CONTINUOUS DEPENDENCE ESTIMATES FOR VISCOSITY SOLUTIONS OF FULLY NONLINEAR DEGENERATE PARABOLIC EQUATIONS

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ABSTRACT. Using the maximum principle for semicontinuous functions [2, 3], we establish here a general "continuous dependence on the nonlinearities" estimate for viscosity solutions of fully nonlinear degenerate parabolic equations with time and space dependent nonlinearities. Our result generalizes a result by Souganidis [10] for first order Hamilton-Jacobi equations and a recent result by Cockburn, Gripenperg, and Londen [1] for a class of degenerate parabolic second order equations. We apply this result to a rather general class of equations and obtain: (i) Explicit continuous dependence estimates. (ii) L^∞ and Hölder regularity estimates. (iii) A rate of convergence for the vanishing viscosity method. Finally, we illustrate the results (i) – (iii) on the Hamilton-Jacobi-Bellman partial differential equation associated with optimal control of a degenerate diffusion process over a finite horizon. For this equation such results are usually derived via probabilistic arguments, which we avoid entirely here. In [8], the basic result given herein is used to derive an explicit rate of convergence for certain numerical approximations.

1. INTRODUCTION

Fully nonlinear degenerate parabolic partial differential equations arise in a variety of applications, ranging from image processing, via optimal stochastic control theory, to the description of evolving interfaces (front propagation problems). Due to a possibly degenerate second order operator, such nonlinear partial differential equations do not in general possess classical solutions and it becomes necessary to interpret them in the sense of viscosity solutions. Here we study viscosity solutions of fully nonlinear degenerate parabolic equations of the type

$$u_t + F(t, x, u, Du, D^2u) = 0 \quad \text{in } Q_T := (0, T) \times \mathbb{R}^N, \tag{1.1}$$

where $u : Q_T \to \mathbb{R}$ is the scalar function that is sought; D denotes the gradient with respect to $x = (x_1, \dots, x_N) \in \mathbb{R}^N$; D^2 denotes the Hessian with respect to x; and the nonlinearity $F = F(t, x, r, p, X)$ is a function that is nonincreasing in its last (matrix) argument X.

Since the introduction [4] of the theory of viscosity solutions for first order Hamilton-Jacobi equations in the early eighties, the theory (existence, uniqueness, stability, regularity, etc.) has by now been intensively studied and extended to a large class of fully nonlinear second order partial differential equations. A part of this theory is an impressive uniqueness (comparison) machinery based on the so-called maximum principle for semicontinuous functions [2, 3]. The uniqueness machinery applies to (1.1) under rather general assumptions on F. We refer to Crandall, Ishii, and Lions [3] for an overview of the viscosity solution theory.

In this paper, we are concerned with the problem of finding an upper bound on the difference between a viscosity subsolution u of (1.1) and a viscosity supersolution v of

$$u_t + G(t, x, u, Du, D^2u) = 0 \quad \text{in } Q_T,$$

where $G = G(t, x, r, p, X)$ is another nonlinearity that is nonincreasing in its last argument. The sought upper bound for $u(t, \cdot) - v(t, \cdot)$ should in one way or another be expressed in terms of the difference between the initial data $u(0, \cdot) - v(0, \cdot)$ and the difference between the nonlinearities "$F - G$". A continuous dependence estimate of the type sought here was obtained by Souganidis in [10, Proposition 1.4] for first order Hamilton-Jacobi equations. For degenerate parabolic second

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order equations, a straightforward applications of the comparison principle [3, p. 50] gives for any $0 \leq t < T$ the estimate

$$\begin{aligned} \sup_{x\in\mathbb{R}^N}\Big(u(t,x)-v(t,x)\Big) \leq& \sup_{x\in\mathbb{R}^N}\Big(u(0,x)-v(0,x)\Big)^+ \\ &+\int_0^t \sup_{(x,r,p,X)\in\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N\times S(N)}\Big(G(s,x,r,p,X)-F(s,x,r,p,X)\Big)^+ ds, \end{aligned} \tag{1.2}$$

where $S(N)$ denotes the set of symmetric $N \times N$ matrices and $a^+ = \max(a, 0)$. This estimate can be applied, for example, when G is of the form $F + h$ for some function $h = h(x)$. In general, this estimate is not particularly useful since the set over which the supremum inside the integral is taken is unbounded. For example, it cannot be used to obtain a convergence rate for viscous approximations $v_t + F(t, x, v, Dv, D^2v) - \nu\Delta v = 0$.

Recently, Cockburn, Gripenperg, and Londen [1] showed how one can improve the continuous dependence estimate in (1.2) for simplified equations of the type

$$u_t + f(u, Du, D^2u) - k(Du)\Delta u = 0 \quad \text{in } Q_T, \tag{1.3}$$

where the nonlinearity $f = f(r, p, X)$ is nondecreasing in its first argument and nonincreasing in its last argument while the "diffusion coefficient" $k = k(p)$ is nonnegative. Note that equation (1.3) can be viewed as a special case of $u_t + f(u, Du, D^2u) = 0$. However, as observed in [1], sharper results are obtained by not doing so. Let u be a viscosity subsolution of (1.3) and let v be a viscosity supersolution of (1.3) with f, k replaced by g, l respectively. Roughly speaking, the result in [1] states that for any $0 \leq t < T$ and $\alpha > 0$

$$\begin{aligned} \sup_{x\in\mathbb{R}^N}\Big(u(t,x)-v(t,x)\Big) \leq& \sup_{x\in\mathbb{R}^N}\Big(u(0,x)-v(0,x)\Big)^+ \\ &+ \sup_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N}\Big(|u(0,x)-u(0,y)|\wedge|v(0,x)-v(0,y)| - \frac{\alpha}{2}|x-y|^2\Big) \\ &+ t\sup_{(r,p,X)\in D^\alpha}\Big(g(r,p,X)-f(r,p,X)+3\alpha N\Big(\sqrt{k(p)}-\sqrt{l(p)}\Big)^2\Big)^+, \end{aligned} \tag{1.4}$$

where $a \wedge b = \min(a, b)$. The second term on the right-hand side in (1.4) measures the "amount of continuity" that the initial values $u(0, \cdot), v(0, \cdot)$ possess. In third term on the right-hand side in (1.4), the supremum is taken over a bounded set $D^\alpha \subset \mathbb{R} \times \mathbb{R}^N \times S(N)$ that depends on the free parameter α. The set D^α becomes unbounded as $\alpha \to \infty$. The idea is that in each particular case one can choose the parameter α in (1.4) so as to obtain optimal results. The proof of (1.4) (as well as (1.6) below) is very similar to the proof of the comparison principle [3] and uses the maximum principle for semicontinuous functions [2, 3].

Motivated by applications, we seek in this paper to generalize the continuous dependence result in [1] to more general equations of the form

$$u_t + f(t, x, u, Du, D^2u) - tr\big[A(t, x, Du)D^2u\big] = 0 \quad \text{in } Q_T, \tag{1.5}$$

where the nonlinearity $f = f(t, x, r, p, X)$ is nonincreasing in its last argument, the $N \times N$ matrix $A = A(t, x, p)$ is of the type $a(t, x, p)a(t, x, p)^T$ for some $N \times P$ matrix $a = a(t, x, p)$, and tr denotes the trace operator. Equation (1.5) generalizes (1.3) in three ways: (i) The nonlinearities are allowed to depend explicitly on the temporal and spatial variables. (ii) The second order operator $tr\big[A(t, x, Du)D^2u\big]$ is rather general and contains the operator $k(Du)\Delta u$ in (1.3) as a simple special case. (iii) $f = f(t, x, r, p, X)$ is not restricted to be monotone in the r variable.

Our main result (Theorem 3.1) is an upper bound on $u - v$ where u is a viscosity subsolution of (1.5) and v is a viscosity supersolution of (1.5) with f, A replaced by g, B respectively, where $B(t, x, p) = b(t, x, p)b(t, x, p)^T$ for another $N \times P$ matrix $b = b(t, x, p)$. Assume for simplicity of notation that $f = f(t, x, r, p, X)$ is nondecreasing in the r variable and that the viscosity sub- and supersolutions are merely semicontinuous (see Section 3 for the general case). Roughly speaking, our main result (Theorem 3.1) then states that for any $0 \leq t < T$ and $\alpha > 0$

(1.6)
$$\sup_{\substack{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N\\ |x-y|\leq C\sqrt{\frac{1}{\alpha}}}} \Big(u(t,x) - v(t,y) - \frac{\alpha}{2}|x-y|^2\Big) \leq \sup_{\substack{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N\\ |x-y|\leq C\sqrt{\frac{1}{\alpha}}}} \Big(u(0,x) - v(0,y) - \frac{\alpha}{2}|x-y|^2\Big)^+$$
$$+ t \sup_{\substack{(\tau,x,y)\in[0,t)\times\mathbb{R}^N\times\mathbb{R}^N\\ |x-y|\leq C\sqrt{\frac{1}{\alpha}},\,(r,p,X)\in D^\alpha}} \Big(g(\tau,y,r,p,X) - f(\tau,x,r,p,X) + 3\alpha c^2\left|a(\tau,x,p) - b(\tau,y,p)\right|^2\Big)^+,$$

where $D^\alpha \subset \mathbb{R}\times\mathbb{R}^N\times S(N)$ is again a bounded subset for each fixed α but becomes unbounded as $\alpha\to\infty$, $C>0$ is a constant independent of α, and $c = N\wedge P$. We note that (1.6) is different from (1.4) in that a quadratic penalization term also occur on the left-hand side of the inequality. In view of their respective proofs, we feel that (1.6) is a more natural statement than (1.4). One can, however, quite easily derive from (1.6) an upper bound that resembles (1.4). Estimate (1.6) can be viewed as a direct generalization to second order equations of Proposition 1.4 in Souganidis [10] for first order equations. The main technical tool that makes this extension possible is of course the maximum principle for semicontinuous functions [2, 3].

Our treatment actually allows us to consider a fully nonlinear version of (1.5). In fact, later we shall state and prove our main result (Theorem 3.1) for the fully nonlinear equation

$$u_t + \sup_{\vartheta\in\Theta}\Big\{f^\vartheta(t,x,u,Du,D^2u) - tr\big[A^\vartheta(t,x,Du)D^2u\big]\Big\} = 0 \quad \text{in } Q_T, \tag{P}$$

where Θ is a given index set and f^ϑ, A^ϑ are of the same type as f, A respectively for each $\vartheta\in\Theta$ (see Section 3 for the precise conditions on f^ϑ, A^ϑ). To illustrate our main result (Theorem 3.1), we apply it to a rather general class of equations and obtain: (i) *Explicit* continuous dependence estimates for continuous viscosity solutions of (P). (ii) A priori L^∞ and x - Hölder regularity estimates for continuous viscosity solutions of (P). (iii) An explicit rate of convergence for vanishing viscosity approximations of x - Hölder continuous viscosity solutions of (P). Using the results mentioned in (ii) we prove also uniform (in the small artifical diffusion parameter) L^∞ and x - Hölder regularity estimates for the vanishing viscosity approximations.

