Blowup in L^∞ for a Class of Genuinely Nonlinear Hyperbolic Systems of Conservation Laws

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Abstract

We construct a class of 3×3 systems of conservation laws with all characteristic fields genuinely nonlinear, and we show the existence of entropy solutions for these that blow up in sup-norm in finite time. The solutions are constructed by considering wave patterns where infinitely many shock waves are produced in finite time. We also consider the role of entropies as a mechanism for preventing this type of singular behavior.

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1 Introduction

Consider the Cauchy problem for $n \times n$ hyperbolic systems of conservation laws of the form

$$\begin{aligned} U_t + F(U)_x &= 0, \\ U(x,0) &= U_0(x), \end{aligned} \tag{1.1}$$

where the flux F is assumed to be smooth. We will construct systems that are strictly hyperbolic, i.e. the eigenvalues $\lambda_1(U), \ldots, \lambda_n(U)$ of the Jacobian $DF(U)$ are real and strictly separated for all values of U. Let $r_1(U), \ldots, r_n(U)$ denote the corresponding eigenvectors. The k-th characteristic field is said to be *genuinely nonlinear* if

$$\nabla\lambda_k(U) \cdot r_k(U) \neq 0 \qquad \text{for all } U,$$

while it is *linearly degenerate* if

$$\nabla\lambda_k(U) \cdot r_k(U) \equiv 0.$$

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For scalar equation ($n = 1$) Kružkov [7] proved that (1.1) admits a unique global (in time) weak entropy solution for any initial data $U_0 \in L^\infty(\mathbb{R})$. The fundamental existence result for systems is given by Glimm's theorem [2] which establishes the existence of a global weak entropy solution to (1.1) provided each characteristic field is either genuinely nonlinear or linearly degenerate and the total variation of the data is sufficiently small. For $n = 2$ the conditions on the data can be relaxed to the much weaker assumption of small *oscillation* [3].

Works by Hunter [4], Joly, Metivier, Rauch [6], and by Young [10] have demonstrated that this stronger result cannot be extended to systems of three or more equations. These authors have given examples of solutions to 3×3-systems with arbitrarily large *magnification* of L^∞-norm and total variation. Furthermore, Bressan and Shen [1] showed that the Cauchy problem for 3×3-systems may be ill-posed when the data are in L^∞ but not in BV.

In [5] it was proved that there are systems of strictly hyperbolic conservation laws of the form (1.1) whose solutions blow up in either sup-norm or total variation in finite time. These solutions were explicitly constructed by setting up wave patterns were infinitely many waves are generated in finite time. The data consist of three discontinuities corresponding to two approaching shocks of the second family on each side of a contact discontinuity of the first family, see Figure 1. When the left 2-wave interacts with the 1-wave a reflected 3-wave is generated. This wave then interacts with the right 2-wave generating a reflected 1-wave, etc. In this way infinitely many waves are generated in finite time. By carefully choosing the flux function F and the initial data one can find solutions for which

- the L^∞-norm of the solution explodes in finite time, or
- the total variation of the solution explodes in finite time while the L^∞-norm remains bounded, see [5] for details.

To the best of our knowledge these are the first examples of this kind and they demonstrate the *possibility* of finite time blowup for systems of conservation laws. However, several natural questions remain open. The systems considered in [5] are rather artificial and have several special properties which simplify the computations. In particular the first and third families are linearly degenerate with constant eigenvalues. A natural next step is to seek examples of blowup for systems where all three fields are genuinely nonlinear.

The aim of this paper is to give such an extension by studying a particular class of systems that contains perturbations of the systems introduced in [5]. However, we have been able to do this only in the case of L^∞ blowup. In the case where the total variation explodes in the unperturbed system, the linear degeneracy implies a particular geometry which makes it possible to have periodicity in state space. The solution visits the same states infinitely many times, thereby driving the variation to infinity. A similar behavior in the case where all fields are genuine nonlinear seems more difficult to produce.

Given the existing results on blowup for systems of equations it is natural to seek conditions that prevent this type of singular behavior. In particular it would be of interest to understand the role of entropies in this context. The existence of a convex entropy implies a certain relation between the "interaction coefficients", see (7.1) below. As observed by Young [11], this relation is violated in all known cases of blowup for systems when the data has large variation and correspondingly small amplitude. We show in Section 7 that it does not exclude the type of blowup presented in this paper. This is done by an explicit computation in the unperturbed case. However, we also include a proof that there is no globally defined, strictly convex entropy in this case. This suggests that the condition of having a strictly convex entropy is strong enough to prevent blowup in L^∞.

2 Outline and Results

We now give a brief outline of the paper and state our main result. In the next section we give a general procedure for constructing a class of systems which contains perturbations of the systems in [5]. The goal is to produce a wave pattern as in Figure 1 where the *reflected* waves (i.e. the 1-waves and 3-waves between the two 2-waves) are shocks and where the *transmitted* waves (i.e. the 1-waves to the left of the left 2-wave and the 3-waves to the right of the right 2-wave) have uniformly large strengths. In Section 4 and Section 5 we compute the wave curves for the first and third families and we establish that the Riemann problems given by $2-1$- and $3-2$- interactions can be solved uniquely. We then perform a detailed analysis of the signs and strengths of the incoming and outgoing waves in these interactions. Finally we use the fact that infinitely many waves are generated to conclude the proof of the following theorem.

Theorem 2.1 *There are 3×3-systems of strictly hyperbolic conservation laws of the form (1.1) which are genuinely nonlinear in all three fields and such that the following holds. There exist piecewise constant initial data $U_0 \in L^\infty(\mathbb{R})\cap BV(\mathbb{R})$ for which the solution $U = U(x,t)$ of (1.1) obtained by solving Riemann problems at each interaction has a wave pattern as in Figure 1, and for which there is a time $T < \infty$ such that*

$$\lim_{t\uparrow T} \|U(\cdot,t)\|_\infty = +\infty.$$

Furthermore, the systems are uniformly strictly hyperbolic along the solution.

We point out that there are *a priori* two ways in which the genuine nonlinearity of the first and third fields could yield a mechanism for preventing blowup. First, since the reflected 1- and 3-waves may be rarefactions, it is not clear that one can produce a wave pattern as in Figure 1 where all waves are shocks. The presence of rarefactions would imply some decay in the solution. Secondly, even when all reflected waves are shocks, there remains the possibility that the transmitted waves interact and send back waves that interact with the 2-shocks. This would complicate the wave pattern further and could prevent blowup. In the following construction we will pay attention to these possibilities.

