AN ERROR ESTIMATE FOR VISCOUS APPROXIMATE SOLUTIONS OF DEGENERATE PARABOLIC EQUATIONS

STEINAR EVJE AND KENNETH H. KARLSEN

Abstract. Relying on recent advances in the theory of entropy solutions for nonlinear (strongly) degenerate parabolic equations, we present a *direct* proof of an L^1 error estimate for viscous approximate solutions of the initial value problem for $\partial_t w + \operatorname{div}\big(V(x)f(w)\big) = \Delta A(w)$, where $V = V(x)$ is a vector field, $f = f(u)$ is a scalar function, and $A'(\cdot) \geq 0$. The viscous approximate solutions are weak solutions of the initial value problem for the uniformly parabolic equation $\partial_t w^\varepsilon + \operatorname{div}\big(V(x)f(w^\varepsilon)\big) = \Delta\big(A(w^\varepsilon) + \varepsilon w^\varepsilon\big)$, $\varepsilon > 0$. The error estimate is of order $\sqrt{\varepsilon}$.

1. Introduction

In this paper, we are interested in certain "viscous" approximations of entropy solutions of the initial value problem

$$
\begin{cases} \partial_t w + \operatorname{div}\big(V(x)f(w)\big) = \Delta A(w), & (x,t) \in Q_T, \\ w(x,0) = w_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.1}
$$

where $Q_T = \mathbb{R}^d \times (0,T)$ with $T > 0$ fixed, $u : Q_T \to \mathbb{R}$ is the sough function, $V : \mathbb{R}^d \to \mathbb{R}$ is a (not necessarily divergence free) velocity field, $f : \mathbb{R} \to \mathbb{R}$ is the convective flux function, and $A : \mathbb{R} \to \mathbb{R}$ is the "diffusion" function. Regarding the diffusion function, the basic assumption is that $A(\cdot)$ is nonincreasing. This condition implies that (1.1) is a (strongly) *degenerate parabolic* problem. For example, the hyperbolic equation $\partial_t w + \operatorname{div}\big(V(x)f(w)\big) = 0$ is a special case of (1.1). Problems such as (1.1) occur in several important applications. We mention here only two examples: flow in porous media (see, e.g., [7]) and sedimentation-consolidation processes [3].

Since $A(\cdot)$ is merely nondecreasing, solutions are not necessarily smooth and weak solutions must be sought. Moreover, as is well-known in the theory of hyperbolic conservation laws, weak solutions are not uniquely determined by their initial data. To have a well-posed problem, we need to consider entropy solutions, i.e., weak solutions that satisfy a Kružkov-Vol'pert type entropy condition. A precise statement is given in Section 2 (see Definition 1). For pure hyperbolic equations, this entropy condition was introduced by Kružkov [14] and Vol'pert [20]. For degenerate parabolic equations, it was introduced by Vol'pert and Hudjaev [21].

Following Carrillo [5], it was proved by Karlsen and Risebro [12] that the entropy solution of (1.1) (as well as a more general equation) is unique. Moreover, in the $L^\infty(0,T;BV(\mathbb{R}^d))$ class of entropy solutions, they proved an L^1 contraction principle. Existence of an $L^\infty(0,T;BV(\mathbb{R}^d))$ entropy solution of (1.1) follows from the results in Vol'pert and Hudjaev [21] or Karlsen and Risebro [11] (the latter dealt with convergence of finite difference methods). The proof in [12] of uniqueness and stability is based on the "doubling of variables" strategy introduced in Carrillo [5], which in turn is a generalization of the pioneering work by Kružkov [14] on hyperbolic equations. Related papers dealing with the "doubling of variables" device for degenerate parabolic equations include, among others, Carrillo [4], Otto [18], Rouvre and Gagneux [19], Cockburn and Gripenberg [6] Bürger, Evje, and Karlsen [1, 2], Ohlberger [17], Mascia, Porretta, and Terracina [16], Eymard, Gallouet, Herbin, and Michel [10], and Karlsen and Ohlberger [13].

Date: February 15, 2001.

Key words and phrases. nonlinear degenerate parabolic equations, velocity field, entropy solution, viscosity method, error estimate.

In this paper, we are interested in certain approximate solutions of (1.1) coming from solving the uniformly parabolic problem

$$\begin{cases} \partial_t w^\varepsilon + \operatorname{div}\big(V(x) f(w^\varepsilon)\big) = \Delta A^\varepsilon(w^\varepsilon), & (x,t) \in Q_T, \\ w^\varepsilon(x,0) = w_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.2}$$

where $A^\varepsilon(w^\varepsilon) = A(w^\varepsilon) + \varepsilon w^\varepsilon$, $\varepsilon > 0$. We refer to w^ε as a *viscous* approximate solution of (1.1). Convergence of w^ε to the unique entropy solution w of (1.1) as $\varepsilon \downarrow 0$ follows from the results in Vol'pert and Hudjaev [21]. Our main interest here is to give an explicit rate of convergence for w^ε as $\varepsilon \downarrow 0$, i.e., an L^1 error estimate for viscous approximate solutions.

There several ways to prove such an error estimate. One way is to view it as a consequence of a continuous dependence estimate. Combining the ideas in [12] with those in Cockburn and Gripenberg [6], who used a variant of Kružkov's "doubling of variables" device for (1.1) with $V \equiv 1$, Evje, Karlsen, and Risebro [8] established an explicit "continuous dependence on the nonlinearities" estimate for entropy solutions of (1.1). A direct consequence of this estimate is the error bound $\|w^\varepsilon - w\|_{L^1|(Q_T)} = \mathcal{O}\big(\sqrt{\varepsilon}\big)$, at least when w^ε, w belong to $L^\infty(0,T;BV(\mathbb{R}^d))$ and V is sufficiently regular. Ufortunately, the techniques employed in [8] require that one works with (smooth) viscous approximations of (1.1). The proof in [8] (as well as the one in [6]) did not exploit the entropy solution "machinery" developed by Carrillo [5].

The main purpose of this work is to show that one can indeed use the "doubling of variables" device to compare *directly* the entropy solution w of (1.1) against the viscous approximation w^ε of (1.2). Hence there is no need to work with approximate solutions of (1.1). Although our proof is of independent interest, it may also shed some light on how to obtain error estimates for numerical methods. Most numerical methods have (1.2) as a "model" problem and, in this context, the size of ε designates the amount of "diffusion" present in the numerical method. A step in the direction of obtaining error estimates for numerical methods has been taken by Ohlberger [17] with his a posteriori error estimate for a finite volume method. We will in future work use the ideas devised herein to derive a priori error estimates for finite difference methods.

The rest of this paper is organized as follows: In Section 2 we state the definition of an entropy solution and the main result (Theorem 1). Section 3 is devoted to the derivation of certain entropy inequalities for the exact entropy solution and its viscous approximation. Equipped with these entropy inequalities, we prove the error estimate (Theorem 1) in Section 4.

Added in process. After the main result of this paper was obtained, we became aware of a paper by Eymard, Gallouet, and Herbin [9] which also proves an error estimate for viscous approximate solutions. They, however, deal with a certain boundary value problem with a divergence free velocity field and obtain an error estimate of order $\varepsilon^{\frac{1}{5}}$. As is the case herein, the proof in [9] does not rely on a continuous dependence estimate.

Acknowledgment. This work was done while the first author (Evje) was visiting the Industrial Mathematics Institute at the University of South Carolina. Part of this work was completed while the second author (Karlsen) was visiting the Department of Mathematics and the Institute for Pure and Applied Mathematics (IPAM) at the University of California, Los Angeles (UCLA).

2. Statement of Result

Following [11, 12], we start by stating sufficient conditions on $V = (V_1, \dots, V_d), f, A, u_0$ to ensure existence of a unique $L^\infty(0,T;BV(\mathbb{R}^d))$ entropy solution of (1.1):

$$\begin{cases} V \in \Big(W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)\Big)^d \bigcap \Big(L^\infty(\mathbb{R}^d)\Big)^d \bigcap \Big(\mathrm{Lip}(\mathbb{R})\Big)^d; \\ \partial_{x_i} V_i \in BV(\mathbb{R}^d),\ i = 1, \dots, d; \\ f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R});\ f(0) = 0; \\ A \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}) \text{ and } A(\cdot) \text{ is nondecreasing with } A(0) = 0; \\ u_0 \in L^\infty(\mathbb{R}^d) \bigcap L^1(\mathbb{R}^d) \bigcap BV(\mathbb{R}^d). \end{cases} \tag{2.1}$$

Equipped with (2.1), we can state the following definition of an entropy solution:

Definition 1 (Entropy Solution). *A function $w(x,t)$ is called an entropy solution of* (1.1) *if:*

(i) $w \in L^1(Q_T) \cap L^\infty(Q_T) \cap C(0,T;L^1(\mathbb{R}^d))$.

(ii) $A(w) \in L^2(0,T;H^1(\mathbb{R}^d))$.

(iii) $w(x,t)$ *satisfies the entropy inequality*

$$\iint_{Q_T} \Big(|w-k|\partial_t\phi + \operatorname{sgn}(w-k)\big[V(x)(f(w)-f(k)) - \nabla A(w)\big]\cdot\nabla\phi \tag{2.2}$$
$$- \operatorname{sgn}(w-k)\operatorname{div}V(x)f(k)\phi\Big)\,dt\,dx \geq 0, \qquad \forall k \in \mathbb{R},$$

for all nonnegative $\phi \in C_0^\infty(Q_T)$.

