Functionals with values in the Non-Archimedean field of Laurent series and Method for Numerical Calculations of Shocks and Soliton like Solutions of Some Conservation Laws[1]

Dedicated to Stefania Radyna

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Abstract

Functionals with values in Non-Archimedean field of Laurent series is considered. Introduced objects are natural generalization of Sobolev-Schwartz distributions. They applied to the definition of generalized solution (in the form of soliton and shock wave) of the Hopf equation and equations of elasticity theory. Calculation method for the profile of infinitely narrow soliton and shock wave is proposed. This method based on the new concept of solution of the conservation laws, functionals with values in Non-Archimedean field of Laurent series and orthogonal system of the Hermite functions. Applying this method, calculations of profiles are reduced to the nonlinear system of algebraic equations in $\mathbf{R}^{n+1}$, $n > 1$. It is shown that there is a possibility to find out some of the solutions of this system using the Newton iteration method. Examples and numerical tests are considered.

KEY WORDS: generalized functions, conservation law, Hopf equation, equations of elasticity theory, soliton, shock wave

1 Introduction

Problems with conservation laws are complicated and require the developed theory, mathematical models which one can use for the numerical calculations. To solve these problems exist several approaches. Further we outline some of them and then consider our point of view on some nonlinear differential equations, their "generalized" solutions and generalized functions.

The simplest hydrodynamic equation and the simplest conservation law is the following equation

$$u_t + uu_x = 0, \tag{1}$$

It is called the Hopf equation [8]. One method of studying (1) was proposed by J. Von Neumann in the paper [18]. He proposed a difference scheme [19] for a calculation of the profile of the shock wave which is described by the equation (1).

The second method of studying the Hopf equation (1) based on its approximation. This idea is to use approximating equation

$$u_t + uu_x = \mu u_{xx} \text{ where } \mu \to 0 \tag{2}$$

for the equation (1) This method is called the "disappearing viscosity" method. It was developed by E.Hopf [8], O.A.Oleinik [21], P.Lax [13], [9]. P.Lax in the paper [9] asserts that the difference scheme of von Neumann, because of the centering of the difference quotients, is *dispersive*; it is this quality that is responsible for the oscillatory nature of the solutions.

There is also the "zero dispersion limit" method developed by Maslov V.P and his collaborators [5], [16].

In the other hand, in the paper [26] by S.L.Sobolev (1936) were introduced mathematical basics of the theory of generalized functions, was developed the idea of a generalized function as a functional and was introduced the concept of generalized solutions of a *linear differential equation*. These generalized functions and generalized solutions were developed by L.Schwartz [25]. Now, they are well known and widely used in various branches of mathematics. However, Sobolev-Schwartz distributions can not applied to nonlinear differential equations. For example, to substitute generalized function $H(x - vt)$ (where H is the Heaviside function) into the (1) one need to define the product of two distributions H and H' for the term uu_x. However, in 1954 L.Schwartz showed that it is impossible to introduce an associate multiplication in the space of distributions.

Starting from 1982 in the works by J.-F. Colombeau [3], M. Oberguggenberger [20], H. Biagioni [2], E. Rozinger [23], A.Y. Le Roux [4], Yu. Egorov [7], J.-A. Marti [15], A. Delcroix, D.Scarpalezos [6], B. Keyfitz [10], A. Antonevich, Ya. Radyno [1], T. Todorov [27], S. Pilipovich [17] and others, a new theory of generalized functions is developed. Such functions form the algebra and contain distributions.

In general, nonlinear generalized functions are classes of equivalent smooth functions. Clearly that, one should pay attention to this approach in order to consider nonlinear differential equations.

However we are now in a position to develop a new point of view on generalized functions and their applications to nonlinear equations. Namely, it is necessary to use an integral nature of a conservation law. Conservation laws are integral expressions from physical point of view and it is natural to consider an integral form of conservation laws. Moreover, we want to develop new point of view on conservation laws using the concept of functionals with values in the Non-Archimedean field of Laurent series. We call such functionals as $\mathbf{R}\langle\varepsilon\rangle$-distributions. In addition, we give the definition of the special kind of solutions of the some conservation laws in the sense of $\mathbf{R}\langle\varepsilon\rangle$-distributions and consider the method for the numerical calculations of the smooth shocks and soliton like solutions of the Hopf equation and equations of elasticity theory in the mentioned sense. This method based on orthogonal system of the Hermite functions as a base for calculation of such solutions (i.e. shocks and infinitely narrow solitons). Calculations of profiles of infinitely narrow soliton and shock wave are reduced to the nonlinear system of algebraic equations in $\mathbf{R}^{n+1}$, $n > 1$. We proved, using the Schauder fixed point theorem [24], that the mentioned system has at least one solution in $\mathbf{R}^{n+1}$. We showed that there is possibility to find out some of the solutions of this system using the Newton iteration method [11]. We considered examples and numerical tests. We also should emphasis that proposed numerical approach do not use a difference scheme.