The general form of (P) implies that many well known partial differential equations drop out as special cases. Quasilinear examples include the equation for mean curvature flow of graphs and the p - Laplace diffusion equation with $p\in[2,\infty)$. One significant fully nonlinear example is the dynamic programming (or Hamilton-Jacobi-Bellman) equation of optimal stochastic control theory. In Section 4, we discuss this equation in particular and present a result about the continuity of the value function (viscosity solution) with respect to the coefficients in the Hamilton-Jacobi-Bellman equations. To best of our knowledge, results of this type have up to now only been available through probabilistic arguments (see, e.g., [5, 9]).

Finally, we mention that our main result (Theorem 3.1) has been used by Jakobsen, Karlsen, and Risebro in [8] to establish a rate of convergence for certain numerical approximations for a class of degenerate parabolic second order equations.

The rest of this paper is organized as follows: In Section 2, we introduce the notation that will be used throughout this paper. Moreover, we recall the notion of viscosity solutions along with the maximum principle for semicontinuous functions. In Section 3, we state our results. In Section 4, we illustrate (apply) our results to the Hamilton-Jacobi-Bellman equation. Finally, the detailed proofs of our results are given in Section 5.

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2. Preliminaries

In this section we introduce some notation (spaces, norms, etc.) that will be used frequently in this paper. We also recall the notions of viscosity solutions as well as the so-called maximum principle for semicontinuous functions.

Let $|\cdot|$ denote the 2-norm in $\mathbb{R}^m$ with $m \in \mathbb{N}$. We also let $|\cdot|$ denote the matrix norm defined by $|C| = \sup_{e\in\mathbb{R}^p} \frac{|Ce|}{|e|}$, where $C \in \mathbb{R}^{m\times p}$ is an $m \times p$ matrix and $m, p \in \mathbb{N}$. The Frobenius norm is defined as $|C|_F^2 = tr[C^T C] = tr[CC^T]$. We recall that there is a constant $c = \min(m, p)$ such that $|C|_F \le c|C|$. The ball with center in $0 \in \mathbb{R}^{m\times p}$ and radius $R > 0$ is the following set, $B_{m\times p}(0, R) := \{x \in \mathbb{R}^{m\times p} : |x| < R\}$. If $p = 0$, we write $B_m(0, R)$. Let $S(m)$ denote the space of $m \times m$ symmetric matrices. On this spaces we have the usual partial ordering $\le$, that is, $X \le Y$ whenever $eXe \le eYe$ for every $e \in \mathbb{R}^m$. By $e_1, \dots, e_m$ we denote the usual unit vectors in $\mathbb{R}^m$.

In what follows, let U be some set. If $f : U \to \mathbb{R}^{m\times p}$, then

$$\|f\| = \sup_{x\in U} |f(x)|.$$

Note that we allow for $\|f\| = \infty$. For a locally bounded function $f : U \to \mathbb{R}^{m\times p}$, the upper and lower semicontinuous envelopes of f are defined respectively as

$$f^*(x) = \limsup_{\substack{y\to x\\ y\in U}} f(y), \qquad f_*(x) = \liminf_{\substack{y\to x\\ y\in U}} f(y).$$

We let $USC(U; \mathbb{R}^{m\times p})$, $LSC(U; \mathbb{R}^{m\times p})$, and $C(U; \mathbb{R}^{m\times p})$ denote the usual spaces of upper semicontinuous, lower semicontinuous, and continuous functions from U to $\mathbb{R}^{m\times p}$ respectively. If $p, m = 1$, we write $USC(U)$, $LSC(U)$, and $C(U)$. Let $f \to I \times \mathbb{R}^N$, $I \subset [0, \infty)$. Then, for $\mu \in (0, 1]$, we define the following Hölder seminorms:

$$[f(t,\cdot)]_\mu = \sup_{\substack{x,y\in\mathbb{R}^N\\ x\neq y}} \frac{|f(t,x) - f(t,y)|}{|x-y|^\mu},$$

$$[f]_\mu = \sup_{\tau\in I} \sup_{\substack{x,y\in\mathbb{R}^N\\ x\neq y}} \frac{|f(\tau,x) - f(\tau,y)|}{|x-y|^\mu}.$$

By $\mathcal{C}^\mu(I \times \mathbb{R}^N)$ we denote the set of functions $f : I \times \mathbb{R}^N \to \mathbb{R}$ for which the norm $\|f\| + [f]_\mu$ is finite. We shall also need the usual Hölder space $C^\mu(\mathbb{R}^N)$ of functions $g : \mathbb{R}^N \to \mathbb{R}$ such that $\|g\| + [g]_\mu$ is finite.

There are several equivalent ways to define viscosity solutions. We will need only one of these definitions in this paper. Consider the following general equation

$$u_t + H(t, x, u, Du, D^2u) = 0 \quad \text{in } Q_T. \tag{2.1}$$

Since the purpose here is only to introduce the notion of viscosity solutions, we only need to assume that $H : [0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times S(N) \to \mathbb{R}$ is locally bounded and nonincreasing in its last argument. We start by introducing the notion of semijets:

Definition 2.1 (Semijets)**.** For a locally bounded function $u \in USC(Q_T)\,(LSC(Q_T))$, the second order parabolic *superjet* (*subjet*) of u at $(t, x) \in Q_T$, which is denoted by $\mathcal{P}^{2,+(-)}u(t, x)$, is defined as the set of $(a, p, X) \in \mathbb{R} \times \mathbb{R}^N \times S(N)$ such that

$$u(s, y) \le (\ge)\, u(t, x) + a(s - t) + \langle p, y - x\rangle + \tfrac{1}{2}\langle X(y - x), y - x\rangle + o(|s - t| + |y - x|^2),$$

as $Q_T \ni (s, y) \to (t, x)$. We define the closure $\overline{\mathcal{P}}^{2,+(-)}u(t, x)$ as the set of $(a, p, X) \in \mathbb{R}\times\mathbb{R}^N \times S(N)$ for which there exists $(t_n, x_n, p_n, X_n) \in \mathbb{R}\times\mathbb{R}^N \times\mathbb{R}^N \times S(N)$ such that $(t_n, x_n, u(x_n, t_n), p_n, X_n) \to (t, x, u(t, x), p, X)$ as $n \to \infty$ and $(a_n, p_n, X_n) \in \mathcal{P}^{2,+(-)}u(t_n, x_n)$ for all n.

Following [3, 6], we state the following general definition of a viscosity solution:

Definition 2.2 (Viscosity Solution)**.** (i) A locally bounded function $u : Q_T \to \mathbb{R}$ is a viscosity subsolution of (2.1) if

$$a + H_*(t, x, u^*(t, x), p, X) \le 0, \qquad \forall (t, x) \in Q_T, \quad \forall (a, p, X) \in \mathcal{P}^{2,+}u^*(t, x). \tag{2.2}$$

(ii) A locally bounded function $u : Q_T \to \mathbb{R}$ is a viscosity supersolution of (2.1) if

$$a + H^*(t,x,u_*(t,x),p,X) \geq 0, \qquad \forall (t,x) \in Q_T, \quad \forall (a,p,X) \in \mathcal{P}^{2,-}u_*(t,x). \tag{2.3}$$

(iii) A function $u : Q_T \to \mathbb{R}$ is a viscosity solution of (2.1) if it is simultaneously a viscosity sub- and supersolution of (2.1).

Remark 2.1. Observe that because H_* and H^* are lower and upper semicontinuous respectively, (2.2) and (2.3) remain true with $\mathcal{P}^{2,+}$ and $\mathcal{P}^{2,-}$ replaced by $\overline{\mathcal{P}}^{2,+}$ and $\overline{\mathcal{P}}^{2,-}$ respectively.

Remark 2.2. In a typical situation, H and the viscosity solution u of (2.1) are continuous functions so that $H_* = H^* = H$ and $u_* = u^* = u$.

For the reader's convenience, we restate here the parabolic version of the maximum principle for semicontinuous functions [2, 3]:

Theorem 2.3 (Crandall, Ishii, and Lions [2, 3])**.** *Let $u_1(t,x), -u_2(t,x)$ belong to $USC(Q_T)$. Let $\phi(t,x,y)$ be once continuously differentiable in $t \in (0,T)$ and twice continuously differentiable in $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$. Suppose $(t_\phi, x_\phi, y_\phi) \in (0,T) \times \mathbb{R}^N \times \mathbb{R}^N$ is a local maximum of the function*

$$(t,x,y) \longrightarrow u_1(t,x) - u_2(t,y) - \phi(t,x,y).$$

Suppose that there is an $r > 0$ such that for every $M > 0$ there is a C such that

$$\begin{cases} a \leq C \text{ whenever } (a,p,X) \in \mathcal{P}^{2,+}u_1(t,x), \ |x - x_\varphi| + |t - t_\varphi| \leq r, \ |u_1(t,x)| + |p| + |X| \leq M, \\ b \geq C \text{ whenever } (b,q,Y) \in \mathcal{P}^{2,-}u_2(t,x), \ |x - x_\varphi| + |t - t_\varphi| \leq r, \ |u_2(t,x)| + |q| + |Y| \leq M. \end{cases}$$

Then for any $\kappa > 0$ there exist two numbers $a, b \in \mathbb{R}$ and two matrices $X, Y \in S(N)$ such that

$$\big(a, D_x\phi(t_\phi,x_\phi,y_\phi), X\big) \in \overline{\mathcal{P}}^{2,+}u_1(x_\phi,t_\phi), \qquad \big(b, -D_y\phi(t_\phi,x_\phi,y_\phi), Y\big) \in \overline{\mathcal{P}}^{2,-}u_2(y_\phi,t_\phi),$$

$$-\left(\frac{1}{\kappa} + \left|D^2\phi(t_\phi,x_\phi,y_\phi)\right|\right) I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq D^2\phi(t_\phi,x_\phi,y_\phi) + \kappa\big[D^2\phi(t_\phi,x_\phi,y_\phi)\big]^2, \tag{2.4}$$

and $a - b = \phi_t(t_\phi,x_\phi,y_\phi)$.

3. Statements of results

In this section we state our main result and several applications of this. The proofs of these results are given in Section 5. We start by specifying the class of equations we consider and then introduce some more notation which is needed for our main result. So in what follows, $N, P \in \mathbb{N}$ are fixed and ϑ always belong to some index set Θ. We will consider equations of the form (P) that satisfy the following conditions:

(C1) For every $R > 0, f^\vartheta \in C([0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times S(N))$ is uniformly continuous on $[0,T] \times \mathbb{R}^N \times [-R,R] \times B_N(0,R) \times B_{N\times N}(0,R)$ uniformly in ϑ.

(C2) For every $t,x,r,p,\vartheta,$ if $X, Y \in S(N), X \leq Y$ then
$f^\vartheta(t,x,r,p,X) \geq f^\vartheta(t,x,r,p,Y)$.
For every t,x,p,X,ϑ and for $R > 0,$ there is $\gamma_R \in \mathbb{R}$ such that
$f^\vartheta(t,x,r,p,X) - f^\vartheta(t,x,s,p,X) \geq \gamma_R(r - s),$ for $-R \leq s \leq r \leq R$.

(C3) For every $t,x,p,\vartheta,$ $A^\vartheta(t,x,p) \geq 0$ and $A^\vartheta(t,x,p) = a^\vartheta(t,x,p)a^\vartheta(t,x,p)^T$ for some matrix $a^\vartheta \in C([0,T] \times \mathbb{R}^N \times \mathbb{R}^N; \mathbb{R}^{N\times P})$. Furthermore for every $R > 0, a^\vartheta$ is uniformly continuous on $[0,T] \times \mathbb{R}^N \times B_N(0,R)$ uniformly in ϑ.