(iv) $\|w(\cdot,t) - w_0\|_{L^1(\mathbb{R}^d)} \to 0$ *as* $t \downarrow 0$ *(essentially)*.

Note that if we take $k > \operatorname{ess\,sup} w(x,t)$ and $k < \operatorname{ess\,inf} w(x,t)$ in (2.2), then an approximation argument will reveal that

$$\iint_{Q_T} \Big(w\phi_t + \big[V(x)f(w) - \nabla A(w)\big]\cdot\nabla\phi\Big)\,dt\,dx = 0 \tag{2.3}$$

holds for all $\phi \in H^1(Q_T)$. Let $\langle\cdot,\cdot\rangle$ denote the usual pairing between $H^{-1}(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d)$. From (2.3), we conclude that

$$\partial_t w \in L^2(0,T;H^{-1}(\mathbb{R}^d)),$$

so that

$$-\int_0^T \langle\partial_t w,\phi\rangle\,dt + \iint_{Q_T}\Big(\big[V(x)f(w) - \nabla A(w)\big]\cdot\nabla\phi\Big)\,dt\,dx = 0, \qquad \forall\phi \in H^1(Q_T). \tag{2.4}$$

In other words, an entropy solution $w(x,t)$ of (1.1) is also a *weak* solution of the same problem.

In this paper, we are interested in comparing the entropy solution w of (1.1) against the weak solution w^ε of the viscous problem (1.2). From the results in Karlsen and Risebro [11] or Vol'pert and Hudjaev [21], there exists a weak solution $w^\varepsilon \in L^\infty(0,T;BV(\mathbb{R}^d))$ of (1.2). Since $A^\varepsilon(\cdot)$ in increasing, the uniqueness result in Karlsen and Risebro [12] (see also Remark 1 herein) tells us that this weak solution is in fact the unique one. Moreover, from the energy estimate we conclude that $w^\varepsilon \in L^2(0,T;H^1(\mathbb{R}^d))$. Of course, if V, f, A, u_0 are smooth enough, one can prove that the weak solution w^ε of (1.2) is actually a classical ($C^{2,1}$) solution, see, e.g., Vol'pert and Hudjaev [21]. Here it will be sufficient to know that w^ε belongs to $L^2(0,T;H^1(\mathbb{R}^d))$ (not $C^{2,1}$).

We are now ready state our main theorem:

Theorem 1 (Error Estimate). *Suppose that the conditions in* (2.1) *hold. Let* $w \in L^\infty(0,T;BV(\mathbb{R}^d))$ *be the unique entropy solution of* (1.1) *and let* $w^\varepsilon \in L^2(0,T;H^1(\mathbb{R}^d)) \bigcap L^\infty(0,T;BV(\mathbb{R}^d))$ *be the unique weak solution of* (1.2). *Then there exists a constant C, independent of ε, such that*

$$\|w^\varepsilon - w\|_{L^1(Q_T)} \leq C\sqrt{\varepsilon}. \tag{2.5}$$

3. Entropy Inequalities

In Section 4, we will follow the uniqueness proof of Carrillo [5] to obtain an estimate of the difference between w^ε and w. To this end, it will be necessary to derive two entropy inequalities for the exact solution w and two approximate entropy inequalities for the viscous solution w^ε. The purpose of this section is to derive these inequalities (see Lemmas 2 and 3 below).

Note that differently from the pure hyperbolic case [14], we need to operate with one additional entropy inequality (actually an equality for the exact solution w) taking into account the parabolic (dissipation) mechanism in the equation. Hence, we shall introduce a set H corresponding to the regions where $A(\cdot)$ is "flat" and (1.1) behaves hyperbolic. More precisely, let $A^{-1}: \mathbb{R} \to \mathbb{R}$ denote the unique left-continuous function which satisfies $A^{-1}(A(u)) = u$ for all $u \in \mathbb{R}$. Then we define

$$H = \Big\{ r \in \mathbb{R} \,:\, A^{-1}(\cdot) \text{ is discontinuous at } r\Big\}.$$

Since $A(\cdot)$ is a monotone function, H is at most countable. The dissipation mechanism in the equation is effective only in the (x,t) region corresponding to the complement of H.

To prove Lemmas 2 and 3 below, we shall need the following "weak" chain rule:

Lemma 1. *Let $u : Q_T \to \mathbb{R}$ be a measurable function satisfying the following four conditions:*
1. $u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C(0,T;L^1(\mathbb{R}^d))$.
2. $u(0,\cdot) = u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.
3. $\partial_t u \in L^2\big(0,T;H^{-1}(\mathbb{R}^d)\big)$.
4. $A(u) \in L^2\big(0,T;H^1(\mathbb{R}^d)\big)$.

For every nonnegative and compactly supported $\phi \in C^\infty(Q_T)$ with $\phi|_{t=0} = \phi|_{t=T} = 0$, we have

$$-\int_0^T \Big\langle \partial_t u, \psi\big(A(u)\big)\phi \Big\rangle \, dt = \iint_{Q_T} \Big(\int_k^u \psi(A(\xi))\, d\xi \Big) \phi_t \, dt\, dx, \qquad k \in \mathbb{R},$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is a nondecreasing and Lipschitz continuous function.

The proof of Lemma 1 is very similar to the proof of the "weak chain" rule in Carrillo [5] and it is therefore omitted (see instead [12]).

The following lemma, which deals with entropy inequalities for the exact entropy solution w, is a direct consequence of the very definition of an entropy solution.

Lemma 2. *The unique entropy solution w of* (1.1) *satisfies:*

(i) *For all $k \in \mathbb{R}$ and all nonnegative $\phi \in C_0^\infty(Q_T)$, we have*

$$E^{\mathrm{hyp}}(w,k,\phi) \ge 0, \tag{3.1}$$

where

$$\begin{aligned} E^{\mathrm{hyp}}(w,k,\phi) := \iint_{Q_T} \Big(&|w-k|\partial_t\phi + \mathrm{sgn}(w-k)\big[V(x)(f(w)-f(k)) - \nabla A(w)\big]\cdot\nabla\phi \\ &- \mathrm{sgn}(w-k)\mathrm{div} V(x) f(k)\phi \Big)\, dt\, dx. \end{aligned} \tag{3.2}$$

We refer to (3.1) *as a hyperbolic entropy inequality.*

(ii) *For all k such that $A(k) \notin H$ and all nonnegative $\phi \in C_0^\infty(Q_T)$, we have*

$$E^{\mathrm{par}}(w,k,\phi) = 0, \tag{3.3}$$

where

$$\begin{aligned} E^{\mathrm{par}}(w,k,\phi) := \iint_{Q_T} \Big(&|w-k|\partial_t\phi + \mathrm{sgn}(w-k)\big[V(x)(f(w)-f(k)) - \nabla A(w)\big]\cdot\nabla\phi \\ &- \mathrm{sgn}(w-k)\mathrm{div} V(x) f(k)\phi \Big)\, dt\, dx \\ &- \lim_{\eta\downarrow 0} \iint_{Q_T} \big|\nabla A(w)\big|^2 \mathrm{sgn}_\eta'\big(A(w)-A(k)\big)\phi\, dt\, dx, \end{aligned} \tag{3.4}$$

In (3.4) *(and elsewhere in this paper), sgn_η is the approximate sign function is defined by*

$$\mathrm{sgn}_\eta(\tau) := \begin{cases} \mathrm{sgn}(\tau) & \text{if } |\tau| > \eta, \\ \tau/\eta & \text{if } |\tau| \le \eta, \end{cases} \qquad \eta > 0. \tag{3.5}$$

We refer to (3.3) *as a parabolic entropy inequality.*

Proof. The first inequality (3.1) is nothing but the entropy condition for the entropy solution w, so there is nothing to prove. Let us turn to the proof of the second inequlity (3.3), which borrows a lot from Carrillo [5] (see also [12]). In what follows, we always let k, ϕ be as in the lemma and the approximate sign function $\mathrm{sgn}_\eta(\cdot)$ is always the one defined in (3.5).

