A PRIORI AND A POSTERIORI ERROR ESTIMATES FOR THE STAGGERED LAX-FRIEDRICHS SCHEME IN MULTI DIMENSIONS FOR SCALAR NONLINEAR CONSERVATION LAWS

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Abstract. Staggered grid finite volume methods (or also called *central schemes*) have been introduced in 1d by Nessyahu and Tadmor in 1990 [19] in order to avoid the necessity to have information on solutions of Riemann problems for the evaluation of numerical fluxes. We consider the general case in multi dimensions and on general staggered grids which have to satisfy an overlap assumption, only. We interpret the staggered Lax-Friedrichs scheme as a three-step method consisting of a prolongation step onto a finer *intersection grid*, a finite volume step with an arbitrarily good numerical flux (e.g. Godunov flux) on the *intersection grid* followed by an averaging step such that the calculation of numerical fluxes reduce to evaluations of the continuous flux. Using this point of view, we prove an a posteriori error estimate and an a priori error estimate in the L^1 norm in space and time which is of order $h^{1/4}$ where h is a mesh-size parameter. Hence, we recover for the staggered Lax-Friedrichs scheme the same order of convergence as for upwind finite volume methods on a fixed grid.

1. Introduction.

In this paper we prove an a priori and a posteriori error estimate for the first order Lax-Friedrichs scheme on general staggered grids for scalar multi-dimensional conservation laws.

Staggered finite volume schemes have been introduced by Nessyahu and Tadmor in 1990 [19]. The main advantage of these schemes is that no information about solutions to local Riemann problems is needed. Using staggered grids one can replace the upwind fluxes by central differences. The price one has to pay is the occurrence of excessive numerical viscosity since the resulting scheme can be interpreted as a Lax-Friedrichs scheme. Therefore, a higher order scheme of MUSCL-type in one spatial dimension is proposed in [19]. Numerical experiments in [19] show the good performance of the algorithm. Later, in [2, 4] the central schemes (of second order) have been generalized to multi-dimensional schemes on unstructured grids and in [13] on 2d tensor product structured grids. In [4] a primal and a dual mesh are used and alternatingly the time evolution is performed on either of those two meshes.

Convergence of finite volume schemes on a fixed grid has been proven in [7, 16, 5]. In case of staggered unstructured grids in multi dimensions there exist only a few convergence results. In [3] convergence of a second order central scheme on special two-dimensional grids has been proven for a linear conservation law. Convergence of the first order Lax-Friedrichs scheme on the same special staggered grids for nonlinear scalar problems has been proven in [11].

We prove an *a priori* and *a posteriori* error estimate on *general* staggered grids for the first order Lax-Friedrichs scheme for scalar nonlinear problems in any spatial dimension, see Theorem 3.3 and Theorem 3.5. We allow that at each time step one may have a new grid. The only *additional* condition on the sequence of staggered grids which we need is an *overlap* assumption of consecutive grids, see Assumption 3.1.

The main idea to prove the error estimates is to re-interpret the Lax-Friedrichs scheme on staggered grids such that the error analysis which is known in case of a fixed grid can be used and modified, see Definition 2.14.

In order to perform a time evolution we need to construct a relation between two grids: the grid at the actual time level and the grid at the next level. A natural candidate for this job is the intersection grid. Then, the staggered Lax-Friedrichs scheme can be decomposed into three steps, a prolongation step of the actual values to values on the intersection grid, a time evolution step on the intersection grid

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with an arbitrary numerical flux (which has to be consistent, conservative, monotone and Lipschitz, see Assumption 2.5 below, for example one can choose the Godunov flux) and an averaging step from the intersection grid to the grid on the next time level. The point is, that due to the properties of the numerical flux (conservativity and consistency) and thanks to the overlap condition on two consecutive grids, the overall scheme (consisting of the three steps described before) reduces to the staggered Lax-Friedrichs scheme.

The technique to prove error estimates for nonlinear hyperbolic equations is quite old and goes back to Kruzkov [17] and Kuznetsov [18] where entropy inequalities are used to establish error estimates. We employ this technique as well, as it is formulated in [8] and [5]. Our a posteriori error estimate is given in the spirit of Kröner and Ohlberger [15] which takes into account the domain of dependence of the error. In that sense, the error estimate is *local*. In [15] an adaptive strategy is developed which can be easily generalized to the situation we consider here.

We want to mention that one can go back from staggered grids to non-staggered grids retaining the simplicity of the central schemes by using a projection step, see [19, 12]. The error analysis of these schemes, as well as the error analysis of higher order schemes on staggered grids constitutes the subject of further ongoing research.

The paper is organized as follows.
In Section 2 we formulate the problem and introduce the finite volume discretisation on a fixed grid and on staggered grids, to show the analogy and difference between these two approaches. In Section 3 we state our *a priori* and *a posteriori* error estimates and give the ideas how to prove these results. The rest of the paper is devoted to the proofs.

2. Finite volume discretisation on staggered grids.

In this Section we present the model problem which we consider in this paper. Then, we define the finite volume scheme on a fixed mesh and on staggered meshes. Finally, we give a re-interpretation of the scheme on staggered meshes and link both methods.

2.1. **Continuous problem.** We consider in this paper the following nonlinear scalar hyperbolic conservation law with some initial condition:

$$\left\{ \begin{array}{rcl} \partial_t u(x,t) + \operatorname{div} F(x,t,u(x,t)) &=& 0, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}^+, \\ u(x,0) &=& u_0(x), \quad x \in \mathbb{R}^d, \end{array} \right. \tag{2.1}$$

where $F : \mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^d$, $d \geq 1$, $(x,t,s) \mapsto F(x,t,s)$.
The following assumptions on data are made.

ASSUMPTION 2.1.

1. $u_0 \in L^\infty(\mathbb{R}^d) \cap BV_{\mathrm{loc}}(\mathbb{R}^d)$ *with* $U_m \leq u_0 \leq U_M$ *almost everywhere.*
2. $F \in C^1(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}; \mathbb{R})$, $(x,t,s) \mapsto F(x,t,s)$ *and* $\partial F/\partial s$ *is locally Lipschitz continuous.*
3. $\operatorname{div}_x F(x,t,s) = \sum_{i=1}^d \partial_{x_i} F(x,t,s) = 0$ *for all* $(x,t,s) \in \mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}$.
4. *For all compact* $K \subset \mathbb{R}$ *there exists a constant* $0 < V_K < +\infty$ *such that*
$$\left| \frac{\partial F}{\partial s}(x,t,s) \right| \leq V_K, \qquad \textit{for almost all } (x,t,s) \in \mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}^d.$$

DEFINITION 2.2.
We say that $u \in L^\infty(\mathbb{R}^d \times]0,\infty[)$ *is an* entropy weak solution *to* (2.1), *if for all* $\kappa \in \mathbb{R}$ *and all* $\varphi \in C_0^1(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R}^+)$

$$\begin{aligned} &\int\limits_{\mathbb{R}^d \times \mathbb{R}^+} \Big\{ |u(x,t) - \kappa| \partial_t \varphi(x,t) + [F(x,t,u(x,t) \top \kappa) - F(x,t,u(x,t) \perp \kappa)] \nabla \varphi(x,t) \Big\} \mathrm{d}t \mathrm{d}x \\ &\qquad + \int\limits_{\mathbb{R}^d} |u_0(x) - \kappa| \varphi(x,0) \mathrm{d}x \geq 0, \end{aligned} \tag{2.2}$$

holds, where $a \top b := \max\{a,b\}$ *and* $a \perp b := \min\{a,b\}$.

Existence and uniqueness of entropy weak solutions to (2.1) have been proven by Kruzkov in 1970 under somewhat stronger assumptions [17]. A proof in case of Assumption 2.1 can be found in [5, Theorem 2].

2.2. **Finite volume discretisation on a fixed grid.** Let $\mathcal{T} = (T_i)_{i\in I}$, $I \subset \mathbb{N}$, be a partition of $\mathbb{R}^d$ consisting of polygonal cells.

NOTATION 2.3.

- *For $i \in I$ let $N(i)$ denote the set of all indices $j \in I$ of cells having a common interface of dimension $d-1$ with T_i.*
- *For $i \in I$ and $j \in N(i)$ let S_{ij} denote the common interface and n_{ij} the unit outer normal to S_{ij} with respect to T_i.*

These notations are illustrated in Figure 1 for the two-dimensional case ($d = 2.$).

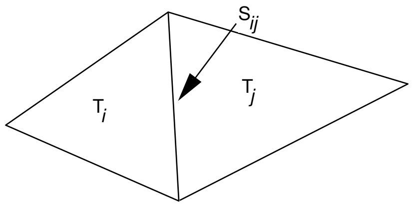


FIGURE 1. Illustration of Notations 2.3.

The time axis $\mathbb{R}^+$ is partitioned into intervals $[t^n, t^{n+1}[$ of length $k^n := t^{n+1} - t^n$. The idea of finite volume schemes is to integrate the partial differential equation in (2.1) over a cell T_i and to apply Gauss' formula. The resulting integrals have to be approximated. For more details we refer to the book of Kröner [14, page 158].
The first order finite volume scheme is defined as follows.

DEFINITION 2.4 (First order finite volume scheme)**.** *Let $i \in I$ and $n \in \mathbb{N}$.*

$$u_i^0 := \frac{1}{|T_i|}\int_{T_i} u_0(x)\mathrm{d}x, \tag{2.3}$$

$$u_i^{n+1} := u_i^n - \frac{k^n}{|T_i|}\sum_{j\in N(i)} g_{ij}^n(u_i^n, u_j^n), \tag{2.4}$$

and

$$u_h(x,t) := u_i^n \quad \text{for } t^n \le t < t^{n+1} \quad \text{and } x \in T_i. \tag{2.5}$$

Here, g_{ij}^n denotes a so-called *numerical flux* which is supposed to lie in $C^0(\mathbb{R}^2;\mathbb{R})$, where $i \in I$, $j \in N(i)$, $n \in \mathbb{N}$.
In order to prove error estimates between u_h, defined in (2.5), and the entropy solution to (2.1) one needs the following assumptions on the mesh and the *numerical fluxes* g_{ij}^n.

ASSUMPTION 2.5.

1. *The mesh $\mathcal{T}$ has to satisfy the following assumption:*
   There exists $\alpha > 0$ (regularity of the mesh) such that for all $i \in I$
   $$\alpha h_i^d \le |T_i|,$$
   $$|\partial T_i| \le \frac{1}{\alpha} h_i^{d-1},$$
   where $h_i := \operatorname{diam}(T_i)$. Set
   $$h := \max_{i\in I} h_i.$$
2. *The numerical fluxes $g_{ij}^n \in C^0(\mathbb{R}^2;\mathbb{R})$ with $i \in I$, $j \in N(i)$, $n \in \mathbb{N}$, have to satisfy the following properties:*

(a) **(Monotony)** *g_{ij}^n is nondecreasing with respect to its first argument and nondecreasing with respect to its second argument.*

(b) **(Conservativity)**

$$g_{ij}^n(v,w) = -g_{ji}^n(w,v) \quad \forall v,w \in \mathbb{R}.$$

(c) *g_{ij}^n is locally Lipschitz continuous, i.e. for every compact interval $I \subset \mathbb{R}$ there exists a constant $L_g > 0$, depending on I and F only, such that for all $u_1, u_2, v_1, v_2 \in I$*

$$|g_{ij}^n(u_1,v_1) - g_{ij}^n(u_2,v_2)| \le L_g |S_{ij}| \left[|u_1 - u_2| + |v_1 - v_2| \right].$$

(d) **(Consistency)**

$$g_{ij}^n(s,s) \quad = \quad \frac{1}{k^n} \int\limits_{t^n}^{t^{n+1}} \int\limits_{S_{ij}} F(\gamma,t,s)\cdot n_{ij} \mathrm{d}\gamma \mathrm{d}t \quad \forall s \in \mathbb{R}$$

Examples for such numerical fluxes can be found for instance in [14, 5]. Then, the following result has been proven in [5, Thm. 4].

THEOREM 2.6. *Let Assumptions 2.1 and 2.5 hold. Assume that the following CFL-condition is met*

$$k^n \le \frac{(1-\xi)\alpha^2 h}{2L_g} \quad \text{with } \xi \in]0,1[. \tag{2.6}$$

Let u be the unique entropy weak solution to (2.1) and u_h its first order finite volume approximation which is defined in Definition 2.4. If $u_0 \in BV_{\mathrm{loc}}(\mathbb{R}^d)$ the following error estimate holds: for any compact set $E \subset \mathbb{R}^d \times \mathbb{R}^+$, there exists a constant $K > 0$, depending on E, F, u_0, L_g, α and ξ only, such that

$$\int\limits_E |u(x,t) - u_h(x,t)| \mathrm{d}x\mathrm{d}t \le K h^{1/4}. \tag{2.7}$$

2.3. **Finite volume discretisation on unstructured staggered grids.** Staggered grid finite volume methods have been introduced by Nessyahu and Tadmor [19] for the one-dimensional case. Later, this method has been generalized to two-dimensional Cartesian grids in [13] and special two-dimensional unstructured grids in [4].

The idea of staggered grid finite volume schemes is to change consecutive grids in such a way that the time evolution corresponds to an integration over the complete Riemann fan. To be more precise, in order to define a new value at time level t^{n+1} one has to evaluate fluxes across edges of the grid at time level t^n and thanks to the use of staggered grid, values of the previous time level are uniquely given on these edges (in contrast to the case of a single grid where one has to use the values on the left and the right of the edge, see Definition 2.4). Hence, no Riemann solvers are needed. The main drawback of the use of staggered grids is the large amount of numerical dissipation which is compensated by the use of higher order methods. However, as a first step we consider in this paper the first order case, only.

We present in this section the extension of the (first order) staggered Lax-Friedrichs scheme on unstructured grids in two spatial dimensions as it has been introduced in [4] for the special case $F(x,t,s) = f(s)$. Let $\mathcal{T}_h$ be a triangular partition of $\mathbb{R}^2$ which is conforming (no hanging nodes) and regular, see Assumption 2.5.1. We use the following notations.

NOTATION 2.7.

1. *The set of vertices of triangles in $\mathcal{T}_h$ is $(a_i)_{i\in I_v}$ with $I_v \subset \mathbb{N}$. The set of indices of vertices which are direct neighbours to a_i is denoted by $N_v(i)$. Note, that the line segment a_ia_j is an edge of a triangle in $\mathcal{T}_h$.*
2. *For $i \in I_v$ and $j \in N_v(i)$ the midpoint of the line segment a_ia_j is denoted by M_{ij}. The center of gravity of the triangle which is left from the (oriented) line a_ia_j is denoted by G_{ij}^+ and the one on the right by G_{ij}^- (see Figure 2).*

From $\mathcal{T}_h$ two additional grids are built as follows.
For the first grid, finite volume cells are the barycentric cells C_i, obtained by joining the midpoints M_{ij} of the sides originating at node a_i to the centroids $G_{ij}^{\pm}$ of the triangles of $\mathcal{T}_h$ which meet at a_i (see Figure 2). For the second grid, the finite volume cells are the quadrilaterals L_{ij} obtained by joining two vertices a_i, a_j to the centroids G_{ij}^{+}, G_{ij}^{-} of the two triangles of $\mathcal{T}_h$ of which $a_i a_j$ is a side. This construction is illustrated in Figure 2.