In what follows, let u^1 and u^2 be bounded sub- and supersolutions respectively of the following two equations:

$$u^i_t + \sup_{\vartheta\in\Theta}\Big\{f^\vartheta_i(t,x,u^i,Du^i,D^2u^i) - tr[A^\vartheta_i(t,x,Du^i)D^2u^i]\Big\} = 0, \quad i = 1,2. \tag{EQ$_i$}$$

Before presenting our main continuous dependence result (Theorem 3.1), we shall need to introduce two sets over which "continuous dependence" is "measured":

$$E^\alpha_{s,t} := \Big\{(\tau,x,y) : s \leq \tau < t, (x,y) \in \Delta^\alpha\Big\} \tag{3.1}$$

and

$$D^{\alpha}_{\gamma,s,t} := \Big\{(\tau,x,y,r,p,X,\vartheta) : p = \alpha(x-y)e^{(\bar\gamma-\gamma)(\tau-s)}, (\tau,x,y)\in E^{\alpha}_{s,t},\\ |r| \le e^{-\gamma(t-s)}\min\left(\|u^1\|,\|u^2\|\right), |X| \le 3\alpha e^{(\bar\gamma-\gamma)(t-s)}, \vartheta\in\Theta\Big\}, \tag{3.2}$$

where $\alpha>0$ is a free parameter, γ and $\bar\gamma$ are constants to be specified in Theorem 3.1 below, and $0\le s\le t\le T$. The set Δ^α appearing in definitions of the set $E^{\alpha}_{s,t}$ depend on the regularity of u^1 and u^2. We give the definition in the different relevant cases.

Case (i): Assume $u^1, -u^2 \in USC(\bar Q_T)$. We then define

$$\Delta^\alpha := \Big\{(x,y)\in\mathbb{R}^{2N} : |x-y| \le \sqrt{2\sup_{\bar Q_T}(u^1-u^2)^+}\ \alpha^{-\frac12}\Big\}.$$

Case (ii): Assume $u^1,u^2\in C(\bar Q_T)$ in the sense that there exist moduli of continuity ω_1,ω_2 such that

$$|u^i(t,x)-u^i(t,y)| \le \omega_i(|x-y|), \quad \forall t\in[0,T], \quad i=1,2. \tag{3.3}$$

We then define

$$\Delta^\alpha := \Big\{(x,y)\in\mathbb{R}^{2N} : \alpha|x-y|^2 - \omega_1(|x-y|) - \omega_2(|x-y|) \le 0\Big\}.$$

Case (iii): Assume $u^1,-u^2\in USC(\bar Q_T)$ and that either u^1 or u^2 lies in $\mathcal{C}^1(\bar Q_T)$. We then define

$$\Delta^\alpha := \Big\{(x,y)\in\mathbb{R}^{2N} : |x-y| \le N\min\left([u^1]_1,[u^2]_1\right)\alpha^{-1}\Big\}.$$

We can now state our main result:

Theorem 3.1. *Assume that conditions (C1) – (C3) hold for f_i^ϑ and A_i^ϑ with constants γ_R^i, for $i=1,2$. Let u^1 and u^2 be bounded viscosity sub- and supersolutions of (EQ_1) and (EQ_2) respectively. Assume that u^1 and u^2 have regularity as stated in one of the Cases (i) – (iii). Set $R:=\max\left(\|u^1\|,\|u^2\|\right)$ and $\gamma=\min\left(\gamma_R^1,\gamma_R^2\right)$. Then for $0\le s\le t\le T$, $\bar\gamma\ge0$, and $\alpha>0$*

$$\begin{aligned}
&\sup_{E^{\alpha}_{s,t}}\Big(e^{\gamma(\tau-s)}\left(u^1(\tau,x)-u^2(\tau,y)\right) - \frac{\alpha}{2}e^{\bar\gamma(\tau-s)}|x-y|^2\Big)\\
&\quad\le \sup_{E^{\alpha}_{s,s}}\Big(u^1(s,x)-u^2(s,y)-\frac{\alpha}{2}|x-y|^2\Big)^+\\
&\qquad + (t-s)\sup_{D^{\alpha}_{\gamma,s,t}}\Big\{e^{\gamma(\tau-s)}\Big\{f_2^\vartheta(\tau,y,r,p,X)-f_1^\vartheta(\tau,x,r,p,X)\Big\}\\
&\qquad\qquad + 3\alpha c^2 e^{\bar\gamma(\tau-s)}\left|a_1^\vartheta(\tau,x,p)-a_2^\vartheta(\tau,y,p)\right|^2 - \frac{\alpha}{2}\bar\gamma e^{\bar\gamma(\tau-s)}|x-y|^2\Big\}^+,
\end{aligned}$$

where the sets $E^{\alpha}_{s,t}$ and $D^{\alpha}_{\gamma,s,t}$ are defined in (3.1) *and* (3.2) *respectively via the set Δ^α defined in Case (i) – Case (iii).*

Remark 3.2. Note that we have introduced an exponential factor in the quadratic penalization term on the left hand side in the above inequality. As a consequence of this, we get a quadratic penalization term on the right hand side also. By appropriately choosing the exponent $\bar\gamma\ge0$, we will see that in most of the following applications we do not need to make any *a priori* regularity assumptions on the solutions. In such cases the set Δ^α does not play any role. However this is not the case when for example $a^\vartheta(t,x,p)$ is only Hölder continuous in x with exponent $\beta<1$. For certain values of β, we can still obtain results, but only when we use the extra information provided by Δ^α. We will not consider this case in this paper.

In the remaining part of this section, we shall see examples of how Theorem 3.1 can be applied. We state some rather general results conserning i) explicit continuous dependence estimates, ii) L^∞ and Hölder estimates for viscosity solutions, and finally iii) a convergence rate for the vanishing viscosity method. In order to obtain these results, we must have stronger assumptions on the data. We shall consider the following conditions:

(C4) There is a constant $C^f > 0$ such that $C^f := \sup_{\Theta\times\bar{Q}_T} |f^\vartheta(t,x,0,0,0)| < \infty$.

(C5) Let $\mu \in (0,1]$ and $f^\vartheta(t,x,r,p,X) = g^\vartheta(t,x,r,p,X) + b^\vartheta(t,x,p)\,p$. For each $R > 0$ there are constants $C^g_R, C^b > 0$ such that
$$|g^\vartheta(t,x,r,p,X) - g^\vartheta(t,y,r,p,X)| \leq C^g_R|x-y|^\mu,$$
$$|b^\vartheta(t,x,p) - b^\vartheta(t,y,p)| \leq C^b|x-y|$$
for $\vartheta \in \Theta$, $t \in [0,T]$, $|r| \leq R$, $x,y,p \in \mathbb{R}^N$, $X \in S(N)$.

(C6) Let $\mu \in (0,1]$. For each $R > 0$ there is a constant $C^f_R > 0$ such that
$$|f^\vartheta(t,x,r,p,X) - f^\vartheta(t,y,r,p,X)| \leq C^f_R\left(|p||x-y| + |x-y|^\mu\right),$$
for $\vartheta \in \Theta$, $t \in [0,T]$, $|r| \leq R$, $x,y,p \in \mathbb{R}^N$, $X \in S(N)$.

(C7) For each $R > 0$ there is a constant $C^f_R > 0$ such that
$$|f^\vartheta(t,x,r,p,X) - f^\vartheta(t,y,r,p,X)| \leq C^f_R|x-y|,$$
for $\vartheta \in \Theta, t \in [0,T], |r|, |p| \leq R, x,y \in \mathbb{R}^N$, $X \in S(N)$.

(C8) For each $R > 0$ there is a constant $C^a_R > 0$ such that
$$|a^\vartheta(t,x,p) - a^\vartheta(t,y,p)| \leq C^a_R|x-y|,$$
for $\vartheta \in \Theta$, $t \in [0,T]$, $x,y \in \mathbb{R}^N$, $|p| \leq R$.

Let us explain why we need assumptions (C1) – (C8). First of all, there are three different assumptions on the x - regularity of f^ϑ. We use the least general (but most explicit) assumption (C5) to derive an explicit continuous dependence estimate without assuming any a priori regularity on the solutions, see Theorem 3.3 a). We do not know how to make such an explicit estimate more general the way we choose to present this theory. However, if we were only interested in regularity estimates as in Theorem 3.4 b), the more general assumption (C6) is sufficient and is probably more or less optimal in our presentation. When we assume a priori that solutions are Lipschitz continuous, we get an explicit comparison theorem using assumption (C7), see Theorem 3.3 b). This assumption implies some sort of local Lipschitz continuity in p and is therefore more general than an assumption like (C6). Note that if we were interested in a priori regularity estimates in this case, then (C7) is too general. In fact, then we need to consider assumption (C6) with $\mu = 1$. We will not prove this here, but just remark that assumption (C6) is the "correct" assumption for first order Hamilton-Jacobi equations, see Souganidis [10].

In the next theorem we state two explicit continuous dependence estimates. In the first one we consider the case with no a priori regularity on the solutions, while in the second one we consider Lipschitz solutions. Note well that in order to get the explicit continuous dependence estimates, it suffices to require that assumptions like (C5) – (C8) hold for only *one* of the two problems being compared. This is the meaning of the assumption "if there are $i,j,k \in \{0,1\}\ldots$" below.

Theorem 3.3 (Continuous Dependence Estimate). *Assume (C1) – (C3) hold for f^ϑ_i and A^ϑ_i with constants γ^i_R for $i = 1,2$. Furthermore assume that $u^1, u^2 \in C(\bar{Q}_T)$ are bounded viscosity solutions of (EQ_1), (EQ_2) respectively. Let $R_0 = \max(\|u^1\|,\|u^2\|)$, $\gamma = \min\left(\gamma^1_{R_0}, \gamma^2_{R_0}\right)$, and D_t be the following set*

$$D_t := \left\{(\tau,x,r,p,\vartheta) : \tau \in [0,t], x \in \mathbb{R}^N, |r| \leq e^{-\gamma t}\min\left(\|u^1\|,\|u^2\|\right), p \in \mathbb{R}^N, X \in S(N), \vartheta \in \Theta\right\}.$$

a) If there are $i,j,k \in \{0,1\}$ such that $u^i(0,\cdot) \in C^\mu(\mathbb{R}^N)$ and f^ϑ_j and a^ϑ_k satisfies (C5) and (C8) respectively, with constants $C^{g_j}_R$, C^{b_j}, and C^{a_k}. Note that C^{a_k} does not depend on R! Then for $0 \leq t \leq T$ there exists a constant M depending only on $T, \gamma, C^{g_j}_R, C^{b_j}, C^{a_k}$ and $[u^i(0,\cdot)]_\mu$ such that

$$e^{\gamma t}\|u^1(t,\cdot) - u^2(t,\cdot)\| \leq \|u^1(0,\cdot) - u^2(0,\cdot)\| + \sup_{D_t}\Big\{te^{\gamma\tau}\left|g^\vartheta_1(\tau,x,r,p,X) - g^\vartheta_2(\tau,x,r,p,X)\right|$$
$$+ Mt^{\mu/2}\left(\left|b^\vartheta_1(\tau,x,p) - b^\vartheta_2(\tau,x,p)\right|^\mu + \left|a^\vartheta_1(\tau,x,p) - a^\vartheta_2(\tau,x,p)\right|^\mu\right)\Big\}.$$

b) Define $\bar{D}_t := \{(\tau, x, r, p, \vartheta) \in D_t : |p| \leq e^{-\gamma t} \min([u^1]_1, [u^2]_1)\}$. *If there are* $i, j, k \in \{0, 1\}$ *such that* $u^i \in \mathcal{C}^1(\bar{Q}_T)$, f_j^ϑ *and* a_k^ϑ *satisfies (C7) and (C8) respectively, with constants* $C_R^{f_j}$ *and* $C_R^{a_k}$. *Then for* $0 \leq t \leq T$ *there exists a constant* M *depending only on* $T, \gamma, C_R^{f_j}, C_R^{a_k}$, *and* $[u^i]_1$ *such that*

$$e^{\gamma t}\|u^1(t,\cdot) - u^2(t,\cdot)\| \leq \|u^1(0,\cdot) - u^2(0,\cdot)\| + \sup_{\bar{D}_t}\Big\{te^{\gamma\tau}\big|f_1^\vartheta(\tau,x,r,p,X) - f_2^\vartheta(\tau,x,r,p,X)\big| + Mt^{1/2}\left|a_1^\vartheta(\tau,x,p) - a_2^\vartheta(\tau,x,p)\right|\Big\}.$$

We next state the regularity and a priori results.