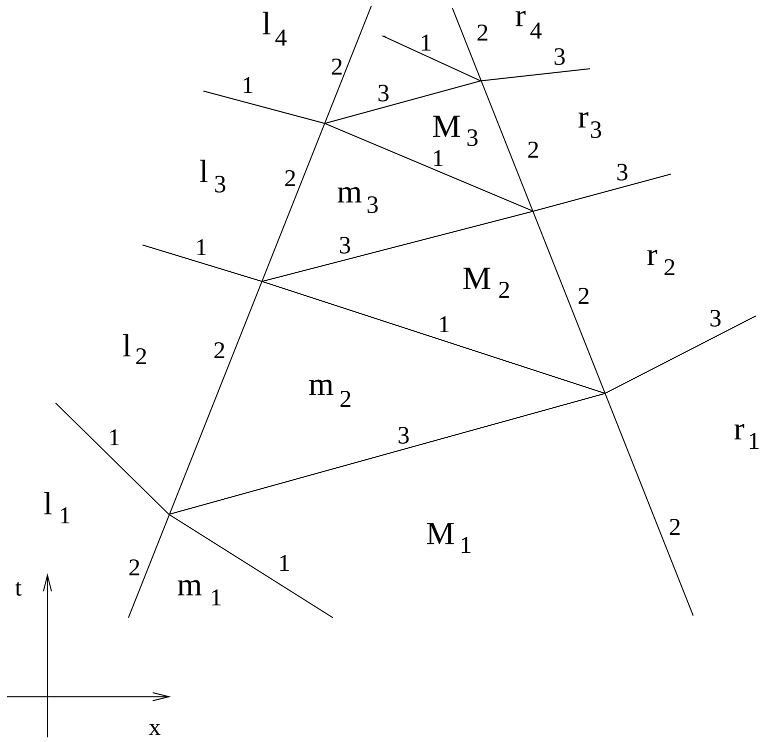

Figure 1: wave pattern

3 The Class of Systems

In this section we introduce the equations and show that these generalize the systems in [5]. We seek genuinely nonlinear systems for which we can find solutions whose L^∞-norm explodes in finite time. We start by observing some key properties of the examples in [5]. These systems have flux functions of the form

$$F(U) \equiv F(u,v,w) = \begin{pmatrix} ua(v)+w \\ \Gamma(v) \\ u\big(\lambda_0^2 - a^2(v)\big) - wa(v) \end{pmatrix}, \tag{3.1}$$

where $\lambda_0 > 0$ is a constant and $a, \Gamma : \mathbb{R} \mapsto \mathbb{R}$. The eigenvalues of the Jacobian DF are

$$\lambda_1 \equiv -\lambda_0, \qquad \lambda_2 = \Gamma'(v), \qquad \lambda_3 \equiv +\lambda_0. \tag{3.2}$$

The first and third eigenvectors are given by

$$r_1 = \begin{pmatrix} 1 \\ 0 \\ -\big(\lambda_0 + a(v)\big) \end{pmatrix}, \qquad r_3 = \begin{pmatrix} 1 \\ 0 \\ \lambda_0 - a(v) \end{pmatrix}. \tag{3.3}$$

The first and third fields are linearly degenerate (constant eigenvalues) and the wave curves in these families are straight lines in planes with $v =$ constant. This implies a fairly simple situation in state space and makes possible explicit computations when resolving interactions. When looking for a generalization to genuinely nonlinear systems we would like to retain as much as possible of this simple situation.

We start by considering 2×2-systems in the (u,w)-variables with the property that the wave curves are straight lines in $\mathbb{R}^2$. However, we keep in mind that the coefficients of this system will depend on v, which in turn will be chosen to be the solution of a decoupled scalar equation. The resulting 3×3-system in the (u,v,w)-variables will then have the property that the wave curves of the first and third families are straight lines in planes with $v =$ constant. We must pay attention to guarantee that the resulting system is strictly hyperbolic and genuinely nonlinear in all families.

A simple way to give a 2×2-system with wave curves that are straight lines in the (u,w)-plane is as follows. Consider two constant vectors

$$R_1 = \begin{pmatrix} 1 \\ a \end{pmatrix}, \qquad R_3 = \begin{pmatrix} 1 \\ b \end{pmatrix},$$

with $a \neq b$, and two scalar functions $\lambda_1(u,w), \lambda_3(u,w)$. We then insist that these are the eigenvectors and eigenvalues of a Jacobian matrix. That is, we ask that the rows of the matrix

$$\begin{aligned} A \;&:=\; [R_1|R_3]\; diag(\lambda_1, \lambda_3)\; [R_1|R_3]^{-1} \\ &=\; \frac{1}{b-a} \begin{bmatrix} b\lambda_1 - a\lambda_3 & \lambda_3 - \lambda_1 \\ ab(\lambda_1 - \lambda_3) & b\lambda_3 - a\lambda_1 \end{bmatrix} \end{aligned}$$

have zero curl. This is the case if and only if

$$\begin{cases} (b\lambda_1 - a\lambda_3)_w = (\lambda_3 - \lambda_1)_u, \\ ab(\lambda_1 - \lambda_3)_w = (b\lambda_3 - a\lambda_1)_u, \end{cases} \tag{3.4}$$

which is equivalent to

$$\begin{cases} (b-a)(\lambda_{1,u} + b\lambda_{1,w}) = 0, \\ (b-a)(\lambda_{3,u} + a\lambda_{3,w}) = 0. \end{cases} \tag{3.5}$$

Thus, by choosing λ_1 as any function of $w - bu$ and λ_3 as any function of $w - au$ we guarantee that A is the Jacobian matrix of some map $\bar{F}(u,w)$. A simple computation shows that $\bar{F}(u,w)$ may be given as

$$\bar{F}(u,w) = \begin{pmatrix} \frac{1}{b-a}\big[H_3(w-au) - H_1(w-bu)\big] \\ \frac{1}{b-a}\big[bH_3(w-au) - aH_1(w-bu)\big] \end{pmatrix}, \tag{3.6}$$

where

$$H_i' = \lambda_i \qquad \text{for} \quad i = 1, 3. \tag{3.7}$$

We can now give a 3×3-system with the property that the wave curves in the first and third families are straight lines in planes with v =constant. We let a and b depend on v with $a(v) \neq b(v)$ for all v, and augment the system with flux $\bar{F}$ by adding a decoupled scalar conservation law for v, $v_t + g(v)_x = 0$, say. The flux function of the resulting full 3×3-system is given by

$$F(U) = F(u,v,w) = \begin{pmatrix} f(u,v,w) \\ g(v) \\ h(u,v,w) \end{pmatrix},$$

where f and h are the first and second components of $\bar{F}$, respectively, with $a = a(v)$ and $b = b(v)$. The eigenvalues of DF are $\lambda_1\big(w - b(v)u\big)$, $\lambda_2 = g'(v)$, and $\lambda_3\big(w - a(v)u\big)$, and the first and third eigenvectors are

$$r_1(v) = \begin{pmatrix} 1 \\ 0 \\ a(v) \end{pmatrix}, \qquad r_3(v) = \begin{pmatrix} 1 \\ 0 \\ b(v) \end{pmatrix}. \tag{3.8}$$

It remains to choose λ_1, g, λ_3 such that the system is strictly hyperbolic and genuinely nonlinear in all three families. We can obtain (uniform) strict hyperbolicity by letting λ_1, g' and λ_3 be bounded with strictly separated ranges. Also, the second characteristic field is genuinely nonlinear provided $g(v)$ is strictly convex. The first and third fields are genuinely nonlinear if and only if

$$\nabla\lambda_1 \cdot r_1 = \big(a(v) - b(v)\big)\,\lambda_1'\big(w - b(v)u\big) \neq 0, \tag{3.9}$$

and

$$\nabla\lambda_3 \cdot r_3 = \big(b(v) - a(v)\big)\,\lambda_3'\big(w - a(v)u\big) \neq 0. \tag{3.10}$$

Since we have assumed $a(v) \neq b(v)$ for all v, it suffices to choose $\lambda_1' \neq 0$ and $\lambda_3' \neq 0$.