Since w satisfies (2.4) and $\big[\mathrm{sgn}_\eta(A(w)-A(k))\phi\big] \in L^2(0,T;H^1(\mathbb{R}^d))$, we have

$$
\begin{aligned}
&-\int_0^T \Big\langle \partial_t w, \mathrm{sgn}_\eta(A(w)-A(k))\phi \Big\rangle \, dt \\
&\qquad + \iint_{Q_T} \Big(\big[V(x)(f(w)-f(k)) - \nabla A(w)\big] \cdot \nabla \big[\mathrm{sgn}_\eta(A(w)-A(k))\phi\big] \\
&\qquad\qquad - \mathrm{div} V(x) f(k) \big[\mathrm{sgn}_\eta(A(w)-A(k))\phi\big] \Big) \, dt\, dx = 0.
\end{aligned}
$$

Introduce the function $\psi_\eta(z) = \mathrm{sgn}_\eta\big(z - A(k)\big)$ and note that Lemma 1 can be applied, so that

$$
-\int_0^T \Big\langle \partial_t w, \mathrm{sgn}_\eta(A(w)-A(k))\phi \Big\rangle \, dt = \iint_{Q_T} \Big(\int_k^w \mathrm{sgn}_\eta(A(\xi)-A(k))\, d\xi \Big) \partial_t \phi \, dt\, dx
$$

and hence

$$
\begin{aligned}
&\iint_{Q_T} \Big(\int_k^w \mathrm{sgn}_\eta(A(\xi)-A(k))\, d\xi \Big) \partial_t \phi \, dt\, dx \\
&\qquad + \iint_{Q_T} \Big(\big[V(x)(f(w)-f(k)) - \nabla A(w)\big] \cdot \nabla \big[\mathrm{sgn}_\eta(A(w)-A(k))\phi\big] \\
&\qquad\qquad - \mathrm{sgn}_\eta(A(w)-A(k)) \mathrm{div} V(x) f(k) \phi \Big) \, dt\, dx = 0.
\end{aligned}
\tag{3.6}
$$

Note that since $A(r) > A(k)$ if and only if $r > k$ (here we make use of the assumption that $k \in$ "parabolic region", i.e., $A(k) \notin H$), $\mathrm{sgn}_\eta(A(r)-A(k)) \to 1$ as $\eta \downarrow 0$ for any $r > k$. Similarly for $r < k$. Consequently, as $\eta \downarrow 0$, $\int_k^w \mathrm{sgn}_\eta(A(\xi)-A(k))\, d\xi \to |w-k|$ a.e. in Q_T. Moreover, we have $\big|\int_k^w \mathrm{sgn}_\eta(A(\xi)-A(k))\, d\xi\big| \le |w-c| \in L^1_{\mathrm{loc}}(Q_T)$, so by Lebegue's dominated convergence theorem

$$
\lim_{\eta\downarrow 0} \iint_{Q_T} \Big(\int_k^w \mathrm{sgn}_\eta(A(\xi)-A(k))\, d\xi \Big) \partial_t \phi \, dt\, dx = \iint_{Q_T} |w-k| \partial_t \phi \, dt\, dx.
$$

Next, we have

$$
\begin{aligned}
&\lim_{\eta\downarrow 0} \iint_{Q_T} \big[V(x)(f(w)-f(k)) - \nabla A(w)\big] \cdot \nabla \big[\mathrm{sgn}_\eta(A(w)-A(k))\phi\big] \, dt\, dx \\
&\quad = \lim_{\eta\downarrow 0} \iint_{Q_T} \big[V(x)(f(w)-f(k)) - \nabla A(w)\big] \cdot \nabla \mathrm{sgn}_\eta(A(w)-A(k))\phi \, dt\, dx \\
&\qquad + \lim_{\eta\downarrow 0} \iint_{Q_T} \big[V(x)(f(w)-f(k)) - \nabla A(w)\big] \cdot \mathrm{sgn}_\eta(A(w)-A(k)) \nabla \phi \, dt\, dx \\
&\quad = \underbrace{\lim_{\eta\downarrow 0} \iint_{Q_T} V(x)(f(w)-f(k)) \mathrm{sgn}_\eta'(A(w)-A(k)) \nabla A(w) \phi \, dt\, dx}_{I_1} \\
&\qquad - \lim_{\eta\downarrow 0} \iint_{Q_T} \big|\nabla A(w)\big|^2 \mathrm{sgn}_\eta'(A(w)-A(k)) \phi \, dt\, dx \\
&\qquad\quad + \underbrace{\lim_{\eta\downarrow 0} \iint_{Q_T} \mathrm{sgn}_\eta(A(w)-A(k)) \big[V(x)(f(w)-f(k)) - \nabla A(w)\big] \cdot \nabla \phi \, dt\, dx}_{I_2} .
\end{aligned}
$$

Note that I_1 can be rewritten as $I_1 = \lim_{\eta\downarrow 0} \iint_{Q_T} V(x)\mathrm{div}\mathcal{Q}_\eta(A(w))\phi\, dt\, dx$, where

$$\mathcal{Q}_\eta(z) := \int_0^z \mathrm{sgn}'_\eta(r - A(k))\Big(f(A^{-1}(r)) - f(A^{-1}(A(k)))\Big)\, dr$$
$$= \frac{1}{\eta}\int_{\min(z,A(k)-\eta)}^{\min(z,A(k)+\eta)} \Big(f(A^{-1}(r)) - f(A^{-1}(A(k)))\Big)\, dr.$$

Since $A(k) \notin H$, it follows that $f \circ A^{-1}$ is Lipschitz continuous and thus $\mathcal{Q}_\eta(z)$ tends to zero as $\eta \downarrow 0$ for all $z \in \mathrm{Range(A)}$. By invoking Lebesgue dominated convergence theorem, we conclude after an integration by parts that

$$I_1 = -\lim_{\eta\downarrow 0}\iint_{Q_T} \Big(\mathcal{Q}_\eta(A(w))V(x)\cdot\nabla\phi + \mathcal{Q}_\eta(A(w))\mathrm{div}V(x)\phi\Big)\, dt\, dx = 0.$$

Using that $\mathrm{sgn}(w-k) = \mathrm{sgn}\big(A(w) - A(k)\big)$ a.e. in Q_T (since $A(k) \notin H$),

$$I_2 = \lim_{\eta\downarrow 0}\iint_{Q_T} \mathrm{sgn}_\eta(A(w) - A(k))\big[V(x)(f(w) - f(k)) - \nabla A(w)\big]\cdot\nabla\phi\, dt\, dx$$
$$= \iint_{Q_T} \mathrm{sgn}(w-k)\big[V(x)(f(w) - f(k)) - \nabla A(w)\big]\cdot\nabla\phi\, dt\, dx.$$

For the same reason, we have that

$$\lim_{\eta\downarrow 0}\iint_{Q_T} \mathrm{sgn}_\eta(A(w) - A(k))\mathrm{div}V(x)f(k)\phi\, dt\, dx = \iint_{Q_T} \mathrm{sgn}(w-k)\mathrm{div}V(x)f(k)\phi\, dt\, dx.$$

Consequently, letting $\eta \downarrow 0$ in (3.6), we obtain (3.3). □

Remark 1. *Observe that if $A(\cdot)$ is increasing, then a weak solution is automatically an entropy solution and hence it is unique.*

The next lemma, which deals with approximate entropy inequalities for the viscous solution w^ε, is a direct consequence of the definition of a weak solution of (1.2).

Lemma 3. *Let E^{hyp} and E^{par} be defined in* (3.2) *and* (3.4) *respectively. Furthermore, define*

$$R_{\mathrm{visc}} := \varepsilon\iint_{Q_T} \big|\nabla w^\varepsilon \cdot \nabla\phi\big|\, dt\, dx. \tag{3.7}$$

The unique weak solution $w^\varepsilon \in L^2(0,T;H^1(\mathbb{R}^d))\bigcap L^\infty(0,T;BV(\mathbb{R}^d))$ of (1.2) *satisfies:*

(i) *For all $k \in \mathbb{R}$ and all nonnegative $\phi \in C_0^\infty(Q_T)$, we have*

$$E^{\mathrm{hyp}}(w^\varepsilon, k, \phi) \geq -R_{\mathrm{visc}}. \tag{3.8}$$

We refer to (3.8) *as an approximate hyperbolic entropy inequality.*

(ii) *For all $k \in \mathbb{R}$ such that $A(k) \notin E$ and all nonnegative $\phi \in C_0^\infty(Q_T)$, we have*

$$E^{\mathrm{par}}(w^\varepsilon, k, \phi) \geq -R_{\mathrm{visc}}. \tag{3.9}$$

We refer to (3.9) *as an approximate parabolic entropy inequality.*

Proof. In what follows, we always let k, ϕ be as indicated by the lemma. The proof of the inequality (3.8) follows the proof of 3.1 rather closely. Since w^ε is a weak solution and $\big[\mathrm{sgn}_\eta(w^\varepsilon - k)\phi\big]$ belongs to $L^2(0,T;H^1(\mathbb{R}^d))$, we have

$$-\int_0^T \Big\langle \partial_t w^\varepsilon, \mathrm{sgn}_\eta(w^\varepsilon - k)\phi\Big\rangle\, dt + \iint_{Q_T}\Big(\big[V(x)(f(w^\varepsilon) - f(k)) - \nabla A^\varepsilon(w^\varepsilon)\big]\cdot\nabla\big[\mathrm{sgn}_\eta(w^\varepsilon) - k)\phi\big]$$
$$- \mathrm{div}V(x)f(k)\big[\mathrm{sgn}_\eta(w^\varepsilon) - k)\phi\big]\Big)\, dt\, dx = 0.$$

By the chain rule, we obviously have

$$
\begin{aligned}
&-\int_0^T \Big\langle \partial_t w^\varepsilon, \mathrm{sgn}_\eta(w^\varepsilon - k)\phi \Big\rangle \, dt \\
&\qquad = \iint_{Q_T} \Big(\int_k^{w^\varepsilon} \mathrm{sgn}_\eta(\xi - k)\, d\xi \Big) \partial_t \phi \, dt\, dx \xrightarrow{\eta\downarrow 0} \iint_{Q_T} |w^\varepsilon - k| \partial_t \phi \, dt\, dx,
\end{aligned}
$$

so that

$$
\begin{aligned}
&\iint_{Q_T} |w^\varepsilon - k|\partial_t\phi\, dt\, dx + \lim_{\eta\downarrow 0} \iint_{Q_T} \Big(\big[V(x)(f(w^\varepsilon) - f(k)) - \nabla A^\varepsilon(w^\varepsilon) \big] \cdot \nabla \big[\mathrm{sgn}_\eta(w^\varepsilon) - k)\phi \big] \\
&\qquad\qquad - \mathrm{sgn}_\eta(w^\varepsilon - k) \mathrm{div} V(x) f(k) \phi \Big) \, dt\, dx = 0.
\end{aligned} \tag{3.10}
$$