Theorem 3.4 (Regularity Estimates). *Assume (C1) – (C3) hold. In addition, let* $u \in C(\bar{Q}_T)$ *be a bounded viscosity solution of (P) with initial data* u_0. *Define* $R := \|u\|$ *and* $\gamma := \gamma_R$. *Then the following statements are true for every* $t \in [0, T]$:

a) If f^ϑ *satisfies (C4), then* $\|u(t,\cdot)\| \leq e^{-\gamma t}\left(\|u_0\| + te^{\gamma^+ t}C^f\right)$.

b) Assume that f^ϑ *and* a^ϑ *satisfy (C6) and (C8) respectively, and the constant in (C8) is independent of* R. *Moreover if* $u_0 \in C^\mu(\mathbb{R}^N)$, *then*

$$[u(t,\cdot)]_\mu \leq Ke^{\tilde{\gamma}t}\left\{[u(0,\cdot)]_\mu + t^{1-\mu/2}e^{\gamma^+ t}C_R^f\right\},$$

where $K \leq 4$ *and* $\tilde{\gamma} = 2\big(C_R^f + 3c^2(C^a)^2 + 1\big) + |\gamma|$.

Finally we turn to the problem of finding a convergence rate for the vanishing viscosity method. The vanishing viscosity method considers the following equation as an approximation to (P):

$$(\mathrm{P}_\nu) \qquad u_t^\nu + \sup_{\vartheta\in\Theta}\Big\{f^\vartheta(t,x,u^\nu,Du^\nu,D^2u^\nu) - tr[A^\vartheta(t,x,Du^\nu)D^2u^\nu]\Big\} = \nu\Delta u^\nu.$$

We are interested in the L^∞ convergence of u^ν to the unique viscosity solution u of (P) as $\nu \to 0$. By now it is classical to use the Barles-Perthame weak limit method (see, e.g., [3]) to prove convergence of the viscous approximations u^ν. The idea is that so-called upper weak limit $\overline{u}$ and the lower weak limit $\underline{u}$, defined by

$$\overline{u}(t,x) = \limsup_{\nu\to 0}{}^* u^\nu(t,x), \qquad \underline{u}(t,x) = \liminf_{\nu\to 0}{}_* u^\nu(t,x),$$

are respectively viscosity sub- and supersolutions of (P). On one hand, we always have $\underline{u} \leq \overline{u}$ in Q_T. On the other hand, the (strong) comparison principle [3] implies that $\overline{u} \leq \underline{u}$ in Q_T and thus $\underline{u} = \overline{v}$ in Q_T. Finally, it is easy to see that this equality implies local L^∞ convergence of u^ν to the function $u := \underline{u} = \overline{u}$ as $\nu \to 0$, which turns out to be the unique bounded continuous viscosity solutions of (1.1).

The advantage of the method of weak limits is that it allows passages to the limits with only an L^∞ estimate on u^ν. The disadvantage is that the method does not say anything about the *rate* of convergence, which is the content of the next theorem.

Theorem 3.5 (Viscous Approximations). *Assume that (C1) – (C4), (C6) and (C8) hold, and that the constant in (C8) is independent of* R. *Furthermore assume that there exists a bounded viscosity solution* $u^\nu \in C(\bar{Q}_T)$ *of* (P_ν) *for each* $\nu > 0$. *Then there exists a viscosity solution* $u \in \mathcal{C}^\mu(\bar{Q}_T)$ *of (P) such that for every* $0 \leq t \leq T$

$$\|u(t,\cdot) - u^\nu(t,\cdot)\| \leq K\left(\|u(0,\cdot) - u^\nu(0,\cdot)\| + \nu^{\frac{\mu}{2}}\right),$$

for some constant K *independent of* ν.

A special case worth mentioning for which the results of this section apply, is the Hamilton-Jacobi-Bellmann equation. The results for this equation will be detailed in the next section.

4. Hamilton-Jacobi-Bellman Equation

Let $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\big)$ be a filtered complete probability space satisfying the usual hypotheses. Let Θ be a closed subset of Euclidian space. On $\Theta\times[0,\infty)\times\mathbb{R}^N$, we are given a $N\times P$ matrix-valued function $\sigma^\vartheta(t,x)$, an $\mathbb{R}^N$ - valued function $b^\vartheta(t,x)$, and $\mathbb{R}$ - valued functions $c^\vartheta(t,x)\geq\gamma\in\mathbb{R}$, $f^\vartheta(t,x)$, $g(x)$. We assume that $\sigma^\vartheta(t,x)$, $b^\vartheta(t,x)$, $c^\vartheta(t,x)$, $f^\vartheta(t,x)$ are bounded and continuous in t,x and θ. Furthermore, we assume that $h=\sigma^\vartheta$ and $h=b^\vartheta$ possess the following Hölder regularity condition with $\delta=1$:

$$|h(t,x)-h(s,y)|\leq \text{Const}\left(|x-y|^\delta+|t-s|^{\delta/2}\right), \tag{4.1}$$

where the constant should be independent of ϑ. Similarly, we assume that $h=c^\vartheta$ and $h=f^\vartheta$ satisfy (4.1) with $\delta=\mu$. Finally we assume $g\in C^\mu(\mathbb{R}^N)$. Let W_t be P - dimensional Wiener process with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ and let $\vartheta=\{\vartheta_t\}_{t\geq 0}$ be an adapted control process taking values in Θ. Consider then the (controlled) stochastic differential equation

$$dX_s=b^{\vartheta_s}(s,X_s)\,ds+\sigma^{\vartheta_s}(s,X_s)\,dW_s,\qquad s>t. \tag{4.2}$$

Under the assumptions given above, there exists a unique solution $X_s=X_s^{\vartheta_s,t,x}$ of (4.2) with initial condition $X_t=x$. For a given $(t,x)\in(0,T)\times\mathbb{R}^N$ and an adapted control process $\vartheta=\{\vartheta_t\}_{t\geq 0}$, the finite horizon optimal stochastic control problems is to maximize the functional

$$\begin{aligned}\mathcal{J}(t,x;\vartheta)=\mathbb{E}^{t,x,\vartheta}\bigg[\int_t^T & f^{\vartheta_s}(s,X_s)\exp\bigg(-\int_t^s c^{\vartheta_r}(r,X_r)\,dr\bigg)\,ds\\ &+g(X_T)\exp\bigg(-\int_t^T c^{\vartheta_r}(r,X_r)\,dr\bigg)\bigg].\end{aligned} \tag{4.3}$$

As usual, to solve this optimization problem we introduce the value function

$$V(t,x)=\sup_\vartheta \mathcal{J}(t,x;\vartheta),\qquad (t,x)\in[0,T]\times\mathbb{R}^N. \tag{4.4}$$

It is well known that the value function $V(t,x)$ is bounded, and satisfies (4.1) with $h=V$ and $\delta=\mu$ (see, e.g., Krylov [9]).

As a consequence of the dynamic programming principle (see, e.g., Fleming and Soner [5]), the value function (4.4) is the unique viscosity solution of the Hamilton-Jacobi-Bellman partial differential equation

$$u_t+\sup_{\vartheta\in\Theta}\Big\{tr\big[A^\vartheta(t,x)D^2u\big]+b^\vartheta(t,x)Du-c^\vartheta(t,x)u+f^\vartheta(t,x)\Big\}=0,\quad u(T,x)=g(x), \tag{4.5}$$

where $A^\vartheta(t,x)=\sigma^\vartheta(t,x)\sigma^\vartheta(t,x)^T$.

Remark 4.1. Note that (4.5) is a terminal value problem. To convert this to an initial value problem of the type studied in this paper, one has to introduce the change of variable $t\mapsto T-t$.

We are interesting in estimating the change in the the value function (4.4) (hence the viscosity solution of (4.5)) when the coefficients in (4.2) and (4.3) (hence in (4.5)) are changed. From Theorem 3.3 a), we immediately get the following result:

Theorem 4.2 (Continuous Dependence Estimate). *Let V be the value function defined in* (4.5). *Let $\overline{V}$ denote the value function obtained by replacing the coefficients σ^ϑ, b^ϑ, c^ϑ, f^ϑ, g in* (4.2) *and* (4.3) *by $\overline{\sigma}^\vartheta$, $\overline{b}^\vartheta$, $\overline{c}^\vartheta$, $\overline{f}^\vartheta$, $\overline{g}$ respectively. Then there exists a constant $K>0$ such that the following estimate holds for $0<t\leq T$:*

$$\begin{aligned}\big\|V(t,\cdot)-\overline{V}(t,\cdot)\big\|\leq&\big\|g-\overline{g}\big\|+K\sup_{\tau\in[T-t,T],x,\vartheta}\Big\{(T-t)\Big(\big|c^\vartheta(\tau,x)-\overline{c}^\vartheta(\tau,x)\big|+\big|f^\vartheta(\tau,x)-\overline{f}^\vartheta(\tau,x)\big|\Big)\\ &+(T-t)^{\mu/2}\Big(\big|b^\vartheta(\tau,x)-\overline{b}^\vartheta(\tau,x)\big|^\mu+\big|\sigma^\vartheta(\tau,x)-\overline{\sigma}^\vartheta(\tau,x)\big|^\mu\Big)\Big\}.\end{aligned}$$

A similar continuous dependence estimate can be proved by means of probabilistic arguments, see, e.g., Fleming and Soner [5, p. 181]. From Theorem 3.5, we also get the following rate of convergence for the vanishing viscosity method:

Theorem 4.3. *Let V be the value function defined in* (4.5). *Let V^ν be the solution of* (4.5) *with a viscosity term $\nu\Delta u$ added on the right-hand side of the equation. Then there exists a constant K independent of ν such that*

$$\|V(t,\cdot)-V^\nu(t,\cdot)\|\le K\nu^{\mu/2}.$$

Also this result is well-known (see, e.g., [5]). Its proof, however, usually relies on probabilistic arguments, which we avoid entirely here.