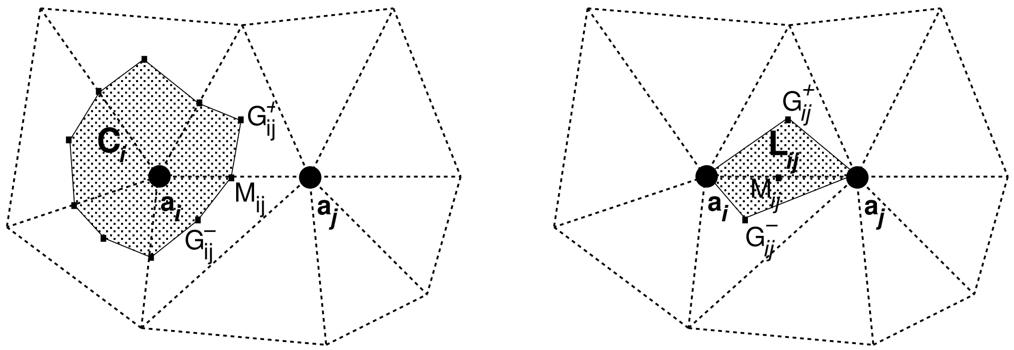


FIGURE 2. Construction of special staggered meshes from triangular mesh: grid consisting of barycentric cells (left) and grid consisting of quadrilaterals (right)

Since we need the fluxes across edges in a finite volume scheme we have to introduce further notations for the (scaled) normals of the barycentric cells which are denoted by $\eta_{ij}^{\pm}$ and normals for quadrilaterals, denoted by $\mu_{ij}^1, \dots, \mu_{ij}^4$. All normals have the length of the corresponding edge. These notations are illustrated in Figure 3. Furthermore, we need the following abbreviation.

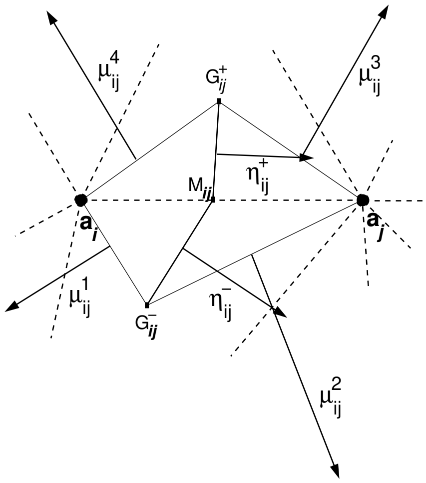


FIGURE 3. Normals on barycentric cells ($\eta_{ij}^{\pm}$) and quadrilaterals ($\mu_{ij}^1, \dots, \mu_{ij}^4$).

NOTATION 2.8.
Define for $i \in I_v$ and $j \in N_v(i)$

$$\theta_{ij} := \mu_{ij}^2 + \mu_{ij}^3. \tag{2.8}$$

REMARK 2.9. *Note, that*

$$\theta_{ij} = -(\mu_{ij}^1 + \mu_{ij}^4) = \eta_{ij}^+ + \eta_{ij}^-.$$

Now, we are in position to state the finite volume algorithm on these staggered grids (note, that the special case $F(x,t,s) = f(s)$ has been considered in [4]). The time axis is partitioned into equidistant intervals of length Δt.

DEFINITION 2.10. *For $i \in I_v$ initial values are given on the barycentric mesh.*

$$u_i^0 = \frac{1}{|C_i|} \int\limits_{C_i} u_0(x) \mathrm{d}x. \tag{2.9}$$

Let $n \in \mathbb{N}$ be an even positive integer and let $(u_i^n)_{i \in I_v}$ be values given on the mesh consisting of barycentric cells.

First step. *For $i \in I_v$ and $j \in N_v(i)$ define values on the mesh consisting of quadrilaterals by*

$$u_{ij}^{n+1} = \frac{1}{2}(u_i^n + u_j^n) - \frac{\Delta t}{|L_{ij}|}\big(f(u_i^n) - f(u_j^n)\big)\theta_{ij}. \tag{2.10}$$

Second step. *For $i \in I_v$ define values on the mesh consisting of barycentric cells by*

$$u_i^{n+2} = \sum_{j \in N_v(i)} \Big[\frac{|L_{ij}|}{2|C_i|} u_{ij}^{n+1} - \frac{\Delta t}{|C_i|} f(u_{ij}^{n+1})\theta_{ij}\Big]. \tag{2.11}$$

The discrete approximation u_h is defined by

$$u_h(x,t) = \begin{cases} u_i^n & \text{if } (x,t) \in C_i \times [t^n, t^{n+1}[, \\ u_{ij}^{n+1} & \text{if } (x,t) \in L_{ij} \times [t^{n+1}, t^{n+2}[. \end{cases} \tag{2.12}$$

In a paper by Haasdonk et al. [11, Theorem 4.1] the following convergence result has been proven.

THEOREM 2.11. *Let $F(x,t,s) = f(s) \in (C^1(\mathbb{R}))^2$, $u_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ with $M := \|u_0\|_\infty$. Let $(\mathcal{T}_{h_k})_{k \in \mathbb{N}}$ be a family of shape regular triangulations satisfying*

$$ah^2 \le |T| \le bh^2 \quad \forall T \in \mathcal{T}_{h_k}$$

uniformly in k with $0 < a \le b$; the sequence of mesh-sizes is supposed to tend to zero, i.e. $h_k \to 0$ for $k \to \infty$. Let β, γ be constants with the relation $0 < \gamma < \beta < a/4$. Let $(\Delta t_k)_{k \in \mathbb{N}}$ be a sequence of time-steps, such that for all $k \in \mathbb{N}$ the following CFL-condition holds

$$\gamma \le \frac{\Delta t_k}{h_k} \max_{s \in [-M,M], i=1,2} |f_i'(s)| \le \beta. \tag{2.13}$$

Then the sequence of approximations $(u_{h_k})_k$, which is defined in Definition 2.10 (with $h = h_k$ and $\Delta t = \Delta_k$). converges strongly in $L^p_{loc}(\mathbb{R}^2 \times \mathbb{R}^+)$ for all $1 \le p < \infty$ towards the unique entropy weak solution of (2.1).

REMARK 2.12. *We prove below an a priori error estimate in a more general framework, see Theorem 3.5, which covers the case treated in this Theorem.*

2.4. **Generalized staggered grid finite volume schemes in multi-dimensions.** We generalize the staggered grid finite volume method introduced in the section afore, (see [1] for a generalization on tetrahedral grids) to general unstructured grids in arbitrary space dimensions. Of course, the grids have to satisfy certain conditions which will be specified later on. The key idea is to give a re-interpretation of the finite volume scheme on staggered grids which is defined in Definition 2.10. As a special case, we recover the staggered grid schemes introduced by Arminjon et al. in [2, 4] for 2d and in [1] for 3d.

To present this general framework we need some more notations.
Here, we consider again an arbitrary partition of the time axis

$$0 = t^0 < t^1 < \ldots < t^n < t^{n+1} < \ldots, \quad k^n := t^{n+1} - t^n.$$

Let $(\mathcal{T}_h^n)_{n \in \mathbb{N}}$ be a sequence of finite volume grids of $\mathbb{R}^d$. Here n corresponds to the time level. That is, for every time-step we may have a different grid. We use the following notations which are a little bit involved since the grids vary from time step to time step.

NOTATION 2.13.

- $\mathcal{T}_h^n = (T_i^n)_{i\in I^n}$, $I^n \subset \mathbb{N}$, *consists of polyhedrons of finite diameter. Set* $h_i^n := \operatorname{diam}(T_i^n)$ *and* $h_n := \max_{i\in I^n} h_i^n$.
- *Define the intersection grid of* $\mathcal{T}_h^n \cap \mathcal{T}_h^{n+1}$ *by* $\mathcal{T}_h^{n,n+1} = (T_i^{n,n+1})_{i\in I^{n,n+1}}$, *where there exist unique indices* $k \in I^n$ *and* $l \in I^{n+1}$ *with* $T_i^{n,n+1} = T_k^n \cap T_l^{n+1} \neq \emptyset$.
- *For* $i \in I^{n,n+1}$ *let* $\mathcal{N}^{n,n+1}(i)$ *denote the set of all neighbouring indices* $j \in I^{n,n+1}$ *with* $S_{ij}^{n,n+1} := \overline{T}_i^{n,n+1} \cap \overline{T}_j^{n,n+1} \neq \emptyset$ *and* $S_{ij}^{n,n+1}$ *being a* $(d-1)$*-dimensional set. We allow* hanging nodes *so that* $S_{ij}^{n,n+1}$ *may consist of a finite number of line segments for* $d = 2$ *and bounded planes for* $d = 3$ *(in general: of a bounded set contained in a* $(d-1)$*-dimensional hyper-plane) since the finite volumes are polyhedrons, i. e.*
$$S_{ij}^{n,n+1} = \bigcup_{l=1}^{L_{ij}} S_{ijl}^{n,n+1}.$$
We denote by $\nu_{ijl}^{n,n+1}$ *the outer normal to* $S_{ijl}^{n,n+1}$ *of length* $|S_{ijl}^{n,n+1}|$. *Set*
$$\nu_{ij}^{n,n+1} := \sum_{l=1}^{L_{ij}} \nu_{ijl}^{n,n+1}.$$
- *(Link between* $\mathcal{T}_h^{n+1}$ *and* $\mathcal{T}_h^{n,n+1}$*) For* $i \in I^{n+1}$ *set*
$$J^{n,n+1}(i) := \{j \in I^{n,n+1} | \, T_j^{n,n+1} \subset T_i^{n+1}\}.$$
For $i \in I^{n+1}$ *and* $j \in J^{n,n+1}(i)$ *set*
$$r_{ij}^{n,n+1} := \frac{|T_j^{n,n+1}|}{|T_i^{n+1}|}.$$
- *We need a link of* $\mathcal{T}_h^n$ *to* $\mathcal{T}_h^{n,n+1}$ *as well. For* $i \in I^n$ *set*
$$K^{n,n+1}(i) := \{j \in I^{n,n+1} | \, T_j^{n,n+1} \subset T_i^n\}.$$

We illustrate these Notations with several pictures. Figure 4 shows the construction of $\mathcal{T}_h^{n+1}$ in the case that $\mathcal{T}_h^n$ consists of barycentric cells and $\mathcal{T}_h^{n+1}$ of quadrilaterals as it has been the case in Definition 2.10.

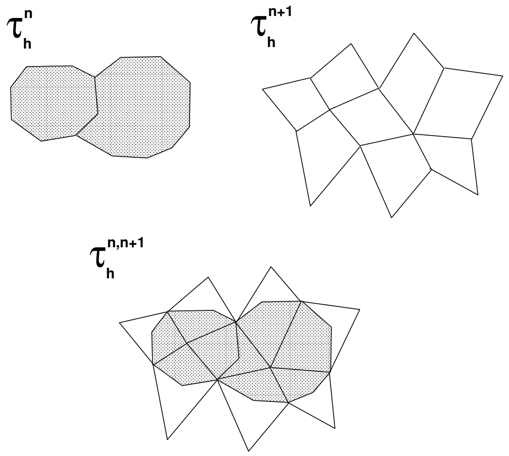


FIGURE 4. Definition of $\mathcal{T}_{h_{n,n+1}}^{n,n+1}$

Figure 5 shows the definition of the index set $J^{n,n+1}(i)$ if $i \in I^{n+1}$ is given.
Finally, Figure 6 shows the index set $K^{n,n+1}(i)$ if $i \in I^n$ is given.

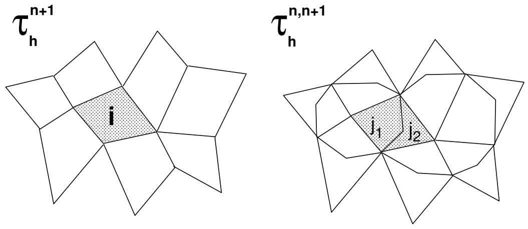


FIGURE 5. The index set $J^{n,n+1}(i)$.

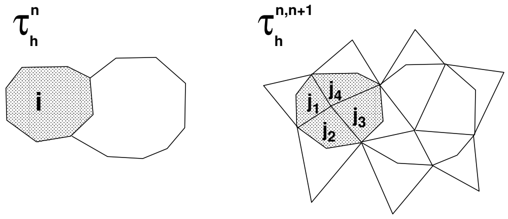


FIGURE 6. The index set $K^{n,n+1}(i)$.

Using these notations the finite volume scheme reads on these staggered grids as follows.

DEFINITION 2.14. *Define an approximation u_h to the solution of* (2.1) *by the following scheme. For $i \in I^0$ set*

$$u_i^0 := \frac{1}{|T_i^0|} \int_{T_i^0} u_0(x)\mathrm{d}x. \tag{2.14}$$

Let $n \in \mathbb{N}$ and u_i^n, $i \in I^n$ be given. Define u_i^{n+1}, $i \in I^{n+1}$, by the following algorithm.

(i) **Prolongation to $\mathcal{T}_h^{n,n+1}$**
For $i \in I^{n,n+1}$ there exists by construction an uniquely defined index $j \in I^n$ such that $i \in K^{n,n+1}(j)$. Set

$$v_i^n := u_j^n. \tag{2.15}$$

(ii) **Time evolution on $\mathcal{T}_h^{n,n+1}$**
For $i \in I^{n,n+1}$ set

$$v_i^{n+1} := v_i^n - \frac{k^n}{|T_i^{n,n+1}|} \sum_{l \in \mathcal{N}^{n,n+1}(i)} g_{il}^{n,n+1}(v_i^n, v_l^n), \tag{2.16}$$

where $g_{il}^{n,n+1}$ is some monotone, consistent, conservative Lipschitz-flux which may vary from time-step to time-step (see Assumption 2.5.2). k^n is the time step which has to meet a CFL-condition to be given below.

(iii) **Averaging to $\mathcal{T}_h^{n+1}$**
For $i \in I^{n+1}$ set

$$u_i^{n+1} = \sum_{j \in J^{n,n+1}(i)} r_{ij}^{n,n+1} v_j^{n+1}. \tag{2.17}$$

Set

$$u_h(x,t) = u_i^n \quad \text{for } t \in [t^n, t^{n+1}[\text{ and } x \in T_i^n. \tag{2.18}$$

Note, that one would implement the overall algorithm and not the single steps. Due to the conservativity of the numerical fluxes g_{il} the fluxes across interior edges of $T_i^{n+1} = \cup_{j \in J^{n,n+1}(i)} T_j^{n,n+1}$ cancel. Hence, fluxes across outer edges of T_i^{n+1} remain. If one assumes that each part of an edge (or face) of elements in $\mathcal{T}_h^{n+1}$ is contained in the interior of an element in $\mathcal{T}_h^n$ then these fluxes across outer edges reduce to evaluations of the function F, see (2.1), which is due to the consistency of the numerical fluxes.

To be more precise, we have to introduce some more notations.

NOTATION 2.15. *(Link between $\mathcal{T}_h^n$ and $\mathcal{T}_h^{n+1}$)*
Let $i \in I^{n+1}$ be given. Denote by $\tilde{K}^{n,n+1}(i)$ those indices $j \in I^n$ which correspond to elements having a nonempty intersection with T_i^{n+1} and by $\tilde{K}_\partial^{n,n+1}(i)$ those indices $j \in I^n$ which correspond to elements intersecting the boundary of T_i^{n+1}, the scaled outer normal associated with those indices by $\tilde{\nu}_{ij}^{n,n+1}$ (compare the construction of the normal in Notation 2.13) and the unit outer normal by

$$\tilde{n}_{ij}^{n,n+1} = \frac{\tilde{\nu}_{ij}^{n,n+1}}{|\tilde{\nu}_{ij}^{n,n+1}|}.$$

The set $\tilde{K}_\partial^{n,n+1}(i)$ of indices is illustrated in Figure 7.

