|_{L^1(\mathbf{R}^d)}.$$

Note that this theorem is a generalization of Theorem 1.2 in Karlsen and Risebro [14].

3. Entropy Dissipation Term

The main purpose of this section is to state and prove a version of an important lemma due to Carrillo [6]. This lemma identifies a certain entropy dissipation term which turns out to be necessary for the uniqueness proof of Kružkov [15] to work in the setting of second order equations.

Let $u = u(x,t)$ be an entropy solution of (1.1). From (2.9), it follows that u satisfies

$$\iint_{\Pi_T} \Big(u\partial_t \phi + \big[f(x,t,u) - K(x,t)\nabla A(u)\big]\cdot \nabla\phi + q(x,t,u)\phi \Big)\, dt\, dx = 0 \tag{3.1}$$

for all $\phi \in C_0^\infty(\Pi_T)$. In view of the assumptions made on f, K, A, q in Section 1, $f(x,t,u)$, $K(x,t)\nabla A(u)$, and $q(x,t,u)$ belong to $L^2(\Pi_T)$. Hence an approximation argument will show that (3.1) actually holds for all

$$\phi \in L^2(0,T;H^1(\mathbf{R}^d)) \bigcap W^{1,1}(0,T;L^\infty(\mathbf{R}^d)).$$

In what follows, we let $\langle \cdot,\cdot\rangle$ denote the usual pairing between $H^{-1}(\mathbf{R}^d)$ and $H^1(\mathbf{R}^d)$. From (3.1), we conclude that

$$\partial_t u \in L^2(0,T;H^{-1}(\mathbf{R}^d)),$$

and hence the following equality holds for all $\phi \in L^2(0,T;H^1(\mathbf{R}^d))$:

$$-\int_0^T \langle \partial_t u, \phi\rangle\, dt + \iint_{\Pi_T} \Big(\big[f(x,t,u) - K(x,t)\nabla A(u)\big]\cdot\nabla\phi + q(x,t,u)\phi\Big)\, dt\, dx = 0. \tag{3.2}$$

We shall later need the following integration by parts/weak chain rule formula:

Lemma 3.1. *Assume that $u : \Pi_T \to \mathbf{R}$ satisfies the following four conditions:*

(1) $u \in L^1(\Pi_T) \bigcap L^\infty(\Pi_T) \bigcap C(0,T;L^1(\mathbf{R}^d))$.
(2) $u(0,\cdot) = u_0 \in L^\infty(\mathbf{R}^d) \bigcap L^1(\mathbf{R}^d)$.
(3) $\partial_t u \in L^2\big(0,T;H^{-1}(\mathbf{R}^d)\big)$.
(4) $A(u) \in L^2\big(0,T;H^1(\mathbf{R}^d)\big)$.

Then for every nonnegative compactly supported $\phi \in C^\infty(\Pi_T)$ such that $\phi|_{t=0} = \phi_{t=T} \equiv 0$,

$$-\int_0^T \Big\langle \partial_t u, \psi\big(A(u)\big)\phi\Big\rangle\, dt = \int_0^T \int_{\mathbf{R}^d} \Big(\int_c^u \psi\big(A(z)\big)\, dz\Big) \partial_t\phi\, dt\, dx, \qquad c \in \mathbf{R},$$

where $\psi : \mathbf{R} \to \mathbf{R}$ is a nondecreasing and Lipschitz continuous function.

Since Lemma 3.1 can be proved more or less in the same way as the integration by parts/weak chain rule formula in Carrillo [6], we omit the proof.

Note that from (**D**.2) and (2.6), it follows that

$$A(u) \in L^2(0,T;H^1(\mathbf{R}^d)), \tag{3.3}$$

which will be used in a fundamental way in the proof of Lemma 3.2. Roughly speaking, the fact that (3.3) is needed in the proof explains why we need to assume that K is bounded away from zero in (2.6).

In what follows, we shall frequently need a continuous approximation of $\operatorname{sign}(\cdot)$. For $\varepsilon > 0$, set

$$\operatorname{sign}_\varepsilon(\tau) = \begin{cases} -1, & \tau < \varepsilon, \\ \tau/\varepsilon, & \varepsilon \le \tau \le \varepsilon, \\ 1 & \tau > \varepsilon. \end{cases}$$

We shall also need to introduce the set

$$E = \Big\{ r : \ A^{-1}(\cdot) \ \text{ discontinuous at } r \Big\},$$

where $A^{-1} : \mathbf{R} \to \mathbf{R}$ is the unique left-continuous function satisfying $A^{-1}(A(u)) = u$ for all $u \in \mathbf{R}$.

We are now ready to state and prove the following lemma:

Lemma 3.2 (Entropy Dissipation Term). *Let u be an entropy solution of* (1.1). *Then, for any non-negative $\phi \in C_0^\infty(\Pi_T)$ and $c \in \mathbf{R}$ such that $A(c) \notin E$, we have*

$$\begin{aligned}
&\iint_{\Pi_T} \Big(|u-c|\partial_t \phi + \operatorname{sign}(u-c)\big[f(x,t,u) - f(x,t,c) - K(x,t)\nabla A(u)\big]\cdot \nabla\phi \\
&\qquad - \operatorname{sign}(u-c)\big(\operatorname{div} f(x,t,c) - q(x,t,u)\big)\phi\Big)\,dt\,dx \\
&= \lim_{\varepsilon \downarrow 0} \iint_{\Pi_T} \big|\sqrt{K(x,t)}\nabla A(u)\big|^2 \operatorname{sign}_\varepsilon'(A(u)-A(c))\,\phi\,dt\,dx.
\end{aligned} \tag{3.4}$$

Proof. The proof is similar to the proof of the corresponding result in Carrillo [6], see also [14].

Before we start, let us remind ourselves of the fact that

$$\operatorname{sign}(u-c) = \operatorname{sign}(A(u)-A(c)) \quad \text{a.e. in } \Pi_T,$$

for each $c \in \mathbf{R}$ such that $A(c) \notin E$. We will use this fact several times below.