This construction yields a large class of systems with the properties we seek. For simplicity we make the following choices in the rest of the paper. We let p be a strictly increasing and bounded function with range contained in the interval $[-\alpha, \alpha]$, and we let

$$g(v) := \int_0^v p(\eta)\,d\eta.$$

Next define

$$\lambda_1(\xi) := \varepsilon p(\xi) - c, \qquad \lambda_3(\xi) = \varepsilon p(\xi) + c,$$

where $\varepsilon > 0$ is a small parameter to be chosen later and $c > (1+\varepsilon)\alpha$. It follows that the ranges of $\lambda_1, \lambda_2 = g'$ and λ_3 do not overlap. We finally let the functions H_1 and H_3 be given as

$$H_1(\xi) := \varepsilon g(\xi) - c\xi, \qquad H_3(\xi) := \varepsilon g(\xi) + c\xi,$$

so that (3.7) is satisfied.

To summarize: we consider systems of the form (1.1) with flux function given by

$$F(U) = \begin{pmatrix} \frac{1}{b-a}\Big\{\varepsilon\big[g(w-au) - g(w-bu)\big] + c\big[2w - (a+b)u\big]\Big\} \\ g(v) \\ \frac{1}{b-a}\Big\{\varepsilon\big[bg(w-au) - ag(w-bu)\big] + c\big[(a+b)w - 2abu\big]\Big\} \end{pmatrix}, \tag{3.11}$$

where we recall that $a = a(v)$, $b = b(v)$. The eigenvalues are

$$\varepsilon p\big(w - b(v)u\big) - c, \qquad g'(v) = p(v), \qquad \varepsilon p\big(w - a(v)u\big) + c,$$

and the 1- and 3-eigenvectors are given by (3.8). The eigenvalues are uniformly strictly separated by the choice of c. This ensures strict hyperbolicity of the system. The monotonicity of p together with choosing $a(v) \neq b(v)$ ensure that the system is genuinely nonlinear in all three families.

To simplify the analysis in the next section we make one more assumption about p:

- **A1**. p is smooth, strictly increasing, with range contained in $[-\alpha, \alpha]$ and with the property that $p(-\xi) = -p(\xi)$.

Note that the functions $a(v)$ and $b(v)$ have not been fixed at this point. The only important thing so far is that $a(v) \neq b(v)$ for all v, so that the two eigenvectors r_1 and r_3 are always linearly independent. For reasons that will be clear from the analysis of interactions in the next section we assume:

A2. $a(v_1) > b(v_2)$ for all v_1 and v_2,

A3. a and b are smooth and globally bounded.

It remains to choose appropriate values for the parameters c and ε. These will be determined in the next section.

It is clear that $\varepsilon = 0$ gives a system that is linearly degenerate in the first and third fields with $\lambda_3(U) = -\lambda_1(U) \equiv c$. It is in this sense that we have constructed a class of systems that contains perturbations of the systems considered in [5]. We will use this fact together with the continuity in ε to give examples of blowup in L^∞ also in the case when $\varepsilon > 0$. We note that for $\varepsilon = 0$ the transmitted 1-waves (or 3-waves) are all parallel (since they all travel with the same speed). This is no longer true when $\varepsilon > 0$ and all the fields are genuinely nonlinear. A priori we may therefore have a situation where transmitted 1-waves (or 3-waves) interact and send back waves that complicate the wave pattern. We shall see that this is not the case for our systems. We refer to $\varepsilon = 0$ as the *unperturbed* case.

4 Wave curves

The wave curves of the first and third families are by construction straight lines and they lie in planes with $v =$constant. It follows that the integral curves of the eigenvector field and the Hugoniot locus coincide in each of these families. Parameterizing the integral curves by σ and τ, and denoting by $D_j[\,\cdot\,; (\bar{u}, \bar{v}, \bar{w})]$ the j-curve starting at the point $(\bar{u}, \bar{v}, \bar{w})$, $j = 1, 3$, we find that

$$D_1[\sigma; (\bar{u}, \bar{v}, \bar{w})] = \begin{pmatrix} \sigma + \bar{u} \\ \bar{v} \\ a(\bar{v})\sigma + \bar{w} \end{pmatrix}, \tag{4.1}$$

$$D_3[\tau; (\bar{u}, \bar{v}, \bar{w})] = \begin{pmatrix} \tau + \bar{u} \\ \bar{v} \\ b(\bar{v})\tau + \bar{w} \end{pmatrix}. \tag{4.2}$$

To decide the parameter ranges corresponding to rarefactions and shocks we recall the assumptions **A1** and **A2**. Using the expressions for the eigenvalues we conclude that

- negative σ-values correspond to 1-shocks,
- positive τ-values correspond to 3-shocks.

The strengths of 1- and 3-waves will be measured by the change in the parameters σ and τ.

We will not need explicit expressions for the wave curves of the second family. We only outline the properties of the 2-shock curve. Recall that the second equation is a decoupled scalar equation. It follows that we can use v

itself as a parameter for the 2-shock curve and that 2-shocks correspond to downward jumps in v (since $p'(\xi) > 0$ implies that $g(v)$ is strictly convex). The equations for the states that can be connected by a 2-shock with speed γ are given by the Rankine-Hugoniot condition $[F(U)] = \gamma[U]$, i.e.

$$\begin{aligned} \gamma[u] &= \big[f(u, v, w)\big], \\ \gamma[v] &= \big[g(v)\big], \\ \gamma[w] &= \big[h(u, v, w)\big], \end{aligned} \tag{4.3}$$

where $[\cdot]$ denotes the jump across the discontinuity. When $[v] = 0$ the second equation in (4.3) is always satisfied and we recover the expressions for the 1- and 3-shock curves. If $[v] \neq 0$ we can solve the second equation for γ as a function of the parameter v and substitute this value into the first and third equations. Solving the so obtained system of two equations yields the u- and w-values along a 2-shock curve as functions of the parameter v.