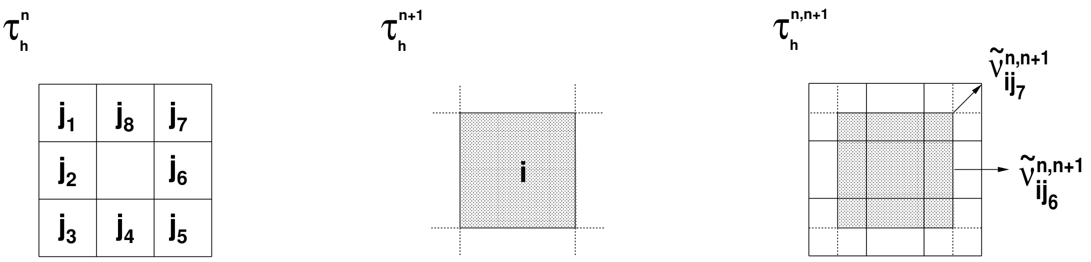


FIGURE 7. For $i \in I^{n+1}$ the set of indices $\tilde{K}_\partial^{n,n+1}(i) \subset I^n$ equals $j_1, \dots, j_8$.

Using these notations, we can define an overall algorithm which one would use in implementations.

DEFINITION 2.16. *Define an approximation u_h to the solution of* (2.1) *by the following scheme. For $i \in I^0$ set*

$$u_i^0 := \frac{1}{|T_i^0|} \int_{T_i^0} u_0(x)\mathrm{d}x. \tag{2.19}$$

For given values u_i^n, $i \in I^n$ define values u_i^{n+1}, $i \in I^{n+1}$ by

$$u_i^{n+1} = \sum_{j \in \tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1} u_j^n - \frac{k^n}{|T_i^{n+1}|} \sum_{j \in \tilde{K}_\partial^{n,n+1}(i)} F_{ij}^{n,n+1}(u_j^n), \tag{2.20}$$

with

$$\tilde{r}_{ij}^{n,n+1} := \frac{|T_j^n \cap T_i^{n+1}|}{|T_i^{n+1}|}$$

and (see also Assumption 2.5)

$$F_{ij}^{n,n+1}(s) := \frac{1}{k^n} \int_{t^n}^{t^{n+1}} \int_{\tilde{S}_{ij}^{n,n+1}} F(\gamma, t, s) \cdot \tilde{n}_{ij} \mathrm{d}\gamma \mathrm{d}t.$$

Set

$$u_h(x,t) = u_i^n \quad \text{for } t \in [t^n, t^{n+1}[\text{ and } x \in T_i^n. \tag{2.21}$$

PROPOSITION 2.17. *Assume that the the numerical fluxes in* (2.16) *satisfy Assumption 2.5 and that the sequence of staggered grids satisfy the condition that for all $n \in \mathbb{N}$, for all $i \in I^{n+1}$ and all $j \in \tilde{K}^{n,n+1}(i)$*

$$\partial T_i^{n+1} \cap \partial T_j^n \quad \text{has dimension at most } d-2 \tag{2.22}$$

holds. Then, the algorithms defined in Definition 2.14 and Definition 2.16 generate the same numerical approximation.

Proof. We simply have to use the properties of the numerical fluxes (see Assumption 2.5) and the *overlap assumption* (2.22). The proof follows by induction over $n \in \mathbb{N}$. For $n = 0$ the assertion follows by construction. Let $n > 0$ be given and let $i \in I^{n+1}$ then

$$\begin{aligned} u_i^{n+1} &\underset{(2.17)}{=} \sum_{j \in J^{n,n+1}(i)} r_{ij}^{n,n+1} v_j^{n+1} \\ &\underset{(2.16)}{=} \sum_{j \in J^{n,n+1}(i)} r_{ij}^{n,n+1} v_j^n - \frac{k^n}{|T_i^{n+1}|} \sum_{j \in J^{n,n+1}(i)} \sum_{l \in \mathcal{N}^{n,n+1}(j)} g_{jl}^{n,n+1}(v_j^n, v_l^n) \\ &\underset{(2.15)}{=} \sum_{j \in \tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1} u_j^n - \frac{k^n}{|T_i^{n+1}|} \sum_{j \in J^{n,n+1}(i)} \sum_{l \in \mathcal{N}^{n,n+1}(j)} g_{jl}^{n,n+1}(v_j^n, v_l^n) \\ &\underset{\text{Ass. 2.5, (2.22)}}{=} \sum_{j \in \tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1} u_j^n - \frac{k^n}{|T_i^{n+1}|} \sum_{j \in \tilde{K}_\partial^{n,n+1}(i)} F_{ij}^{n,n+1}(u_j^n), \end{aligned}$$

where we have used in the last step that fluxes across interior edges of $\mathcal{T}_h^{n,n+1}$ in T_i^{n+1} vanish due to the conservativity of the numerical fluxes and that fluxes across outer edges reduce to F-evaluations thanks to the overlap-condition (2.22) and the consistency of the numerical fluxes. This concludes the proof. □

REMARK 2.18.

1. *This result shows that the staggered Lax-Friedrichs scheme in Definition 2.16 can be viewed as an upwind finite volume scheme with an arbitrary good numerical flux (e.g. with Godunov flux) on a finer grid (the intersection grid $\mathcal{T}_h^{n,n+1}$) followed by a projection step.*
2. *In the error analysis we will use both point of views of the staggered Lax-Friedrichs scheme. The latter one in Definition 2.16 is used to establish an estimate for the entropy dissipation (see Proposition 4.2) while the algorithm in its decomposed form (see Definition 2.14) is used to prove a continuous entropy estimate for the approximate solution. In this proof (see Theorem 5.2 below) we can use parts of the argumentation of the corresponding result for the single-grid case (i.e. $\mathcal{T}_h^n \equiv \mathcal{T}_h$ for each time step $n \in \mathbb{N}$).*

One main demand on numerical methods for conservation laws is that these methods are mass conservative. This property is satisfied by the method in Definition 2.14 as well.

PROPOSITION 2.19 (Conservation)**.**
Let u_h be given in Definition 2.14 and let $\operatorname{supp}(u_0)$ *be compact. Then the total mass is conserved, i. e.*

$$\int_{\mathbb{R}^d} u_h(x,t)\mathrm{d}x = \int_{\mathbb{R}^d} u_0(x)\mathrm{d}x \quad \textit{for all } t > 0. \tag{2.23}$$

Proof. By definition we have

$$\int_{\mathbb{R}^d} u_h(x,0)\mathrm{d}x = \sum_{i \in I^0} |T_i^0| u_i^0 = \int_{\mathbb{R}^d} u_0(x)\mathrm{d}x.$$

The assertion follows now by induction if we prove

$$\sum_{i \in I^{n+1}} |T_i^{n+1}| u_i^{n+1} = \sum_{i \in I^n} |T_i^n| u_i^n.$$

This is a simple consequence of the definition of u_h and due to the conservativity of the numerical fluxes.

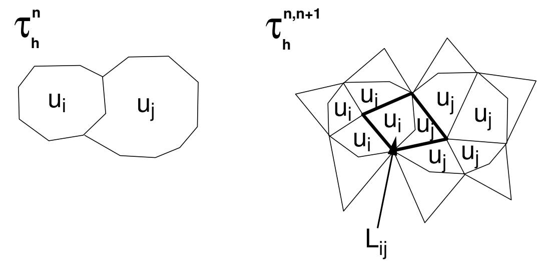
$$
\begin{aligned}
\sum_{i\in I^{n+1}} |T_i^{n+1}|\, u_i^{n+1} &= \sum_{i\in I^{n+1}} \sum_{j\in I^{n,n+1}(i)} |T_j^{n,n+1}|\, v_j^{n+1} && \text{(see (2.17))}\\
&= \sum_{j\in I^{n,n+1}} |T_j^{n,n+1}|\, v_j^{n+1}\\
&= \sum_{j\in I^{n,n+1}} |T_j^{n,n+1}|\, v_j^{n} && \text{(conservation of the fluxes)}\\
&= \sum_{i\in I^{n}} \sum_{j\in K^{n,n+1}} |T_j^{n,n+1}|\, v_j^{n}\\
&= \sum_{i\in I^{n}} \sum_{j\in K^{n,n+1}} |T_j^{n,n+1}|\, u_i^{n} && \text{(see (2.15))}\\
&= \sum_{i\in I^{n}} |T_i^{n}|\, u_i^{n}.
\end{aligned}
$$

□

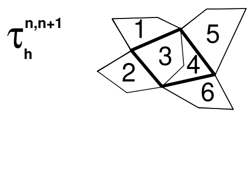
In the rest of this section we show that the staggered grid finite volume scheme defined in [4, 11] which we have stated in Definition 2.10 can be rewritten in the form introduced in Definition 2.16. In order to do so, we simply have to identify the corresponding index sets. However, since our notations are a little bit involved and in order to give some insight into the proof of Proposition 2.17 we show that that (2.10) and (2.11) can be rewritten in the form given in Definition 2.14. The main technical difficulty is the identification of index sets. Therefore, we give a more instructive "proof" by drawing pictures.
We show that the first step of the Algorithm in Definition 2.10, see (2.10), can be rewritten in the framework of Definition 2.14, i.e. assume that $n \in \mathbb{N}$ is even and that values $(u_i^n)_{i\in I}$ are given on barycentric cells, see Figure 2. We have to show that the three steps in Definition 2.14 lead to values $(u_{ij}^{n+1})_{i\in I, j\in N_v(i)}$ given by

$$
u_{ij}^{n+1} = \frac{1}{2}(u_i^n + u_j^n) - \frac{\Delta t}{|L_{ij}|}\big(f(u_i^n) - f(u_j^n)\big)\theta_{ij}.
$$

We use the same grids as in Figure 4.
Prolongation. (see (2.15)). Values $(u_i^n)_{i\in I^n}$ are trivially prolongated onto the intersection grid $\mathcal{T}_h^{n,n+1}$.



Time evolution on $\mathcal{T}_h^{n,n+1}$. For the time evolution on $\mathcal{T}_h^{n,n+1}$ we need the index set $I^{n,n+1}$. We assume an enumeration which is depicted in the following figure.



Identifying $(v_i^n)_{i\in I^{n,n+1}}$ with the corresponding values $(u_i^n)_{i\in I^n}$ we get (using the definition of the normals in Figure 3 and noting that $F(x,t,s)=f(s))$ with $i\in I_v$ and $j\in N_v(i)$

$$\begin{aligned} v_3^{n+1} &= u_i^n - \frac{\Delta t}{|L_{ij}\cap C_i|}\Big(f(u_i^n)(\mu_{ij}^4+\mu_{ij}^1)+g_{34}^n(u_i^n,u_j^n)\Big) \\ v_4^{n+1} &= u_j^n - \frac{\Delta t}{|L_{ij}\cap C_j|}\Big(f(u_j^n)(\mu_{ij}^3+\mu_{ij}^2)+g_{43}^n(u_j^n,u_i^n)\Big). \end{aligned}$$

Averaging to $\mathcal{T}_h^{n+1}$. In the situation we consider here, (2.17) reads

$$\begin{aligned} u_{ij}^{n+1} &= \frac{|L_{ij}\cap C_i|}{|L_{ij}|}u_i^n + \frac{|L_{ij}\cap C_j|}{|L_{ij}|}u_j^n + \frac{\Delta t}{|L_{ij}|}\Big[g_{34}^n(u_i^n,u_j^n)+g_{43}^n(u_j^n,u_i^n) \\ &\quad + f(u_i^n)(\mu_{ij}^4+\mu_{ij}^1)+f(u_j^n)(\mu_{ij}^3+\mu_{ij}^2)\Big]. \end{aligned}$$

Using the conservativity of the numerical flux (see Assumption 2.5), the Definition of θ_{ij} (see (2.8) and Remark 2.9) and that by construction

$$\frac{|L_{ij}\cap C_i|}{|L_{ij}|}=\frac{|L_{ij}\cap C_j|}{|L_{ij}|}=\frac{1}{2}$$

we end up with (2.11) what we wanted to show.
The proof that the second step (2.12) in Definition 2.10 fits in the framework of Definition 2.14 is similar and is left to the reader.

3. Main results and the ideas of its proofs.

In this section we state an a posteriori and an a priori error estimate between the entropy weak solution of (2.1) and the numerical approximation defined in Definition 2.14 in the L^1-norm in space and time. Then we explain the guideline of the proofs of these results.
First, we need additional assumptions on the CFL-number and on the overlap between two consecutive finite volume grids.

ASSUMPTION 3.1.

1. **Regularity of the meshes.** *Assume that $(\mathcal{T}_h^n)_{n\in\mathbb{N}}$ and $(\mathcal{T}_h^{n,n+1})_{n\in\mathbb{N}}$ satisfy the mesh regularity assumption 2.5.1 uniformly in n, i.e. the mesh regularity parameter α does not depend on h_n or $h_{n,n+1}$ for all $n\in\mathbb{N}$.*
2. **CFL-condition.** *For $n\in\mathbb{N}$ the time-step k^n is chosen such that for all $i\in I^{n,n+1}$*

$$k^n V_{[U_m,U_M]} \le \frac{1}{2}(1-\xi)\alpha^2 h_i^{n,n+1} \quad \xi\in]0,1[, \tag{3.1}$$

where $V_{[U_m,U_M]}$ is defined in Assumption 2.1. In particular, we have

$$k^n V_{[U_m,U_M]}\frac{|\partial T_i^{n,n+1}|}{|T_i^{n,n+1}|} \le \frac{1}{2}(1-\xi).$$

3. **Inverse CFL-condition.** *There exists a constant $\eta>0$ independent of $(h_{n,n+1})_{n\in\mathbb{N}}$ and $(k^n)_{n\in\mathbb{N}}$ such that for all $n\in\mathbb{N}$ and all $i\in I^{n,n+1}$ the estimate holds*

$$\eta \le \frac{k^n V_{[U_m,U_M]}}{\alpha^2 h_i^{n,n+1}}. \tag{3.2}$$

4. **Overlap condition.** *There exists a constant $C_{\mathrm{ov}}>0$ such that for all $n\in\mathbb{N}$, $i\in I^{n+1}$, $j\in\tilde{K}^{n,n+1}(i)$ the estimate holds*

$$C_{\mathrm{ov}} \le \frac{|T_j^n\cap T_i^{n+1}|}{|T_i^{n+1}|} \le 1. \tag{3.3}$$

Furthermore, suppose that for all $n\in\mathbb{N}$, for all $i\in I^{n+1}$ and all $j\in\tilde{K}^{n,n+1}(i)$

$$\partial T_i^{n+1}\cap\partial T_j^n \quad \text{has dimension at most } d-2. \tag{3.4}$$

Note, that the overlap condition states that the mesh-sizes of $\mathcal{T}_h^{n,n+1}$ and $\mathcal{T}_h^{n+1}$ should be comparable and that by (3.4)

$$\frac{|T_j^n \cap T_i^{n+1}|}{|T_i^{n+1}|} <\neq 1.$$

First, we state the a posteriori error estimate in space and time which is the analogue to the single grid result given in Kröner et al. [15, Thm. 2.11].