First, we have

$$
\lim_{\eta\downarrow 0} \iint_{Q_T} \mathrm{sgn}_\eta(w^\varepsilon) - k) \mathrm{div} V(x) f(k) \phi \, dt\, dx = \iint_{Q_T} \mathrm{sgn}(w^\varepsilon - k) \mathrm{div} V(x) f(k)\phi\, dt\, dx.
$$

Next, we have

$$
\begin{aligned}
&\lim_{\eta\downarrow 0} \iint_{Q_T} \big[V(x)(f(w^\varepsilon) - f(k)) - \nabla A^\varepsilon(w^\varepsilon) \big] \cdot \nabla \big[\mathrm{sgn}_\eta(w^\varepsilon - k)\phi \big] \, dt\, dx \\
&\quad = \lim_{\eta\downarrow 0} \iint_{Q_T} \big[V(x)(f(w^\varepsilon) - f(k)) - \nabla A^\varepsilon(w^\varepsilon) \big] \cdot \nabla \mathrm{sgn}_\eta(w^\varepsilon - k)\phi\, dt\, dx \\
&\qquad + \lim_{\eta\downarrow 0} \iint_{Q_T} \big[V(x)(f(w^\varepsilon) - f(k)) - \nabla A^\varepsilon(w^\varepsilon) \big] \cdot \mathrm{sgn}_\eta(w^\varepsilon - k) \nabla\phi\, dt\, dx \\
&\quad = \underbrace{\lim_{\eta\downarrow 0} \iint_{Q_T} V(x)(f(w^\varepsilon) - f(k)) \mathrm{sgn}_\eta'(w^\varepsilon - k) \nabla A^\varepsilon(w^\varepsilon)\phi\, dt\, dx}_{I_1} \\
&\qquad - \lim_{\eta\downarrow 0} \iint_{Q_T} (A^\varepsilon)'(w^\varepsilon) \big|\nabla w^\varepsilon\big|^2 \mathrm{sgn}_\eta'(w^\varepsilon - k)\phi\, dt\, dx \\
&\qquad\quad + \iint_{Q_T} \mathrm{sgn}(w^\varepsilon - k) \big[V(x)(f(w^\varepsilon) - f(k)) - \nabla A^\varepsilon(w^\varepsilon) \big] \cdot \nabla\phi\, dt\, dx.
\end{aligned}
$$

Note that I_1 can be rewritten as $I_1 = \lim_{\eta\downarrow 0} \iint_{Q_T} V(x) \mathrm{div} \mathcal{Q}_\eta(w^\varepsilon)\phi\, dt\, dx$, where

$$
\mathcal{Q}_\eta(z) := \int_0^z \mathrm{sgn}_\eta'(r - k)\big(f(r) - f(k)\big)\, dr = \frac{1}{\eta} \int_{\min(z, k-\eta)}^{\min(z, k+\eta)} \big(f(r) - f(k)\big)\, dr \to 0 \text{ as } \eta \downarrow 0.
$$

By invoking Lebesgue dominated convergence theorem, we conclude that

$$
I_1 = -\lim_{\eta\downarrow 0} \iint_{Q_T} \Big(\mathcal{Q}_\eta(A(w^\varepsilon)) V(x) \cdot \nabla\phi + \mathcal{Q}_\eta(A(w^\varepsilon)) \mathrm{div} V(x)\phi \Big) \, dt\, dx = 0.
$$

Summing up, we have

$$
\begin{aligned}
&\iint_{Q_T} \Big(|w^\varepsilon - k|\phi_t + \mathrm{sgn}(w^\varepsilon - k) \big[V(x)(f(w^\varepsilon) - f(k)) - \nabla A^\varepsilon(w^\varepsilon) \big] \cdot \nabla\phi \\
&\qquad\qquad - \mathrm{sgn}(w^\varepsilon - k) \mathrm{div} V(x) f(k)\phi \Big) \, dt\, dx \\
&\quad = \lim_{\varepsilon\downarrow 0} \iint_{Q_T} (A^\varepsilon)'(w^\varepsilon) \big|\nabla w^\varepsilon\big|^2 \mathrm{sgn}_\eta(w^\varepsilon - k)\phi\, dt\, dx \ge 0,
\end{aligned} \tag{3.11}
$$

for any $0 \le \phi \in C_0^\infty(Q_T)$ and any $k \in \mathbb{R}$. From this we conclude easily that (3.8) holds

It remains to prove the parabolic entropy inequality (3.9). Let $0 \le \phi \in C_0^\infty(Q_T)$, and $k \in \mathbb{R}$ be such that $A(k) \notin H$. Starting off by choosing $\left[\operatorname{sgn}_\eta(A(w^\varepsilon) - A(k))\phi\right]$ as a test function in the weak formulation and then continuing exactly as in the proof of (3.3), we end up with

$$E^{\text{par}}(w^\varepsilon, k, \phi) = \lim_{\eta \downarrow 0} \iint_{Q_T} \varepsilon \nabla w^\varepsilon \cdot \nabla \left[\operatorname{sgn}_\eta\big(A(w^\varepsilon) - A(k)\big)\phi\right] dt\, dx.$$

The right-hand side of this equality can be expanded into

$$\lim_{\eta \downarrow 0} \iint_{Q_T} \Big(\varepsilon A'(w^\varepsilon)\big|\nabla w^\varepsilon\big|^2 \operatorname{sgn}_\eta'(A(w^\varepsilon) - A(k))\phi + \varepsilon \operatorname{sgn}_\eta(A(w^\varepsilon) - A(k))\nabla w^\varepsilon \cdot \nabla\phi\Big)\, dt\, dx$$

$$\ge \lim_{\eta \downarrow 0} \iint_{Q_T} \varepsilon \operatorname{sgn}_\eta(A(w^\varepsilon) - A(k))\nabla w^\varepsilon \cdot \nabla\phi\, dt\, dx \ge -\varepsilon \iint_{Q_T} \big|\nabla w^\varepsilon \cdot \nabla\phi\big|\, dt\, dx.$$

This concludes the proof of (3.9). □

4. Proof of Theorem 1

Following Carrillo [5] (see also [12]), we shall in this section use Lemmas 2 and 3 to prove Theorem 1. Let $w^\varepsilon = w^\varepsilon(x,t)$ solve (1.1) and $w = w(y,s)$ solve (1.2). Following Kružkov [14] and Kuznetsov [15], we now specify a nonnegative test function $\phi = \phi(t,x,s,y)$ defined on $Q_T \times Q_T$. To this end, let $\rho \in C_0^\infty(\mathbb{R})$ be a function satisfying

$$\operatorname{supp}(\rho) \subset \{\sigma \in \mathbb{R} : |\sigma| \le 1\}, \quad \rho(\sigma) \ge 0\, \forall \sigma \in \mathbb{R}, \quad \int_{\mathbb{R}} \rho(\sigma)\, d\sigma = 1.$$

For $x \in \mathbb{R}^d, t \in \mathbb{R}$ and $r, r_0 > 0$, let $\omega_r(x) = \frac{1}{r}\rho\left(\frac{x_1}{r}\right)\cdots\frac{1}{r}\rho\left(\frac{x_d}{r}\right)$ and $\rho_{r_0}(t) = \frac{1}{r_0}\rho\left(\frac{t}{r_0}\right)$. Pick any two points $\nu, \tau \in (0,T)$, $\nu < \tau$. For any $\alpha_0 > 0$, define

$$\psi_{\alpha_0}(t) = H_{\alpha_0}(t-\nu) - H_{\alpha_0}(t-\tau), \quad H_{\alpha_0}(t) = \int_{-\infty}^t \rho_{\alpha_0}(\xi)\, d\xi.$$

With $0 < r_0 < \min(\nu, T-\tau)$ and $\alpha_0 \in \big(0, \min(\nu - r_0, T - \tau - r_0)\big)$, we then set

$$\phi(x,t,y,s) := \psi_{\alpha_0}(t)\omega_r(x-y)\rho_{r_0}(t-s). \tag{4.1}$$

Note that $\operatorname{supp}(\phi(x,\cdot,y,s)) \subset (r_0, T-r_0)$ for all $x,y \in \mathbb{R}^d, s \in (0,T)$ and $\operatorname{supp}(\phi(x,t,y,\cdot)) \subset (0,T)$ for all $x,y \in \mathbb{R}^d, t \in (0,T)$. Consequently, $(x,t) \mapsto \phi(x,t,y,s)$ belongs to $C_0^\infty(Q_T)$ for each fixed $(y,s) \in Q_T$ and $(y,s) \mapsto \phi(x,t,y,s)$ belongs to $C_0^\infty(Q_T)$ for each fixed $(x,t) \in Q_T$.