5 Interactions

In order to set up an initial value problem with a wave pattern as in Figure 1 where all waves are shocks we must study $2-1$-interactions and $3-2$-interactions in detail. Since the second equation is a decoupled scalar equation it follows that the v-component is constant across 1-waves and 3-waves. Thus there are only three v-values present in this setup: the left, middle and right states which we denote by v_-, v_0 and v_+, respectively. Also, since the flux $g(v)$ is strictly convex, the 2-waves will be shocks provided

$$v_- > v_0 > v_+.$$

We introduce the parameter $V > 0$ and put $v_- := V$, $v_0 := 0$, $v_+ := -V$. Let γ be the constant speed of the left 2-wave, i.e.

$$\gamma = \frac{g(0) - g(V)}{-V} = \frac{g(V)}{V} > 0.$$

Note that since p is an odd function (by **A1**) it follows that the speed of the right 2-wave is $-\gamma$.

5.1 Solvability of $2-1$-interactions and $3-2$-interactions

We first show that the Riemann problems given by the interactions have unique solutions when the perturbation parameter ε is sufficiently small. We introduce the following notation. We write the first and third components of the flux F as

$$f = \phi + \varepsilon S, \quad h = \theta + \varepsilon T,$$

where

$$\phi = \frac{c}{b-a}\big(2w - (a+b)u\big), \quad S = \frac{1}{b-a}\big(g(w-au) - g(w-bu)\big),$$

and

$$\theta = \frac{c}{b-a}\big((a+b)w - 2abu\big), \quad T = \frac{1}{b-a}\big(bg(w-au) - ag(w-bu)\big),$$

where we recall that c is a constant while a and b are functions of v.

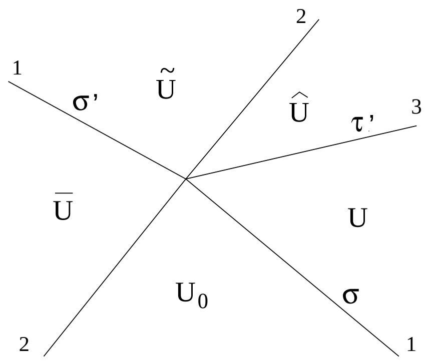

Figure 2: $2-1$-interaction

Consider first the case when a 2-shock with positive speed γ interacts with a 1-wave. Denote by $\bar{U}$, U_0 and U, respectively the left, middle and right state before the interaction, see Figure 2. Let σ be the strength of the incoming 1-wave. We then have the Riemann problem $(\bar{U}, U)$ and we want to show that it can be resolved by a 1-wave of strength σ' connecting $\bar{U}$ to some state $\tilde{U}$, a 2-shock connecting $\tilde{U}$ to some state $\hat{U}$ and a 3-wave with strength τ' connecting $\hat{U}$ with U. Thus the Riemann problem $(\bar{U}, U)$ has a solution if and only if there exist $\sigma', \tau' \in \mathbb{R}$ such that the two states $\tilde{U} = \bar{U} + \sigma' r_1(V)$ and $\hat{U} = U_0 + \sigma r_1(0) - \tau' r_3(0)$ satisfy the Rankine-Hugoniot condition for a 2-shock with speed γ.

Using the notation introduced above we can write the first and third of the Rankine-Hugoniot equations in the form

$$\begin{cases} \gamma(\hat{u} - \tilde{u}) = \big(\phi(\hat{U}) - \phi(\tilde{U})\big) + \varepsilon\big(S(\hat{U}) - S(\tilde{U})\big), \\ \gamma(\hat{w} - \tilde{w}) = \big(\theta(\hat{U}) - \theta(\tilde{U})\big) + \varepsilon\big(T(\hat{U}) - T(\tilde{U})\big). \end{cases} \tag{5.1}$$

By assumption $\bar{U}$ is connected to U_0 by a 2-shock so that

$$\begin{cases} \gamma(u_0 - \bar{u}) = \big(\phi(U_0) - \phi(\bar{U})\big) + \varepsilon\big(S(U_0) - S(\bar{U})\big), \\ \gamma(w_0 - \bar{w}) = \big(\theta(U_0) - \theta(\bar{U})\big) + \varepsilon\big(T(U_0) - T(\bar{U})\big). \end{cases} \tag{5.2}$$

Subtracting (5.2) from (5.1) we obtain

$$\begin{aligned}
\gamma\big((\hat{u}-u_0)-(\tilde{u}-\bar{u})\big) &= \big\{\big(\phi(\hat{U})-\phi(U_0)\big)-\big(\phi(\tilde{U})-\phi(\bar{U})\big)\big\} \\
&\quad +\varepsilon\big\{\big(S(\hat{U})-S(U_0)\big)-\big(S(\tilde{U})-S(\bar{U})\big)\big\}, \\
\gamma\big((\hat{w}-w_0)-(\tilde{w}-\bar{w})\big) &= \big\{\big(\theta(\hat{U})-\theta(U_0)\big)-\big(\theta(\tilde{U})-\theta(\bar{U})\big)\big\} \\
&\quad +\varepsilon\big\{\big(T(\hat{U})-T(U_0)\big)-\big(T(\tilde{U})-T(\bar{U})\big)\big\}.
\end{aligned}$$

Since $\hat{U}-U_0=\sigma r_1(0)-\tau' r_3(0)$ and $\tilde{U}-\bar{U}=\sigma' r_1(V)$ we obtain from (3.8) that

$$\hat{u}-u_0=\sigma-\tau', \quad \hat{w}-w_0=a_0\sigma-b_0\tau', \quad \tilde{u}-\bar{u}=\sigma', \quad \tilde{w}-\bar{w}=\bar{a}\sigma',$$

where we have put $a_0=a(0)$, $b_0=b(0)$ and $\bar{a}=a(V)$. Recalling the expressions for ϕ and θ, the equations can be written as

$$(\gamma-c)\tau'+(\gamma+c)\sigma'-(\gamma+c)\sigma+\varepsilon\mathcal{S}(\sigma',\tau';\sigma,\bar{U},V)=0 \tag{5.3}$$

and

$$b_0(\gamma-c)\tau'+\bar{a}(\gamma+c)\sigma'-a_0(\gamma+c)\sigma+\varepsilon\mathcal{T}(\sigma',\tau';\sigma,\bar{U},V)=0, \tag{5.4}$$

where

$$\mathcal{S}(\sigma',\tau';\sigma,\bar{U},V)=\big(S(\hat{U})-S(U_0)\big)-\big(S(\tilde{U})-S(\bar{U})\big),$$

and

$$\mathcal{T}(\sigma',\tau';\sigma,\bar{U},V)=\big(T(\hat{U})-T(U_0)\big)-\big(T(\tilde{U})-T(\bar{U})\big).$$

For $\varepsilon=0$ equations (5.3)-(5.4) reduce to the linear system

$$\begin{aligned}
(\gamma+c)\sigma'+(\gamma-c)\tau' &= (\gamma+c)\sigma, &(5.5)\\
\bar{a}(\gamma+c)\sigma'+b_0(\gamma-c)\tau' &= a_0(\gamma+c)\sigma, &(5.6)
\end{aligned}$$

which always has a unique solution σ_0', τ_0' since $b_0-\bar{a}\neq 0$ by **A2** and $\gamma\pm c\neq 0$ by the properties of p and c.