Lemma 3.1 can be applied with $\psi(z) = \operatorname{sign}_\varepsilon(z - A(c))$, so that

$$-\int_0^T \Big\langle \partial_t u, \operatorname{sign}_\varepsilon(A(u)-A(c))\,\phi\Big\rangle\,dt = \iint_{\Pi_T}\Big(\int_c^u \operatorname{sign}_\varepsilon(A(z)-A(c))\,dz\Big)\partial_t\phi\,dt\,dx.$$

Since (3.3) holds, $(\operatorname{sign}_\varepsilon(A(u)-A(c))\,\phi)$ belongs to $L^2(0,T;H^1(\mathbf{R}))$ and can thus act as a test function in (3.2). Consequently, we have

$$\begin{aligned}
&-\int_0^T \Big\langle \partial_t u, \operatorname{sign}_\varepsilon(A(u)-A(c))\,\phi\Big\rangle\,dt \\
&\quad + \iint_{\Pi_T}\Big(\big[f(x,t,u)-f(x,t,c)-K(x,t)\nabla A(u)\big]\cdot\nabla\big(\operatorname{sign}_\varepsilon(A(u)-A(c))\,\phi\big) \\
&\qquad - \big(\operatorname{div} f(x,t,c) - q(x,t,u)\big)\big(\operatorname{sign}_\varepsilon(A(u)-A(c))\,\phi\big)\Big)\,dt\,dx = 0,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\iint_{\Pi_T}\Big(\int_c^u \operatorname{sign}_\varepsilon(A(z)-A(c))\,dz\Big)\partial_t\phi\,dt\,dx \\
&\quad + \iint_{\Pi_T}\Big(\big[f(x,t,u)-f(x,t,c)-K(x,t)\nabla A(u)\big]\cdot\nabla\big(\operatorname{sign}_\varepsilon(A(u)-A(c))\,\phi\big) \\
&\qquad - \operatorname{sign}_\varepsilon(A(u)-A(c))\big(\operatorname{div} f(x,t,c) - q(x,t,u)\big)\phi\Big)\,dt\,dx = 0.
\end{aligned} \tag{3.5}$$

For c such that $A(c) \notin E$, we have (see [6, 14])

$$\lim_{\varepsilon\downarrow 0}\iint_{\Pi_T}\Big(\int_c^u \operatorname{sign}_\varepsilon(A(z)-A(c))\,dz\Big)\partial_t\phi\,dt\,dx = \iint_{\Pi_T} |u-c|\partial_t\phi\,dt\,dx.$$

For c such that $A(c) \notin E$, we can calculate as follows (see [6, 14]):

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \iint\limits_{\Pi_T} \big[f(x,t,u) - f(x,t,c) - K(x,t)\nabla A(u)\big] \cdot \nabla\big(\mathrm{sign}_\varepsilon\left(A(u) - A(c)\right)\phi\big)\,dt\,dx \\
&\quad = \lim_{\varepsilon\downarrow 0} \iint\limits_{\Pi_T} \mathrm{sign}_\varepsilon\left(A(u) - A(c)\right)\big[f(x,t,u) - f(x,t,c) - K(x,t)\nabla A(u)\big] \cdot \nabla\phi\,dt\,dx \\
&\qquad + \lim_{\varepsilon\downarrow 0} \iint\limits_{\Pi_T} \big[f(x,t,u) - f(x,t,c) - K(x,t)\nabla A(u)\big] \cdot \nabla \mathrm{sign}_\varepsilon\left(A(u) - A(c)\right)\phi\,dt\,dx \\
&\quad = \lim_{\varepsilon\downarrow 0} \iint\limits_{\Pi_T} \mathrm{sign}_\varepsilon\left(A(u) - A(c)\right)\big[f(x,t,u) - f(x,t,c) - K(x,t)\nabla A(u)\big] \cdot \nabla\phi\,dt\,dx \\
&\qquad + \underbrace{\lim_{\varepsilon\downarrow 0} \iint\limits_{\Pi_T} \mathrm{sign}'_\varepsilon\left(A(u) - A(c)\right)\big(f(x,t,u) - f(x,t,c)\big) \cdot \nabla A(u)\phi\,dt\,dx}_{\to 0 \text{ as } \varepsilon \downarrow 0} \\
&\qquad\quad - \lim_{\varepsilon\downarrow 0} \iint\limits_{\Pi_T} \big|\sqrt{K(x,t)}\nabla A(u)\big|^2 \mathrm{sign}'_\varepsilon\left(A(u) - A(c)\right)\phi\,dt\,dx \\
&\quad = \iint\limits_{\Pi_T} \mathrm{sign}\left(u - c\right)\big[f(x,t,u) - f(x,t,c) - K(x,t)\nabla A(u)\big] \cdot \nabla\phi\,dt\,dx \\
&\qquad - \lim_{\varepsilon\downarrow 0} \iint\limits_{\Pi_T} \big|\sqrt{K(x,t)}\nabla A(u)\big|^2 \mathrm{sign}'_\varepsilon\left(A(u) - A(c)\right)\phi\,dt\,dx.
\end{aligned}
$$

In addition, for c such that $A(c) \notin E$,

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \iint\limits_{\Pi_T} \mathrm{sign}_\varepsilon\left(A(u) - A(c)\right)\big(\mathrm{div} f(x,t,c) - q(x,t,u)\big)\phi\,dt\,dx \\
&\qquad = \iint\limits_{\Pi_T} \mathrm{sign}\left(u - c\right)\big(\mathrm{div} f(x,t,c) - q(x,t,u)\big)\phi\,dt\,dx.
\end{aligned}
$$

Summing up, sending $\varepsilon \downarrow 0$ in (3.5) yields the desired equation (3.4). □

4. Proof of Theorem 2.1

We now set out to prove Theorem 2.1 by properly adapting Kružkov's "doubling of variables" device to deal with a spatially dependent second order differential operator. We refer to Carrillo [6] for the proof when the first and second order differential operators do not depend on x, t and Karlsen and Risebro [14] for the proof when the first order differential operator is x, t dependent.