5. Proofs of Results

5.1. **Proof of Theorem 3.1.** We begin with giving a lemma that will be needed in the proof.

Lemma 5.1. *Let $f:\mathbb{R}^n\to\mathbb{R}$ be a bounded function and define $m,m_\varepsilon\ge 0$ and $x_\varepsilon,y_\varepsilon\in\mathbb{R}^n$ as follows: $m_\varepsilon=\max_{x\in\mathbb{R}^n}\{f(x)-\varepsilon|x|^2\}=f(x_\varepsilon)-\varepsilon|x_\varepsilon|^2$ and $m=\sup_{x\in\mathbb{R}^n}f(x)$. Then as $\varepsilon\to 0$, $m_\varepsilon\to m$ and $\varepsilon|x_\varepsilon|^2\to 0$.*

Proof. Choose an $\eta>0$. By the definition of supremum there is an $x'\in\mathbb{R}^N$ such that $f(x')\ge m-\eta$. Pick an ε' so small that $\varepsilon'|x'|^2<\eta$, then the first part follows since

$$m\ge m_{\varepsilon'}=f(x_{\varepsilon'})-\varepsilon'|x_{\varepsilon'}|^2\ge f(x')-\varepsilon'|x'|^2\ge m-2\eta.$$

Now define $k_\varepsilon=\varepsilon|x_\varepsilon|^2$. This quantity is bounded by the above calculations since f is bounded. Pick a converging subsequence $\{k_\varepsilon\}_\varepsilon$ and call the limit k (≥ 0). Note that $f(x_\varepsilon)-k_\varepsilon\le m-k_\varepsilon$, so going to the limit yields $m\le m-k$. This means that $k\le 0$, that is $k=0$. Now we are done since if every subsequence converges to 0, the sequence has to converge to 0 as well. ☐

Case $\gamma=0$. Now $f_i^\vartheta(\tau,x,r,p,X)$ is nondecreasing in r for $i=1,2$. We let $M:=\sup_{\bar{Q}_T}\left(u^1-u^2\right)^+$, $\mathcal{F}_1^\vartheta(\tau,x,r,p,X):=f_1^\vartheta(\tau,x,r,p,X)-tr[A_1^\vartheta(\tau,x,p)X]$ and $\mathcal{F}_2^\vartheta(\tau,x,r,p,X):=f_2^\vartheta(\tau,x,r,p,X)-tr[A_2^\vartheta(\tau,x,p)X]$. For $\varepsilon>0$, define

$$E_0:=\sup_{E^\alpha_{s,s}}\left(u^1(s,x)-u^2(s,y)-\frac{\alpha}{2}|x-y|^2\right)^+,\ \text{and} \tag{5.1}$$

$$\sigma:=\sup_{E^\alpha_{s,t}}\left\{u^1(\tau,x)-u^2(\tau,y)-\left\{\frac{\alpha}{2}e^{\bar{\gamma}(\tau-s)}|x-y|^2+\frac{\varepsilon}{2}\left(|x|^2+|y|^2\right)+\frac{\varepsilon}{t-\tau}\right\}\right\}-E_0. \tag{5.2}$$

We shall derive an (positive) upper bound on σ, so we may assume that $\sigma>0$. Let $\delta\in(0,1)$, choose $e^{\bar{\gamma}(t-s)}\alpha>5\varepsilon$, and define

$$\psi(\tau,x,y):=u^1(\tau,x)-u^2(\tau,y)-\frac{\delta(\tau-s)}{t-s}\sigma-\left\{\frac{\alpha}{2}e^{\bar{\gamma}(\tau-s)}|x-y|^2+\frac{\varepsilon}{2}\left(|x|^2+|y|^2\right)+\frac{\varepsilon}{t-\tau}\right\}.$$

Note that if $f,g:U\to\mathbb{R}$ are functions on some set U and $\sup_U f<\infty$, then $\sup_U(f-g)\ge\sup_U f-\sup_U g$. Let $g:=\frac{\delta(\tau-s)}{t-s}\sigma$, $f:=\psi+g$, and $U:=E^\alpha_{s,t}$. Then we get

$$\sup_{E^\alpha_{s,t}}\psi(\tau,x,y)\ge\sigma+E_0-\delta\sigma=(1-\delta)\sigma+E_0. \tag{5.3}$$

Since u^1 and u^2 are bounded, and since ψ tends to $-\infty$ as τ tends to t and $|x|,|y|$ tend to ∞, $\sup\psi$ is obtained on a compact in $[s,t)\times\mathbb{R}^N\times\mathbb{R}^N$. It follows that there is a point $(\tau_0,x_0,y_0)\in[s,t)\times\mathbb{R}^N\times\mathbb{R}^N$ such that

$$\psi(\tau_0,x_0,y_0)\ge\psi(\tau,x,y),\qquad\forall(\tau,x,y)\in[s,t)\times\mathbb{R}^N\times\mathbb{R}^N.$$

On the other hand, we have by (5.3) since $E_0\ge 0$ and $\sigma>0$ that

$$0\le\psi(\tau_0,x_0,y_0)\le\sup_{\bar{Q}_T}\left(u^1-u^2\right)^+-\left\{\frac{\alpha}{2}e^{\bar{\gamma}(\tau_0-s)}|x_0-y_0|^2+\frac{\varepsilon}{2}\left(|x_0|^2+|y_0|^2\right)\right\},$$

so with $M:=\sup_{\bar{Q}_T}\left(u^1-u^2\right)^+$ it follows that

$$|x_0-y_0|\le\left(\frac{2M}{\alpha}\right)^{1/2},\qquad|x_0|,|y_0|\le\left(\frac{2M}{\varepsilon}\right)^{1/2}, \tag{5.4}$$

which corresponds to Case (i). If u^1, u^2 are more regular, we can get better estimates. By considering the inequality $2\psi(\tau_0, x_0, y_0) \geq \psi(\tau_0, x_0, x_0) + \psi(\tau_0, y_0, y_0)$, we find

$$\alpha e^{\bar{\gamma}(\tau_0 - s)}|x_0 - y_0|^2 \leq u^1(\tau_0, x_0) - u^1(\tau_0, y_0) + u^2(\tau_0, x_0) - u^2(\tau_0, y_0).$$

Using (3.3), we get

$$\alpha|x_0 - y_0|^2 - \omega_1(|x_0 - y_0|) - \omega_2(|x_0 - y_0|) \leq 0, \tag{5.5}$$

which corresponds to Case (ii). Finally, let u^1 or u^2 belong to $\mathcal{C}^1(\bar{Q}_T)$. Since $\psi(\tau_0, x, y_0)$ has its maximum in x_0, there is a $\bar{\delta} > 0$ such that $\psi(\tau_0, x, y_0) \leq \psi(\tau_0, x_0, y_0)$ for $x \in B_R(x_0, \bar{\delta})$. Using this and letting $e_i \in \mathbb{R}^N$ be a basis vector and h a real number, we find that

$$\begin{aligned} \frac{\varepsilon}{2}(|x_0|^2 - |x_0 + e_i h|^2) + \frac{\alpha}{2} e^{\bar{\gamma}(\tau_0 - s)}(|x_0 - y_0|^2 - |x_0 + e_i h - y_0|^2) \\ \leq u^1(\tau_0, x_0) - u^1(x_0 + e_i h, \tau_0) \leq [u^1]_1 |h|. \end{aligned}$$

Taking the limits as $h \to 0^{\pm}$, we get $|\varepsilon x_{0i} + \alpha e^{\bar{\gamma}(\tau_0 - s)}(x_{0i} - y_{0i})| \leq [u^1]_1$. Similarly we use $\psi(\tau_0, x_0, y)$ to get $|\varepsilon y_{0i} - \alpha e^{\bar{\gamma}(\tau_0 - s)}(x_{0i} - y_{0i})| \leq [u^2]_1$. Summing up, we have shown that

$$|p| = |\alpha e^{\bar{\gamma}(\tau_0 - s)}(x_0 - y_0)| \leq N \min\{[u^1]_1, [u^2]_1\} + N\,(2M\varepsilon)^{1/2}, \tag{5.6}$$

which corresponds to Case (iii) plus an error term of order $\varepsilon^{1/2}/\alpha$.

Note that the bounds on x_0 and y_0 in (5.4) can be improved using Lemma 5.1. Because by this Lemma there is a continuous nondecreasing function $m : [0, \infty) \to [0, \infty)$ satisfying $m(0) = 0$, such that

$$|x_0|, |y_0| \leq \varepsilon^{-1/2} m(\varepsilon). \tag{5.7}$$

Now we prove that $\tau_0 > s$. Suppose $\tau_0 = s$, then by (5.3), (5.4), and (5.1)

$$E_0 + (1 - \delta)\sigma \leq \psi(s, x_0, y_0) \leq \sup_{E^{\alpha}_{s,s}} \left(u^1(s, x) - u^2(s, y) - \frac{\alpha}{2}|x - y|^2 \right)^+ = E_0.$$

This means that $\sigma \leq 0$, which contradicts the assumption that $\sigma > 0$. So we have $\tau_0 > s$.

We define our test function

$$\phi(\tau, x, y) := \frac{\delta(\tau - s)}{t - s}\sigma + \left\{ \frac{\alpha}{2} e^{\bar{\gamma}(\tau - s)}|x - y|^2 + \frac{\varepsilon}{2}\left(|x|^2 + |y|^2\right) + \frac{\varepsilon}{t - \tau} \right\}.$$

We can now apply Theorem 2.3 to conclude that there are numbers a, b and symmetric matrices X, Y such that

$$(a, D_x\phi(\tau_0, x_0, y_0), X) \in \overline{\mathcal{P}}^{2,+} u^1(\tau_0, x_0), \qquad (b, -D_y\phi(\tau_0, x_0, y_0), Y) \in \overline{\mathcal{P}}^{2,-} u^2(\tau_0, y_0)$$

such that $a - b = \phi_t(\tau_0, x_0, y_0)$ and for $A = \begin{pmatrix} D^2_{xx}\phi & D^2_{xy}\phi \\ D^2_{yx}\phi & D^2_{yy}\phi \end{pmatrix}_{(\tau_0, x_0, y_0)}$ and $\nu > 0$, the following holds

$$-\left(\frac{1}{\nu} + |A|\right)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \nu A^2.$$

Let $\bar{\alpha} := e^{\bar{\gamma}(\tau_0 - s)}\alpha$ and $\nu = \bar{\alpha}^{-1}$. Then, after some calculations, we get

$$-(3\bar{\alpha} + \varepsilon)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \left\{ (3\bar{\alpha} + 2\varepsilon)\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \begin{pmatrix} (\frac{\varepsilon^2}{\bar{\alpha}} + \varepsilon)I & 0 \\ 0 & (\frac{\varepsilon^2}{\bar{\alpha}} + \varepsilon)I \end{pmatrix} \right\}. \tag{5.8}$$

By the definition of viscosity sub- and supersolutions,

$$\begin{aligned} a + \sup_{\Theta} \mathcal{F}^{\vartheta}_1(\tau_0, x_0, u^1(\tau_0, x_0), D_x\phi(\tau_0, x_0, y_0), X) \leq 0, \\ b + \sup_{\Theta} \mathcal{F}^{\vartheta}_2(\tau_0, y_0, u^2(\tau_0, y_0), -D_y\phi(\tau_0, x_0, y_0), Y) \geq 0. \end{aligned}$$