NOTATION 3.2.
Let $\omega = V_{[U_m,U_M]}$ and R, T be given. Set

$$\begin{aligned} I_0 &:= \Big\{ n \in \mathbb{N} |\, 0 \leq t^n \leq \min\Big\{ \frac{R+1}{\omega}, T \Big\} \Big\}, \\ N_0 &:= \max\{n|\, n \in I_0\}, \\ D \equiv D_{R+1}(x_0) &:= \{(x,t) \in \mathbb{R}^d \times \mathbb{R}_+ |\, |x - x_0| + \omega t < R+1\}, \end{aligned}$$

and for $n \in I_0$

$$I_D^{n+1} := \{ i \in I^{n+1} |\, T_i^{n+1} \times \{t^n\} \subset D_{R+1}(x_0)\}.$$

The oriented set of edges in $\mathcal{T}_h^{n,n+1}$ is defined as

$$\mathcal{E}^{n,n+1} := \{(j,l)|\, j \in I^{n,n+1},\ l \in \mathcal{N}^{n,n+1}(j) \quad \text{and } v_j^n > v_l^n\}.$$

The set of edges which are contained in D is denoted by $\mathcal{E}_D^{n,n+1}$.
Furthermore, we need the oriented set of edges $S_{jl}^n = \partial(T_j^n \cap T_l^n)$ in $\mathcal{T}_h^n$ which is defined as

$$\mathcal{E}^n := \{(j,l)|\, j \in I^n,\ l \in \mathcal{N}^n(j) \quad \text{and } u_j^n > u_l^n\}.$$

The set of edges which are contained in D is denoted by $\mathcal{E}_D^n$. Finally, we need portions of edges in $\mathcal{E}^n$ which are contained in T_i^{n+1} where $i \in I^{n+1}$. For $i \in I^{n+1}$ set

$$\tilde{\mathcal{E}}^{n,n+1}(i) := \{(j,l) \in \mathcal{E}^n | \tilde{S}_{jl}^{n,n+1}(i) := S_{jl}^n \cap T_i^{n+1} \neq \emptyset\}.$$

Note that I_D^{n+1} contains those elements which are contained in $D_{R+1}(x_0)$ at time t^n since those elements are used to evolve in time from t^n to t^{n+1}.

THEOREM 3.3 (A posteriori error estimate)**.**
Assume that the Assumptions 2.1 and 2.5 and the CFL-condition (3.1) *hold. Let $K \subset\subset \mathbb{R}^d \times \mathbb{R}_+$, $\omega = V_{[U_m,U_M]}$ and choose $T, R > 0$ and $x_0 \in \mathbb{R}^d$ such that $T \in]0, R/\omega[$ and*

$$K \subset \bigcup_{0 \leq t \leq T} B_{R-\omega t}(x_0) \times \{t\}.$$

Then we have

$$\int_K |u - u_h| \leq T \Big[\int_{|x-x_0|<R+1} |u_0 - u_h(.,0)| + aQ + \sqrt{bcQ} \Big], \tag{3.5}$$

where

$$\begin{aligned} a &:= 2\omega + \frac{1}{T} + 2, \\ b &:= 4 + 2^{d+2}, \\ c &:= \|u\|_{\mathrm{BV}} \big[2(2\omega + \frac{1}{T}) + V_{[U_m,U_M]}(8 + 2^{d+5}) \big] \\ &\quad + \|u_0\|_{\mathrm{BV}} \big[2^{d+4} V_{[U_m,U_M]} + 1 \big] + 2V_{[U_m,U_M]} \max\{U_m, U_M\} \left[|B_{R+1}(0)| - |B_R(0)| \right] T, \end{aligned}$$

and

$$
\begin{aligned}
Q \;=\; & \frac{1}{2}\sum_{n=0}^{N_0}\sum_{i\in I_D^{n+1}} h_i^{n+1}|T_i^{n+1}| \sum_{j,l\in\tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1}\tilde{r}_{il}^{n,n+1}|u_j^n-u_l^n| \\
& +\sum_{n=0}^{N_0-1} k^n \sum_{i\in I_D^{n+1}} |T_i^{n+1}| \left|u_i^{n+1}-\sum_{j\in\tilde{K}^{n,n+1}(i)}\tilde{r}_{ij}^{n,n+1}u_j^n\right| \\
& +6V_{[U_m,U_M]}\sum_{n=0}^{N_0} k^n \sum_{i\in I_D^{n+1}} (h_i^{n+1}+k^n) \sum_{(j,l)\in\tilde{\mathcal{E}}^{n,n+1}(i)} |\tilde{S}_{jl}^{n,n+1}(i)|\;|u_j^n-u_l^n| \\
& +C_{F,R,T,u_0}\sum_{n=0}^{N_0}\sum_{(j,l)\in\mathcal{E}_D^{n,n+1}} k^n|S_{jl}^{n,n+1}|\left[\,\mathrm{diam}(S_{jl}^{n,n+1})+k^n\right]^2,
\end{aligned}
$$

where C_{F,R,T,u_0} is defined in (5.12)*, see also Lemma 6.7.*

REMARK 3.4.

1. *Note, that the domain of dependence of the error is taken into account since the set K is embedded into the characteristic cone $D_{R+1}(x_0)$.*
2. *If $F(x,t,s) = f(s)$ then one can choose $C_{F,R,T,u_0} = 0$ (see* (5.12) *in the proof of Theorem 5.2 below).*

From the a posteriori error estimate one gets the following a priori error estimate.

THEOREM 3.5 (A priori error estimate)**.**
Let Assumptions 2.1, 2.5 and 3.1 hold. Let u be the unique entropy solution to (2.1) *and let u_h be given by Definition 2.14. Then, for all compact $K \subset\subset \mathbb{R}^d\times\mathbb{R}^+$ there exists a constant $C_e>0$, which depends on K, F, u_0 and the geometrical constants given above, only, such that*

$$
\int_K |u_h(x,t)-u(x,t)|\,\mathrm{d}x\mathrm{d}t \le C_e h^{1/4}, \tag{3.6}
$$

where

$$
h := \sup_{0\le n\le N_0}\max\{h_n, h_{n,n+1}\}.
$$

The proof of this result relies on a technique introduced by Kruzkov [17] which has been used by Kuznetsov [18] to prove an error estimate for first order finite volume schemes in one space dimension. The key idea is to use the concept of entropy, see Definition 2.2, and the so-called technique of *doubling the variables*. This part can be done on an abstract level and is contained in the Approximation Lemma 3.6 below.

Then all the conditions in this Approximation Lemma have to be verified. The main difficulty is to prove a continuous entropy inequality for the discrete solution.

For a simple introduction concerning error estimation for hyperbolic problems we refer to Cockburn [6].

We will employ the following Approximation Lemma [5, Lemma 10] in which error terms are expressed with the help of some positive Radon measures. For $\Omega=\mathbb{R}^d$ or $\Omega=\mathbb{R}^d\times\mathbb{R}^+$ we denote by $\mathcal{M}(\Omega)$ the set of positive Radon measures, i.e. the set of positive continuous linear forms on $C_0^0(\Omega)$. For $\mu\in\mathcal{M}(\Omega)$ we set

$$
\langle\mu,g\rangle = \int_\Omega g\,\mathrm{d}\mu, \quad g\in C_0^0(\Omega).
$$

LEMMA 3.6 (Approximation Lemma)**.** *Assume 2.1 and $u_0\in BV_{\mathrm{loc}}(\mathbb{R}^d)$. Let $\tilde{u}\in L^\infty(\mathbb{R}^d\times\mathbb{R}^+)$. Assume that there exist measures $\mu\in\mathcal{M}(\mathbb{R}^d\times\mathbb{R}^+)$ and $\mu_0\in\mathcal{M}(\mathbb{R}^d)$ such that for all $\kappa\in\mathbb{R}$ and all*

$\phi \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^+, \mathbb{R}^+)$

$$\int\limits_{\mathbb{R}^d\times\mathbb{R}^+} \Big\{|\tilde u(x,t)-\kappa|\phi_t(x,t) + \big[F(x,t,\tilde u(x,t)\top\kappa) - F(x,t,\tilde u(x,t)\bot\kappa)\big]\cdot\nabla\phi(x,t)\Big]\mathrm{d}x\mathrm{d}t$$
$$+\int\limits_{\mathbb{R}^d} |u_0(x)-\kappa|\phi(x,0)\mathrm{d}x$$
$$\geq -\int\limits_{\mathbb{R}^d\times\mathbb{R}^+} \big[|\phi_t(x,t)| + |\nabla\phi(x,t)|\big]\mathrm{d}\mu(x,t) - \int\limits_{\mathbb{R}^d}\phi(x,0)\mathrm{d}\mu_0(x). \tag{3.7}$$

Let u be the unique entropy solution to (2.1)*, i.e. for all $\kappa \in \mathbb{R}$ and all $\phi \in C_0^\infty(\mathbb{R}^d\times\mathbb{R}^+,\mathbb{R}^+)$ the estimate holds*

$$\int\limits_{\mathbb{R}^d\times\mathbb{R}^+} \Big\{|u(x,t)-\kappa|\partial_t\varphi(x,t) + [F(x,t,u(x,t)\top\kappa) - F(x,t,u(x,t)\bot\kappa)]\nabla\varphi(x,t)\Big\}\mathrm{d}t\mathrm{d}x$$
$$+\int\limits_{\mathbb{R}^d}|u_0(x)-\kappa|\varphi(x,0)\mathrm{d}x \geq 0, \tag{3.8}$$

Then, for all compact sets $E \subset \mathbb{R}^d\times\mathbb{R}^+$, there exist positive $C_{E,F,u_0} > 0$, R and T which depend on E, F and u_0 only, such that the following error estimate holds:

$$\int\limits_E |\tilde u(x,t) - u(x,t)|\mathrm{d}x\mathrm{d}t \quad\leq\quad C_{E,F,u_0}\Big(\mu_0(B_R(0)) + \mu(B_R(0)\times[0,T])$$
$$+[\mu(B_R(0)\times[0,T])]^{1/2}\Big). \tag{3.9}$$

In order to apply this Lemma we have to prove a L^∞-bound for the discrete solution (see Proposition 4.1 below) and we have to prove inequality (3.7) for the approximate solution (see Theorem 5.2) which follows from a discrete entropy inequality. Then, we have to control the behaviour of the measures which are involved. To do this, we need some regularity estimates of the discrete solution, namely BV-regularity. One can prove a strong BV-regularity result for the solution of (2.1), see Theorem 3 in [5]

$$\|u\|_{BV_{\mathrm{loc}}(\mathbb{R}^d\times\mathbb{R}^+)} \leq C(F,u_0).$$

In case of $F(x,t,s) = f(s)$ this can be sharpened as follows, see [10, Thm. 2.3.1]: for all $t > 0$

$$|u(.,t)|_{\mathrm{BV}(\mathbb{R}^d)} \leq |u_0|_{\mathrm{BV}(\mathbb{R}^d)}.$$

Unfortunately, it is so far not possible to prove such a strong BV-estimate for the discrete approximation on arbitrary unstructured grids. However, it is sufficient to prove a *weak* BV-estimate which is payed for by a loss in the convergence order (see Proposition 4.4 below).
The proofs of these results are given in the rest of the paper and then the proofs of Theorem 3.3 and 3.5 consists in putting together these results and to apply Lemma 6.1 below which is a refined version of Lemma 3.6.

4. Stability results.

We prove a maximum principle and an estimate of the entropy production from which weak BV-estimates of the approximate solution can be deduced.

PROPOSITION 4.1 (Maximum Principle)**.**
Let $u_0 \in L^\infty(\mathbb{R}^d)$ with $U_m \leq u(x) \leq U_M$ almost everywhere. Let u_h be defined in Definition 2.14 and suppose that Assumption 2.5 holds. Then

$$U_m \leq u_i^n \leq U_M, \quad \forall n\in\mathbb{N},\ \forall i\in I^n, \tag{4.1}$$

and

$$\|u_h\|_{L^\infty(\mathbb{R}^d\times[0,T])} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}, \quad \forall T > 0. \tag{4.2}$$

Proof. This is a proof by induction over $n \in \mathbb{N}$. Note that due to the convex combination in (2.17) we have

$$\min_{j \in J^{n,n+1}(i)} v_j^{n+1} \le u_i^{n+1} \le \max_{j \in J^{n,n+1}(i)} v_j^{n+1}.$$

For the v-values on $\mathcal{T}_h^{n,n+1}$ a maximum principle is well-known under condition (3.1), see [5, Lemma 1]. Taking into account that the first step of the algorithm (2.15) is just a trivial prolongation where the function is not changed, concludes the proof. ☐

The next result is a key result for proving the a priori error estimate. This result is a straightforward generalization of [11, Prop. 4.4 and Lemma 4.8]. For the sake of completeness we include its proof here.

Unfortunately, we cannot work with the Kruzkov entropies but have to deal with quadratic entropies. Let $U(x) = x^2$ and Φ the associated entropy flux, i. e.

$$\partial_s \Phi(x,t,.) = U'(.)\partial_s F(x,t,.) \quad \text{a.e.} \quad .$$

We set for $i \in I^{n+1}$ and $j \in \tilde{K}_\partial^{n,n+1}(i)$

$$\Phi_{ij}^{n,n+1}(s) := \frac{1}{k^n} \int_{t^n}^{t^{n+1}} \int_{\tilde{S}_{ij}^{n,n+1}} \Phi(\gamma,t,s) \cdot \tilde{n}_{ij}^{n,n+1} \mathrm{d}\gamma \mathrm{d}t. \tag{4.3}$$

PROPOSITION 4.2 (Entropy production estimate).
Assume that Assumptions 2.1, 2.5 and 3.1 hold where ξ in (3.1) has to fulfill the restriction $\xi \in]0, C_{\mathrm{ov}}[$. Then for $i \in I^{n+1}$ the following estimate holds

$$\begin{aligned} U(u_i^{n+1}) - \sum_{j \in \tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1} U(u_j^n) - \frac{k^n}{|T_i^{n+1}|} \sum_{j \in \tilde{K}_\partial^{n,n+1}(i)} \Phi_{ij}^{n,n+1}(u_j^n) \\ + \xi^2 \sum_{j \in \tilde{K}^{n,n+1}(i)} \sum_{l \in \tilde{K}^{n,n+1}(i)} U(u_j^n - u_l^n) \le 0. \end{aligned} \tag{4.4}$$

Proof. Let $n \in \mathbb{N}$ and $i \in I^{n+1}$ be given. Enumerate the values u_j^n, $j \in \tilde{K}^{n,n+1}(i)$ by size. To be more precise, set $m := |\tilde{K}^{n,n+1}(i)|$ and define a bijection $\alpha : \{1, \dots, m\} \to \tilde{K}^{n,n+1}(i)$ with $u_l := u_{i\alpha(l)}^n$ and

$$u_1 \le u_2 \le \dots \le u_m.$$

Analogously, set $r_l := \tilde{r}_{i\alpha(l)}^{n,n+1}$ and for $\alpha(l) \in \tilde{K}_\partial^{n,n+1}(i)$ we set $F_l(s) = F_{i\alpha(l)}^{n,n+1}(s)$ and $\Phi_l = \Phi_{i\alpha(l)}^{n,n+1}(s)$. To simplify notation set

$$\delta_l := \begin{cases} 1 & \alpha(l) \in \tilde{K}_\partial^{n,n+1}(i), \\ 0 & \text{else.} \end{cases}$$

Using these notations we can rewrite (2.20) as follows

$$u_i^{n+1} = \sum_{l=1}^m \left[r_l u_l - \frac{k^n}{|T_i^{n+1}|} F_l(u_l)\delta_l \right] =: q(u_1, \dots, u_m). \tag{4.5}$$

Set

$$p(t_1, \dots, t_m) := U(q(t_1, \dots, t_m)) - \sum_{l=1}^m \left[r_l U(t_l) - \frac{k^n}{|T_i^{n+1}|} \Phi_l(t_l)\delta_l \right] + \xi^2 \sum_{j,l=1}^m U(t_l - t_j).$$

We have to prove that

$$p(u_1, \dots, u_m) \le 0.$$

Clearly, p is differentiable. Define values $P_1, \ldots, P_m \in \mathbb{R}^m$ as follows

$$\begin{aligned} P_1 &:= (u_1, \ldots, u_1), \\ P_2 &:= (u_1, u_2, \ldots, u_2), \\ &\vdots \\ P_{m-1} &:= (u_1, \ldots, u_{m-1}, u_{m-1}), \\ P_m &:= (u_1, \ldots, u_m). \end{aligned}$$

We prove that

i) $p(P_1) = 0$.

ii) For $k = 2, \ldots, m$ and for all $x \in I(P_{k-1}, P_k)$ we have

$$\nabla P(x)(P_k - P_{k-1}) \leq 0,$$

where $I(P_{k-1}, P_k)$ denotes the straight line between the points P_{k-1} and P_k.