Since $\nabla_{(\sigma',\tau')}\mathcal{S}$ and $\nabla_{(\sigma',\tau')}\mathcal{T}$ are *uniformly* bounded (by the properties of a, b and p), it follows that the same is true for (5.3)-(5.4) whenever ε is sufficiently small. That is, for ε sufficiently small, a $2-1$-interaction for the nonlinear system (5.3)-(5.4) can always be resolved uniquely in terms of $1-$, $2-$ and $3-$waves. Furthermore, by the Mean Value Theorem, we can rewrite (5.3)-(5.4) in the form

$$\big((\gamma+c)+\varepsilon A\big)\sigma'+\big((\gamma-c)+\varepsilon B\big)\tau'=\big((\gamma+c)+\varepsilon Q\big)\sigma, \tag{5.7}$$

and

$$\big(\bar{a}(\gamma+c)+\varepsilon D\big)\sigma'+\big(b_0(\gamma-c)+\varepsilon E\big)\tau'=\big(a_0(\gamma+c)+\varepsilon R\big)\sigma. \tag{5.8}$$

where A, B, Q, D, E, R are uniformly bounded functions of $(\sigma', \tau'; \sigma, \bar{U}, V)$. A direct computation now shows that there is a constant K, independent of $(\sigma', \tau'; \sigma, \bar{U}, V)$, such that the solutions (σ', τ') of (5.3)- (5.4) and (σ_0', τ_0') of (5.5)-(5.6) satisfy

$$|\sigma' - \sigma_0'|, |\tau' - \tau_0'| < K\varepsilon|\sigma|. \tag{5.9}$$

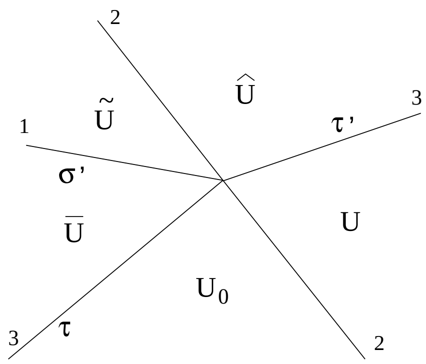

Figure 3: $3-2$-interaction

A completely similar analysis applies to $3-2$-interactions along the right 2-shock. We consider a 3-shock of strength $\tau > 0$ which collides with a 2-shock with negative speed $-\gamma$. Let the left, middle and right states be $\bar{U}, U_0$, and U, respectively (see Figure 3). Let σ' and τ' denote the strengths of the outgoing 1- and 3-waves. We let σ_0' and τ_0' denote the outgoing strengths in the unperturbed case $\varepsilon = 0$. Arguing as above we conclude that the Riemann problem $(\bar{U}, U)$ has a unique solution whenever ε is sufficiently small. Furthermore, (5.9) holds also in this case with τ instead of σ and for a suitable constant K which does not depend on $(\sigma', \tau'; \tau, \bar{U}, V)$.

5.2 Sign and strength of outgoing waves

We proceed to analyze the signs and strengths of the outgoing waves. We want to construct a solution with a wave pattern as in Figure 1, where all waves are shocks. By construction the two approaching 2-waves are shocks. We will now carefully choose the parameters such that $2-1$- and $3-2$-interactions where the incoming waves are shocks, always produce outgoing shocks in all families. Furthermore, to guarantee blowup in L^∞, we will make sure that each reflected wave is stronger than the incoming wave that produced it.

As a first step we establish this for the unperturbed system corresponding to $\varepsilon = 0$. Consider a $2-1$-interaction as above. We assume that the incoming 1-wave is a shock, that is $\sigma < 0$. The solution in this case is given by

$$\sigma_0' = \frac{(a_0 - b_0)}{(\bar{a} - b_0)}\sigma, \tag{5.10}$$

$$\tau_0' = \frac{(\gamma + c)}{(\gamma - c)} \frac{(\bar{a} - a_0)}{(\bar{a} - b_0)} \sigma. \tag{5.11}$$

It follows from **A2** and (5.10) that $sign\, \sigma_0' = sign\, \sigma$, so that the transmitted 1-wave is again a shock. Observe that the assumptions on p implies that

$$0 < \gamma < \alpha.$$

As $c > \alpha$ it follows that the first coefficient $(\gamma + c)/(\gamma - c)$ in (5.11) is negative and with absolute value strictly greater than 1. Furthermore, by choosing $a(v)$ and $b(v)$ with $\bar{a} > a_0$ and $a_0 - b_0$ sufficiently small, it is clear that the second coefficient $(\bar{a} - a_0)/(\bar{a} - b_0)$ can be made arbitrarily close to 1 (without violating **A2**). For suitable choices of the parameters we can therefore make $\tau_0' > 0$, so that the reflected 3-wave is a shock. And we can do this in such a way that $|\tau_0'| \geq L_1|\sigma|$, where $L_1 > 1$. That is, the reflected wave is stronger than the incoming wave.

Next we perform the same analysis for $3 - 2$-interactions in the unperturbed case $\varepsilon = 0$. We consider a 3-shock of strength $\tau > 0$ which collides with a 2-shock with negative speed $-\gamma$ as described above. The strengths of the outgoing 1- and 3-waves are now given by

$$\sigma_0' = \frac{(\gamma + c)}{(\gamma - c)} \frac{(b_0 - \bar{b})}{(a_0 - \bar{b})} \tau, \tag{5.12}$$

$$\tau_0' = \frac{(a_0 - b_0)}{(a_0 - \bar{b})} \tau, \tag{5.13}$$

respectively, where $\bar{b} = b(-V)$. We see that **A2** implies $sign\, \tau_0' = sign\, \tau$ such that the transmitted 3-wave is again a shock. As above the first coefficient in (5.12) is negative with absolute value strictly larger than 1. Again, by choosing $a(v)$ and $b(v)$ with $\bar{b} < b_0$ and $a_0 - b_0$ sufficiently small, it is clear that the second coefficient $(b_0 - \bar{b})/(a_0 - \bar{b})$ can be made arbitrarily close to 1 (without violating **A2** or the choices made above). It follows that we can make choices for the parameters in such a way that both the outgoing waves are shocks. And we can do this in such a way that $|\sigma_0'| \geq L_2|\tau|$ where $L_2 > 1$, i.e. the reflected wave is stronger than the incoming wave.

Note that in both cases above, once the values for a and b have been appropriately chosen, the strength of the transmitted wave is larger than the strength of the incoming wave multiplied by a fixed constant M, where

$$M := \min\left\{\frac{a_0 - b_0}{\bar{a} - b_0}, \frac{a_0 - b_0}{a_0 - \bar{b}}\right\} > 0.$$

6 Proof of Theorem 2.1

In this section we summarize the results for the linear (unperturbed) system and use this to demonstrate the existence of solutions that blow up in L^∞ for systems where all three fields are genuinely nonlinear.