Subtracting the above two inequalities gives us

$$\tag{5.9}\begin{aligned}\phi_t(\tau_0,x_0,y_0) \le \sup_{\Theta}\Big\{&\mathcal{F}_2^{\vartheta}(\tau_0,y_0,u^2(\tau_0,y_0),-D_y\phi(\tau_0,x_0,y_0),Y)\\ &-\mathcal{F}_1^{\vartheta}(\tau_0,x_0,u^1(\tau_0,x_0),D_x\phi(\tau_0,x_0,y_0),X)\Big\}.\end{aligned}$$

By (5.3) we must have $u^1(\tau_0,x_0)\ge u^2(\tau_0,y_0)$. We can now use (C2) to rewrite (5.9) in terms of either $u^1(\tau_0,x_0)$ or $u^2(\tau_0,y_0)$. The argument is symmetric, and we rewrite the inequality in terms of the quantity with smallest norm. Assuming $\|u^1\|\le\|u^2\|$, we get

$$f_2^{\vartheta}(\tau,x,u^2(\tau_0,y_0),p,X)\le f_2^{\vartheta}(\tau,x,u^1(\tau_0,x_0),p,X).$$

We use this expression to rewrite inequality (5.9) in the following way:

$$\tag{5.10}\begin{aligned}\phi_t(\tau_0,x_0,y_0) \le \sup_{\Theta}\Big\{&\mathcal{F}_2^{\vartheta}(\tau_0,y_0,u^1(\tau_0,x_0),-D_y\phi(\tau_0,x_0,y_0),Y)\\ &-\mathcal{F}_1^{\vartheta}(\tau_0,x_0,u^1(\tau_0,x_0),D_x\phi(\tau_0,x_0,y_0),X)\Big\}.\end{aligned}$$

Then we estimate the left hand side:

$$\tag{5.11}\phi_t(\tau_0,x_0,y_0)=\frac{\delta\sigma}{t-s}+\frac{\alpha}{2}\bar\gamma e^{\bar\gamma(\tau_0-s)}|x_0-y_0|^2+\frac{\varepsilon}{(t-\tau_0)^2}\ge\frac{\delta\sigma}{t-s}+\frac{\alpha}{2}\bar\gamma e^{\bar\gamma(\tau_0-s)}|x_0-y_0|^2.$$

Let $(\tau,x,y,r,p,\bar p)\in[s,t)\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$. From (5.8) it follows that

$$\tag{5.12} X\le Y+4\varepsilon I,\qquad |Y|,|X|\le 3\alpha e^{\bar\gamma(\tau_0-s)}+4\varepsilon.$$

Using this and (C2) and get

$$\tag{5.13} f_2^{\vartheta}(\tau,x,r,p,Y)\le f_2^{\vartheta}(\tau,x,r,p,X-4\varepsilon I).$$

Following Ishii [7], let $C,D\in\mathbb{R}^{N\times P}$ be two matrices and note that the following $2N\times 2N$ matrix is symmetric and positive semidefinite:

$$B=\begin{pmatrix}CC^T & DC^T\\ CD^T & DD^T\end{pmatrix}.$$

Consider $B\begin{pmatrix}X & 0\\ 0 & -Y\end{pmatrix}$ and use inequality (5.8) to obtain the following estimate

$$tr[CC^TX-DD^TY]\le(3\alpha e^{\bar\gamma(\tau_0-s)}+2\varepsilon)|C-D|_F^2+2\varepsilon\left(|C|_F^2+|D|_F^2\right).$$

Recall that there is a constant c such that $|\cdot|_F\le c|\cdot|$, see Section 2. If we let $C=a_1^{\vartheta}(\tau,x,p)$ and $D=a_2^{\vartheta}(\tau,y,\bar p)$, then we get

$$\tag{5.14}\begin{aligned}&tr[A_1^{\vartheta}(\tau,x,p)X]-tr[A_2^{\vartheta}(\tau,y,\bar p)Y]\\ &\quad\le(3\alpha e^{\bar\gamma(\tau_0-s)}+2\varepsilon)c^2|a_1^{\vartheta}(\tau,x,p)-a_2^{\vartheta}(\tau,y,\bar p)|^2+2\varepsilon c^2\left(|a_1^{\vartheta}(\tau,x,p)|^2+|a_2^{\vartheta}(\tau,y,\bar p)|^2\right).\end{aligned}$$

Let $p:=\alpha e^{\bar\gamma(\tau_0-s)}(x_0-y_0)$, $p^x:=\varepsilon x_0$, and $p^y:=\varepsilon y_0$. Then we define the following set

$$\tag{5.15}\begin{aligned}F_{s,t}^{\alpha,\varepsilon}:=\Big\{&(\tau,x,y,z^x,z^y,r,p,p^x,p^y,X,\vartheta):(\tau,x,y,r,p,X,\vartheta)\in D_{0,s,t}^{\alpha},\\ &\varepsilon^{1/2}|x|,\varepsilon^{1/2}|y|\le m(\varepsilon),\alpha|z^x|,\alpha|z^y|,|p^x|,|p^y|\le(N+1)\left(2M\varepsilon\right)^{1/2}\Big\},\end{aligned}$$

where $D^{\alpha}_{0,s,t}$ is defined in (3.2) via (3.1) and Δ^{α} as defined in Cases (i) – (iii). Now from (5.4) – (5.7), (5.11) – (5.14) using (5.10) we obtain the upper bound on σ

$$\begin{aligned}\frac{\delta\sigma}{t-s} \leq \sup_{F^{\alpha,\varepsilon}_{s,t}} \Bigg\{ & f_2^{\vartheta}(\tau, y+z^y, r, p-p^y, X-4\varepsilon I) - f_1^{\vartheta}(\tau, x+z^x, r, p+p^x, X) \\ & + (3\alpha e^{\bar\gamma(\tau-s)} + 2\varepsilon)c^2 \left|a_1^{\vartheta}(\tau, x+z^x, p+p^x) - a_2^{\vartheta}(\tau, y+z^y, p-p^y)\right|^2 \\ & - \bar\gamma e^{\bar\gamma(\tau-s)}\frac{\alpha}{2}|x-y+z^x-z^y|^2 \\ & + 2\varepsilon c^2\Big(\left|a_1^{\vartheta}(\tau, x+z^x, p+p^x)\right|^2 + \left|a_2^{\vartheta}(\tau, y+z^y, p-p^y)\right|^2\Big)\Bigg\}^+ .\end{aligned}$$

By the definition of $F^{\alpha,\varepsilon}_{s,t}$ (5.15) and by the uniform continuity assumed in (C1) and (C3), there exists a modulus of continuity ω such that

$$\begin{aligned}\frac{\delta\sigma}{t-s} \leq \sup_{F^{\alpha,\varepsilon}_{s,t}} \Bigg\{ & f_2^{\vartheta}(\tau, y, r, p, X) - f_1^{\vartheta}(\tau, x, r, p, X) + 3\alpha e^{\bar\gamma(\tau-s)}c^2 \left|a_1^{\vartheta}(\tau, x, p) - a_2^{\vartheta}(\tau, y, p)\right|^2 \\ & - \bar\gamma e^{\bar\gamma(\tau-s)}\frac{\alpha}{2}|x-y|^2 + \omega(|z^x| + |z^y| + |p^x| + |p^y| + \varepsilon) \\ & + \varepsilon\mathrm{Const}\Big(\left|a_1^{\vartheta}(\tau, x+z^x, p+p^x)\right|^2 + \left|a_2^{\vartheta}(\tau, y+z^y, p-p^y)\right|^2\Big)\Bigg\}^+ .\end{aligned}$$

Let $(\tau, x, y) \in E^{\alpha}_{s,t}$. By the definition of σ (see (5.2)), we have

$$u^1(\tau,x) - u^2(\tau,y) - \frac{\alpha}{2}e^{\bar\gamma(\tau-s)}|x-y|^2 \leq \sigma + E_0 + \varepsilon\left\{\frac{1}{t-\tau} + \frac{1}{2}\left(|x|^2+|y|^2\right)\right\}.$$

Combining the two previous inequalities gives

(5.16)

$$u^1(\tau,x) - u^2(\tau,y) - \frac{\alpha}{2}e^{\bar\gamma(\tau-s)}|x-y|^2 \leq E_0 + \frac{t-s}{\delta}\sup_{F^{\alpha,\varepsilon}_{s,t}}\Big\{\cdots\Big\}^+ + \varepsilon\left\{\frac{1}{t-\tau} + \frac{1}{2}\left(|x|^2+|y|^2\right)\right\}.$$

Sending $\varepsilon \to 0$ in (5.16), the only questionable terms are those of the form $\varepsilon|a_1^{\vartheta}(\tau, x+z^x, p+p^x)|^2$, where (τ, x, z^x, p, p^x) comes from $F^{\alpha,\varepsilon}_{s,t}$. But uniform continuity (C3) and (5.15) implies a linear growth condition in the x-variable, so with $\varepsilon \leq 1$ we get

$$\varepsilon|a_1^{\vartheta}(\tau, x+z^x, p+p^x)|^2 \leq \varepsilon\mathrm{Const}\,(1+|x+z^x|)^2 \leq \mathrm{Const}\,\varepsilon(1+\varepsilon^{-1}m^2(\varepsilon)) \leq \mathrm{Const}\,m^2(\varepsilon).$$

Since m is continuous and $m(0)=0$ these terms tend to 0 as $\varepsilon \to 0$. So by first letting $\varepsilon \to 0$ and then letting $\delta \to 1$ in (5.16), we have proved:

Lemma 5.2. *Assume that conditions (C1) – (C3) holds for f_i^{ϑ} and A_i^{ϑ} with constants $\gamma_R^i = 0$ for $i = 1, 2$. Let u^1 and u^2 be bounded viscosity sub- and supersolutions of (EQ_1) and (EQ_2) respectively. Assume that u^1 and u^2 have regularity as stated in one of the Cases (i)–(iii). Then for $0 \leq s \leq t \leq T$, $\bar\gamma \geq 0$ and $\alpha > 0$*

$$\begin{aligned}&\sup_{E^{\alpha}_{s,t}}\Big(u^1(\tau,x) - u^2(\tau,y) - \frac{\alpha}{2}e^{\bar\gamma(\tau-s)}|x-y|^2\Big) \\ &\quad\leq \sup_{E^{\alpha}_{s,s}}\Big(u^1(s,x) - u^2(s,y) - \frac{\alpha}{2}|x-y|^2\Big)^+ \\ &\qquad + (t-s)\sup_{D^{\alpha}_{0,s,t}}\Big\{f_2^{\vartheta}(\tau,y,r,p,X) - f_1^{\vartheta}(\tau,x,r,p,X) \\ &\qquad\qquad + 3\alpha c^2 e^{\bar\gamma(\tau-s)}\left|a_1^{\vartheta}(\tau,x,p) - a_2^{\vartheta}(\tau,y,p)\right|^2 - \bar\gamma e^{\bar\gamma(\tau-s)}\frac{\alpha}{2}|x-y|^2\Big\}.\end{aligned}$$

Case $\gamma \neq 0$. Let $v^i(\tau,x) = e^{\gamma(\tau-s)}u^i(\tau,x)$, $i=1,2$. Then v^i, $i=1,2$ are viscosity sub- and supersolutions respectivly of the following equations:

$$\begin{aligned} v^i_t - \gamma v^i + e^{\gamma(\tau-s)} \sup_{\vartheta\in\Theta}\Big\{ f^\vartheta_i(\tau,x,e^{-\gamma(\tau-s)}v^i, e^{-\gamma(\tau-s)}Dv^i, e^{-\gamma(\tau-s)}D^2v^i) \\ - e^{-\gamma(\tau-s)} tr[A^\vartheta(\tau,x,e^{-\gamma(\tau-s)}Dv^i)D^2v^i]\Big\} = 0, \quad i=1,2. \end{aligned} \tag{5.17}$$

The idea is now to apply Lemma 5.2 to v^i, $i=1,2$.