Observe that with the choice of ϕ as in (4.1), we have

$$\partial_t\phi + \partial_s\phi = \left[\rho_{\alpha_0}(t-\nu) - \rho_{\alpha_0}(t-\tau)\right]\omega_r(x-y)\rho_{r_0}(t-s), \qquad \nabla_x\phi + \nabla_y\phi = 0. \tag{4.2}$$

Before continuing, we need to introduce the two "hyperbolic" sets

$$\mathcal{H}^\varepsilon = \Big\{(x,t) \in Q_T : A(w^\varepsilon(x,t)) \in H\Big\}, \qquad \mathcal{H} = \Big\{(y,s) \in Q_T : A(w(y,s)) \in H\Big\},$$

and notice that

$$\nabla_x A(w^\varepsilon) = 0 \text{ a.e. in } \mathcal{H}^\varepsilon \quad \text{and} \quad \nabla_y A(w) = 0 \text{ a.e. in } \mathcal{H}, \tag{4.3}$$

$$\operatorname{sgn}(w^\varepsilon - w) = \operatorname{sgn}\big(A(w^\varepsilon) - A(w)\big) \text{ a.e. in } \Big[(Q_T \setminus \mathcal{H}) \times Q_T\Big] \cup \Big[Q_T \times (Q_T \setminus \mathcal{H}^\varepsilon)\Big]. \tag{4.4}$$

Using the approximate hyperbolic entropy inequality (3.8) for the viscous solution $w^\varepsilon = w^\varepsilon(x,t)$ with $k = w(y,s)$, we get for $(y,s) \in Q_T$

$$\begin{aligned} \iint_{Q_T} \Big(&|w^\varepsilon - w|\partial_t\phi + \operatorname{sgn}(w^\varepsilon - w)\left[V(x)(f(w^\varepsilon) - f(w)) - \nabla_x A(w^\varepsilon)\right] \cdot \nabla_x\phi \\ &- \operatorname{sgn}(w^\varepsilon - w)\operatorname{div}_x V(x) f(w)\phi\Big)\, dt\, dx\, ds\, dy \ge -\overline{R}_{\text{visc}}. \end{aligned} \tag{4.5}$$

Using the approximate parabolic entropy inequality (3.9) for the viscous solution $w^\varepsilon = w^\varepsilon(x,t)$ with $k = w(y,s)$, we get for $(y,s) \in Q_T \setminus \mathcal{H}$

$$\begin{aligned}
&\iint_{Q_T} \Big(|w^\varepsilon - w|\partial_t \phi + \operatorname{sgn}(w^\varepsilon - w)\big[V(x)(f(w^\varepsilon) - f(w)) - \nabla_x A(w^\varepsilon)\big] \cdot \nabla_x \phi \\
&\qquad\qquad\qquad\qquad - \operatorname{sgn}(w^\varepsilon - w)\operatorname{div}_x V(x) f(w)\phi \Big)\, dt\, dx \qquad (4.6)\\
&\quad \geq \lim_{\eta \downarrow 0} \iint_{Q_T} \big|\nabla_x A(w^\varepsilon)\big|^2 \operatorname{sgn}'_\eta(A(w^\varepsilon) - A(w))\phi \, dt\, dx - \overline{R}_{\text{visc}}.
\end{aligned}$$

Next we would like to integrate (4.5) and (4.6) over $(y,s) \in Q_T$ and $(y,s) \in Q_T \setminus \mathcal{H}$ respectively. To this end, we need to know that the involved functions are (y,s) integrable. Consider first $(y,s) \mapsto \iint_{Q_T} \operatorname{sgn}(v-u)\nabla_x A(w^\varepsilon) \cdot \nabla_x \phi \, dt\, dx$. We denote this function by $D(y,s)$.

To see that $D(\cdot,\cdot)$ is integrable on Q_T, we observe that for each fixed $(y,s) \in Q_T$

$$\operatorname{sgn}(v-u)\nabla_x A(w^\varepsilon) = \nabla_x \left|A(w^\varepsilon) - A(w)\right| \quad \text{for a.e. } (x,t) \in Q_T,$$

and hence

$$D(y,s) = \iint_{Q_T} \Big[\nabla_x \big|A(w^\varepsilon) - A(w)\big|\Big] \cdot \nabla_x \phi \, dt\, dx.$$

Since the function $(x,t) \mapsto \phi(x,t,y,s)$ belongs to $C_0^\infty(Q_T)$ for each fixed $(y,s) \in Q_T$, an integration by parts in x gives

$$D(y,s) = -\iint_{Q_T} \big|A(w^\varepsilon) - A(w)\big| \Delta_x \phi \, dt\, dx.$$

Integrating over $(y,s) \in Q_T$ and estimating yield

$$\left| \iint_{Q_T} D(y,s)\, ds\, dy \right| \leq \iiiint_{Q_T \times Q_T} \Big(|A(w^\varepsilon(x,t))| + |A(w(y,s))| \Big) \Delta_x \phi(x,y,t,s)\, dt\, dx\, ds\, dy.$$

By changing the variables ($z := x - y$, $\tau = t - s$) and taking into account that $w^\varepsilon, w \in L^1(Q_T)$,

$$\begin{aligned}
\left| \iint_{Q_T} D(y,s)\, ds\, dy \right| &\leq \iiiint |A(w^\varepsilon(x,t))| \psi_{\alpha_0}(t) |\Delta_z \omega_r(z)|\, \rho_{r_0}(\tau)\, dt\, dx\, d\tau\, dz \\
&\quad + \iiiint |A(w(x-z,t-\tau))| \psi_{\alpha_0}(t) |\Delta_z \omega_r(z)|\, \rho_{r_0}(\tau)\, dt\, dx\, d\tau\, dz \\
&\leq \|A(w^\varepsilon)\|_{L^1(Q_T)} \|\Delta_z \omega_r\|_{L^1(\mathbb{R}^d)} + \|A(w)\|_{L^1(Q_T)} \|\Delta_z \omega_r\|_{L^1(\mathbb{R}^d)} < \infty.
\end{aligned}$$

Hence we have that $D(\cdot,\cdot)$ is integrable on Q_T.

In a similar vain, one can also show the integrability of $(y,s) \mapsto \iint_{Q_T} |w^\varepsilon - w|\partial_t \phi \, dt\, dx$, $(y,s) \mapsto \iint_{Q_T} \operatorname{sgn}(w^\varepsilon - w)V(x)(f(w^\varepsilon) - f(w)) \cdot \nabla_x \phi \, dt\, dx$, $(y,s) \mapsto \iint_{Q_T} \operatorname{sgn}(w^\varepsilon - w)\operatorname{div}_x V(x) f(w)\phi \, dt\, dx$, and $(y,s) \mapsto \overline{R}_{\text{visc}}$.

It remains to consider the integrability of the function $Q_T \setminus \mathcal{H} \ni (y,s) \mapsto \lim_{\eta\downarrow 0} \iint_{Q_T} \big|\nabla_x A(w^\varepsilon)\big|^2 \operatorname{sgn}'_\eta(A(w^\varepsilon) - A(w))\phi \, dt\, dx$, but this follows from (4.6).

We have by Lebesgue' dominated convergence theorem and the first part of (4.3)

$$
\begin{aligned}
&\iint_{Q_T\setminus\mathcal{H}} \left(\lim_{\eta\downarrow 0}\iint_{Q_T} \left|\nabla_x A(w^\varepsilon)\right|^2 \operatorname{sgn}_\eta'(A(w^\varepsilon)-A(w))\phi\,dt\,dx\right)ds\,dy \\
&\quad= \lim_{\eta\downarrow 0}\iiiint_{(Q_T\setminus\mathcal{H})\times Q_T} \left|\nabla_x A(w^\varepsilon)\right|^2 \operatorname{sgn}_\eta'(A(w^\varepsilon)-A(w))\phi\,dt\,dx\,ds\,dy. \\
&\quad= \lim_{\eta\downarrow 0}\iiiint_{(Q_T\setminus\mathcal{H})\times(Q_T\setminus\mathcal{H})} \left|\nabla_x A(w^\varepsilon)\right|^2 \operatorname{sgn}_\eta'(A(w^\varepsilon)-A(w))\phi\,dt\,dx\,ds\,dy.
\end{aligned}
\tag{4.7}
$$

Let us now integrate (4.5) over $(y,s)\in Q_T$ and (4.6) over $(y,s)\in Q_T\setminus\mathcal{H}$. Adding the two resulting inequalities yields

$$
\begin{aligned}
&\iiiint_{Q_T\times Q_T}\Big(|w^\varepsilon-w|\partial_t\phi+\operatorname{sgn}(w^\varepsilon-w)\big[V(x)(f(w^\varepsilon)-f(w))-\nabla_x A(w^\varepsilon)\big]\cdot\nabla_x\phi \\
&\qquad\qquad -\operatorname{sgn}(w^\varepsilon-w)\operatorname{div}_x V(x)f(w)\phi\Big)\,dt\,dx\,ds\,dy \\
&=\iiiint_{(Q_T\setminus\mathcal{H})\times Q_T}\Big(|w^\varepsilon-w|\partial_t\phi+\operatorname{sgn}(w^\varepsilon-w)\big[V(x)(f(w^\varepsilon)-f(w))-\nabla_x A(w^\varepsilon)\big]\cdot\nabla_x\phi \\
&\qquad\qquad -\operatorname{sgn}(w^\varepsilon-w)\operatorname{div}_x V(x)f(w)\phi\Big)\,dt\,dx\,ds\,dy \\
&\quad+\iiiint_{\mathcal{H}\times Q_T}\Big(|w^\varepsilon-w|\partial_t\phi+\operatorname{sgn}(w^\varepsilon-w)\big[V(x)(f(w^\varepsilon)-f(w))-\nabla_x A(w^\varepsilon)\big]\cdot\nabla_x\phi \\
&\qquad\qquad -\operatorname{sgn}(w^\varepsilon-w)\operatorname{div}_x V(x)f(w)\phi\Big)\,dt\,dx\,ds\,dy \\
&\geq \lim_{\eta\downarrow 0}\iiiint_{(Q_T\setminus\mathcal{H})\times(Q_T\setminus\mathcal{H}^\varepsilon)} \left|\nabla_x A(w^\varepsilon)\right|^2 \operatorname{sgn}_\eta'(A(w^\varepsilon)-A(w))\phi\,dt\,dx\,ds\,dy-\overline{R}_{\mathrm{visc}},
\end{aligned}
\tag{4.8}
$$

where $\overline{R}_{\mathrm{visc}} := \iint_{Q_T} R_{\mathrm{visc}}\,ds\,dy$ and we have used (4.7).