Let $\phi \in C^\infty(\Pi_T \times \Pi_T)$, $\phi \geq 0$, $\phi = \phi(x,t,y,s)$, $v = v(x,t)$, and $u = u(y,s)$. Before we continue, let us specify our choice of ϕ to be used in the remaining part of this paper. Let $\delta \in C_0^\infty(\mathbf{R})$ be a nonnegative function satisfying

$$
\delta(\sigma) = \delta(-\sigma), \quad \delta(\sigma) \equiv 0 \text{ for } |\sigma| \geq 1, \quad \int_{\mathbf{R}} \delta(\sigma)\,d\sigma = 1.
$$

For $\rho_0 > 0$, let $\delta_{\rho_0}(\sigma) = \frac{1}{\rho_0}\delta\left(\frac{\sigma}{\rho_0}\right)$. Pick two (arbitrary but fixed) Lebesgue points $\nu, \tau \in (0, T)$ of $\|v(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbf{R}^d)}$. For any $\alpha_0 \in \big(0, \min(\nu, T - \tau)\big)$, let

$$
W_{\alpha_0}(t) = H_{\alpha_0}(t - \nu) - H_{\alpha_0}(t - \tau), \quad H_{\alpha_0}(t) = \int_{-\infty}^{t} \delta_{\alpha_0}(s)\,ds.
$$

For $\rho > 0$, $\sigma \in \mathbf{R}$, $x \in \mathbf{R}^d$, let $\delta_\rho(\sigma) = \frac{1}{\rho}\delta\left(\frac{\sigma}{\rho}\right)$, and $\omega_\rho(x) = \delta_\rho(x_1) \cdots \delta_\rho(x_d)$.

Following [15, 16], we take ϕ to be of the form

$$\phi(x,t,y,s) = W_{\alpha_0}(t)\,\omega_\rho(x-y)\delta_{\rho_0}(t-s), \tag{4.1}$$

so that the derivatives of ϕ which are singular in the limit $\rho, \rho_0 \downarrow 0$ cancel:

$$\nabla_x\phi + \nabla_y\phi = 0, \qquad \partial_t\phi + \partial_s\phi = \big[\delta_{\alpha_0}(t-\nu) - \delta_{\alpha_0}(t-\tau)\big]\omega_\rho(x-y)\delta_{\rho_0}(t-s). \tag{4.2}$$

We shall need the "hyperbolic" sets

$$\mathcal{E}_v = \Big\{(x,t) \in \Pi_T : A(v(x,t)) \in E\Big\}, \qquad \mathcal{E}_u = \Big\{(y,s) \in \Pi_T : A(u(y,s)) \in E\Big\}.$$

For later use, observe that we have $\nabla_x A(v) = 0$ a.e. in $\mathcal{E}_v$ and $\nabla_y A(u) = 0$ a.e. in $\mathcal{E}_u$ as well as $\operatorname{sign}(v-u) = \operatorname{sign}(A(v) - A(u))$ a.e. in

$$\Big[(\Pi_T \setminus \mathcal{E}_u) \times \Pi_T\Big] \bigcup \Big[\Pi_T \times (\Pi_T \setminus \mathcal{E}_v)\Big].$$

With $v = v(x,t)$ and $c = u(y,s)$, we use the definition of entropy solution (2.9) and Lemma 3.2 to get

(4.3)
$$\begin{aligned}
&-\iiiint\limits_{\Pi_T\times\Pi_T} \Big(|v-u|\partial_t\phi + \operatorname{sign}(v-u)\left[f(x,t,v) - f(x,t,u) - K(x,t)\nabla_x A(v)\right]\cdot\nabla_x\phi \\
&\qquad\qquad - \operatorname{sign}(v-u)\left(\operatorname{div}_x f(x,t,u) - q(x,t,v)\right)\phi\Big)\,dt\,dx\,ds\,dy \\
&= -\iiiint\limits_{(\Pi_T\setminus\mathcal{E}_u)\times\Pi_T} \Big(|v-u|\partial_t\phi + \operatorname{sign}(v-u)\left[f(x,t,v) - f(x,t,u) - K(x,t)\nabla_x A(v)\right]\cdot\nabla_x\phi \\
&\qquad\qquad - \operatorname{sign}(v-u)\left(\operatorname{div}_x f(x,t,u) - q(x,t,v)\right)\phi\Big)\,dt\,dx\,ds\,dy \\
&\quad - \iiiint\limits_{\mathcal{E}_u\times\Pi_T} \Big(|v-u|\partial_t\phi + \operatorname{sign}(v-u)\left[f(x,t,v) - f(x,t,u) - K(x,t)\nabla_x A(v)\right]\cdot\nabla_x\phi \\
&\qquad\qquad - \operatorname{sign}(v-u)\left(\operatorname{div}_x f(x,t,u) - q(x,t,v)\right)\phi\Big)\,dt\,dx\,ds\,dy \\
&\le -\lim_{\varepsilon\downarrow 0} \iiiint\limits_{(\Pi_T\setminus\mathcal{E}_u)\times\Pi_T} \left|\sqrt{K(x,t)}\nabla_x A(v)\right|^2 \operatorname{sign}'_\varepsilon\left(A(v) - A(u)\right)\phi\,dt\,dx\,ds\,dy \\
&= -\lim_{\varepsilon\downarrow 0} \iiiint\limits_{(\Pi_T\setminus\mathcal{E}_u)\times(\Pi_T\setminus\mathcal{E}_v)} \left|\sqrt{K(x,t)}\nabla_x A(v)\right|^2 \operatorname{sign}'_\varepsilon\left(A(v) - A(u)\right)\phi\,dt\,dx\,ds\,dy.
\end{aligned}$$