If we introduce the functions $\bar f^\vartheta_i(\tau,x,r,p,X) = -\gamma r + e^{\gamma(\tau-s)} f^\vartheta_i(\tau,x,e^{-\gamma(\tau-s)}r, e^{-\gamma(\tau-s)}p, e^{-\gamma(\tau-s)}X)$ and $\bar A^\vartheta_i(\tau,x,p) = A^\vartheta(\tau,x,e^{-\gamma(\tau-s)}p)$, then we can write (5.17) in the following way

$$v^i_t + \sup_{\vartheta\in\Theta}\Big\{ \bar f^\vartheta_i(\tau,x,v^i,Dv^i,D^2v^i) - tr[\bar A^\vartheta(\tau,x,Dv^i)D^2v^i]\Big\} = 0, \quad i=1,2.$$

Note that $\bar f^\vartheta_i$ and $\bar A^\vartheta_i$ satisfy conditions (C2) and (C3) respectively, with constants $\bar\gamma^i_R = 0$ and $\bar Q^i_R \le Q^i_{e^{-\gamma(t-s)}R}$, for $i=1,2$. So by using Lemma 5.2 we get

$$\begin{aligned} &\sup_{E^\alpha_{s,t}}\Big(v^1(\tau,x) - v^2(\tau,y) - \frac{\alpha}{2}e^{\bar\gamma(\tau-s)}|x-y|^2\Big) \\ &\le \sup_{E^\alpha_{s,s}}\Big(v^1(s,x) - v^2(s,y) - \frac{\alpha}{2}|x-y|^2\Big)^+ \\ &\quad + (t-s)\sup_{D^\alpha_{0,s,t}}\Big\{ \bar f^\vartheta_2(\tau,y,r,p,X) - \bar f^\vartheta_1(\tau,x,r,p,X) \\ &\quad + 3\alpha c^2 e^{\bar\gamma(\tau-s)}\left|\bar a^\vartheta_1(\tau,x,p) - \bar a^\vartheta_2(\tau,y,p)\right|^2 - \bar\gamma e^{\bar\gamma(\tau-s)}\frac{\alpha}{2}|x-y|^2\Big\}. \end{aligned}$$

Back-substitution now yields

$$\begin{aligned} &\sup_{E^\alpha_{s,t}}\Big(e^{\gamma(\tau-s)}u^1(\tau,x) - e^{\gamma(\tau-s)}u^2(\tau,y) - \frac{\alpha}{2}e^{\bar\gamma(\tau-s)}|x-y|^2\Big) \\ &\le \sup_{E^\alpha_{s,s}}\Big(u^1(s,x) - u^2(s,y) - \frac{\alpha}{2}|x-y|^2\Big)^+ \\ &\quad + (t-s)\sup_{D^\alpha_{0,s,t}}\Big\{ e^{\gamma(\tau-s)}\Big\{ f^\vartheta_2(\tau,y,e^{-\gamma(\tau-s)}r, e^{-\gamma(\tau-s)}p, e^{-\gamma(\tau-s)}X) \\ &\quad - f^\vartheta_1(\tau,x,e^{-\gamma(\tau-s)}r, e^{-\gamma(\tau-s)}p, e^{-\gamma(\tau-s)}X)\Big\} \\ &\quad + 3\alpha c^2 e^{\bar\gamma(\tau-s)}\left|a^\vartheta_1(\tau,x,e^{-\gamma(\tau-s)}p) - a^\vartheta_2(\tau,y,e^{-\gamma(\tau-s)}p)\right|^2 - \bar\gamma e^{\bar\gamma(\tau-s)}\frac{\alpha}{2}|x-y|^2\Big\} \\ &\le \sup_{E^\alpha_{s,s}}\Big(u^1(s,x) - u^2(s,y) - \frac{\alpha}{2}|x-y|^2\Big)^+ \\ &\quad + (t-s)\sup_{D^\alpha_{\gamma,s,t}}\Big\{ e^{\gamma(\tau-s)}\Big\{ f^\vartheta_2(\tau,y,r,p,X) - f^\vartheta_1(\tau,x,r,p,X)\Big\} \\ &\quad + 3\alpha c^2 e^{\bar\gamma(\tau-s)}\left|a^\vartheta_1(\tau,x,p) - a^\vartheta_2(\tau,y,p)\right|^2 - \bar\gamma e^{\bar\gamma(\tau-s)}\frac{\alpha}{2}|x-y|^2\Big\}, \end{aligned}$$

which completes the proof of Theorem 3.1.

5.2. **Proof of Theorem 3.3.** a) Note well that in this proof the indices i,j,k are fixed as defined in the statement of the result. Let us start by using Theorem 3.1 to compare u^1 and u^2. To this end, notice that

$$e^{\gamma t}\sup_{\mathbb{R}^N}\big(u^1(t,x) - u^2(t,x)\big) \le \sup_{E^\alpha_{0,t}}\Big(e^{\gamma\tau}u^1(\tau,x) - e^{\gamma\tau}u^2(\tau,y) - \frac{\alpha}{2}e^{\bar\gamma\tau}|x-y|^2\Big),$$

$$\sup_{E^\alpha_{0,0}}\Big(u^1(0,x) - u^2(0,y) - \frac{\alpha}{2}|x-y|^2\Big)^+ \le \|u^1(0,\cdot) - u^2(0,\cdot)\| + \frac{1}{2}\alpha^{-\frac{\mu}{2-\mu}}[u^i(0,\cdot)]_\mu^{\frac{2}{2-\mu}}.$$

The first inequality is obvious, while the second inequality follows from maximizing the function $h(r) = [u^i(0,\cdot)]_\mu r^\mu - \frac{\alpha}{2}r^2$ with $r = |x-y|$. An application of Theorem 3.1 together with conditions (C5) and (C8) (with constant independent of R) now yields for every $0 \le t \le T$

$$
\begin{aligned}
e^{\gamma t}\sup_{\mathbb{R}^N}\big(u^1(t,\cdot)-u^2(t,\cdot)\big) &\le \|u^1(0,\cdot)-u^2(0,\cdot)\| + \frac{1}{2}\alpha^{-\frac{\mu}{2-\mu}}[u^i(0,\cdot)]_\mu^{\frac{2}{2-\mu}} \\
&+ t\sup_{D^\alpha_{\gamma,0,t}}\Big\{e^{\gamma\tau}|f_1^\vartheta(\tau,x,r,p,X)-f_2^\vartheta(\tau,x,r,p,X)| + e^{\gamma\tau}C^{g_j}_{R_0}|x-y|^\mu \\
&+ 2\alpha e^{\bar\gamma\tau}\left|b_1^\vartheta(\tau,x,p)-b_2^\vartheta(\tau,x,p)\right|^2 + 2\alpha e^{\bar\gamma\tau}|x-y|^2 + \alpha e^{\bar\gamma\tau}C^{b_j}|x-y|^2 \\
&+ 6\alpha c^2 e^{\bar\gamma\tau}\left|a_1^\vartheta(\tau,x,p)-a_2^\vartheta(\tau,x,p)\right|^2 + 6\alpha c^2 e^{\bar\gamma\tau}C^{a_k\,2}|x-y|^2 - \bar\gamma\frac{\alpha}{2}e^{\bar\gamma\tau}|x-y|^2\Big\}^+,
\end{aligned}
\tag{5.18}
$$

where $p := \alpha(x-y)e^{(\bar\gamma-\gamma)\tau}$ and we have used the following estimates:

$$|x-y||b_1^\vartheta(\tau,x,p)-b_2^\vartheta(\tau,x,p)| \le 2|x-y|^2 + 2|b_1^\vartheta(\tau,x,p)-b_2^\vartheta(\tau,x,p)|^2,$$
$$|a_1^\vartheta(\tau,x,p)-a_2^\vartheta(\tau,y,p)|^2 \le 2|a_1^\vartheta(\tau,x,p)-a_2^\vartheta(\tau,x,p)|^2 + 2|a_2^\vartheta(\tau,x,p)-a_2^\vartheta(\tau,y,p)|^2.$$

In (5.18), we collect all terms involving $\alpha|x-y|^2e^{\bar\gamma\tau}$. Then by choosing $\bar\gamma$ appropriately, we see that

$$\alpha|x-y|^2e^{\bar\gamma\tau}\left(C^{g_j}_{R_0} + (2+C^{b_j}) + 6c^2(C^{a_k})^2 - \frac{1}{2}\bar\gamma\right) = -\frac{\alpha}{2}|x-y|^2e^{\bar\gamma\tau}.$$

The remaining "unwanted" terms inside the supremum we treat in a similar way as we treated the initial data:

$$e^{\gamma\tau}C^{f_j}_{R_0}|x-y|^\mu - \frac{\alpha}{2}|x-y|^2e^{\bar\gamma\tau} \le \frac{1}{2}\alpha^{-\frac{\mu}{2-\mu}}e^{\bar\gamma\tau}\left(e^{(\gamma-\bar\gamma)\tau}C^{f_j}_{R_0}\right)^{\frac{2}{2-\mu}}.$$

Summing up what we have done with (5.18) so far, the terms with explicit dependence in α read

$$\text{Const}\ \alpha^{-\frac{\mu}{2-\mu}} + \alpha t\,\text{Const}\sup_{D^\alpha_{\gamma,0,t}} e^{\bar\gamma\tau}\big(\left|b_1^\vartheta(\tau,x,p)-b_2^\vartheta(\tau,x,p)\right|^2 + \left|a_1^\vartheta(\tau,x,p)-a_2^\vartheta(\tau,x,p)\right|^2\big).$$

Note that the minimum of $h(r) = C_1r^{\mu/(2-\mu)} + C_2r$ is less than or equal to $2C_1^{(2-\mu)/2}C_2^{\mu/2}$. So let $r=\alpha$, $C_1 = \text{Const}$, and

$$C_2 = t\,\text{Const}\sup_{D^\alpha_{\gamma,0,t}} e^{\bar\gamma\tau}\big(\left|b_1^\vartheta(\tau,x,p)-b_2^\vartheta(\tau,x,p)\right|^2 + \left|a_1^\vartheta(\tau,x,p)-a_2^\vartheta(\tau,x,p)\right|^2\big).$$

Then we obtain

$$e^{\gamma t}\sup_{\mathbb{R}^N}\big(u^1(t,\cdot)-u^2(t,\cdot)\big) \le \|u^1(0,\cdot)-u^2(0,\cdot)\| + t\sup_{D_{\gamma,0,t}} e^{\gamma\tau}|f_2-f_1| + 2C_1^{(2-\mu)/2}C_2^{\mu/2}.$$

After an application of the following inequality in $\mathbb{R}$, $(a^2+b^2)^{\mu/2} \le |a|^\mu + |b|^\mu$, this proves Theorem 3.3 a) since the argument is symmetric in u^1 and u^2.

b) Note that the indices i,j,k are predefined and fixed, see the statement of this result. Now let $\bar\gamma = 0$, $L = e^{-\gamma T}[u^i]_1$, and $R = \max(R_0, L)$. As in Case (i), we use Theorem 3.1 and estimate the different terms. After an application of conditions (C7) – (C8) and substitution of the bounds for $|x-y|$ (in Case (iii)), we get

$$
\begin{aligned}
e^{\gamma t}\sup_{\mathbb{R}^N}\big(u^1(t,\cdot)-u^2(t,\cdot)\big) &\le \|u^1(0,\cdot)-u^2(0,\cdot)\| + \frac{1}{2\alpha}[u^i(0,\cdot)]_1^2 \\
&+ t\sup_{D_{\gamma,0,t}}\Big\{e^{\gamma\tau}|f_1^\vartheta(\tau,x,r,p,X)-f_2^\vartheta(\tau,x,r,p,X)| + C^{f_j}_R\frac{N[u^i]_1}{\alpha} \\
&\qquad + 6\alpha c^2\left|a_1^\vartheta(\tau,x,p)-a_2^\vartheta(\tau,x,p)\right|^2 + 6c^2\frac{1}{\alpha}(C^{a_k}_R)^2N^2[u^i]_1^2,\Big\}.
\end{aligned}
$$

Note that all the terms which explicitly depends on α can be written as Const α + Const α^{-1}. This can be minimized with respect to α as in a). We thus obtain a constant M such that the result holds.