Similarly, using the hyperbolic, parabolic entropy inequalities (3.1), (3.3) for the exact entropy solution $w=w(y,s)$ with $k=w^\varepsilon(x,t)$ and then integrating over $(x,t)\in Q_T$, we get

$$
\begin{aligned}
&\iiiint_{Q_T\times Q_T}\Big(|w-w^\varepsilon|\partial_s\phi+\operatorname{sgn}(w-w^\varepsilon)\big[V(y)(f(w)-f(w^\varepsilon))-\nabla_y A(w)\big]\cdot\nabla_y\phi \\
&\qquad\qquad -\operatorname{sgn}(w-w^\varepsilon)\operatorname{div}_y V(y)f(w^\varepsilon)\phi\Big)\,dt\,dx\,ds\,dy \\
&\quad\geq \lim_{\eta\downarrow 0}\iiiint_{(Q_T\setminus\mathcal{H}^\varepsilon)\times(Q_T\setminus\mathcal{H})} \left|\nabla_y A(w)\right|^2 \operatorname{sgn}_\eta'(A(w)-A(w^\varepsilon))\phi\,dt\,dx\,ds\,dy.
\end{aligned}
\tag{4.9}
$$

Using (4.3) and (4.4), we find that

$$
\begin{aligned}
&-\iiiint_{Q_T\times Q_T} \operatorname{sgn}(w_\varepsilon - w)\nabla_x A(w^\varepsilon)\cdot\nabla_y\phi\,dt\,dx\,ds\,dy\\
&\quad= -\iiiint_{Q_T\times(Q_T\setminus\mathcal{H}^\varepsilon)} \operatorname{sgn}(A(w^\varepsilon)-A(w))\nabla_x A(w^\varepsilon)\cdot\nabla_y\phi\,dt\,dx\,ds\,dy\\
&\quad= -\lim_{\eta\downarrow 0}\iiiint_{Q_T\times(Q_T\setminus\mathcal{H}^\varepsilon)} \operatorname{sgn}_\eta(A(w^\varepsilon)-A(w))\nabla_x A(w^\varepsilon)\cdot\nabla_y\phi\,dt\,dx\,ds\,dy\\
&\quad= -\lim_{\eta\downarrow 0}\iiiint_{Q_T\times(Q_T\setminus\mathcal{H}^\varepsilon)} \nabla_y A(w)\cdot\nabla_x A(w^\varepsilon)\operatorname{sgn}_\eta'(A(w^\varepsilon)-A(w))\phi\,dt\,dx\,ds\,dy.\\
&\quad= -\lim_{\eta\downarrow 0}\iiiint_{(Q_T\setminus\mathcal{H})\times(Q_T\setminus\mathcal{H}^\varepsilon)} \nabla_y A(w)\cdot\nabla_x A(w^\varepsilon)\operatorname{sgn}_\eta'(A(w^\varepsilon)-A(w))\phi\,dt\,dx\,ds\,dy.
\end{aligned}
\tag{4.10}
$$

Similarly, again using (4.3) and (4.4), we find that

$$
\begin{aligned}
&-\iiiint_{Q_T\times Q_T} \operatorname{sgn}(w - w^\varepsilon)\nabla_y A(w)\cdot\nabla_x\phi\,dt\,dx\,ds\,dy\\
&\quad= -\lim_{\eta\downarrow 0}\iiiint_{(Q_T\setminus\mathcal{H})\times(Q_T\setminus\mathcal{H}^\varepsilon)} \nabla_x A(w^\varepsilon)\cdot\nabla_y A(w)\operatorname{sgn}_\eta'(A(w)-A(w^\varepsilon))\phi\,dt\,dx\,ds\,dy.
\end{aligned}
\tag{4.11}
$$

Using the second part of (4.2) when adding (4.8) and (4.10) yields

$$
\begin{aligned}
&\iiiint_{Q_T\times Q_T}\Big(|w^\varepsilon - w|\partial_t\phi + \operatorname{sgn}(w^\varepsilon - w)\big[V(x)(f(w^\varepsilon)-f(w))\big]\cdot\nabla_x\phi\\
&\qquad\qquad - \operatorname{sgn}(w^\varepsilon - w)\operatorname{div}_x V(x)f(w)\phi\Big)\,dt\,dx\,ds\,dy\\
&\geq \lim_{\eta\downarrow 0}\iiiint_{(Q_T\setminus\mathcal{H}^\varepsilon)\times(Q_T\setminus\mathcal{H})}\Big(\big|\nabla_x A(w^\varepsilon)\big|^2 - \nabla_y A(w)\cdot\nabla_x A(w^\varepsilon)\Big)\\
&\qquad\qquad \times\operatorname{sgn}_\eta'(A(w^\varepsilon)-A(w))\phi\,dt\,dx\,ds\,dy - R_{\text{visc}}.
\end{aligned}
\tag{4.12}
$$

Similarly, adding (4.9) and (4.11) yields

$$
\begin{aligned}
&\iiiint_{Q_T\times Q_T}\Big(|w - w^\varepsilon|\partial_s\phi + \operatorname{sgn}(w - w^\varepsilon)\big[V(y)(f(w)-f(w^\varepsilon)\big]\cdot\nabla_y\phi\\
&\qquad\qquad - \operatorname{sgn}(w - w^\varepsilon)\operatorname{div}_y V(y)f(w^\varepsilon)\phi\Big)\,dt\,dx\,ds\,dy\\
&\geq \lim_{\eta\downarrow 0}\iiiint_{(Q_T\setminus\mathcal{H}^\varepsilon)\times(Q_T\setminus\mathcal{H})}\Big(\big|\nabla_y A(w)\big|^2 - \nabla_x A(w^\varepsilon)\cdot\nabla_y A(w)\Big)\\
&\qquad\qquad \times\operatorname{sgn}_\eta'(A(w)-A(w^\varepsilon))\phi\,dt\,dx\,ds\,dy.
\end{aligned}
\tag{4.13}
$$

Following Karlsen and Risebro [12], we write

$$
\begin{aligned}
&\operatorname{sgn}(w^\varepsilon - w)V(x)(f(w^\varepsilon)-f(w))\cdot\nabla_x\phi - \operatorname{sgn}(w^\varepsilon - w)\operatorname{div}_x V(x)f(w)\phi\\
&\quad= \operatorname{sgn}(w^\varepsilon - w)\big(V(x)f(w^\varepsilon)-V(y)f(w)\big)\cdot\nabla_x\phi + \operatorname{sgn}(w^\varepsilon - w)\operatorname{div}_x\big[\big(V(y)f(w)-V(x)f(w)\big)\phi\big],
\end{aligned}
$$

$$
\begin{aligned}
&\operatorname{sgn}(w - w^\varepsilon)V(y)(f(w)-f(w^\varepsilon))\cdot\nabla_y\phi - \operatorname{sgn}(w - w^\varepsilon)\operatorname{div}_y V(y)f(w^\varepsilon)\phi\\
&\quad= \operatorname{sgn}(w^\varepsilon - w)\big(V(x)f(w^\varepsilon)-V(y)f(w)\big)\cdot\nabla_y\phi - \operatorname{sgn}(w^\varepsilon - w)\operatorname{div}_y\big[\big(V(x)f(w^\varepsilon)-V(y)f(w^\varepsilon)\big)\phi\big].
\end{aligned}
$$

When adding (4.12) and (4.13), we use the second part of (4.2) and the identities

$$\operatorname{sgn}(-r) = -\operatorname{sgn}(r) \text{ a.e. in } \mathbb{R}, \qquad \operatorname{sgn}'_\eta(-r) = \operatorname{sgn}'_\eta(r) \text{ a.e. in } \mathbb{R}.$$

The final result takes the form

$$-\iiiint_{Q_T\times Q_T} |w^\varepsilon - w|(\partial_t\phi + \partial_s\phi)\,dt\,dx\,ds\,dy \le R_{\mathrm{diss}} + \overline{R}_{\mathrm{visc}} + R_{\mathrm{conv}} \le \overline{R}_{\mathrm{visc}} + R_{\mathrm{conv}}, \tag{4.14}$$

where the expression for $\partial_t\phi + \partial_s\phi$ is written out in (4.2), $R_{\mathrm{conv}} := \iiiint_{Q_T\times Q_T} I_{\mathrm{conv}}\,dt\,dx\,ds\,dy$,

$$I_{\mathrm{conv}} := \operatorname{sgn}(w^\varepsilon - w)\Big(\operatorname{div}_x\big[\big(V(y)f(w) - V(x)f(w)\big)\phi\big] - \operatorname{div}_y\big[\big(V(x)f(w^\varepsilon) - V(y)f(w^\varepsilon)\big)\phi\big]\Big),$$

and $R_{\mathrm{diss}} := -\lim_{\eta\downarrow 0} \iiiint_{(Q_T\setminus\mathcal{H}^\varepsilon)\times(Q_T\setminus\mathcal{H})} \big|\nabla_x A(w^\varepsilon) - \nabla_y A(w)\big|^2 \operatorname{sgn}'_\eta\big(A(w^\varepsilon) - A(w)\big)\phi\,dt\,dx\,ds\,dy \le 0.$