Similarly, with $u = u(y,s)$ and $c = v(x,t)$, we use (2.9) and Lemma 3.2 to get

$$\begin{aligned}
&-\iiiint\limits_{\Pi_T\times\Pi_T} \Big(|u-v|\partial_t\phi + \operatorname{sign}(u-v)\left[f(y,s,u) - f(y,s,v) - K(y,s)\nabla_y A(u)\right]\cdot\nabla_y\phi \\
&\qquad\qquad - \operatorname{sign}(u-v)\left(\operatorname{div}_y f(y,s,v) - q(y,s,u)\right)\phi\Big)\,dt\,dx\,ds\,dy \\
&\le -\lim_{\varepsilon\downarrow 0} \iiiint\limits_{(\Pi_T\setminus\mathcal{E}_u)\times(\Pi_T\setminus\mathcal{E}_v)} \left|\sqrt{K(y,s)}\nabla_y A(u)\right|^2 \operatorname{sign}'_\varepsilon\left(A(u) - A(v)\right)\phi\,dt\,dx\,ds\,dy.
\end{aligned} \tag{4.4}$$

Observe that

$$
\begin{aligned}
&\iiiint\limits_{\Pi_T\times\Pi_T} \operatorname{sign}(v-u)\, K(x,t)\nabla_x A(v)\cdot\nabla_y\phi\, dt\, dx\, ds\, dy\\
&\qquad = \iiiint\limits_{\Pi_T\times(\Pi_T\setminus\mathcal{E}_v)} \operatorname{sign}(A(v)-A(u))\, K(x,t)\nabla_x A(v)\cdot\nabla_y\phi\, dt\, dx\, ds\, dy\\
&\qquad = \lim_{\varepsilon\downarrow 0} \iiiint\limits_{(\Pi_T\times(\Pi_T\setminus\mathcal{E}_v)} K(x,t)\nabla_y A(u)\cdot\nabla_x A(v)\operatorname{sign}_\varepsilon'(A(v)-A(u))\,\phi\, dt\, dx\, ds\, dy.\\
&\qquad = \lim_{\varepsilon\downarrow 0} \iiiint\limits_{(\Pi_T\setminus\mathcal{E}_u)\times(\Pi_T\setminus\mathcal{E}_v)} K(x,t)\nabla_y A(u)\cdot\nabla_x A(v)\operatorname{sign}_\varepsilon'(A(v)-A(u))\,\phi\, dt\, dx\, ds\, dy.
\end{aligned}
\tag{4.5}
$$

Similarly, observe that

$$
\begin{aligned}
&\iiiint\limits_{\Pi_T\times\Pi_T} \operatorname{sign}(A(u)-A(v))\, K(y,s)\nabla_y A(u)\cdot\nabla_x\phi\, dt\, dx\, ds\, dy\\
&\qquad = \lim_{\varepsilon\downarrow 0} \iiiint\limits_{(\Pi_T\setminus\mathcal{E}_u)\times(\Pi_T\setminus\mathcal{E}_v)} K(y,s)\nabla_x A(u)\cdot\nabla_y A(v)\operatorname{sign}_\varepsilon'(A(v)-A(u))\,\phi\, dt\, dx\, ds\, dy.
\end{aligned}
\tag{4.6}
$$

Keeping (4.2) in mind, adding (4.3) and (4.5) yields

$$
\begin{aligned}
&-\iiiint\limits_{\Pi_T\times\Pi_T}\Big(|v-u|\partial_t\phi + \operatorname{sign}(v-u)\Big[\big(f(x,t,v)-f(x,t,u)\big)\cdot\nabla_x\phi\\
&\qquad\qquad - \operatorname{sign}(v-u)\big(\operatorname{div}_x f(x,t,u) - q(x,t,v)\big)\phi\Big)\, dt\, dx\, ds\, dy\\
&\qquad \le -\lim_{\varepsilon\downarrow 0}\iiiint\limits_{(\Pi_T\setminus\mathcal{E}_u)\times(\Pi_T\setminus\mathcal{E}_v)}\Big(\big|\sqrt{K(x,t)}\nabla_x A(v)\big|^2 - K(x,t)\nabla_y A(u)\cdot\nabla_x A(v)\Big)\\
&\qquad\qquad\qquad \times \operatorname{sign}_\varepsilon'(A(v)-A(u))\,\phi\, dt\, dx\, ds\, dy.
\end{aligned}
\tag{4.7}
$$

Similarly, adding (4.4) and (4.6) yields

$$
\begin{aligned}
&-\iiiint\limits_{\Pi_T\times\Pi_T}\Big(|u-v|\partial_s\phi + \operatorname{sign}(u-v)\Big[\big(f(y,s,u)-f(y,s,v)\big)\cdot\nabla_y\phi\\
&\qquad\qquad - \operatorname{sign}(u-v)\big(\operatorname{div}_y f(y,s,v) - q(y,s,u)\big)\phi\Big)\, dt\, dx\, ds\, dy\\
&\qquad \le -\lim_{\varepsilon\downarrow 0}\iiiint\limits_{(\Pi_T\setminus\mathcal{E}_u))\times(\Pi_T\setminus\mathcal{E}_v)}\Big(\big|\sqrt{K(y,s)}\nabla_y A(u)\big|^2 - K(y,s)\nabla_x A(u)\cdot\nabla_y A(u)\Big)\\
&\qquad\qquad\qquad \times \operatorname{sign}_\varepsilon'(A(u)-A(v))\,\phi\, dt\, dx\, ds\, dy.
\end{aligned}
\tag{4.8}
$$

Following [14], we write

$$
\begin{aligned}
&\operatorname{sign}(v-u)\big(f(x,t,v)-f(x,t,u)\big)\cdot\nabla_x\phi - \operatorname{sign}(v-u)\operatorname{div}_x f(x,t,u)\phi\\
&\qquad = \operatorname{sign}(v-u)\big(f(x,t,v)-f(y,s,u)\big)\cdot\nabla_x\phi + \operatorname{sign}(v-u)\operatorname{div}_x\big[\big(f(y,s,u)-f(x,t,u)\big)\phi\big]
\end{aligned}
$$

and

$$
\begin{aligned}
&\operatorname{sign}(u-v)\big(f(y,s,u)-f(y,s,v)\big)\cdot\nabla_y\phi - \operatorname{sign}(u-v)\operatorname{div}_y f(y,s,v)\phi\\
&\qquad = \operatorname{sign}(v-u)\big(f(x,t,v)-f(y,s,u)\big)\cdot\nabla_y\phi - \operatorname{sign}(v-u)\operatorname{div}_y\big[\big(f(x,t,v)-f(y,s,v)\big)\phi\big].
\end{aligned}
$$