In the previous section we showed that the assumptions **A1**-**A3**, $\varepsilon = 0$ and the choices

- $v_\pm = \mp V$, $v_0 = 0$,
- $c > (1+\varepsilon)\alpha$,
- $a_0 - b_0 = a(0) - b(0)$ small, $\bar{a} - a_0 = a(V) - a(0) > 0$, $b_0 - \bar{b} = b(0) - b(-V) > 0$

give a system for which there are constants $L := \min\{L_1, L_2\} > 1, M > 0$ such that the following holds.

- Each $2-1$ interaction where the incoming waves are shocks produces outgoing shocks in all three families. Denoting the strengths of the incoming 1-shock, transmitted 1-wave and the reflected 3-wave by σ, σ_0' and τ_0', respectively, we have $\sigma < 0$,

$$\sigma_0' < 0, \qquad |\sigma_0'| > M|\sigma|,$$

and

$$\tau_0' > 0, \qquad |\tau_0'| > L|\sigma|.$$

- Each $3-2$ interaction where the incoming waves are shocks produces outgoing shocks in all three families. Denoting the strengths of the incoming 3-shock, transmitted 3-wave and the reflected 1-wave by τ, τ_0' and σ_0', respectively, we have $\tau > 0$,

$$\tau_0' > 0, \qquad |\tau_0'| > M|\tau|,$$

and

$$\sigma_0' < 0, \qquad |\sigma_0'| > L|\tau|.$$

We now use the uniform continuity with respect to ε in (5.9) to conclude that

$$|\tau'| \geq |\tau_0'| - |\tau' - \tau_0'| \geq L|\sigma| - K\varepsilon|\sigma|,$$

for $2-1$-interactions with incoming 1-strength σ and outgoing 3-strength τ'. Thus, for $\varepsilon > 0$ sufficiently small there exists a constant $C > 1$ such that

$$|\tau'| \geq C|\sigma|. \tag{6.1}$$

By a similar argument (possibly changing C but keeping $C > 1$) we have the corresponding estimate

$$|\sigma'| \geq C|\tau|, \tag{6.2}$$

for $3-2$-interactions with incoming 3-strength τ and outgoing 1-strength σ'. We conclude that for sufficiently small $\varepsilon > 0$ we can set up a wave pattern

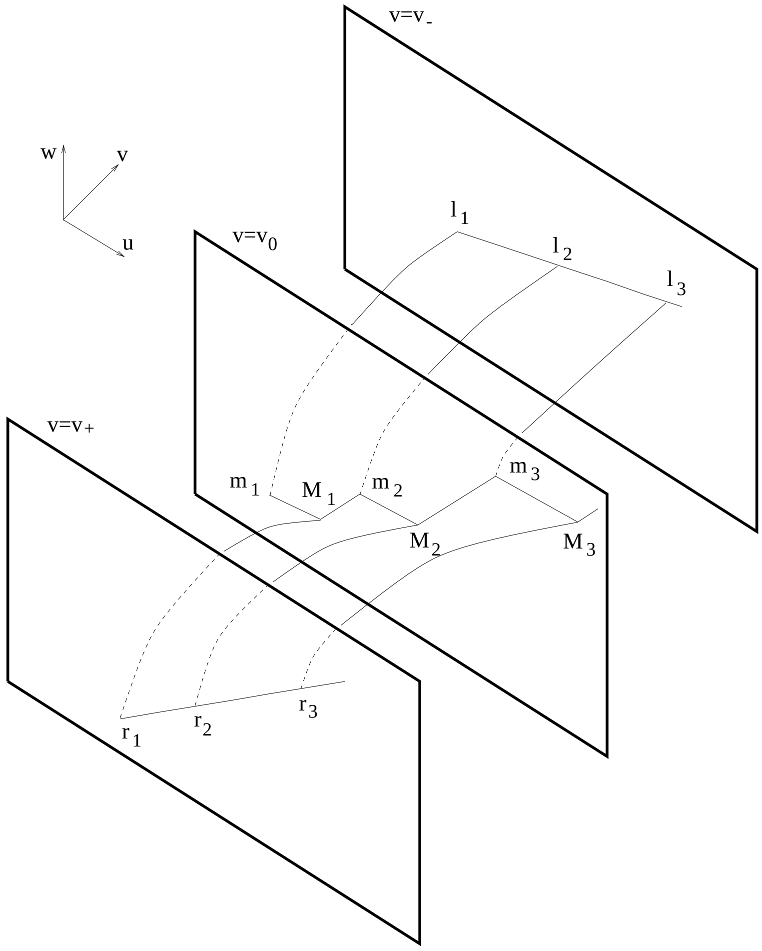

Figure 4: solution in state-space

as in Figure 1 and with the property that each reflected wave is stronger than the incoming wave that produced it.

Thus, if we choose the parameters as above, with $\varepsilon > 0$ small enough, and give initial data that yield two approaching 2-shocks with a 1-shock in between as in Figure 1, then the solution will contain infinitely many shock waves. For a representation of the solution in state-space compare Figures 1 and 4. If we consider the states $l_1, l_2, \ldots$, say, to the left of the left 2-wave, then these will all lie on the same straight line in the plane $v \equiv V$ (see Figure 4). Since the strength of a reflected wave between the two 2-waves is always stronger than the incoming wave which produced it, and since the strength of a transmitted wave is bounded from below by a multiple of the strength of the incoming wave, it follows that the states $l_1, l_2, \ldots$ lie outside balls with radii tending to $+\infty$.

We observe that the transmitted 1-waves may well interact due to the genuine nonlinearity and that this may happen before the 2-shocks meet. However, when two 1-shocks interact to the left of the left 2-shock, all that comes out of the interaction is a single 1-wave of strength equal to the sum of the strengths of the incoming 1-waves. This follows since the wave curve of the first family is a straight line in the plane $v \equiv V$. Clearly, a similar statement holds for any 3-waves that may interact to the right of the right 2-shock. Consequently, these interactions do not change the L^∞-norm of the solution. Since the 2-shocks meet in finite time this means that the L^∞-norm of the solution blows up at some positive time $T < \infty$. This completes the proof of Theorem 2.1.

Remark 6.1 Note that blowup occurs also in the limiting unperturbed case when $\varepsilon = 0$. We will make use of this fact in the last section.

Remark 6.2 The variation of the initial data can be "small" in the construction above. However, this does not contradict Glimm's theorem: the bound on the initial variation which guarantees global existence depends on the flux. What we do here is to first give the data and then find a suitable flux.

Remark 6.3 For the particular example presented above we only need to prove the unique solvability of some special $2-1$- and $3-2$-interactions. However, it is easy to prove that, provided ε is sufficiently small, *every* Riemann problem can be solved in a unique way in terms of $1, 2$ and 3 waves. The 2-wave curves are globally defined and the system is uniformly strictly hyperbolic even for arbitrarily large data. This shows that the singular behavior is due neither to the lack of existence of solutions for some Riemann problems, nor to a loss of strict hyperbolicity at infinity.