The assertion $i)$ follows directly. For proving $ii)$ let $l \in \{1, \ldots, m\}$ be given. Using $\partial_s \Phi(x,t,.) = U'(.)\partial_s F(x,t,.)$ a.e. and $U'(s) = 2s$ we calculate

$$\partial_l p(t_1, \ldots, t_m) = 2 \sum_{k=1}^{m} \left\{ \left[r_l - \frac{k^n}{|T_i^{n+1}|} F_l'(t_l)\delta_l \right] \left[r_l - \frac{k^n}{|T_i^{n+1}|} F_l'(\eta_{kl})\delta_l \right] - \xi^2 \right\}_1 (t_k - t_l),$$

where we used that

$$q(t_1, \ldots, t_m) - t_l = \sum_{k=1}^{m} \left[r_k(t_k - t_l) - \frac{k^n}{|T_i^{n+1}|} F_k'(\eta_{kl})\delta_k(t_k - t_l) \right]$$

and η_{kl} is a value between t_k and t_l. Using the CFL–condition (3.1) and $\xi \in]0, C_{\text{ov}}[$ one gets that $\{\ldots\}_1 \geq 0$. Finally, with the definition of the values P_k it is now easy to verify $ii)$ which concludes the proof. □

REMARK 4.3. *Using this entropy production estimate we are in position to prove weak BV-estimates. These estimate are different to the corresponding results on a fixed grid in the following respect. First, these results are weak BV-estimates since half an order of convergence is lost from optimality* [20]. *The estimates are stronger than those on fixed grids (cf. e.g.* [5, Lemma 2]*) since those have the form (with notations obviously changed for the case of a fixed grid)*

$$\sum_{n=0}^{N_0} k^n \sum_{(j,l)\in\mathcal{E}_D} C_{jl}(u_j^n, u_l^n)|u_j^n - u_l^n| \leq Ch^{-1/2}$$

where

$$C_{jl}(u,v) = \frac{g_{jl}(u,u) - 2g_{jl}(u,v) - g_{jl}(v,v)}{u - v}.$$

To make the point, in our BV estimate we have a factor 1 *instead of C_{jl} in front of differences of values $(u_i^n)_{i\in I^n}$. This factor* 1 *is essentially needed since it appears naturally in the continuous entropy estimate for the approximate solution (see Theorem 5.2 below) which is due to the averaging step* (2.17) *in the numerical scheme in Definition 2.14. It is possible to generalize the result of Chainais-Hillairet* [5, Lemma 2] *to the situation we consider here but it is not possible to estimate the factors C_{jl}* uniformly *from below.*

PROPOSITION 4.4 (Weak local space BV estimate)**.**
Assume that the conditions of Proposition 4.2 hold. Then there exists a constant $C > 0$, depending on the characteristic cone D, on data F, u_0, the geometry parameter α and the CFL-condition parameter ξ only, such that

$$\sum_{n=0}^{N_0} k^n \sum_{i\in I_D^{n+1}} |T_i^{n+1}| \sum_{j,l\in\tilde{K}^{n,n+1}(i)} |u_j^n - u_l^n| \leq Ch^{1/2}. \tag{4.6}$$

where

$$h = \max_{n=0,\dots,N_T} h_n. \tag{4.7}$$

Proof. First, we note that for $i \in I^{n+1}$ by Assumption 3.1

$$|\tilde{K}^{n,n+1}(i)| = \sum_{j\in\tilde{K}^{n,n+1}(i)} \frac{|T_j^n \cap T_i^{n+1}|}{|T_j^n \cap T_i^{n+1}|} \le \frac{1}{\min_{j\in\tilde{K}^{n,n+1}(i)} |T_j^{n,n+1}|}|T_i^{n+1}| \le \frac{1}{C_{\mathrm{ov}}}.$$

Using Cauchy's inequality we see that

$$\begin{aligned}
&\sum_{n=0}^{N_0} k^n \sum_{i\in I_D^{n+1}} |T_i^{n+1}| \sum_{j,l\in\tilde{K}^{n,n+1}(i)} |u_j^n - u_l^n| \\
\le\ & C_{\mathrm{ov}}^{-1/2} \sum_{n=0}^{N_0} k^n \sum_{i\in I_D^{n+1}} |T_i^{n+1}| \Big(\sum_{j,l\in\tilde{K}^{n,n+1}(i)} |u_j^n - u_l^n|^2\Big)^{1/2} \\
\le\ & C_{\mathrm{ov}}^{-1/2} \Big[\sum_{n=0}^{N_0} k^n \sum_{i\in I_D^{n+1}} |T_i^{n+1}|\Big]^{1/2} \Big[\sum_{n=0}^{N_0} k^n \sum_{i\in I_D^{n+1}} |T_i^{n+1}| \sum_{j,l\in\tilde{K}^{n,n+1}(i)} |u_j^n - u_l^n|^2\Big]^{1/2}
\end{aligned}$$

$$\le \Big(\frac{|D_{R+1}(x_0)|}{C_{\mathrm{ov}}}\Big)^{1/2} \Big[\sum_{n=0}^{N_0} k^n \sum_{i\in I_D^{n+1}} |T_i^{n+1}| \sum_{j,l\in\tilde{K}^{n,n+1}(i)} |u_j^n - u_l^n|^2\Big]^{1/2}. \tag{4.8}$$

In order to estimate the last factor we multiply (4.4) by T_i^{n+1} and sum over $i \in I_D^{n+1}$ and get

$$\begin{aligned}
&\|u_h(.,t^{n+1})\|_{L^2(D_{R+1}\cap(B_{R+1}\times\{t^{n+1}\}))} - k^n \sum_{i\in I_D^{n+1}} \sum_{j\in\tilde{K}_\partial^{n,n+1}(i)} \Phi_{ij}^{n,n+1}(u_j^n) \\
&+\xi^2 \sum_{i\in I_D^{n+1}} |T_i^{n+1}| \sum_{j,l\in\tilde{K}^{n,n+1}(i)} |u_j^n - u_l^n|^2 \\
&\qquad \le \|u_h(.,t^n)\|_{L^2(D_{R+1}\cap(B_{R+1}\times\{t^{n+1}\}))} \\
&\qquad \le \|u_h(.,t^n)\|_{L^2(D_{R+1}\cap(B_{R+1}\times\{t^{n}\}))}.
\end{aligned}$$

Let us consider the term containing the numerical entropy fluxes $\Phi_{ij}^{n,n+1}$. Obviously, these fluxes are conservative in the sense of Assumption 2.5 and hence the sum over all edges (i,j) with $i \in I_D^{n+1}$ and $j \in \tilde{K}_\partial^{n,n+1}(i)$ reduces to the sum over all edges lying on the boundary of the characteristic cone $D_{R+1} \cap (B_{R+1} \times \{t^{n+1}\})$. The number of such edges is bounded by $C(\alpha)h^{d-1}$. Furthermore, since we have the maximum principle, see Proposition 4.1, we know that there exists a constant $C > 0$, depending on D, F, u_0, U only such that

$$|\Phi_{ij}^{n,n+1}(s)| \le C|S_{ij}^{n,n+1}| \quad \forall s \in [U_m, U_M].$$

Using this we can continue our estimate and get

$$\begin{aligned}
&\|u_h(.,t^{n+1})\|_{L^2(D_{R+1}\cap(B_{R+1}\times\{t^{n+1}\}))} + \xi^2 \sum_{i\in I_D^{n+1}} |T_i^{n+1}| \sum_{j,l\in\tilde{K}^{n,n+1}(i)} |u_j^n - u_l^n|^2 \\
&\qquad \le \|u_h(.,t^n)\|_{L^2(D_{R+1}\cap(B_{R+1}\times\{t^{n}\}))} + Ck^n.
\end{aligned}$$

Summing over $n \in \{0, \dots, N_0\}$ we end up with

$$\sum_{n=0}^{N_0} \sum_{i\in I_D^{n+1}} |T_i^{n+1}| \sum_{j,l\in\tilde{K}^{n,n+1}(i)} |u_j^n - u_l^n|^2 \le \frac{1}{\xi^2}\|u_0\|_{L^2(B_{R+1}(x_0))} + C.$$

Using this and applying the CFL-condition (3.1) in inequality (4.8) concludes the proof. $\square$

Finally, we are in position to prove a weak time BV-estimate.

PROPOSITION 4.5. *Let the Assumptions of Proposition 4.4 hold. Then there exists a constant $C > 0$, depending on the characteristic cone D, on data F, u_0, the geometry parameter α and the CFL-condition parameter ξ only, such that*

$$\sum_{n=0}^{N_0-1} k^n \sum_{i\in I_R^{n+1}} |T_i^{n+1}| \left| u_i^{n+1} - \sum_{j\in \tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1} v_j^n \right| \le C h^{1/2} \tag{4.9}$$

Proof. The assertion of this proposition follows from Proposition 4.4. Note, that by definition of the index sets $J^{n,n+1}(i)$ and $\tilde{K}^{n,n+1}(i)$ (Notation 2.13 and Notation 2.15) we have

$$\sum_{j\in J^{n,n+1}(i)} r_{ij} v_j^n = \sum_{j\in \tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1} u_j^n.$$

Using the definition of u_i^{n+1}, see (2.17), and the Lipschitz continuity of the numerical fluxes (Assumption 2.5) we get

$$\begin{aligned}
&\left| u_i^{n+1} - \sum_{j\in \tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1} u_j^n \right| \\
\le\ & \sum_{j\in J^{n,n+1}(i)} r_{ij}^{n,n+1} |v_j^{n+1} - v_j^n| \\
\le\ & \sum_{j\in J^{n,n+1}(i)} r_{ij}^{n,n+1} \frac{k^n}{|T_j^{n,n+1}|} \sum_{l\in \mathcal{N}^{n,n+1}(j)} |g_{jl}^{n,n+1}(v_j^n, v_l^n) - g_{jl}^{n,n+1}(v_j^n, v_j^n)| \\
\le\ & \sum_{j\in J^{n,n+1}(i)} r_{ij}^{n,n+1} \frac{k^n}{|T_j^{n,n+1}|} L_g \sum_{l\in \mathcal{N}^{n,n+1}(j)} |S_{jl}^{n,n+1}|\, |v_j^n - v_l^n| \\
\le\ & \frac{1}{2}(1-\xi) \sum_{j,l\in J^{n,n+1}(i)} |v_j^n - v_l^n| \\
=\ & \frac{1}{2}(1-\xi) \sum_{j,l\in \tilde{K}^{n,n+1}(i)} |u_j^n - u_l^n|,,
\end{aligned}$$

where we have used (3.3) and the CFL-condition (3.1). □

REMARK 4.6. *Note, that we have estimated in the previous proof the Lipschitz constant of the numerical fluxes by*

$$L_g \le V_{[U_m, U_M]}.$$

Since in practice no numerical flux is needed and since we use these numerical fluxes just as a tool for our analysis we have freedom to choose the numerical flux. The aforementioned inequality holds for example if $g_{jl}^{n,n+1}$ is chosen as Godunov-flux, see e.g. [5, Section 2.1].

5. Continuous entropy estimate for the approximate solution

In this section we prove a continuous entropy estimate for the approximate solution. Therefore, we need a *discrete entropy inequality* for the approximate solution. For the concept of entropy solutions and discrete entropy inequalities we refer to [14, 21, 9].

LEMMA 5.1 (Discrete entropy inequalities). *Let u_h be given in Definition 2.14 and assume that the Assumptions 2.1 and 2.5 and that the CFL-condition* (3.1) *hold. For $j, l \in I^{n,n+1}$ define a numerical entropy flux $G_{jl}^{n,n+1}$ by*

$$G_{jl}^{n,n+1}(u, v, \kappa) := g_{jl}^{n,n+1}(u\top\kappa, v\top\kappa) - g_{jl}^{n,n+1}(u\bot\kappa, v\bot\kappa), \tag{5.1}$$

where

$$a\top b := \max\{a, b\} \quad \textit{and } a\bot b := \min\{a, b\}$$

and $g_{jl}^{n,n+1}$ is the numerical flux in (2.16).

Then the following discrete entropy inequalities hold for all $n \in \mathbb{N}$

$$
\begin{aligned}
&|T_j^{n+1}| \frac{|v_j^{n+1} - \kappa| - |v_j^n - \kappa|}{k^n} \\
&\quad + \sum_{l \in \mathcal{N}^{n,n+1}(j)} G_{jl}^{n,n+1}(v_j^n, v_l^n, \kappa) \leq 0, \quad \forall j \in I^{n,n+1},
\end{aligned} \tag{5.2}
$$

$$
\begin{aligned}
&\frac{1}{k^n}\Big[|T_i^{n+1}|\,|u_i^{n+1} - \kappa| - \sum_{j \in J^{n,n+1}(i)} |T_j^{n,n+1}|\,|v_j^n - \kappa|\Big] \\
&\quad + \sum_{j \in J^{n,n+1}(i)} \sum_{l \in \mathcal{N}^{n,n+1}(j)} G_{jl}^{n,n+1}(v_j^n, v_l^n, \kappa) \leq 0, \quad \forall i \in I^{n+1}.
\end{aligned} \tag{5.3}
$$

Proof. The assertion (5.2) follows from the monotony of the second step in Definition 2.14 which is due to the monotony of the numerical fluxes $g_{jl}^{n,n+1}$ (see Assumption 2.5). The details can be found in [5, Lemma 3]. The second statement follows from the first one using that u_i^{n+1} is a convex combination of $(v_j^{n+1})_{j \in J^{n,n+1}(i)}$, see (2.17). □

For $\Omega = \mathbb{R}^d$ or $\mathbb{R}^d \times \mathbb{R}_+$ we denote by $\mathcal{M}(\Omega)$ the set of positive Radon measures on Ω. The following theorem is an adaption of [9, Theorem 6.1] to our case of varying grids.