Having in mind the first part of (4.2), we get by the triangle inequality

$$-\iiiint_{Q_T\times Q_T} |w^\varepsilon(x,t) - w(y,s)|(\partial_t\phi + \partial_s\phi)\,dt\,dx\,ds\,dy \le R_{w^\varepsilon,w} + R_{w,x} + R_{w,t},$$

where

$$R_{w^\varepsilon,w} := -\iiiint_{Q_T\times Q_T} |w^\varepsilon(x,t) - w(x,t)|\big[\rho_{\alpha_0}(t-\nu) - \rho_{\alpha_0}(t-\tau)\big]\omega_r(x-y)\rho_{r_0}(t-s)\,dt\,dx\,ds\,dy,$$

$$R_{w,x} := -\iiiint_{Q_T\times Q_T} |w(x,t) - w(y,t)|\big[\rho_{\alpha_0}(t-\nu) - \rho_{\alpha_0}(t-\tau)\big]\omega_r(x-y)\rho_{r_0}(t-s)\,dt\,dx\,ds\,dy,$$

$$R_{w^\varepsilon,t} := -\iiiint_{Q_T\times Q_T} |w(y,t) - w(y,s)|\big[\rho_{\alpha_0}(t-\nu) - \rho_{\alpha_0}(t-\tau)\big]\omega_r(x-y)\rho_{r_0}(t-s)\,dt\,dx\,ds\,dy.$$

First of all, a standard L^1 continuity argument gives $\lim_{r_0\downarrow 0} R_{w,t} = 0$. Next,

$$\begin{aligned}
\lim_{\alpha_0\downarrow 0} R_{w,x} &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Big(|w(x,\tau) - w(y,\tau)| - |w(x,\nu) - w(y,\nu)|\Big)\omega_r(x-y)\,dx\,dy \\
&\overset{(z:=x-y)}{\le} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |w(y+z,\tau) - w(y,\tau)|\omega_r(z)\,dy\,dz \\
&\le |w|_{L^\infty(0,T;BV(\mathbb{R}^d))} \int_{\mathbb{R}^d} |z|\omega_r(z)\,dz \le C_1 r,
\end{aligned}$$

where $C_1 := |w|_{L^\infty(0,T;BV(\mathbb{R}^d))}$. Finally, we have

$$\lim_{\alpha_0\downarrow 0} R_{w^\varepsilon,w} = \int_{\mathbb{R}^d} |w^\varepsilon(x,\tau) - w(x,\tau)|\,dx - \int_{\mathbb{R}^d} |w^\varepsilon(x,\nu) - w(x,\nu)|\,dx.$$

Summing up, from (4.14) we therefore get the following approximation inequality

$$\int_{\mathbb{R}^d} |w^\varepsilon(x,\tau) - w(x,\tau)|\,dx \le \int_{\mathbb{R}^d} |w^\varepsilon(x,\nu) - w(x,\nu)|\,dx + C_1 r + \lim_{r_0,\alpha_0\downarrow 0}\Big(\overline{R}_{\mathrm{visc}} + R_{\mathrm{conv}}\Big). \tag{4.15}$$

We start with the estimation of $\overline{R}_{\mathrm{visc}}$, which can be done as follows:

$$\begin{aligned}
\overline{R}_{\mathrm{visc}} &\le \varepsilon \sum_{i=1}^d \iiiint_{Q_T\times Q_T} \big|\partial_{x_i} w^\varepsilon\big|\,\psi_{\alpha_0}(t)\big|\partial_{x_i}\omega_r(x-y)\big|\rho_{r_0}(t-s)\,dt\,dx\,ds\,dy \\
&\overset{\alpha_0\downarrow 0}{\longrightarrow} \sum_{i=1}^d \int_\nu^\tau \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big|\partial_{x_i} w^\varepsilon\big|\,\big|\partial_{x_i}\omega_r(x-y)\big|\,dx\,dy\,dt \\
&\le \varepsilon K/r \sum_{i=1}^d \int_\nu^\tau \int_{\mathbb{R}^d} \big|\partial_{x_i} w^\varepsilon\big|\,dt\,dx \le \varepsilon TK/r|w^\varepsilon|_{L^\infty(0,T;BV(\mathbb{R}^d))} \le C_2 T\varepsilon/r,
\end{aligned} \tag{4.16}$$

where $K := \int_{\mathbb{R}^d} |\delta'(\sigma)|\, d\sigma$ and $C_2 := K|w^\varepsilon|_{L^\infty(0,T;BV(\mathbb{R}^d))}$.

Before we continue with the estimation of R_{conv}, let us write $I_{\text{conv}} = I^1_{\text{conv}} + I^2_{\text{conv}}$, where

$$
\begin{aligned}
I^1_{\text{conv}} &= \operatorname{sgn}(w^\varepsilon - w)\Big[\big(V(y)f(w) - V(x)f(w)\big)\cdot\nabla_x\phi - \big(V(x)f(w^\varepsilon) - V(y)f(w^\varepsilon)\big)\cdot\nabla_y\phi\Big],\\
I^2_{\text{conv}} &= \operatorname{sgn}(w^\varepsilon - w)\big(\operatorname{div}_y V(y)f(w^\varepsilon) - \operatorname{div}_x V(x)f(w)\big)\phi,
\end{aligned}
$$

so that $R_{\text{conv}} = R^1_{\text{conv}} + R^2_{\text{conv}}$, $R^1_{\text{conv}} = \iiiint\limits_{Q_T\times Q_T} I^1_{\text{conv}}\, dt\, dx\, ds\, dy$, $R^2_{\text{conv}} = \iiiint\limits_{Q_T\times Q_T} I^2_{\text{conv}}\, dt\, dx\, ds\, dy$.

Let us start by estimating R^1_{conv}. To this end, introduce $F(w^\varepsilon, w) := \operatorname{sgn}(w^\varepsilon - w)\big[f(w^\varepsilon) - f(w)\big]$ and observe that since $\nabla_y\phi = -\nabla_x\phi$,

$$
R^1_{\text{conv}} = \iiiint\limits_{Q_T\times Q_T} \Big((V(x) - V(y))F(w^\varepsilon, w)\Big)\cdot\nabla_x\phi\, dt\, dx\, ds\, dy.
$$

The function $F(\cdot,\cdot)$ is locally Lipschitz continuous in both variables and the common Lipschitz constant equals $\operatorname{Lip}(f)$. Since $w^\varepsilon \in L^\infty(Q_T)\cap L^\infty(0,T;BV(\mathbb{R}^d))$, $\nabla_x F(w^\varepsilon, w)$ is a finite measure and $\iint\limits_{Q_T} \big|\partial_{x_i}F(w^\varepsilon, w)\big|\, dt\, dx \le \operatorname{Lip}(f)\iint\limits_{Q_T}\big|\partial_{x_i}w^\varepsilon\big|\, dt\, dx$, $i = 1,\dots,d$. Integrating by parts thus gives

$$
\begin{aligned}
R^1_{\text{conv}} = &\underbrace{-\iiiint\limits_{Q_T\times Q_T}\Big(\operatorname{div}_x V(x)F(w^\varepsilon, w)\psi_{\alpha_0}(t)\omega_r(x-y)\rho_{r_0}(t-s)\, dt\, dx\, ds\, dy}_{R^{1,1}_{\text{conv}}}\\
&\underbrace{-\iiiint\limits_{Q_T\times Q_T}(V(x) - V(y))\cdot\nabla_x F(w^\varepsilon, w)\psi_{\alpha_0}(t)\omega_r(x-y)\rho_{r_0}(t-s)\, dt\, dx\, ds\, dy}_{R^{1,2}_{\text{conv}}}.
\end{aligned}
$$

For $R^{1,2}_{\text{conv}}$, we calculate as follows:

$$
\begin{aligned}
\big|R^{1,2}_{\text{conv}}\big| &\le \operatorname{Lip}(f)\sum_{i=1}^d \iiiint\limits_{Q_T\times Q_T}\big|V_i(x) - V_i(y)\big|\,\big|\partial_{x_i}w^\varepsilon\big|\psi_{\alpha_0}(t)\omega_r(x-y)\rho_{r_0}(t-s)\, dt\, dx\, ds\, dy\\
&\overset{\alpha_0\downarrow 0}{\longrightarrow} \operatorname{Lip}(f)\sum_{i=1}^d\int_\nu^\tau\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big|V_i(x) - V_i(y)\big|\,\big|\partial_{x_i}w^\varepsilon\big|\,\omega_r(x-y)\, dx\, dy\, dt\\
&\overset{(z:=x-y)}{=} \operatorname{Lip}(f)\sum_{i=1}^d\int_\nu^\tau\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big|V_i(y+z) - V_i(y)\big|\,\big|\partial_{y_i}w^\varepsilon(y+z,t)\big|\omega_r(z)\, dz\, dy\, dt\\
&\le \operatorname{Lip}(V)\operatorname{Lip}(f)\sum_{i=1}^d\int_\nu^\tau\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|z|\big|\partial_{y_i}w^\varepsilon(y+z,t)\big|\omega_r(z)\, dz\, dy\, dt\\
&\le T\operatorname{Lip}(V)\operatorname{Lip}(f)|w^\varepsilon|_{L^\infty(0,T;BV(\mathbb{R}^d))}\int_{\mathbb{R}^d}|z|\omega_r(z)\, dz\\
&\le rT\operatorname{Lip}(V)\operatorname{Lip}(f)|w^\varepsilon|_{L^\infty(0,T;BV(\mathbb{R}^d))} \le C_3 Tr,
\end{aligned}
$$

where $\operatorname{Lip}(V) := \max_{i=1,\dots,d}\operatorname{Lip}(V_i)$ and $C_3 := \operatorname{Lip}(V)\operatorname{Lip}(f)|w^\varepsilon|_{L^\infty(0,T;BV(\mathbb{R}^d))}$. Note that we have used the Lipschitz regularity of the velocity field V (see (2.1)) to get the desired result.