First, let us consider a bit of theory which we will apply to conservation laws.

2 Non-Archimedean field of Laurent series and $\mathbf{R}\langle\varepsilon\rangle$–distributions.

The theory of Non-Archimedean fields was considered in the book by A.H.Lightstone and A.Robinson [14].

Definition 2.1 *A* Laurent series *is a formal object*

$$\sum_{n=0}^{\infty} \xi_{n+k}\varepsilon^{n+k}$$

where k is a fixed (i.e., fixed for this Laurent series), each $\xi_i \in \mathbf{R}$, and either $\xi_k \neq 0$ or each $\xi_i = 0$.

The Laurent series $\sum_{n=0}^{\infty} \xi_n \varepsilon^n$, where $\xi_0 = 1$ and $\xi_n = 0$ if $n > 0$, is denoted by 1. It is easy to see that the Laurent series is a field. Let us denote it by $\mathbf{R}\langle\varepsilon\rangle$. The norm on the field of Laurent series can define

$$|x|_\nu = e^{-\nu(x)} \quad \text{for} \quad \text{each} \quad x \in \mathbf{R}\langle\varepsilon\rangle$$

(in place of e can use any number greater then 1). The function $\nu(x)$ is a Non-Archimedean valuation. Define

$$\nu(0) = \infty \quad \text{and} \quad \nu\left(\sum_{n=0}^{\infty} \xi_{n+k}\varepsilon^{n+k}\right) = k \quad \text{if} \quad \sum_{n=0}^{\infty} \xi_{n+k}\varepsilon^{n+k} \neq 0, \quad \xi_k \neq 0.$$

The norm $|\cdot|_\nu$ have properties

1. $|x|_\nu = 0 \quad$ if and only if $\quad x = 0$,
2. $|xy|_\nu = |x|_\nu \cdot |y|_\nu$,
3. $|x + y|_\nu \leq \max\{|x|_\nu, |y|_\nu\}$.

Here, we propose a general construction of the $\mathbf{R}\langle\varepsilon\rangle$–valued generalized functions [22]. These objects are a natural generalization of Sobolev-Schwartz distributions. We call them as $\mathbf{R}\langle\varepsilon\rangle$–distributions.

1. Consider all functions $f(x,\varepsilon) \in C^\infty(\mathbf{R} \times (0,1))$ such that integrals

$$\int_{-\infty}^{+\infty} f(x,\varepsilon)\psi(x)dx$$

exist for any ε and for all $\psi(x)$ from a given class of functions $\mathcal{X}$ ($\mathcal{X}$ can be $C_0^\infty(\mathbf{R})$, $\mathcal{S}(\mathbf{R})$ and etc.).

2. Suppose also that $\int\limits_{-\infty}^{+\infty} f(x,\varepsilon)\psi(x)dx$ is a number $a_{f,\varepsilon}(\psi)$ from the field of Laurent series $\mathbf{R}\langle\varepsilon\rangle$.

3. The two functions $f(x,\varepsilon)$ and $g(x,\varepsilon)$ call equivalent with respect to test functions $\mathcal{X}$ if and only if

$$\int\limits_{-\infty}^{+\infty} f(x,\varepsilon)\psi(x)dx = a_{f,\varepsilon}(\psi) = a_{g,\varepsilon}(\psi) = \int\limits_{-\infty}^{+\infty} g(x,\varepsilon)\psi(x)dx.$$

   The equality means in sense of the field of Laurent series $\mathbf{R}\langle\varepsilon\rangle$ for all functions $\psi \in \mathcal{X}$. Classes of equivalent functions call $\mathbf{R}\langle\varepsilon\rangle$*–functions.* The expression

$$\int\limits_{-\infty}^{+\infty} f(x,\varepsilon)\psi(x)dx$$

   associates a number from $\mathbf{R}\langle\varepsilon\rangle$ with every ψ. Such a quantity is called a functional. In this case a linear functional map $\mathcal{X}$ into the Non-Archimedean field $\mathbf{R}\langle\varepsilon\rangle$. Call these functionals as $\mathbf{R}\langle\varepsilon\rangle$*-distributions.*

Thus,

Proposition 2.1 $\mathbf{R}\langle\varepsilon\rangle$*–function* $f(x,\varepsilon) = 0$ *if and only if*

$$\int\limits_{-\infty}^{+\infty} f(x,\varepsilon)\psi(x)dx = 0 \in \mathbf{R}\langle\varepsilon\rangle$$

for every ψ *from* $\mathcal{X}$.