Keeping (4.2) in mind, adding (4.7) and (4.8) yields

$$
-\iiiint\limits_{\Pi_T\times\Pi_T}\Big(|v-u|\big(\partial_t\phi+\partial_s\phi\big) + I_{\mathrm{conv}} + I_{\mathrm{sour}}\Big)\, dt\, dx\, ds\, dy \le \mathrm{RHS},
\tag{4.9}
$$

where the expression for $\partial_t\phi + \partial_s\phi$ is written out in (4.2) and

$$
\begin{aligned}
I_{\mathrm{conv}} &= \operatorname{sign}(v-u)\Big[\operatorname{div}_x\big[\big(f(y,s,u)-f(x,t,u)\big)\phi\big] - \operatorname{div}_y\big[\big(f(x,t,v)-f(y,s,v)\big)\phi\big]\Big],\\
I_{\mathrm{sour}} &= \operatorname{sign}(v-u)\big(q(x,t,v)-q(y,s,u)\big)\phi,\\
\mathrm{RHS} &= -\lim_{\varepsilon\downarrow 0} \iiiint\limits_{(\Pi_T\setminus\mathcal{E}_u)\times(\Pi_T\setminus\mathcal{E}_v)} \Bigg(\Big|\sqrt{K(x,t)}\nabla_x A(v) - \sqrt{K(y,s)}\nabla_y A(u)\Big|^2\\
&\qquad\qquad - \Big(\sqrt{K(x,t)} - \sqrt{K(y,s)}\Big)^2 \nabla_x A(v)\cdot\nabla_y A(u)\Bigg)\\
&\qquad\qquad\qquad \times \operatorname{sign}_\varepsilon'\left(A(v)-A(u)\right)\phi\,dt\,dx\,ds\,dy.
\end{aligned}
$$

Recall that $\Big(\sqrt{K(x,t)}-\sqrt{K(y,s)}\Big)^2 = \operatorname{diag}\Big\{\Big(\sqrt{k_i(x,t)}-\sqrt{k_i(y,s)}\Big)^2\,\Big|\, i=1,\dots,d\Big\}$.

Concerning the term RHS, we notice that

$$
\begin{aligned}
\mathrm{RHS} &\le \lim_{\varepsilon\downarrow 0} \iiiint\limits_{(\Pi_T\setminus\mathcal{E}_u)\times(\Pi_T\setminus\mathcal{E}_v)} \Big(\sqrt{K(x,t)}-\sqrt{K(y,s)}\Big)^2 \nabla_x A(v)\cdot\nabla_y A(u)\\
&\qquad\qquad \times \operatorname{sign}_\varepsilon'\left(A(v)-A(u)\right)\phi\,dt\,dx\,ds\,dy,\\
&= \lim_{\varepsilon\downarrow 0} \iiiint\limits_{\Pi_T\times\Pi_T} \Big(\sqrt{K(x,t)}-\sqrt{K(y,s)}\Big)^2 \nabla_x A(v)\cdot\nabla_y A(u)\\
&\qquad\qquad \times \operatorname{sign}_\varepsilon'\left(A(v)-A(u)\right)\phi\,dt\,dx\,ds\,dy.
\end{aligned}
$$

For fixed y,s, we use integration by parts in the x variable to produce

$$
\begin{aligned}
&\int_{\mathbf{R}^d}\Big(\sqrt{K(x,t)}-\sqrt{K(y,s)}\Big)^2 \nabla_x A(v)\cdot\nabla_y A(u)\operatorname{sign}_\varepsilon'\left(A(v)-A(u)\right)\phi\,dx\\
&\quad = \int_{\mathbf{R}^d}\Big(\sqrt{K(x,t)}-\sqrt{K(y,s)}\Big)^2 \nabla_y A(u)\cdot\nabla_x \operatorname{sign}_\varepsilon\left(A(v)-A(u)\right)\phi\,dx\\
&\quad = -\int_{\mathbf{R}^d}\operatorname{sign}_\varepsilon\left(A(v)-A(u)\right)\Bigg(\Big(\sqrt{K(x,t)}-\sqrt{K(y,s)}\Big)^2\nabla_y A(u)\cdot\nabla_x\phi\\
&\qquad\qquad + \phi\,\nabla_x\cdot\Big[\Big(\sqrt{K(x,t)}-\sqrt{K(y,s)}\Big)^2\nabla_y A(u)\Big]\Bigg)\,dx,
\end{aligned}
$$

from which we easily derive the upper bound

$$
\mathrm{RHS} \le E_{\mathrm{diff}}^1 + E_{\mathrm{diff}}^2,
$$

where

$$
\begin{aligned}
E_{\mathrm{diff}}^1 &= \iiiint\limits_{\Pi_T\times\Pi_T}\Big|\big(\sqrt{K(x,t)}-\sqrt{K(y,s)}\big)^2\nabla_y A(u)\cdot\nabla_x\phi\Big|\,dt\,dx\,ds\,dy,\\
E_{\mathrm{diff}}^2 &= \iiiint\limits_{\Pi_T\times\Pi_T}\Big|\nabla_x\cdot\Big[\Big(\sqrt{K(x,t)}-\sqrt{K(y,s)}\Big)^2\nabla_y A(u)\Big]\Big|\,\phi\,dt\,dx\,ds\,dy.
\end{aligned}
$$