THEOREM 5.2. *Assume that Assumptions 2.1, 2.5 and the CFL-condition* (3.1)*hold. Let* u_h *be given by Definition 2.14. Then there exist measures* $\mu_h \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}_+)$ *and* $\mu_0 \in \mathcal{M}(\mathbb{R}^d)$ *such that for all* $\kappa \in \mathbb{R}$ *and all* $\phi \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+)$ *the following estimate holds*

$$
\begin{aligned}
&\int_{\mathbb{R}^+}\int_{\mathbb{R}^d} \big(|u_h(x,t) - \kappa|\phi_t(x,t) + \big[F(x,t,u_h(x,t)\top\kappa) - F(x,t,u_h(x,t)\bot\kappa)\big]\nabla\phi(x,t)\big)\mathrm{d}x\mathrm{d}t \\
&+ \int_{\mathbb{R}^d} |u_0(x) - \kappa|\phi(x,0)\mathrm{d}x \\
&\geq - \int_{\mathbb{R}^d \times \mathbb{R}_+} \left(|\phi_t(x,t)| + |\nabla\phi(x,t)|\right)\mathrm{d}\mu_h(x,t) - \int_{\mathbb{R}^d} \phi(x,0)\mathrm{d}\mu_0(x).
\end{aligned} \tag{5.4}
$$

The measures are defined as follows. Let $\psi \in C_0(\mathbb{R}^d)$. *Then*

$$
\langle \mu_0, \psi \rangle \equiv \int_{\mathbb{R}^d} \psi(x)\mathrm{d}\mu_0(x) := \int_{\mathbb{R}^d} |u_h(x,0) - u_0(x)|\psi(x,0)\mathrm{d}x. \tag{5.5}
$$

The measure μ_h *is given by* $\mu_h = \mu_{h1} + \mu_{h2} + \mu_{h3} + \mu_{h4}$, *where the four parts are defined as (with* $\psi \in C_0(\mathbb{R}^d \times \mathbb{R}_+)$*)*

$$
\begin{aligned}
\langle \mu_{h1}, \psi \rangle &:= \frac{1}{2}\sum_{n \in \mathbb{N}} \sum_{i \in I^{n+1}} \sum_{j,l \in \tilde{K}^{n,n+1}(i)} |u_j^n - u_l^n| \\
&\quad \times \frac{h_i^{n+1}}{|T_i^{n+1}|} \int_{T_l^n \cap T_i^{n+1}} \int_{T_j^n \cap T_i^{n+1}} \int_0^1 \psi(x + \theta(y-x), t^{n+1})\mathrm{d}\theta\mathrm{d}y\mathrm{d}x,
\end{aligned} \tag{5.6}
$$

$$
\begin{aligned}
\langle \mu_{h2}, \psi \rangle &:= \sum_{n \in \mathbb{N}} \sum_{i \in I^{n+1}} \Big|u_i^{n+1} - \sum_{j \in \tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1} u_j^n\Big| \\
&\quad \times \int_{t^n}^{t^{n+1}} \int_{T_i^{n+1}} \int_0^1 \psi(x, t + \theta(t^{n+1} - t))\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{5.7}
$$

$$\begin{aligned}\langle \mu_{h3}, \psi\rangle &:= 2\sum_{n\in\mathbb{N}} k^n \sum_{i\in I^{n+1}} \sum_{j\in J^{n,n+1}(i)} \sum_{l\in\mathcal{N}^{n,n+1}(j)} \left|g_{jl}^{n,n+1}(v_j^n, v_l^n) - g_{jl}^{n,n+1}(v_j^n, v_j^n)\right| \\ &\quad \times \frac{h_i^{n+1}}{k^n\,|T_i^{n+1}|\,|T_j^{n,n+1}|} \int_{t^n}^{t^{n+1}} \int_{T_i^{n+1}} \int_{T_j^{n,n+1}} \int_0^1 \psi(x+\theta(y-x),t)\mathrm{d}\theta\mathrm{d}x\mathrm{d}y\mathrm{d}t, \end{aligned} \tag{5.8}$$

$$\begin{aligned}\langle \mu_{h4}, \psi\rangle &:= 2\sum_{n\in\mathbb{N}} k^n \sum_{(j,l)\in\mathcal{E}^{n,n+1}} \Big\{ \left|g_{jl}^{n,n+1}(v_j^n, v_l^n) - g_{jl}^{n,n+1}(v_j^n, v_j^n)\right| \langle \mu_{jl}^{n,n+1}, \psi\rangle \\ &\quad + \left|g_{jl}^{n,n+1}(v_j^n, v_l^n) - g_{jl}^{n,n+1}(v_l^n, v_l^n)\right| \langle \mu_{lj}^{n,n+1}, \psi\rangle \Big\} \\ &\quad + 4C_{F,\phi,u_0} \sum_{n\in\mathbb{N}} \sum_{(j,l)\in\mathcal{E}^{n,n+1}} \langle \lambda_{jl}^{n,n+1}, \psi\rangle \end{aligned} \tag{5.9}$$

where

$$\begin{aligned}\langle \mu_{jl}^{n,n+1}, \psi\rangle &:= \frac{h_j^{n,n+1}+k^n}{(k^n)^2} \int_{t^n}^{t^{n+1}} \int_{t^n}^{t^{n+1}} \frac{1}{|T_j^{n,n+1}|} \int_{T_j^{n,n+1}} \frac{1}{|S_{jl}^{n,n+1}|} \int_{S_{jl}^{n,n+1}} \int_0^1 \\ &\quad \psi(x+\theta(y-x), t+\theta(s-t))\mathrm{d}\theta\mathrm{d}\gamma(y)\mathrm{d}x\mathrm{d}s\mathrm{d}t, \end{aligned} \tag{5.10}$$

$$\begin{aligned}\langle \lambda_{jl}^{n,n+1}, \psi\rangle &:= \frac{(\,\mathrm{diam}\,(S_{jl}^{n,n+1}) + k^n)^2}{k^n |S_{jl}^{n,n+1}|} \int_{t^n}^{t^{n+1}} \int_{t^n}^{t^{n+1}} \int_{S_{jl}^{n,n+1}} \int_{S_{jl}^{n,n+1}} \int_0^1 \\ &\quad \psi(\xi+\theta(\gamma-\xi), t+\theta(s-t))\mathrm{d}\theta\mathrm{d}\xi\mathrm{d}\gamma\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{5.11}$$

Proof. Let $\phi \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+)$ be given. Set

$$T_{10} := -\int_{\mathbb{R}^+} \int_{\mathbb{R}^d} |u_h(x,t) - \kappa|\phi_t(x,t)\mathrm{d}x\mathrm{d}t - \int_{\mathbb{R}^d} |u_0(x) - \kappa|\phi(x,0)\mathrm{d}x,$$

$$T_{20} := -\int_{\mathbb{R}^+} \int_{\mathbb{R}^d} \left[F(x,t,u_h(x,t)\top\kappa) - F(x,t,u_h(x,t)\perp\kappa)\right]\nabla\phi(x,t)\big)\mathrm{d}x\mathrm{d}t.$$

Let us consider the first term T_{10}. Using the definition of u_h in (2.18) we get

$$\begin{aligned} T_{10} &= -\sum_{n\in\mathbb{N}} \sum_{i\in I^n} |u_i^n - \kappa| \int_{T_i^n} \left[\phi(x,t^{n+1}) - \phi(x,t^n)\right] \mathrm{d}x - \int_{\mathbb{R}^d} |u_0(x) - \kappa|\phi(x,0)\mathrm{d}x \\ &= \sum_{n\in\mathbb{N}} \Big\{ \sum_{i\in I^{n+1}} |u_i^{n+1} - \kappa| \int_{T_i^{n+1}} \phi(x,t^{n+1})\mathrm{d}x - \sum_{i\in I^n} |u_i^n - \kappa| \int_{T_i^n} \phi(x,t^{n+1})\mathrm{d}x \Big\} \\ &\quad + \sum_{i\in I^0} |u_i^0 - \kappa| \int_{T_i^0} \phi(x,t^0)\mathrm{d}x - \int_{\mathbb{R}^d} |u_0(x) - \kappa|\phi(x,0)\mathrm{d}x \\ &=: T_{11} + T_{12}. \end{aligned}$$

Using the triangle inequality we simply have that

$$\begin{aligned} T_{12} &= \sum_{i\in I^0} |u_i^0 - \kappa| \int_{T_i^0} \phi(x,0)\mathrm{d}x - \int_{\mathbb{R}^d} |u_0(x) - \kappa|\phi(x,0)\mathrm{d}x \\ &\le \int_{\mathbb{R}^d} |u_h(x,0) - u_0(x)|\phi(x,0)\mathrm{d}x \\ &= \langle \mu_0, \phi(.,0)\rangle. \end{aligned}$$

Now, let us consider the term T_{11}. We note that using the first step of Definition 2.14, see (2.15), we have that

$$\begin{aligned}\sum_{i\in I^n}|u_i^n-\kappa|\int_{T_i^n}\phi(x,t^{n+1})\mathrm{d}x &= \sum_{i\in I^n}|u_i^n-\kappa|\sum_{j\in K^{n,n+1}(i)}\int_{T_j^{n,n+1}}\phi(x,t^{n+1})\mathrm{d}x\\ &= \sum_{j\in I^{n,n+1}}|v_j^n-\kappa|\int_{T_j^{n,n+1}}\phi(x,t^{n+1})\mathrm{d}x\\ &= \sum_{i\in I^{n+1}}\sum_{j\in J^{n,n+1}(i)}|v_j^n-\kappa|\int_{T_j^{n,n+1}}\phi(x,t^{n+1})\mathrm{d}x.\end{aligned}$$

Using this and inserting an additional term which is due to the definition of u_i^{n+1} in (2.17) we get

$$\begin{aligned}T_{11} &= \sum_{n\in\mathbb{N}}\Big\{\sum_{i\in I^{n+1}}|u_i^{n+1}-\kappa|\int_{T_i^{n+1}}\phi(x,t^{n+1})\mathrm{d}x-\sum_{i\in I^n}|u_i^n-\kappa|\int_{T_i^n}\phi(x,t^{n+1})\mathrm{d}x\Big\}\\ &= \sum_{n\in\mathbb{N}}\Big[\sum_{i\in I^{n+1}}|u_i^{n+1}-\kappa|-\sum_{j\in J^{n,n+1}(i)}\frac{|T_j^{n,n+1}|}{|T_i^{n+1}|}|v_j^n-\kappa|\Big]\int_{T_i^{n+1}}\phi(x,t^{n+1})\mathrm{d}x\\ &\quad+\sum_{n\in\mathbb{N}}\Big[\sum_{i\in I^{n+1}}\sum_{j\in J^{n,n+1}(i)}\frac{|T_j^{n,n+1}|}{|T_i^{n+1}|}|v_j^n-\kappa|\int_{T_i^{n+1}}\phi(x,t^{n+1})\mathrm{d}x\\ &\quad-\sum_{i\in I^{n+1}}\sum_{j\in J^{n,n+1}(i)}|v_j^n-\kappa|\int_{T_j^{n,n+1}}\phi(x,t^{n+1})\mathrm{d}x\Big]\\ &=: T_{13}+T_{14}.\end{aligned}$$

For the term T_{14} we use a symmetrization trick.

$$\begin{aligned}T_{14} &= \sum_{n\in\mathbb{N}}\Big[\sum_{i\in I^{n+1}}\sum_{j\in J^{n,n+1}(i)}\frac{|T_j^{n,n+1}|}{|T_i^{n+1}|}|v_j^n-\kappa|\int_{T_i^{n+1}}\phi(x,t^{n+1})\mathrm{d}x\\ &\quad-\sum_{i\in I^n}\sum_{j\in J^{n,n+1}(i)}|v_j^n-\kappa|\int_{T_j^{n,n+1}}\phi(x,t^{n+1})\mathrm{d}x\Big]\\ &= \sum_{n\in\mathbb{N}}\Big[\sum_{i\in I^{n+1}}\sum_{j\in J^{n,n+1}(i)}\sum_{l\in J^{n,n+1}(i)}\frac{|T_j^{n,n+1}|}{|T_i^{n+1}|}|v_j^n-\kappa|\int_{T_l^{n,n+1}}\phi(x,t^{n+1})\mathrm{d}x\\ &\quad-\sum_{i\in I^n}\sum_{j\in J^{n,n+1}(i)}\sum_{l\in J^{n,n+1}(i)}\frac{|T_l^{n,n+1}|}{|T_i^{n+1}|}|v_j^n-\kappa|\int_{T_j^{n,n+1}}\phi(x,t^{n+1})\mathrm{d}x\Big]\\ &= \frac{1}{2}\sum_{n\in\mathbb{N}}\sum_{i\in I^{n+1}}\sum_{j,l\in J^{n,n+1}(i)}\Big\{|v_j^n-\kappa|\Big[\frac{|T_j^{n,n+1}|}{|T_i^{n+1}|}\int_{T_l^{n,n+1}}\phi(x,t^{n+1})\mathrm{d}x\\ &\quad-\frac{|T_l^{n,n+1}|}{|T_i^{n+1}|}\int_{T_j^{n,n+1}}\phi(x,t^{n+1})\mathrm{d}x\Big]\\ &\quad-|v_l^n-\kappa|\Big[\frac{|T_j^{n,n+1}|}{|T_i^{n+1}|}\int_{T_l^{n,n+1}}\phi(x,t^{n+1})\mathrm{d}x-\frac{|T_l^{n,n+1}|}{|T_i^{n+1}|}\int_{T_j^{n,n+1}}\phi(x,t^{n+1})\mathrm{d}x\Big]\Big\}\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}\sum_{n\in\mathbb{N}}\sum_{i\in I^{n+1}}\sum_{j,l\in J^{n,n+1}(i)}|v_j^n-v_l^n| \\
&\quad\times\Big|\frac{|T_j^{n,n+1}|}{|T_i^{n+1}|}\int_{T_l^{n,n+1}}\phi(x,t^{n+1})\mathrm{d}x-\frac{|T_l^{n,n+1}|}{|T_i^{n+1}|}\int_{T_j^{n,n+1}}\phi(x,t^{n+1})\mathrm{d}x\Big| \\
&\le \frac{1}{2}\sum_{n\in\mathbb{N}}\sum_{i\in I^{n+1}}\sum_{j,l\in\tilde K^{n,n+1}(i)}|u_j^n-u_l^n| \\
&\quad\times\frac{h_i^{n+1}}{|T_i^{n+1}|}\int_{T_l^n\cap T_i^{n+1}}\int_{T_j^n\cap T_i^{n+1}}\int_0^1|\nabla\phi(x+\theta(y-x),t^{n+1})|\mathrm{d}\theta\mathrm{d}y\mathrm{d}x \\
&= \langle\mu_{h1},|\nabla\phi|\rangle,
\end{aligned}
$$

where we have used

$$
\sum_{j,l\in J^{n,n+1}(i)}|v_j^n-v_l^n|=\sum_{j,l\in\tilde K^{n,n+1}(i)}|u_j^n-u_l^n|.
$$