The set of all $\mathbf{R}\langle\varepsilon\rangle$–distributions *denote by* $\mathcal{R}(\mathcal{X})$

Remark 2.1 *Recall that the idea of representation of a function* $f \in L^1_{loc}(\mathbf{R})$ *in terms of a linear functional*

$$C_0^\infty(\mathbf{R}) \ni \psi \longmapsto \int\limits_{-\infty}^{+\infty} f(x)\psi(x)dx \in \mathbf{R}$$

based on well-known proposition that if $f \in L^1_{loc}(\mathbf{R})$ *and* $\int\limits_{-\infty}^{+\infty} f(x)\psi(x)dx = 0$ *for any* $\psi \in C_0^\infty(\mathbf{R})$ *then* $f = 0$ *almost everywhere.*

Let us consider an example of the $\mathbf{R}\langle\varepsilon\rangle$-distribution.

Example 2.2 *Take* $\mathcal{X} = C_0^\infty(\mathbf{R})$ *and* $f(x,\varepsilon) = \varphi(x/\varepsilon)$, $\varphi(x) \in C_0^\infty(\mathbf{R})$ *then* $\mathbf{R}\langle\varepsilon\rangle$*–distribution can write in the following form.*

$$\int\limits_{-\infty}^{+\infty} \varphi(x/\varepsilon)\psi(x)dx = \varepsilon \int\limits_{-\infty}^{+\infty} \varphi(x)dx\psi(0) + \varepsilon^2 \int\limits_{-\infty}^{+\infty} x\varphi(x)dx\frac{\psi'(0)}{1!} + \dots.$$

Note that $\varphi(x/\varepsilon)$ *converges to the function*

$$u(x) = \begin{cases} \varphi(0), & \text{if } x = 0, \\ 0, & \text{if } x \neq 0. \end{cases}$$

Last function almost everywhere equals to zero.

Like Sobolev-Schwartz distributions we can differentiate $\mathbf{R}\langle\varepsilon\rangle$*–distributions. For example,*

$$\int_{-\infty}^{+\infty}\frac{d}{dx}\varphi(x/\varepsilon)\psi(x)dx = -\int_{-\infty}^{+\infty}\varphi(x/\varepsilon)\frac{d}{dx}\psi(x)dx,$$

$$-\int_{-\infty}^{+\infty}\varphi(x/\varepsilon)\frac{d}{dx}\psi(x)dx = -\varepsilon\int_{-\infty}^{+\infty}\varphi(x)dx\psi'(0) - \varepsilon^2\int_{-\infty}^{+\infty}x\varphi(x)dx\frac{\psi''(0)}{1!} - \dots .$$

It is evident that $\mathbf{R}\langle\varepsilon\rangle$–distributions are more general objects than Sobolev-Schwartz distributions [25], [26].

3 Conservation laws. Non-Archimedean approach.

A conservation law asserts that the rate of change of the total amount of substance contained in a fixed domain G is equal to the flux of that substance across the boundary of G. Denoting the density of that substance by u, and the flux by f, the conservation law is

$$\frac{d}{dt}\int_G u(t,x)dx = -\int_{\partial G} f\cdot\vec{n}dS.$$

Applying the divergence theorem and taking d/dt under the integral sign we obtain

$$\int_G (u_t + \mathbf{div} f)dx = 0.$$

Dividing by vol (G) and shrinking G to a point where all partial derivatives of u and f are continuous we obtain the differential conservation law

$$u_t(t,x) + \mathbf{div} f(u(t,x)) = 0.$$

Note, that if $f(u) = u^2/2$ then we obtained the Hopf equation (1). In general, previous calculations based on the following well known proposition.