Sending $\alpha_0,\rho_0\downarrow 0$ in E_{diff}^1 and E_{diff}^2, we obtain

$$
\begin{aligned}
\lim_{\alpha_0,\rho_0\downarrow 0} E_{\mathrm{diff}}^1 &= \int_\nu^\tau\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}\Big|\big(\sqrt{K(x,t)}-\sqrt{K(y,t)}\big)^2\nabla_y A(u)\cdot\nabla_x\omega_\rho(x-y)\Big|\,dx\,dy\,dt,\\
\lim_{\alpha_0,\rho_0\downarrow 0} E_{\mathrm{diff}}^2 &= \int_\nu^\tau\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}\Big|\nabla_x\cdot\Big[\Big(\sqrt{K(x,t)}-\sqrt{K(y,t)}\Big)^2\nabla_y A(u)\Big]\Big|\,\omega_\rho(x-y)\,dx\,dy\,dt.
\end{aligned}
$$

In what follows, we set $z = x - y$, and $C_A = \max_{i=1,\dots,d} \|\partial_{y_i} A(u)\|_{L^2(\Pi_T)} < \infty$. We also note that (2.5) implies the existence of a finite constant C such that for all $i = 1, \dots, d$

$$\begin{aligned} &\left\| \sqrt{k_i(\cdot + z, \cdot)} - \sqrt{k_i(\cdot, \cdot)} \right\|_{L^\infty(\Pi_T)} \leq C|z|, \qquad \forall z \in \mathbf{R}^d, \\ &\left\| \sqrt{k_i(\cdot + z, \cdot)} - \sqrt{k_i(\cdot, \cdot)} \right\|_{L^2(\Pi_T)} \leq C|z|, \qquad \forall z \in \mathbf{R}^d. \end{aligned} \tag{4.10}$$

Using Hölder's inequality, we estimate as follows:

$$\begin{aligned} \lim_{\alpha_0, \rho_0 \downarrow 0} E^1_{\text{diff}} &\leq \sum_{i=1}^{d} \int_\nu^\tau \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \left(\sqrt{k_i(x,t)} - \sqrt{k_i(y,t)} \right)^2 \left| \partial_{y_i} A(u) \right| \left| \partial_{x_i} \omega_\rho(x-y) \right| dx\, dy\, dt \\ &\leq \sum_{i=1}^{d} \int_\nu^\tau \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \left(\sqrt{k_i(y+z,t)} - \sqrt{k_i(y,t)} \right)^2 \left| \partial_{y_i} A(u) \right| dy \left| \partial_{z_i} \omega_\rho(z) \right| dz\, dt \\ &\leq C_A \sum_{i=1}^{d} \int_{\mathbf{R}^d} \left\| \left(\sqrt{k_i(\cdot + z, \cdot)} - \sqrt{k_i(\cdot, \cdot)} \right)^2 \right\|_{L^2(\Pi_T)} \left| \partial_{z_i} \omega_\rho(z) \right| dz \\ &\leq C_A C^1_K \sum_{i=1}^{d} \int_{\mathbf{R}^d} |z|^2 \left| \partial_{z_i} \omega_\rho(z) \right| dz, \end{aligned}$$

for some finite constant C^1_K depending only on K. To derive the fourth inequality from the third, we have used (4.10) in combination with the estimate

$$\begin{aligned} &\left\| \left(\sqrt{k_i(\cdot + z, \cdot + \tau)} - \sqrt{k_i(\cdot, \cdot)} \right)^2 \right\|_{L^2(\Pi_T)} \\ &\quad \leq \left\| \sqrt{k_i(\cdot + z, \cdot + \tau)} - \sqrt{k_i(\cdot, \cdot)} \right\|_{L^\infty(\Pi_T)} \left\| \sqrt{k_i(\cdot + z, \cdot + \tau)} - \sqrt{k_i(\cdot, \cdot)} \right\|_{L^2(\Pi_T)}. \end{aligned}$$

As ω_ρ is supported in $\{ x \in \mathbf{R}^d \mid x_i \leq \rho \text{ for all } i = 1, \dots, d \}$ and

$$\int_{\mathbf{R}^d} \left| \partial_{z_i} \omega_\rho(z) \right| dz \leq \frac{1}{\rho} \int_{\mathbf{R}^d} \left| \partial_{z_i} \omega(z) \right| dz,$$

straightforward calculations will reveal that

$$\lim_{\rho \downarrow 0} \int_{\mathbf{R}^d} |z|^2 \left| \partial_{z_i} \omega_\rho(z) \right| dz \leq \lim_{\rho \downarrow 0} \int_{\mathbf{R}^d} d\rho^2 \frac{1}{\rho} \left| \partial_{z_i} \omega(z) \right| dz \leq C \lim_{\rho \downarrow 0} \rho = 0.$$

We therefore conclude that E^1_{diff} tends to zero as $\alpha_0, \rho, \rho_0 \downarrow 0$.

In a similar way, we can show that also E^2_{diff} tends to zero as $\alpha_0, \rho, \rho_0 \downarrow 0$. To this end, let

$$C^2_K = \max_{i=1,\dots,d} \left\| \frac{\partial_{x_i} k_i}{\sqrt{k_i}} \right\|_{L^\infty(\Pi_T)},$$

and note that $C^2_K < \infty$ in view of (2.6) and (4.10). Using Hölder's inequality and (4.10), we get

$$\begin{aligned} \lim_{\alpha_0, \rho_0 \downarrow 0} E^2_{\text{diff}} &\leq \sum_{i=1}^{d} \int_\nu^\tau \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \left| \sqrt{k_i(y+z,t)} - \sqrt{k_i(y,t)} \right| \left| \frac{\partial_{x_i} k_i(y+z,t)}{\sqrt{k_i(y+z,t)}} \right| \left| \partial_{y_i} A(u) \right| dy\, \omega_\rho(z)\, dz\, dt \\ &\leq C^2_K \sum_{i=1}^{d} \int_\nu^\tau \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \left| \sqrt{k_i(y+z,t)} - \sqrt{k_i(y,t)} \right| \left| \partial_{y_i} A(u) \right| dy\, \omega_\rho(z)\, dz\, dt \\ &\leq C_A C^2_K \sum_{i=1}^{d} \int_{\mathbf{R}^d} \left\| \sqrt{k_i(\cdot + z, \cdot)} - \sqrt{k_i(\cdot, \cdot)} \right\|_{L^2(\Pi_T)} \omega_\rho(z)\, dz \\ &\leq C_A C^2_K \sum_{i=1}^{d} \int_{\mathbf{R}^d} |z| \omega_\rho(z)\, dz \to 0 \quad \text{as } \rho \downarrow 0. \end{aligned}$$