5.3. **Proof of Theorem 3.4.** *(a)* This result is not a consequence of Theorem 3.1. But the proof is very similar. What we need to do is to go through the proof of Theorem 3.1 with 0 and u as sub- and supersolutions.

We assume first that $\gamma = 0$. In the first case, we get from (5.10) and (5.11) with $u^1 = 0$, $s = 0$, $\bar{\gamma} = 0$, and $\varepsilon \in (0,1]$ that

$$\frac{\delta\sigma}{t} \leq \sup_{\vartheta}\Big\{f^{\vartheta}(\tau_0, y_0, 0, \text{Const }\varepsilon^{1/2}, X - \text{Const }\varepsilon I) - tr[A^{\vartheta}(\tau_0, y_0, \text{Const }\varepsilon^{1/2})(X - \text{Const }\varepsilon I)]\Big\},$$

where $(0, X) \in \mathcal{P}^{2,+}0$. The gradient is Const $\varepsilon^{1/2}$ by (5.6). By (C2) and (5.12), we have replaced Y by $X - \text{Const }\varepsilon I$.

The fact that $(0, X) \in \mathcal{P}^{2,+}0$ means $X \geq 0$. If we use the monotonicity properties of $f^{\vartheta}(t, x, r, p, \cdot)$ and $tr[A^{\vartheta}(t, x, p)\cdot]$, we get

$$\frac{\delta\sigma}{t} \leq \sup_{\vartheta}\Big\{f^{\vartheta}(\tau_0, y_0, 0, \text{Const }\varepsilon^{1/2}, -\text{Const }\varepsilon I) + \text{Const } m(\varepsilon)\Big\}.$$

The last term follows from the growth condition in (C3) and (5.7). Now we continue as in the proof of Theorem 3.1. The result after having taken the limits $\varepsilon \to 0$ and $\delta \to 1$ is the following

$$-\inf_{\mathbb{R}^N} u(t, \cdot) \leq \|u_0\| + t \sup_{\Theta \times Q_t} |f^{\vartheta}(\tau, x, 0, 0, 0)|.$$

In a similar way, by interchanging the roles of 0 and u, we get

$$\sup_{\mathbb{R}^N} u(t, \cdot) \leq \|u_0\| + t \sup_{\Theta \times Q_t} |f^{\vartheta}(\tau, x, 0, 0, 0)|.$$

This completes the proof of part (a) for the case $\gamma = 0$. The case $\gamma \neq 0$ follows from the case $\gamma = 0$ as in the proof of Theorem 3.1.

(b) Let $f_1^{\vartheta} = f_2^{\vartheta} = f^{\vartheta}$, $a_1^{\vartheta} = a_2^{\vartheta} = a^{\vartheta}$, and $u^1 = u^2 = u$ and apply Theorem 3.1. This proof consists of simplifying the resulting expression using assumptions (C6) and (C8). At the end there should appear an inequality like

$$u(x) - u(y) \leq k_1 \alpha^{-\frac{\mu}{2-\mu}} + k_2|x - y|^2\alpha, \tag{5.19}$$

for arbitrary $x, y \in \mathbb{R}^N$. Then we are done since

$$\inf_{\alpha}\{k_1 \alpha^{-\frac{\mu}{2-\mu}} + k_2|x - y|^2\alpha\} \leq 2k_1^{1-\mu/2}k_2^{\mu/2}|x - y|^{\mu}, \tag{5.20}$$

and the argument is symmetric in x and y.

Now let us prove (5.19). First choose $\bar{\gamma} = 2(C_R^f + 3c^2(C^a)^2 + 1) + \gamma^+$. Then using (C6) and (C8), remembering that $p = \alpha(x - y)e^{(\bar{\gamma}-\gamma)\tau}$, we get

$$\begin{aligned}
&e^{\gamma\tau}(f(\tau, x, r, p, X) - f(\tau, y, r, p, X)) + 3c^2 e^{\bar{\gamma}\tau}|a(\tau, x, p) - a((\tau, y, p)|^2 - \frac{\alpha}{2}\bar{\gamma}|x - y|^2 e^{\bar{\gamma}\tau} \\
&\leq e^{\gamma\tau}C_R^f|x - y|^{\mu} + \alpha|x - y|^2 e^{\bar{\gamma}\tau}\big(C_R^f + 3c^2(C^a)^2 - \frac{\bar{\gamma}}{2}\big) \\
&\leq e^{\gamma\tau}C_R^f|x - y|^{\mu} - \alpha|x - y|^2 e^{\bar{\gamma}\tau} \leq e^{\gamma t}(C_R^f)^{\frac{2}{2-\mu}}\alpha^{-\frac{\mu}{2-\mu}}.
\end{aligned}$$

The last inequality follows from $\sup_{r\geq 0}\{c_1 r^{\mu} - c_2 r^2\} \leq c_1^{2/(2-\mu)}c_2^{-\mu/(2-\mu)}$ for $c_1, c_2 > 0$. Using the same result on the initial data yields

$$u(0, x) - u(0, y) - \frac{\alpha}{2}|x - y|^2 \leq 2[u(0, \cdot)]_{\mu}^{\frac{2}{2-\mu}}\alpha^{-\frac{\mu}{2-\mu}}.$$

Now fix $x, y \in \mathbb{R}^N$ and $0 \leq t \leq T$. An application of Theorem 3.1 now yields

$$e^{-\gamma^- t}(u(t, x) - u(t, y)) - e^{\bar{\gamma}t}\frac{\alpha}{2}|x - y|^2 \leq \Big(2[u(0, \cdot)]_{\mu}^{\frac{2}{2-\mu}} + te^{\gamma^+ t}(C_R^f)^{\frac{2}{2-\mu}}\Big)\alpha^{-\frac{\mu}{2-\mu}}.$$

So we have an inequality like (5.19). Now the finial simplifications are

$$\left(\frac{e^{\bar{\gamma}t}}{2}\right)^{\mu/2} \leq e^{\bar{\gamma}t} \quad \text{and} \quad \Big(2[u(0, \cdot)]_{\mu}^{\frac{2}{2-\mu}} + te^{\gamma^+ t}(C_R^f)^{\frac{2}{2-\mu}}\Big)^{\frac{2-\mu}{2}} \leq 2[u(0, \cdot)]_{\mu} + e^{\gamma^+ t}t^{1-\mu/2}C_R^f.$$

5.4. **Proof of Theorem 3.5.** The existence of a bounded viscosity solution follow from Barles-Perthames weak limit procedure, as discussed after Theorem 3.3. Furthermore it follows from Theorem 3.4 that the functions u and u^ν are in $\mathcal{C}^\mu(\bar{Q}_T)$ with bounds that are uniform in ν.

It remains to prove the convergence rate. This result is a consequence of the continuous dependence result in Theorem 3.1. Consider first u as a subsolution and u^ν as a supersolution. In this case $f_2^\vartheta(\tau,x,r,p,X) = f^\vartheta(\tau,x,r,p,X) - \nu\ tr[X]$, $f_1^\vartheta = f^\vartheta$, $A_i^\vartheta = A^\vartheta$, $i=1,2$. Let $R = e^{-\gamma T}\max(\|u\|, \sup_\nu \|u^\nu\|)$. We estimate the non zero terms after the application of Theorem 3.1. As in the proof of Theorem 3.3 we get

$$\begin{aligned}
&e^{\gamma t}\sup_{\mathbb{R}^N}(u(t,\cdot)-u^\nu(t,\cdot)) \le \sup_{E^\alpha_{0,t}}\Big(e^{\gamma\tau}u(\tau,x) - e^{\gamma\tau}u^\nu(\tau,y) - \frac{\alpha}{2}|x-y|^2\Big),\\
&\sup_{E^\alpha_{0,0}}\Big(u(0,x)-u^\nu(0,y)-\frac{\alpha}{2}|x-y|^2\Big)^+ \le \|u(0,\cdot)-u^\nu(0,\cdot)\| + \text{Const}\ \alpha^{-\frac{\mu}{2-\mu}}.
\end{aligned}$$

By Youngs inequality, $|x-y|^\mu \le \frac{2-\mu}{2}\alpha^{-\mu/(2-\mu)} + \frac{\mu}{2}\alpha|x-y|^2$. Moreover, using (C6), (C8), and $p = \text{Const}\,\alpha|x-y|^2$, we obtain

$$\begin{aligned}
f_1(\tau,x,r,p,X) - f_2(\tau,y,r,p,X) &\le C_R^f\left(|p||x-y| + |x-y|^\mu\right) + \nu|tr[X]|\\
&\le \text{Const}\left(\alpha^{-\frac{\mu}{2-\mu}} + \alpha|x-y|^2 + \nu|tr[X]|\right).
\end{aligned}$$

Since $|a^\vartheta(\tau,x,p) - a^\vartheta(\tau,y,p)| \le C^a|x-y|$ by (C8), this term contributes with a term of the form $\text{Const}\,\alpha|x-y|^2$. Choosing $\bar{\gamma}$ appropriately eliminates all terms of the form $\text{Const}\,\alpha|x-y|^2$. Using the bounds X in $D^\alpha_{\gamma,0,t}$, we see that $\nu|tr[X]| \le \text{Const}\,\alpha\nu$. Consequently, an application of Theorem 3.1 yields

$$\sup_{\mathbb{R}^N}(u(t,\cdot)-u^\nu(t,\cdot)) \le e^{-\gamma t}\|u(0,\cdot)-u^\nu(0,\cdot)\| + \text{Const}\left(\alpha^{-\frac{\mu}{2-\mu}} + \nu\alpha\right).$$

The result now follows by setting $\alpha = \nu^{-\frac{2-\mu}{2}}$ and then reversing the roles of u and u^ν.

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