Proposition 3.1 *If* $G \in L^1_{loc}(\mathbf{R})$ *and* $\int_{-\infty}^{+\infty} G(x)\psi(x)dx = 0$ *for any* $\psi \in C_0^\infty(\mathbf{R})$ *then* $G = 0$ *almost everywhere.*

Definition 3.1 *Let us consider two sets of the smooth functions, depending on a small parameter* $\varepsilon \in (0,1]$*. Let us take all functions* $v(t,x,\varepsilon)$ *which have the type*

$$v(t,x,\varepsilon) = l_0 + \Delta l\varphi\left(\frac{x-ct}{\varepsilon}\right),$$

$l_0, \Delta l, c$ *are real numbers,* $\Delta l \neq 0$ *and* $\varphi \in \mathcal{S}(\mathbf{R})$, $\int_{-\infty}^{+\infty}\varphi(y)dy = 1$*. We denote this set of functions by* I*. We call* I *as a set of infinetely narrow solitons.*

Definition 3.2 *Now, let us take all functions* $w(t,x,\varepsilon)$ *which have the type*

$$w(t,x,\varepsilon) = h_0 + \Delta h H\left(\frac{x-at}{\varepsilon}\right),$$

$h_0, \Delta h, a$ *are real numbers,* $\Delta h \neq 0$ *and* $H(x) = \int_{-\infty}^{x}\theta(y)dy$, $\int_{-\infty}^{+\infty}\theta(y)dy = 1$ *and* $\theta \in \mathcal{S}(\mathbf{R})$*. We denote this set of functions by* J*. We call* J *as a set of shock waves.*

It is natural to consider conservation laws as an integral expressions which contain the time t as parameter. Therefore, we introduce the following concept.

Definition 3.3 *The function $v \in I$ (or $w \in J$) will be a solution of the Hopf equation up to e^{-p}, $p \in \mathbf{N}_0$ in the sense of $\mathbf{R}\langle\varepsilon\rangle$–distributions if for any $t \in [0,T]$*

$$\int\limits_{-\infty}^{+\infty} \{v_t(t,x,\varepsilon) + v(t,x,\varepsilon)v_x(t,x,\varepsilon)\}\,\psi(x)dx = \sum_{k=p}^{+\infty} \xi_k \varepsilon^k \in \mathbf{R}\langle\varepsilon\rangle, \tag{3}$$

$$\int\limits_{-\infty}^{+\infty} \{w_t(t,x,\varepsilon) + w(t,x,\varepsilon)w_x(t,x,\varepsilon)\}\,\psi(x)dx = \sum_{k=p}^{+\infty} \eta_k \varepsilon^k \in \mathbf{R}\langle\varepsilon\rangle \tag{4}$$

for every $\psi \in \mathcal{S}(\mathbf{R})$. In case when p is equal to $+\infty$ the function $v(t,x,\varepsilon)$ (or $w(t,x,\varepsilon)$) exactly satisfies the Hopf equation in the sense of $\mathbf{R}\langle\varepsilon\rangle$–distributions.

Certainly, one can consider instead of the Hopf equation some conservation law.

From mathematical point of view, we deal with a infinitely differentiable functions in definitions 3.1 and 3.2, so that we avoid the problem of distribution multiplication. From physical point of view, functions from the set I or J can describe fast processes. Mathematical models of such processes based on functions from I or J may give additional information and take in account a short zone where physical system make a jump from one position to another.

Thus, we will consider solutions of the Hopf equation which are infinitely narrow solitons or shock waves. It easy to see that

$$v(t,x,\varepsilon) \longrightarrow \begin{cases} l_0 + \Delta l\varphi(0), & if \quad x = ct, \\ l_0, & if \quad x \neq ct. \end{cases} \qquad \text{as} \quad \varepsilon \to 0$$

$$w(t,x,\varepsilon) \longrightarrow h_0 + \Delta h H(x - at), \quad \text{as} \quad \varepsilon \to 0$$

H is Heaviside function.

4 Method for the numerical calculations of the microscopic profiles of soliton like solutions of the Hopf equation in the sense of $\mathbf{R}\langle\varepsilon\rangle$–distributions.

Thus, conservation laws are integral expressions. Therefore, it is natural, that one can interpret the Hopf equation in the sense of the definition 3.3.

We will seek a solution of the Hopf equation in the type of infinitely narrow soliton, i.e. let us $v \in I$. Substitute $v(t,x,\varepsilon)$ into integral expression (3) using the following formulas

$$\int\limits_{-\infty}^{+\infty} \frac{\partial}{\partial t}\left\{\varphi\left(\frac{x-ct}{\varepsilon}\right)\right\}\psi(x)dx = \sum_{k=0}^{+\infty} c\varepsilon^{k+1} m_k \frac{1}{k!}\psi^{(k+1)}(ct), \tag{5}$$

$$\int\limits_{-\infty}^{+\infty} \frac{\partial}{\partial x}\left\{\frac{1}{2}\varphi^2\left(\frac{x-ct}{\varepsilon}\right)\right\}\psi(x)dx = \sum_{k=0}^{+\infty} -\varepsilon^{k+1} g_k \frac{1}{k!}\psi^{(k+1)}(ct). \tag{6}$$