Summing up, we have shown that

$$\lim_{\alpha_0,\rho,\rho_0\downarrow 0}\Big(E^1_{\text{diff}}+E^2_{\text{diff}}\Big)=0. \tag{4.11}$$

Equipped with (4.2) and (4.11), we can now send $\alpha_0,\rho,\rho_0\downarrow 0$ in (4.9). The result reads

$$\|v(\cdot,\tau)-u(\cdot,\tau)\|_{L^1(\mathbf{R}^d)}\le\|v(\cdot,\nu)-u(\cdot,\nu)\|_{L^1(\mathbf{R}^d)}+\lim_{\alpha_0,\rho,\rho_0\downarrow 0}\Big(E_{\text{conv}}+E_{\text{sour}}\Big), \tag{4.12}$$

where $E_{\text{conv}}=\iiiint\limits_{\Pi_T\times\Pi_T} I_{\text{conv}}\,dt\,dx\,ds\,dy$ and $E_{\text{sour}}=\iiiint\limits_{\Pi_T\times\Pi_T} I_{\text{sour}}\,dt\,dx\,ds\,dy$.

In what follows, we shall estimate E_{conv} and E_{sour} separately, starting with E_{conv}. This part of the proof follows [14] closely. We first write $I_{\text{conv}}=I^1_{\text{conv}}+I^2_{\text{conv}}$, where

$$\begin{aligned}
I^1_{\text{conv}}&=\operatorname{sign}(v-u)\Big[\big(f(y,s,u)-f(x,t,u)\big)\cdot\nabla_x\phi-\big(f(x,t,v)-f(y,s,v)\big)\cdot\nabla_y\phi\Big],\\
I^2_{\text{conv}}&=\operatorname{sign}(v-u)\big(\operatorname{div}_y f(y,s,v)-\operatorname{div}_x f(x,t,u)\big)\phi.
\end{aligned}$$

Letting $E^1_{\text{conv}}=\iiiint\limits_{\Pi_T\times\Pi_T} I^1_{\text{conv}}\,dt\,dx\,ds\,dy$ and $E^2_{\text{conv}}=\iiiint\limits_{\Pi_T\times\Pi_T} I^2_{\text{conv}}\,dt\,dx\,ds\,dy$, we hence write

$$E_{\text{conv}}=E^1_{\text{conv}}+E^2_{\text{conv}}.$$

Using (2.2) and (2.8), we see immediately that

$$\lim_{\alpha_0,\rho,\rho_0\downarrow 0}\Big(E^2_{\text{conv}}+E_{\text{sour}}\Big)\le C\int_\nu^\tau\|v(\cdot,t)-u(\cdot,t)\|_{L^1(\mathbf{R}^d)}\,dt,$$

for some constant C.

It remains to study E^1_{conv}. Using $\nabla_y\phi=-\nabla_x\phi$, the integrand I^1_{conv} can be rewritten as

$$I^1_{\text{conv}}=\Big(F(x,t,v,u)-F(y,s,v,u)\Big)\cdot\nabla_x\phi=\sum_{i=1}^d\Big(F_i(x,t,v,u)-F_i(y,s,v,u)\Big)\partial_{x_i}\phi,$$

where F_i is defined in (2.4). Sending $\alpha_0,\rho_0\downarrow 0$ in E^1_{conv}, we obtain

$$\sum_{i=1}^d\int_\nu^\tau\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}\Big(F_i(x,t,v(x,t),u(y,t))-F_i(y,t,v(x,t),u(y,t))\Big)\partial_{x_i}\omega_\rho(x-y)\,dy\,dx\,dt,$$

where $F=(F_1,\dots,F_d)$ and F_i is defined in (2.4). Taking (2.3) into account, we get

$$\begin{aligned}
&\Big(F_i(x,t,v(x,t),u(y,t))-F_i(y,t,v(x,t),u(y,t))\Big)\partial_{x_i}\omega_\rho(x-y)\\
&\qquad\le\gamma|v(x,t)-u(y,t)|\,\frac{|x-y|}{\rho}\,\frac{1}{\rho}\delta\left(\frac{x_1-y_1}{\rho}\right)\cdots\frac{1}{\rho}\left|\delta'\left(\frac{x_i-y_i}{\rho}\right)\right|\cdots\frac{1}{\rho}\delta\left(\frac{x_d-y_d}{\rho}\right)\\
&\qquad\le C_{F,\delta}\,|v(x,t)-u(y,t)|\,\frac{1}{\rho}\mathbf{1}_{|x_1-y_1|<\rho}\cdots\frac{1}{\rho}\mathbf{1}_{|x_i-y_i|<\rho}\cdots\frac{1}{\rho}\mathbf{1}_{|x_d-y_d|<\rho},
\end{aligned}$$

where $C_{F,\delta}=\gamma\max\Big(\max_\sigma|\delta(\sigma)|\,,\,\max_\sigma|\delta'(\sigma)|\Big)$. With t fixed, we have that the limit

$$\lim_{\rho\downarrow 0}\left\{\frac{1}{\rho}\int_{|x_1-y_1|<\rho}\cdots\frac{1}{\rho}\int_{|x_i-y_i|<\rho}\cdots\frac{1}{\rho}\int_{|x_d-y_d|<\rho}|v(x,t)-u(y,t)|\,dy_1\cdots dy_i\cdots dy_d\right\}$$

equals $|v(x,t)-u(x,t)|$ for a.e. $x\in\mathbf{R}^d$. Consequently, we obtain the estimate

$$\lim_{\alpha_0,\rho,\rho_0\downarrow 0}E^1_{\text{conv}}\le C\int_\nu^\tau\|v(\cdot,t)-u(\cdot,t)\|_{L^1(\mathbf{R}^d)}\,dt,$$

for some constant C.