It remains to consider the terms T_{13} and T_{20}. In order to do this we insert the discrete entropy inequality (5.3). Define

$$
\begin{aligned}
T_1 &:= \sum_{n\in\mathbb{N}}\sum_{i\in I^{n+1}}\Big[|u_i^{n+1}-\kappa|-\sum_{j\in J^{n,n+1}(i)}\frac{|T_j^{n,n+1}|}{|T_i^{n+1}|}|v_j^n-\kappa|\Big]\frac{1}{k^n}\int_{t^n}^{t^{n+1}}\int_{T_i^{n+1}}\phi(x,t)\mathrm{d}x\,\mathrm{d}t, \\
T_2 &:= \sum_{n\in\mathbb{N}}k^n\sum_{i\in I^{n+1}}\sum_{j\in J^{n,n+1}(i)}\sum_{l\in\mathcal{N}^{n,n+1}(j)}\big[G_{jl}^{n,n+1}(v_j^n,v_l^n,\kappa)-G_{jl}^{n,n+1}(v_j^n,v_j^n,\kappa)\big] \\
&\quad\times\frac{1}{k^n}\int_{t^n}^{t^{n+1}}\frac{1}{|T_i^{n+1}|}\int_{T_i^{n+1}}\phi(x,t)\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Since $\phi\ge 0$ we have by Lemma 5.1

$$
T_1+T_2\le 0.
$$

We now compare T_{13} with T_1 and T_{20} with T_2. This part is almost identically with the proof of Theorem 1 in [5]. Let us start with the first pair. We have

$$
\begin{aligned}
T_{13}-T_1 &= \sum_{n\in\mathbb{N}}\sum_{i\in I^{n+1}}\Big[|u_i^{n+1}-\kappa|-\sum_{j\in J^{n,n+1}(i)}\frac{|T_j^{n,n+1}|}{|T_i^{n+1}|}|v_j^n-\kappa|\Big] \\
&\quad\times\Big[\frac{1}{k^n}\int_{t^n}^{t^{n+1}}\int_{T_i^{n+1}}\phi(x,t^{n+1})\mathrm{d}x\,\mathrm{d}t-\frac{1}{k^n}\int_{t^n}^{t^{n+1}}\int_{T_i^{n+1}}\phi(x,t)\mathrm{d}x\,\mathrm{d}t\Big] \\
&\le \sum_{n\in\mathbb{N}}\sum_{i\in I^{n+1}}\Big|u_i^{n+1}-\sum_{j\in\tilde K^{n,n+1}(i)}\frac{|T_j^n\cap T_i^{n+1}|}{|T_i^{n+1}|}u_j^n\Big| \\
&\quad\times\int_{t^n}^{t^{n+1}}\int_{T_i^{n+1}}\int_0^1|\phi_t(x,t+\theta(t^{n+1}-t))|\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t \\
&= \langle\mu_{h2},|\phi_t|\rangle.
\end{aligned}
$$

It remains to compare T_{20} and T_2. We first simplify T_{20} using the definition of u_h, see (2.18) and (2.15), and that $\operatorname{div}_x F=0$, see Assumption 2.1.

$$
\begin{aligned}
T_{20} &= -\int\limits_{\mathbb{R}^+}\int\limits_{\mathbb{R}^d} \big[F(x,t,u_h(x,t)\top\kappa) - F(x,t,u_h(x,t)\bot\kappa)\big]\nabla\phi(x,t)\big)\mathrm{d}x\mathrm{d}t \\
&= \sum_{n\in\mathbb{N}}\sum_{i\in I^n}\sum_{j\in K^{n,n+1}(i)} \int\limits_{t^n}^{t^{n+1}}\int\limits_{T_j^{n,n+1}} \big[F(x,t,u_h(x,t)\top\kappa) - F(x,t,u_h(x,t)\bot\kappa)\big]\nabla\phi(x,t)\big)\mathrm{d}x\mathrm{d}t \\
&= \sum_{n\in\mathbb{N}}\sum_{i\in I^n}\sum_{j\in K^{n,n+1}(i)}\sum_{l\in\mathcal{N}^{n,n+1}(j)} \int\limits_{t^n}^{t^{n+1}}\int\limits_{S_{jl}^{n,n+1}} \\
&\qquad\qquad \big[F(\gamma,t,v_j^n\top\kappa) - F(\gamma,t,v_j^n\bot\kappa)\big]n_{jl}^{n,n+1}\phi(\gamma,t)\big)\mathrm{d}\gamma\mathrm{d}t \\
&= \sum_{n\in\mathbb{N}}\sum_{(j,l)\in\mathcal{E}^{n,n+1}} \int\limits_{t^n}^{t^{n+1}}\int\limits_{S_{jl}^{n,n+1}} \Big\{\big[F(\gamma,t,v_l^n\top\kappa) - F(\gamma,t,v_l^n\bot\kappa)\big] \\
&\quad -\big[F(\gamma,t,v_j^n\top\kappa) - F(\gamma,t,v_j^n\bot\kappa)\big]\Big\}n_{jl}^{n,n+1}\phi(\gamma,t)\big)\mathrm{d}\gamma\mathrm{d}t.
\end{aligned}
$$

Due to the consistency of the numerical fluxes, see Assumption 2.5, we insert some terms such that

$$
T_{20} = T_{200} + T_{201},
$$

where

$$
\begin{aligned}
T_{200} &:= -\sum_{n\in\mathbb{N}} k^n \sum_{(j,l)\in\mathcal{E}^{n,n+1}} \big[G_{jl}^{n,n+1}(v_j^n,v_j^n) - G_{jl}^{n,n+1}(v_j^n,v_l^n) \\
&\qquad + G_{jl}^{n,n+1}(v_j^n,v_l^n) - G_{jl}^{n,n+1}(v_l^n,v_l^n)\big]\frac{1}{k^n\,|S_{jl}^{n,n+1}|}\int\limits_{t^n}^{t^{n+1}}\int\limits_{S_{jl}^{n,n+1}}\phi(\gamma,t)\mathrm{d}\gamma\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
T_{201} &:= \sum_{n\in\mathbb{N}}\sum_{(j,l)\in\mathcal{E}^{n,n+1}} \int\limits_{t^n}^{t^{n+1}}\int\limits_{S_{jl}^{n,n+1}} \Big\{\Big(\big[F(\gamma,t,v_l^n\top\kappa) - F(\gamma,t,v_l^n\bot\kappa)\big]n_{jl}^{n,n+1} \\
&\quad -\frac{1}{|S_{jl}^{n,n+1}|}G_{jl}^{n,n+1}(v_l^n,v_l^n,\kappa)\Big) - \big[F(\gamma,t,v_j^n\top\kappa) - F(\gamma,t,v_j^n\bot\kappa)\big]n_{jl}^{n,n+1} \\
&\quad -\frac{1}{|S_{jl}^{n,n+1}|}G_{jl}^{n,n+1}(v_j^n,v_j^n,\kappa)\Big)\Big\}\phi(\gamma,t)\mathrm{d}\gamma\mathrm{d}t.
\end{aligned}
$$

First, we estimate the consistency error T_{201} of the numerical flux. Due to the Definition of $G_{jl}^{n,n+1}$, see (5.1), we have

$$
\begin{aligned}
T_{201} &= \sum_{n\in\mathbb{N}}\sum_{(j,l)\in\mathcal{E}^{n,n+1}} \frac{1}{k^n|S_{jl}^{n,n+1}|}\int\limits_{t^n}^{t^{n+1}}\int\limits_{t^n}^{t^{n+1}}\int\limits_{S_{jl}^{n,n+1}}\int\limits_{S_{jl}^{n,n+1}} \Big\{ \\
&\quad \Big[\big(F(\gamma,t,v_l^n\top\kappa)n_{jl}^{n,n+1} - \frac{1}{|S_{jl}^{n,n+1}|}g_{jl}^{n,n+1}(v_l^n\top\kappa,v_l^n\top\kappa)\big) \\
&\quad -\big(F(\gamma,t,v_l^n\bot\kappa)n_{jl}^{n,n+1} - \frac{1}{|S_{jl}^{n,n+1}|}g_{jl}^{n,n+1}(v_l^n\bot\kappa,v_l^n\bot\kappa)\big)\Big] \\
&\quad -\Big[\big(F(\gamma,t,v_j^n\top\kappa)n_{jl}^{n,n+1} - \frac{1}{|S_{jl}^{n,n+1}|}g_{jl}^{n,n+1}(v_j^n\top\kappa,v_j^n\top\kappa)\big)
\end{aligned}
$$

$$-\big(F(\gamma,t,v_j^n\perp\kappa)n_{jl}^{n,n+1}-\frac{1}{|S_{jl}^{n,n+1}|}g_{jl}^{n,n+1}(v_j^n\perp\kappa,v_j^n\perp\kappa)\big)\Big]\Big\}$$
$$\times\big[\phi(\gamma,t)-\phi(\xi,s)\big]\mathrm{d}\xi\mathrm{d}\gamma\mathrm{d}x\mathrm{d}t.$$

Note, that the term involving $\phi(\xi,s)$ equals zero due to the consistency of the numerical fluxes, see Assumption 2.5. Since $F\in C^1$ (see Assumption 2.1) there exists a constant C_{F,ϕ,u_0} such that for all $(\gamma,s,v)\in\operatorname{supp}(\phi)\cap(\mathbb{R}^d\times[t^n,t^{n+1}[)\times[U_m,U_M]$

$$\Big|F(\gamma,s,v)n_{jl}^{n,n+1}-\frac{1}{|S_{jl}^{n,n+1}|}g_{jl}^{n,n+1}(v,v)\Big|\le C_{F,\phi,u_0}(\operatorname{diam}(S_{jl}^{n,n+1})+k^n). \tag{5.12}$$

Moreover, we have for $(\gamma,\xi,t,s)\in(S_{jl}^{n,n+1})^2\times[t^n,t^{n+1}[^2$

$$\big|\phi(\gamma,t)-\phi(\xi,s)\big|\le(\operatorname{diam}(S_{jl}^{n,n+1})+k^n)\int_0^1(|\nabla\phi|+|\phi_t|)(\xi+\theta(\gamma-\xi),t+\theta(s-t))\mathrm{d}\theta.$$

Hence, we can estimate

$$|T_{201}|\le 4C_{F,\phi,u_0}\sum_{n\in\mathbb{N}}\sum_{(j,l)\in\mathcal{E}^{n,n+1}}\langle\lambda_{jl}^{n,n+1},|\nabla\phi|+|\phi_t|\rangle.$$

Now, let us consider T_2.

$$\begin{aligned}
T_2 \;=\;& \sum_{n\in\mathbb{N}}k^n\sum_{i\in I^{n+1}}\sum_{j\in J^{n,n+1}(i)}\sum_{l\in\mathcal{N}^{n,n+1}(j)}\big[G_{jl}^{n,n+1}(v_j^n,v_l^n,\kappa)-G_{jl}^{n,n+1}(v_j^n,v_j^n,\kappa)\big]\\
&\times\Big[\frac{1}{k^n}\int_{t^n}^{t^{n+1}}\frac{1}{|T_i^{n+1}|}\int_{T_i^{n+1}}\phi(x,t)\mathrm{d}x\,\mathrm{d}t-\frac{1}{k^n}\int_{t^n}^{t^{n+1}}\frac{1}{|T_j^{n,n+1}|}\int_{T_j^{n,n+1}}\phi(x,t)\mathrm{d}x\,\mathrm{d}t\Big]\\
&+\sum_{n\in\mathbb{N}}k^n\sum_{i\in I^{n+1}}\sum_{j\in J^{n,n+1}(i)}\sum_{l\in\mathcal{N}^{n,n+1}(j)}\big[G_{jl}^{n,n+1}(v_j^n,v_l^n,\kappa)-G_{jl}^{n,n+1}(v_j^n,v_j^n,\kappa)\big]\\
&\times\frac{1}{k^n}\int_{t^n}^{t^{n+1}}\frac{1}{|T_j^{n,n+1}|}\int_{T_j^{n,n+1}}\phi(x,t)\mathrm{d}x\,\mathrm{d}t\\
=:\;& T_3+T_4.
\end{aligned}$$

Note, that due to the monotony of the numerical fluxes $g_{jl}^{n,n+1}$ (see Assumption 2.5) and due to the definition of $G_{jl}^{n,n+1}$, see (5.1), we have for all $\kappa\in\mathbb{R}$

$$\begin{aligned}
&|G_{jl}^{n,n+1}(v_j^n,v_l^n,\kappa)-G_{jl}^{n,n+1}(v_j^n,v_j^n,\kappa)|\\
&\le\; 2|g_{jl}^{n,n+1}(v_j^n,v_l^n)-g_{jl}^{n,n+1}(v_j^n,v_j^n)|.
\end{aligned}\tag{5.13}$$

Using this we get for T_3

$$
\begin{aligned}
T_3 &= \sum_{n\in\mathbb{N}} k^n \sum_{i\in I^{n+1}} \sum_{j\in J^{n,n+1}(i)} \sum_{l\in\mathcal{N}^{n,n+1}(j)} \left[G_{jl}^{n,n+1}(v_j^n, v_l^n, \kappa) - G_{jl}^{n,n+1}(v_j^n, v_j^n, \kappa)\right] \\
&\quad \times\Big[\frac{1}{k^n|T_i^{n+1}|}\int\limits_{t^n}^{t^{n+1}}\int\limits_{T_i^{n+1}} \phi(x,t)\mathrm{d}x\,\mathrm{d}t - \frac{1}{k^n|T_j^{n,n+1}|}\int\limits_{t^n}^{t^{n+1}}\int\limits_{T_j^{n,n+1}} \phi(x,t)\mathrm{d}x\,\mathrm{d}t\Big] \\
&\leq 2\sum_{n\in\mathbb{N}} k^n \sum_{i\in I^{n+1}} \sum_{j\in J^{n,n+1}(i)} \sum_{l\in\mathcal{N}^{n,n+1}(j)} \left|g_{jl}^{n,n+1}(v_j^n, v_l^n) - g_{jl}^{n,n+1}(v_j^n, v_j^n)\right| \\
&\quad \times \frac{h_i^{n+1}}{k^n\,|T_i^{n+1}|\,|T_j^{n,n+1}|}\int\limits_{t^n}^{t^{n+1}}\int\limits_{T_i^{n+1}}\int\limits_{T_j^{n,n+1}}\int\limits_0^1 |\nabla\phi(x+\theta(y-x),t)|\mathrm{d}\theta\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&= \langle \mu_{h3}, |\nabla\phi|\rangle.
\end{aligned}
$$

Now, we write the term T_4 as a sum over edges.

$$
\begin{aligned}
T_4 &= \sum_{n\in\mathbb{N}} k^n \sum_{i\in I^{n+1}} \sum_{j\in J^{n,n+1}(i)} \sum_{l\in\mathcal{N}^{n,n+1}(j)} \left[G_{jl}^{n,n+1}(v_j^n, v_l^n, \kappa) - G_{jl}^{n,n+1}(v_j^n, v_j^n, \kappa)\right] \\
&\quad \times \frac{1}{k^n|T_j^{n,n+1}|}\int\limits_{t^n}^{t^{n+1}}\int\limits_{T_j^{n,n+1}} \phi(x,t)\mathrm{d}x\,\mathrm{d}t \\
&= \sum_{n\in\mathbb{N}} k^n \sum_{(j,l)\in\mathcal{E}^{n,n+1}} \Big[G_{jl}^{n,n+1}(v_j^n, v_l^n)\phi_j^{n,n+1} - G_{jl}^{n,n+1}(v_j^n, v_j^n)\phi_j^{n,n+1} \\
&\quad + G_{jl}^{n,n+1}(v_l^n, v_l^n)\phi_l^{n,n+1} - G_{jl}^{n,n+1}(v_j^n, v_l^n)\phi_l^{n,n+1}\Big],
\end{aligned}
$$

where

$$
\phi_j^{n,n+1} := \frac{1}{k^n|T_j^{n,n+1}|}\int\limits_{t^n}^{t^{n+1}}\int\limits_{T_j^{n,n+1}} \phi(x,t)\mathrm{d}x\,\mathrm{d}t
$$

for $j \in I^{n,n+1}$.