Regarding the term R^2_{conv}, let us first rewrite it as

$$
R^2_{\text{conv}} = \underbrace{\iiiint\limits_{Q_T\times Q_T}\operatorname{div}_x V(x)F(w^\varepsilon, w)\psi_{\alpha_0}(t)\omega_r(x-y)\rho_{r_0}(t-s)\, dt\, dx\, ds\, dy}_{R^{2,1}_{\text{conv}}}
$$

$$+\underbrace{\iiiint_{Q_T\times Q_T}\operatorname{sgn}(w^\varepsilon-w)\big(\operatorname{div}_y V(y)-\operatorname{div}_x V(x)\big)f(w^\varepsilon)\psi_{\alpha_0}(t)\omega_r(x-y)\rho_{r_0}(t-s)\,dt\,dx\,ds\,dy}_{R^{2,2}_{\mathrm{conv}}}.$$

We estimate $R^{2,2}_{\mathrm{conv}}$ as follows:

$$\begin{aligned}
\left|R^{2,2}_{\mathrm{conv}}\right| &\le \|f(w^\varepsilon)\|_{L^\infty(Q_T)}\sum_{i=1}^d\iiiint_{Q_T\times Q_T}\left|\partial_{y_i}V_i(y)-\partial_{x_i}V_i(x)\right|\psi_{\alpha_0}(t)\omega_r(x-y)\rho_{r_0}(t-s)\,dt\,dx\,ds\,dy\\
&\xrightarrow{\alpha_0\downarrow 0}\|f(w^\varepsilon)\|_{L^\infty(Q_T)}\sum_{i=1}^d\int_\nu^\tau\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\partial_{y_i}V_i(y)-\partial_{x_i}V_i(x)\right|\omega_r(x-y)\,dx\,dy\,dt\\
&\overset{(z=x-y)}{=}\|f(w^\varepsilon)\|_{L^\infty(Q_T)}\sum_{i=1}^d\int_\nu^\tau\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\partial_{y_i}V_i(y)-\partial_{y_i}V_i(y+z)\right|\omega_r(z)\,dz\,dy\,dt\\
&\le T\|f(w^\varepsilon)\|_{L^\infty(Q_T)}\sum_{i=1}^d\left|\partial_{x_i}V_i\right|_{BV(\mathbb{R}^d)}\int_{\mathbb{R}^d}|z|\omega_r(z)\,dz\le C_4Tr,
\end{aligned}$$

where $C_4 := \|f(w^\varepsilon)\|_{L^\infty(Q_T)}\sum_{i=1}^d\left|\partial_{x_i}V\right|_{BV(\mathbb{R}^d)}$. Note that we have used the BV regularity of $\partial_{x_i}V$, $i=1,\dots,d$, to get the desired result. Since $R^{1,1}_{\mathrm{conv}}=R^{2,1}_{\mathrm{conv}}$, we end up with

$$R_{\mathrm{conv}}=R^1_{\mathrm{conv}}+R^2_{\mathrm{conv}}\le C_5Tr,\qquad C_5=\max(C_3,C_4). \tag{4.17}$$

Set $C_6=\max(C_1,C_2,C_5)$. Then from (4.15), (4.16), and (4.17), we get

$$\begin{aligned}
&\int_{\mathbb{R}^d}|w^\varepsilon(x,\tau)-w(x,\tau)|\,dx\\
&\quad\le\int_{\mathbb{R}^d}|w^\varepsilon(x,\nu)-w(x,\nu)|\,dx+C_6\left((1+T)r+\frac{Tr}{\varepsilon}\right)\xrightarrow{\nu\downarrow 0}C_6\left((1+T)r+\frac{r}{\varepsilon}\right).
\end{aligned}\tag{4.18}$$

By choosing $r=\sqrt{T\varepsilon}$, we immediately obtain

$$\int_{\mathbb{R}^d}|w^\varepsilon(x,\tau)-w(x,\tau)|\,dx\le C_7\sqrt{T\varepsilon}, \tag{4.19}$$

for some constant C_7 independent of ε. To obtain (2.5), we simply integrate (4.19) over $\tau\in(0,T)$.

References

[1] R. Bürger, S. Evje, and K. H. Karlsen. On strongly degenerate convection-diffusion problems modeling sedimentation-consolidation processes. *J. Math. Anal. Appl.*, 247(2):517–556.

[2] R. Bürger and K. H. Karlsen. A strongly degenerate convection-diffusion problem modeling centrifugation of flocculated suspensions. Proc. Hyp 2000.

[3] M. C. Bustos, F. Concha, R. Bürger, and E. M. Tory. *Sedimentation and Thickening: Phenomenological Foundation and Mathematical Theory.* Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999.

[4] J. Carrillo. On the uniqueness of the solution of the evolution dam problem. *Nonlinear Anal.*, 22(5):573–607, 1994.

[5] J. Carrillo. Entropy solutions for nonlinear degenerate problems. *Arch. Rational Mech. Anal.*, 147(4):269–361, 1999.

[6] B. Cockburn and G. Gripenberg. Continuous dependence on the nonlinearities of solutions of degenerate parabolic equations. *J. Differential Equations*, 151(2):231–251, 1999.

[7] M. S. Espedal and K. H. Karlsen. Numerical solution of reservoir flow models based on large time step operator splitting algorithms. In *Filtration in Porous Media and Industrial Applications*, 9–77, Lecture Notes in Math., 1734, Springer, Berlin, 2000.

[8] S. Evje, K. H. Karlsen, and N. H. Risebro. A continuous dependence result for nonlinear degenerate parabolic equations with spatially dependent flux function. Proc. Hyp 2000.

[9] R. Eymard, T. Gallouet, and R. Herbin. Error estimate for approximate solutions of a nonlinear convection-diffusion problem. Preprint, 2000.

[10] R. Eymard, T. Gallouet, R. Herbin, and A. Michel. Convergence of a finite volume scheme for nonlinear degenerate parabolic equations. Preprint, 2000.

[11] K. H. Karlsen and N. H. Risebro. Convergence of finite difference schemes for viscous and inviscid conservation laws with rough coefficients. Mathematical Modelling and Numerical Analysis. To appear.

[12] K. H. Karlsen and N. H. Risebro. On the uniqueness and stability of entropy solutions of nonlinear degenerate parabolic equations with rough coefficients. Preprint, 2000.

[13] K. H. Karlsen and M. Ohlberger. A note on the uniqueness of entropy solutions of nonlinear degenerate parabolic equations. Preprint, 2001.

[14] S. N. Kružkov. First order quasi-linear equations in several independent variables. *Math. USSR Sbornik*, 10(2):217–243, 1970.

[15] N. N. Kuznetsov. Accuracy of some approximative methods for computing the weak solutions of a first-order quasi-linear equation. *USSR Comput. Math. and Math. Phys. Dokl.*, 16(6):105–119, 1976.

[16] C. Mascia, A. Porretta, and A. Terracina. Nonhomogeneous Dirichlet problems for degenerate parabolic-hyperbolic equations. Preprint, 2000.

[17] M. Ohlberger. A posteriori error estimates for vertex centered finite volume approximations of convection-diffusion-reaction equations. Preprint, Mathematische Fakultät, Albert-Ludwigs-Universität Freiburg, 2000.

[18] F. Otto. L^1-contraction and uniqueness for quasilinear elliptic-parabolic equations. *J. Differential Equations*, 131(1):20–38, 1996.

[19] É. Rouvre and G. Gagneux. Solution forte entropique de lois scalaires hyperboliques-paraboliques dégénérées. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(7):599–602, 1999.

[20] A. I. Vol'pert. The spaces BV and quasi-linear equations. *Math. USSR Sbornik*, 2(2):225–267, 1967.

[21] A. I. Vol'pert and S. I. Hudjaev. Cauchy's problem for degenerate second order quasilinear parabolic equations. *Math. USSR Sbornik*, 7(3):365–387, 1969.

[22] A. I. Volpert. Generalized solutions of degenerate second-order quasilinear parabolic and elliptic equations. *Adv. Differential Equations*, 5(10-12):1493–1518, 2000.

(Steinar Evje)
RF-Rogaland Research
Thormøhlensgt. 55
N–5008 Bergen, Norway
E-mail address: Steinar.Evje@rf.no

(Kenneth H. Karlsen)
Department of Mathematics
University of Bergen
Johs. Brunsgt. 12
N–5008 Bergen, Norway
E-mail address: kennethk@math.uib.no
URL: http://www.mi.uib.no/~kennethk/