Summing up, we have proved that

$$\|v(\cdot,\tau)-u(\cdot,\tau)\|_{L^1(\mathbf{R}^d)}\le\|v(\cdot,\nu)-u(\cdot,\nu)\|_{L^1(\mathbf{R}^d)}+C\int_\nu^\tau\|v(\cdot,t)-u(\cdot,t)\|_{L^1(\mathbf{R}^d)}\,dt,$$

for some constant $C > 0$ depending on f, q, and the test function. Sending $\nu \downarrow 0$ and then using Gronwall's lemma, we get

$$\|v(\cdot,\tau) - u(\cdot,\tau)\|_{L^1(\mathbf{R}^d)} \le e^{C\tau} \|v(\cdot,0) - u(\cdot,0)\|_{L^1(\mathbf{R}^d)} \equiv 0. \tag{4.13}$$

Since this inequality holds for a.e. $\tau \in (0,T)$, we can conclude that $v = u$ a.e. in Π_T. This concludes the proof of Theorem 2.1.

5. Proof of Theorem 2.2

Next we restrict ourselves to problems of the form (1.2). Let $v, u \in L^\infty(0,T;BV(\mathbf{R}^d))$ be two entropy solutions of (1.2) with initial data $v_0, u_0 \in L^1(\mathbf{R}^d) \bigcap L^\infty(\mathbf{R}^d) \bigcap BV(\mathbf{R}^d)$, respectively. Following [14] closely, we repeat everything up to (4.12) and find that

$$\|v(\cdot,\tau) - u(\cdot,\tau)\|_{L^1(\mathbf{R}^d)} \le \|v(\cdot,\nu) - u(\cdot,\nu)\|_{L^1(\mathbf{R}^d)} + \lim_{\alpha_0,\rho,\rho_0\downarrow 0} \Big(E^1_{\mathrm{conv}} + E^2_{\mathrm{conv}} + E_{\mathrm{sour}}\Big), \tag{5.1}$$

where

$$E^1_{\mathrm{conv}} = \iiiint_{\Pi_T\times\Pi_T} F(v,u)\big(V(x,t) - V(y,s)\big)\cdot\nabla_x\phi\,dt\,dx\,ds\,dy,$$

$$E^2_{\mathrm{conv}} = \iiiint_{\Pi_T\times\Pi_T} \mathrm{sign}\,(v-u)\,\big(\mathrm{div}_y V(y,s) f(v) - \mathrm{div}_x V(x,t) f(u)\big)\phi\,dt\,dx\,ds\,dy,$$

$$E_{\mathrm{sour}} = \iiiint_{\Pi_T\times\Pi_T} \mathrm{sign}\,(v-u)\,\big(q(x,t,v) - q(y,s,u)\big)\phi\,dt\,dx\,ds\,dy,$$

the test function $\phi = \phi(x,t,y,s)$ is defined in (4.1), and $F(v,u) := \mathrm{sign}\,(v-u)\,\big[f(v) - f(u)\big]$.

We start by estimating E^1_{conv}. Note that the function $F(v,u)$ is locally Lipschitz continuous in v and u with Lipschitz constant that of f. Now since $v(\cdot,t) \in L^\infty(\mathbf{R}^d)\bigcap BV(\mathbf{R}^d)$ for each t, $\nabla_x F(v,u)$ is a finite measure. After an integration by parts, we thus get

$$E^1_{\mathrm{conv}} = -\iiiint_{\Pi_T\times\Pi_T} \Big(\mathrm{div}_x V(x,t) F(v,u) + \big(V(x,t) - V(y,s)\big)\cdot\nabla_x F(v,u)\Big)\phi\,dt\,dx\,ds\,dy.$$

Since $\nabla_x F(v,u)$ is a finite measure, it follows that

$$\iiiint_{\Pi_T\times\Pi_T} \big(V(x,t) - V(y,s)\big)\cdot\nabla_x F(v,u)\phi\,dt\,dx\,ds\,dy \to 0 \text{ as } \alpha_0\rho, \rho_0 \downarrow 0.$$

Consequently, we end up with

$$\lim_{\alpha_0\rho,\rho_0\downarrow 0} E^1_{\mathrm{conv}} = -\int_\nu^T \int_{\mathbf{R}^d} \mathrm{div}_x V(x,t) F(v(x,t),u(x,t))\,dx\,dt.$$

Regarding E^2_{conv}, it is easy to see that

$$\lim_{\alpha_0,\rho,\rho_0\downarrow 0} E_2 = \int_\nu^\tau \int_{\mathbf{R}^d} \mathrm{div}_x V(x,t) F(v(x,t),u(x,t))\,dt\,dx \equiv -\lim_{\alpha_0\rho,\rho_0\downarrow 0} E^1_{\mathrm{conv}}.$$

Finally, we have as before

$$\lim_{\alpha_0,\rho,\rho_0\downarrow 0} E_{\mathrm{sour}} \le C\int_\nu^\tau \|v(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbf{R}^d)}\,dt,$$

for some constant C.

From (5.1), we hence get

$$\|v(\cdot,\tau) - u(\cdot,\tau)\|_{L^1(\mathbf{R}^d)} \le \|v(\cdot,\nu) - u(\cdot,\nu)\|_{L^1(\mathbf{R}^d)} + C\int_\nu^\tau \|v(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbf{R}^d)}\,dt.$$

Sending $\nu \downarrow 0$, using Gronwall's lemma, and recalling that $\tau \in (0,T)$ was an arbitrary Lebesgue point of $\|v(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbf{R}^d)}$, we obtain the L^1 stability property claimed in Theorem 2.2.

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