7 Entropies and Conditions Preventing Blowup

As noted in the introduction, the systems we constructed above are artificial in the sense that they are not motivated by physical considerations. Every physical $n \times n$ system of conservation laws $z_t + G(z)_x = 0$ admits a non-trivial convex entropy, i.e. a strictly convex function $\eta : \mathbb{R}^n \mapsto \mathbb{R}$ together with an entropy-flux $q : \mathbb{R}^n \mapsto \mathbb{R}$ such that $\eta(z)_t + q(z)_x = 0$ for every smooth solution z of $z_t + G(z)_x = 0$. This means that η and q satisfy the relation $Dq(u) = D\eta(u)DG(u)$. For systems with more than two equations this yields an over-determined system of equations for the entropy. In general the existence of a strictly convex entropy is equivalent to the existence of a strictly positive definite change of variables which symmetrizes the system, see [8].

In [11] Young shows that a *necessary* condition for a system to admit a convex entropy is that

$$\frac{\Lambda_i^{jk}}{\lambda_j - \lambda_k} + \frac{\Lambda_j^{ki}}{\lambda_k - \lambda_i} + \frac{\Lambda_k^{ij}}{\lambda_i - \lambda_j} = 0, \tag{7.1}$$

when $i, j, k = 1, \ldots, n$ are all distinct. Here λ_i is the i-th characteristic speed and Λ_i^{jk} are the *interaction coefficients* defined by

$$\Lambda_i^{jk} = l_i \cdot [r_j, r_k],$$

where l_i, r_i are the i-th left and right eigenvectors, respectively, and $[X, Y]$ denotes the Lie-bracket of two vector fields X, Y. As observed by Young, the constraint (7.1) is enough to rule out all known examples of blowup in situations where the data has large total variation and correspondingly small amplitude, [4, 6, 10].

In this section we will first show that the condition (7.1) is *not* enough to prevent the type of blowup presented above. We do this by considering the degenerate case $\varepsilon = 0$. However, we also show that there are no nontrivial, globally defined entropies in this case.

7.1 Condition on interaction coefficients

For a 3×3 system, relation (7.1) takes the form

$$\frac{\Lambda_1^{32}}{\lambda_3 - \lambda_2} + \frac{\Lambda_2^{31}}{\lambda_3 - \lambda_1} + \frac{\Lambda_3^{21}}{\lambda_2 - \lambda_1} = 0. \tag{7.2}$$

We compute the interaction coefficients in the case $\varepsilon = 0$. The eigenvalues are

$$\lambda_1 \equiv -c, \qquad \lambda_2 = g'(v) = p(v), \qquad \lambda_3 \equiv c.$$

The first and third eigenvectors are given by (3.8) while a straightforward calculation shows that the second eigenvector is given as $r_2 = (r_2^1, r_2^2, r_2^3)^{\mathrm{tr}}$ where

$$\begin{aligned} r_2^1 &= 2c\Big[\big(ab'(p-c) - ba'(p+c)\big)u + \big(a'(p+c) - b'(p-c)\big)w\Big], \\ r_2^2 &= (a-b)^2\big(p^2 - c^2\big), \\ r_2^3 &= 2c\Big[\big(a^2b'(p-c) - b^2a'(p+c)\big)u + \big(ba'(p+c) - ab'(p-c)\big)w\Big]. \end{aligned}$$

We first observe that $[r_1, r_3] = 0$ so that $\Lambda_2^{31} = 0$. Alternatively we can observe that any function of v alone is a 2-Riemann coordinate for our system. This illustrates the general fact that if there is a Riemann coordinate for the ith family, then $\Lambda_i^{j,k} = 0$ for all j, k, see [9]. For $[r_1, r_2]$ and $[r_3, r_2]$ we have

$$\begin{aligned}
[r_1,r_2]^1 &= \frac{\partial r_2^1}{\partial u} + a\frac{\partial r_2^1}{\partial w} &&= 2c(a-b)a'(p+c), \\
[r_1,r_2]^2 &= \frac{\partial r_2^2}{\partial u} + a\frac{\partial r_2^2}{\partial w} &&= 0, \\
[r_1,r_2]^3 &= \frac{\partial r_2^3}{\partial u} + a\frac{\partial r_2^3}{\partial w} - a'r_2^2 &&= 2c(a-b)ba'(p+c) - a'(a-b)^2(p^2-c^2), \\
[r_3,r_2]^1 &= \frac{\partial r_2^1}{\partial u} + b\frac{\partial r_2^1}{\partial w} &&= 2c(a-b)b'(p-c), \\
[r_3,r_2]^2 &= \frac{\partial r_2^2}{\partial u} + b\frac{\partial r_2^2}{\partial w} &&= 0, \\
[r_3,r_2]^3 &= \frac{\partial r_2^3}{\partial u} + b\frac{\partial r_2^3}{\partial w} - b'r_2^2 &&= 2c(a-b)ab'(p-c) - b'(a-b)^2(p^2-c^2).
\end{aligned}$$

To compute the interaction coefficients we need the left eigenvectors l_1 and l_3. These are assumed to be normalized so that $l_1 \cdot r_1 = l_3 \cdot r_3 = 1$. It follows that l_1 and l_3 have the form

$$l_1 = \frac{1}{a-b}\begin{pmatrix} -b(v) \\ l_1^2 \\ 1 \end{pmatrix}, \qquad l_3 = \frac{1}{a-b}\begin{pmatrix} a(v) \\ l_3^2 \\ -1 \end{pmatrix}.$$

Note that the second components of $[r_1, r_2]$ and $[r_3, r_2]$ are both zero so that we do not need the expressions for l_1^2 and l_3^2 to compute the interaction coefficients. We have

$$\Lambda_3^{21} = -l_3 \cdot [r_1, r_2] = \frac{1}{a-b}\left(-a[r_1,r_2]^1 + [r_1,r_2]^3\right) = a'(b-a)(p+c)^2,$$

and

$$\Lambda_1^{32} = l_1 \cdot [r_3, r_2] = \frac{1}{a-b}\left(-b[r_3,r_2]^1 + [r_3,r_2]^3\right) = b'(b-a)(p-c)^2.$$

Using these expressions we get that (7.2) in the unperturbed case reduces to

$$b'(v) = -a'(v)\frac{c+p(v)}{c-p(v)}. \tag{7.3}$$

We now recall from Section 6 that the only constraints we have on $a(v)$ and $b(v)$ in order to have blowup, are that their ranges are bounded and separate, that $a(0) - b(0)$ is sufficiently small, and that $\bar{a} - a_0 = a(V) - a(0) > 0$, $b_0 - \bar{b} = b(0) - b(-V) > 0$. It is easy to see that these conditions do not contradict (7.3). We conclude that the relation (7.1) is not enough to rule out blowup in L^∞.