Now, we can compare T_{200} with T_4.

$$
\begin{aligned}
T_{200} - T_4 &= \sum_{n\in\mathbb{N}} k^n \sum_{(j,l)\in\mathcal{E}^{n,n+1}} \Big\{\left[G_{jl}^{n,n+1}(v_j^n, v_l^n) - G_{jl}^{n,n+1}(v_j^n, v_j^n)\right]\left[\phi_{jl}^{n,n+1} - \phi_j^{n,n+1}\right] \\
&\quad - \left[G_{jl}^{n,n+1}(v_j^n, v_l^n) - G_{jl}^{n,n+1}(v_l^n, v_l^n)\right]\left[\phi_{jl}^{n,n+1} - \phi_l^{n,n+1}\right], \Big\}
\end{aligned}
$$

where we have used the abbreviation

$$
\phi_{jl}^{n,n+1} := \frac{1}{k^n}\int\limits_{t^n}^{t^{n+1}} \frac{1}{|S_{jl}^{n,n+1}|}\int\limits_{S_{jl}^{n,n+1}} \phi(\gamma,t)\mathrm{d}\gamma\mathrm{d}t.
$$

Using the mean value theorem we get

$$
\begin{aligned}
|\phi_{jl}^{n,n+1} - \phi_j^{n,n+1}| &\leq \frac{h_j^{n,n+1} + k^n}{(k^n)^2|T_j^{n,n+1}|\,|S_{jl}^{n,n+1}|}\int\limits_{t^n}^{t^{n+1}}\int\limits_{t^n}^{t^{n+1}}\int\limits_{T_j^{n,n+1}}\int\limits_{S_{jl}^{n,n+1}}\int\limits_0^1 \\
&\quad |\nabla\phi(x+\theta(\gamma-x), t+\theta(s-t))| + \\
&\quad +|\phi_t(x+\theta(\gamma-x), t+\theta(s-t))|\mathrm{d}\theta\mathrm{d}\gamma\mathrm{d}x\mathrm{d}s\mathrm{d}t \\
&= \langle \mu_{jl}^{n,n+1}, |\nabla\phi| + |\phi_t|\rangle.
\end{aligned}
$$

Using this and (5.13) we get that

$$T_{20} - T_4 \le \langle \mu_{h4}, |\nabla\phi| + |\phi_t| \rangle,$$

which concludes the proof.

□

6. Proof of the a posteriori error estimate.

It is well-known that the Kruzkov technique for proving error estimates leads to *a posteriori* error estimates, cf. [6]. The main idea to prove Theorem 3.3 consists in estimating the constant in the Approximation Lemma 3.6, to take care of the domain of dependence of the error and to estimate the measures appropriately. In case of a fixed grid, this has been done in [15] where an adaptive strategy is introduced as well.

A more refined version of the Approximation Lemma 3.6 contains test-functions. These test-functions are chosen specifically and can be interpreted as a solution of a linearized dual problem to (2.1). This is the content of the next lemma.

LEMMA 6.1. [15, Lemma 2.10]
Let the Assumptions in Lemma 3.6 hold. Let $\omega, R \in \mathbb{R}_+$ be given and let $\rho \in C_0^1(\mathbb{R}_+;[0,1])$ be such that $\rho' \le 0$ and

$$\begin{aligned} \rho &= 1 \quad \text{on } [0,R], \\ \rho &= 0 \quad \text{on } [R+1,\infty[. \end{aligned}$$

Set

$$\begin{aligned} \psi(x,t) &:= \frac{T-t}{T}\rho(|x-x_0| + \omega t) \quad \text{on } \mathbb{R}^d \times [0,T], \\ \psi(x,t) &:= 0 \quad \text{on } \mathbb{R}^d \times [T,\infty[, \end{aligned}$$

where $x_0 \in \mathbb{R}^d$ is given.
Then we have

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}_+} \Big\{ |\tilde{u}(x,t) - u(x,t)|\psi_t(x,t) + \big[F(x,t\tilde{u}(x,t)\top u(x,t)) - F(x,t,\tilde{u}(x,t)\bot u(x,t))\big]\nabla\psi(x,t) \Big\} \mathrm{d}t\mathrm{d}x$$

$$\ge -\mu_0(\{\psi(.,0) \ne 0\}) - 2\sqrt{bc}[\mu(\{\psi \ne 0\})]^{1/2} - a\mu(\{\psi \ne 0\})\Big], \tag{6.1}$$

with

$$\begin{aligned} a &:= 2\omega + \frac{1}{T} + 2 \\ b &:= 4 + 2^{d+2} \\ c &:= \|u\|_{\mathrm{BV}}\big[2(2\omega + \frac{1}{T}) + V_{[U_m,U_M]}(8 + 2^{d+5})\big] \\ &\quad + \|u_0\|_{\mathrm{BV}}\big[2^{d+4}V_{[U_m,U_M]} + 1\big] + 2V_{[U_m,U_M]}\max\{U_m,U_M\}\,[|B_{R+1}(0)| - |B_R(0)|]\,T. \end{aligned}$$

REMARK 6.2. *Note, that due to Theorem 3 in* [5] *we have under Assumption 2.1*

$$\|u\|_{BV} := \|u\|_{BV(D_{R+1}(x_0))} \le C(F,u_0,R,T).$$

In the special case where $F(x,t,s) = f(s)$ one has the estimate [10, Thm. 2.3.1]

$$\|u(.,t)\|_{BV(\mathbb{R}^d)} \le \|u_0\|_{BV(\mathbb{R}^d)} \quad \forall t > 0.$$

In the next few lemmata we estimate the measures which are defined in Theorem 5.2. Putting all these results together then gives the assertion of our a posteriori error estimate in Theorem 3.3.

LEMMA 6.3. *Let μ_{h1} be defined in* (5.6). *Then*

$$\mu_{h1}(D_{R+1}(x_0)) \le \frac{1}{2}\sum_{n=0}^{N_0}\sum_{i\in I_D^{n+1}} h_i^{n+1}|T_i^{n+1}| \sum_{j,l\in\tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1}\tilde{r}_{il}^{n,n+1}|u_j^n - u_l^n|.$$

Proof. Let $\chi_{D_{R+1}(x_0)}$ denote the characteristic function of the set $D_{R+1}(x_0)$. By definition of μ_{h1} we have

$$\begin{aligned}\mu_{h1}(D_{R+1}(x_0)) &= \frac{1}{2}\sum_{n=0}^{N_0}\sum_{i\in I_D^{n+1}}\sum_{j,l\in\tilde{K}^{n,n+1}(i)}|u_j^n-u_l^n| \\ &\quad\times\frac{h_i^{n+1}}{|T_i^{n+1}|}\int\limits_{T_l^n\cap T_i^{n+1}}\int\limits_{T_j^n\cap T_i^{n+1}}\int\limits_0^1\chi_{D_{R+1}(x_0)}(x+\theta(y-x),t^{n+1})\mathrm{d}\theta\mathrm{d}y\mathrm{d}x.\end{aligned}$$

For $n\le N_0$ and $i\in I_D^{n+1}$ we have that $T_l^n\cap T_i^{n+1}, T_j^n\cap T_i^{n+1}\subset T_i^{n+1}$ for $j,l\in\tilde{K}^{n,n+1}(i)$. Therefore, we estimate $\chi_{D_{R+1}}(x+\theta(y-x),t^{n+1})\le 1$ which gives the assertion. □

LEMMA 6.4. *Let μ_{h2} be defined in* (5.7). *Then*

$$\mu_{h2}(D_{R+1}(x_0))\le\sum_{n=0}^{N_0-1}k^n\sum_{i\in I_D^{n+1}}|T_i^{n+1}|\left|u_i^{n+1}-\sum_{j\in\tilde{K}^{n,n+1}(i)}\tilde{r}_{ij}^{n,n+1}u_j^n\right|.$$

The proof is obvious and therefore omitted.

LEMMA 6.5. *Let μ_{h3} be defined in* (5.8). *Then*

$$\mu_{h3}(D_{R+1}(x_0))\le 2V_{[U_m,U_M]}\sum_{n=0}^{N_0}k^n\sum_{i\in I_D^{n+1}}h_i^{n+1}\sum_{(j,l)\in\tilde{\mathcal{E}}^{n,n+1}(i)}|\tilde{S}_{jl}^{n,n+1}(i)|\,|u_j^n-u_l^n|.$$

Proof. Note, that analogously to the proof of Lemma 6.3 we have

$$\frac{h_i^{n+1}}{k^n\,|T_i^{n+1}|\,|T_j^{n,n+1}|}\int\limits_{t^n}^{t^{n+1}}\int\limits_{T_i^{n+1}}\int\limits_{T_j^{n,n+1}}\int\limits_0^1\chi_{D_{R+1}(x_0)}(x+\theta(y-x),t)\mathrm{d}\theta\mathrm{d}x\mathrm{d}y\mathrm{d}t\le h_i^{n+1}.$$

If indices $j,l\in\mathcal{E}^{n,n+1}$ correspond to an edge which lies on ∂T_i^{n+1} then by construction, see (2.15),

$$v_j^n=v_l^n$$

and hence the flux difference vanishes. Therefore, only flux differences across interior edges of elements T_i^{n+1} remain which are estimated using the Lipschitz property of the numerical fluxes, see Assumption 2.5, and Remark 4.6. Hence, we have

$$\mu_{h3}(D_{R+1}(x_0))\le 2V_{[U_m,U_M]}\sum_{n=0}^{N_0}k^n\sum_{i\in I_D^{n+1}}h_i^{n+1}\sum_{(j,l)\in\mathcal{E}^{0,n,n+1}(i)}|S_{jl}^{n,n+1}|\,|v_j^n-v_l^n|,$$

where

$$\mathcal{E}^{0,n,n+1}(i):=\{(j,l)\in\mathcal{E}^{n,n+1}|\,T_j^{n,n+1}\subset T_i^{n+1}\quad\text{and } T_l^{n,n+1}\subset T_i^{n+1}\}$$

denotes the set edges in $\mathcal{T}_h^{n,n+1}$ which lie in the interior of T_i^{n+1}. Using the identification of indices due to the prolongation step, see (2.15), and Notation 3.2 we get the assertion. □

Using similar arguments as before one can prove the following two lemmata.

LEMMA 6.6. *Let $\mu_{jl}^{n,n+1}$ and $\lambda_{jl}^{n,n+1}$ be defined in* (5.10) *and* (5.11), *respectively. Then*

$$\begin{aligned}\mu_{jl}^{n,n+1}(D_{R+1}(x_0)) &\le h_j^{n,n+1}+k^n\\ \lambda_{jl}^{n,n+1}(D_{R+1}(x_0)) &\le k^n|S_{jl}^{n,n+1}|\left[\mathrm{diam}(S_{jl}^{n,n+1})+k^n\right]^2.\end{aligned}$$

LEMMA 6.7. *Let μ_{h4} be defined in* (5.9). *Then*

$$\begin{aligned}\mu_{h4}(D_{R+1}(x_0)) \quad &\leq \quad 4V_{[U_m,U_M]}\sum_{n=0}^{N_0} k^n \sum_{i\in I_D^{n+1}} (h_i^{n+1}+k^n) \sum_{(j,l)\in\tilde{\mathcal{E}}^{n,n+1}(i)} |\tilde{S}_{jl}^{n,n+1}(i)|\,|u_j^n-u_l^n| \\ &\quad + C_{F,R,T,u_0}\sum_{n=0}^{N_0}\sum_{(j,l)\in\mathcal{E}_D^{n,n+1}} k^n|S_{jl}^{n,n+1}|\left[\mathrm{diam}(S_{jl}^{n,n+1})+k^n\right]^2,\end{aligned}$$

where C_{F,R,T,u_0} is defined in (5.12).

7. Proof of the a priori error estimate.

The proof of the a priori error estimate in Theorem 3.5 follow from the a posteriori error estimate in Theorem 3.3 by estimating the term Q. In order to do so we manipulate the various parts in such a way that we can apply the BV-regularity results given in Section 4, see Proposition 4.4 and Proposition 4.5.

Proof. We have to estimate $Q = Q_1+Q_2+Q_3+Q_4$, where

$$\begin{aligned}Q_1 &= \frac{1}{2}\sum_{n=0}^{N_0}\sum_{i\in I_D^{n+1}} h_i^{n+1}|T_i^{n+1}| \sum_{j,l\in\tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1}\tilde{r}_{il}^{n,n+1}|u_j^n-u_l^n| \\ Q_2 &= \sum_{n=0}^{N_0-1} k^n \sum_{i\in I_D^{n+1}} |T_i^{n+1}|\left|u_i^{n+1} - \sum_{j\in\tilde{K}^{n,n+1}(i)} \tilde{r}_{ij}^{n,n+1}u_j^n\right| \\ Q_3 &= 6V_{[U_m,U_M]}\sum_{n=0}^{N_0} k^n \sum_{i\in I_D^{n+1}} (h_i^{n+1}+k^n) \sum_{(j,l)\in\tilde{\mathcal{E}}^{n,n+1}(i)} |\tilde{S}_{jl}^{n,n+1}(i)|\,|u_j^n-u_l^n| \\ Q_4 &= C_{F,R,T,u_0}\sum_{n=0}^{N_0}\sum_{(j,l)\in\mathcal{E}_D^{n,n+1}} k^n|S_{jl}^{n,n+1}|\left[\,\mathrm{diam}(S_{jl}^{n,n+1})+k^n\right]^2.\end{aligned}$$

Now, we need geometrical estimates to apply the stability estimates of Section 4.
Using the mesh regularity (Assumption 3.1 and 2.5) and the Overlap-condition (Assumption 3.1) we get

$$\begin{aligned}h_i^{n+1} &\leq (C_{\mathrm{ov}}\alpha)^{-1/d}h_j^{n,n+1} \quad \forall i\in I^{n+1},\ \forall j\in J^{n,n+1}(i), \\ |S_{jl}^{n,n+1}| &\leq |\partial T_j^{n,n+1}| \leq \frac{1}{\alpha}(h_j^{n,n+1})^{d-1} \quad \forall (j,l)\in\mathcal{E}_D^{n,n+1}.\end{aligned}$$

Using these estimates together with the CFL-condition and the inverse CFL-condition, see Assumption 3.1, we get the following estimates

$$\begin{aligned}Q_1 &\leq (C_{\mathrm{ov}}\alpha)^{-1/d}\frac{V_{[U_m,U_M]}}{2\alpha^2\eta}\sum_{n=0}^{N_0} k^n \sum_{i\in I_D^{n+1}} |T_i^{n+1}| \sum_{j,l\in\tilde{K}^{n,n+1}(i)} |u_j^n-u_l^n| \\ Q_3 &\leq 6V_{[U_m,U_M]}\frac{1}{\alpha^2}\Big[(C_{\mathrm{ov}}\alpha)^{-1/d}+\frac{1}{2}(1-\xi)\alpha^2\frac{1}{V_{[U_m,U_M]}}\Big]\sum_{n=0}^{N_0} k^n \sum_{i\in I_D^{n+1}} |T_i^{n+1}| \sum_{j,l\in\tilde{K}^{n,n+1}(i)} |u_j^n-u_l^n| \\ Q_4 &\leq C_{F,R,T,u_0}\big[1+\frac{1}{2}(1-\xi)\frac{\alpha^2}{V_{[U_m,U_M]}}\big]^2\frac{1}{\alpha^2}|D_{R+1}(x_0)|\,h.\end{aligned}$$

Estimating further Q_1 and Q_3 by Proposition 4.4 and Q_2 by Proposition 4.5 and applying Theorem 3.3 concludes the proof. ☐

Note, that the constant C_e in Theorem 3.5 can be specified explicitly.

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