Remark 7.1 As observed in [11], condition (7.2) implies that not all the strengths of the incoming waves can be magnified after an interaction. Although this is true only up to second order in wave strengths, we note that it holds also in our example where the waves are large: it follows from (5.10) and (5.13), and the assumptions on a and b, that a transmitted wave is weaker than the incoming wave that produced it. Nevertheless the strength of the successive transmitted waves $l_1, l_2, \ldots, l_m, \ldots$ (see Figures 1 and 4) increases as $m \to \infty$.

7.2 Nonexistence of entropies

In this last part we prove that there is no global, strictly convex entropy for the case $\varepsilon = 0$. This suggests that the existence of an entropy is enough to exclude the type of blowup described by Theorem 2.1.

Before starting we observe that our systems possess many trivial entropies that are convex but not *strictly* convex. This is clear since the second equation in (1.1) is a decoupled scalar equation, so that any convex scalar function of v alone is a (nonstrictly) convex entropy for (1.1).

Assume, for contradiction, that there was a globally defined and strictly convex entropy $\eta(U)$ for (1.1) with flux

$$F(U) = \begin{pmatrix} \frac{1}{b-a}\big[2w - (a+b)u\big] \\ g(v) \\ \frac{1}{b-a}\big[(a+b)w - 2abu\big] \end{pmatrix} = \begin{pmatrix} f(u,v,w) \\ g(v) \\ h(u,v,w) \end{pmatrix}, \tag{7.4}$$

where we have chosen $c = 1$ for simplicity. Since $D\eta DF$ is the gradient of the corresponding entropy flux we must have

$$\mathrm{curl}\,(D\eta DF) \equiv 0, \tag{7.5}$$

and this gives three equations for η. Now,

$$D\eta DF = \begin{pmatrix} -\left(\frac{a+b}{b-a}\right)\eta_u - \left(\frac{2ab}{b-a}\right)\eta_w \\ \\ f_v\eta_u + p\eta_v + h_v\eta_w \\ \\ \left(\frac{2}{b-a}\right)\eta_u + \left(\frac{a+b}{b-a}\right)\eta_w \end{pmatrix}^{tr}, \tag{7.6}$$

from which it follows that the second component of $\mathrm{curl}\,(D\eta DF)$ is zero if and only if

$$\eta_{uu} + (a+b)\eta_{uw} + ab\eta_{ww} = 0, \tag{7.7}$$

hence we may write η in the general form

$$\eta(u,v,w) = A\big(a(v)u - w, v\big) + B\big(b(v)u - w, v\big). \tag{7.8}$$

To obtain a contradiction we proceed to show that this form of η together with the two other equations from (7.5), the assumptions on a and b, and the necessary condition (7.3) yield a contradiction. The two remaining equations from (7.5) are

$$\left[-\left(\frac{a+b}{b-a}\right)\eta_u - \left(\frac{2ab}{b-a}\right)\eta_w\right]_v = \left[f_v\eta_u + p\eta_v + h_v\eta_w\right]_u, \tag{7.9}$$

and

$$\left[\left(\frac{2}{b-a}\right)\eta_u + \left(\frac{a+b}{b-a}\right)\eta_w\right]_v = \left[f_v\eta_u + p\eta_v + h_v\eta_w\right]_w. \tag{7.10}$$

Let $\xi(u,v,w) = A(a(v)u - w, v)$ and $\chi(u,v,w) = B(b(v)u - w, v)$, so that $\eta = \xi + \chi$. Substituting into the left hand sides of (7.9) and (7.10) we observe that these can be written as $(\xi - \chi)_{uv}$ and $(\xi - \chi)_{wv}$, respectively. Integration with respect to u in (7.9) and with respect to w in (7.10) thus yields

$$\xi_v - \chi_v = f_v\eta_u + p\eta_v + h_v\eta_w + \phi(v,w),$$

and

$$\xi_v - \chi_v = f_v\eta_u + p\eta_v + h_v\eta_w + \psi(u,v),$$

for some functions ϕ, ψ. It is clear from this that ϕ, ψ are functions of v alone, $\theta(v)$ say, and that both equations reduce to the single equation

$$(1-p)\xi_v - f_v\xi_u - h_v\xi_w = (1+p)\chi_v + f_v\chi_u + h_v\chi_w + \theta(v).$$

Substituting back the expressions for ξ and χ we get

$$\begin{aligned}&\Big[(1-p)a'u - af_v + h_v\Big]\partial_1 A(au - w, v) + (1-p)\partial_2 A(au - w, v)\\ &= \Big[(1+p)b'u + bf_v - h_v\Big]\partial_1 B(bu - w, v) + (1+p)\partial_2 B(bu - w, v) + \theta(v),\end{aligned}$$

where ∂_i denotes differentiation with respect to the i-th variable. Using the expressions for f and h and introducing the variables $y = bu - w$, $z = au - w$, yields

$$\begin{aligned}-a'\big[(1-p)z + (1+p)y\big]&\partial_1 A(z,v) + (1-p)(b-a)\partial_2 A(z,v)\\ = b'\big[(1-p)z + (1+p)y\big]&\partial_1 B(y,v) + (1+p)(b-a)\partial_2 B(y,v)\\ &+ (b-a)\theta(v).\end{aligned} \tag{7.11}$$

Taking the y-derivative on each side and then fixing v and y while z varies, we conclude that $\partial_1 A(z,v)$ is a linear function of z for fixed v. Similarly, we see that $\partial_1 B(y,v)$ is a linear function of y for fixed v. Let $\alpha(v) = \partial_{1,1}A(z,v)$

and $\beta(v) = \partial_{1,1}B(y,v)$. Taking double derivatives of (7.11) then gives the following three equations for α and $\beta(v)$,

$$\begin{aligned}
a'\alpha &= \frac{b-a}{2}\alpha' \\
-b'\beta &= \frac{b-a}{2}\beta' \\
a'(1+p)\alpha + b'(1-p)\beta &= 0
\end{aligned} \tag{7.12}$$

Using the two first equations in the last equation, we see that

$$(1+p)\alpha' = (1-p)\beta'. \tag{7.13}$$

Now, from the assumptions on a we know that the set $\mathcal{I} := \{v \mid a'(v) \neq 0\}$ is nonempty and open. Note that, by the necessary relation (7.3), we have $\mathcal{I} = \{v \mid b'(v) \neq 0\}$. From the last equation in (7.12) together with the necessary relation (7.3), we conclude that $\alpha = \beta$ on $\mathcal{I}$. Hence, $\alpha' = \beta'$ on $\mathcal{I}$ and, according to (7.13), this is possible only if $\alpha' = \beta' \equiv 0$ on $\mathcal{I}$. Substituting this back into the first two equations in (7.12) we finally conclude that $\alpha \equiv 0$ and $\beta \equiv 0$ on $\mathcal{I}$. But this implies that $\eta_{ww} = \alpha + \beta \equiv 0$ on $\mathcal{I}$, contradicting the strict convexity of η. This shows that our systems do not admit globally defined and strictly convex entropies.

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