We denote

$$m_k(\varphi) = \int\limits_{-\infty}^{+\infty} y^k \varphi(y) dy, \; g_k(\varphi) = \int\limits_{-\infty}^{+\infty} y^k \frac{\varphi^2(y)}{2} dy, \; k = 0, 1, 2, \dots . \tag{7}$$

Thus, we obtain

$$\int\limits_{-\infty}^{+\infty} \{v_t + v v_x\} \psi dx = \sum_{k=0}^{+\infty} \{\Delta l (c - l_0) m_k - (\Delta l)^2 g_k\} \varepsilon^{k+1} \frac{\psi^{(k+1)}(ct)}{k!}. \tag{8}$$

From the last expression we have conditions for the function $\varphi(x)$. Namely,

$$g_k(\varphi) - \frac{c - l_0}{\Delta l} m_k(\varphi) = 0, \; k = 0, 1, 2 \dots . \tag{9}$$

From the first $(k = 0)$ we have

$$\frac{c - l_0}{\Delta l} = \frac{g_0}{m_0} = \frac{1}{2} \int\limits_{-\infty}^{+\infty} \varphi^2(x) dx. \tag{10}$$

Hence, we can rewrite conditions (9) as follows.

$$\int\limits_{-\infty}^{+\infty} \varphi^2(x) dx \cdot \int\limits_{-\infty}^{+\infty} x^k \varphi(x) dx = \int\limits_{-\infty}^{+\infty} x^k \varphi^2(x) dx, \; k = 0, 1, 2 \dots . \tag{11}$$

Now, let us prove the following lemma.

Lemma 4.1 *For any non-negative integer n exists such function $\varphi \in \mathcal{S}(\mathbf{R})$, $\varphi \not\equiv 0$ which satisfies the following system of non-linear equations:*

$$\int\limits_{-\infty}^{+\infty} x^k \varphi(x) dx = \int\limits_{-\infty}^{+\infty} x^k \varphi^2(x) dx \Big/ \int\limits_{-\infty}^{+\infty} \varphi^2(x) dx \quad k = 0, 1, 2 \dots n. \tag{12}$$

Proof. First, we will seek function $\varphi(x)$ in the following type:

$$\varphi(x) = c_0 h_0(x) + c_1 h_1(x) + \dots + c_n h_n(x), \tag{13}$$

where

$$h_k(x) = \frac{H_k(x)}{\sqrt{2^k k!} \sqrt[4]{\pi}} e^{-x^2/2} \text{ are Hermit functions.} \tag{14}$$

Then we substitute the expression (13) into conditions (12). After that we will have nonlinear system of $n + 1$ equations with $n + 1$ unknowns $(c_0, c_1, c_2, \dots, c_n)$. We write this system by the following way.

$$A\vec{x} = \mathcal{N}(\vec{x}), \; \vec{x} = (c_0, c_1, \dots, c_n) \tag{15}$$

A is a matrix with elements

$$A_{kj} = \int\limits_{-\infty}^{+\infty} x^k h_j(x) dx = (-\mathbf{i})^j \mathbf{i}^k \sqrt{2\pi} h_j^{(k)}(0), \; \mathbf{i} = \sqrt{-1}, \; k, j = 0, 1, 2, \dots n$$

$\mathcal{N}$ is nonlinear map such that

$$\mathcal{N}(\vec{x}) = \frac{1}{\|\vec{x}\|^2}\sum_{k=0}^{n}(N(k)\vec{x},\vec{x})\vec{e}_k \equiv \sum_{k=0}^{n} f_k(\vec{x})\vec{e}_k \tag{16}$$

Vector $\vec{e}_k = (e_0, e_1, \dots, e_n)$ such that $e_k = 1$ and $e_j = 0$ for all $j \neq k$. $N(k)$ are matrices with elements

$$N_{ij}(k) = \int_{-\infty}^{+\infty} x^k h_i(x) h_j(x) dx, \quad i,j,k = 0,1,2\dots n \tag{17}$$

and functions

$$f_k(\vec{x}) = \frac{(N(k)\vec{x},\vec{x})}{\|\vec{x}\|^2}.$$

Note that functions $f_k(\vec{x})$ are continuous everywhere except $\vec{x} = 0$ and $|f_k(\vec{x})| \leq \|N(k)\|$ due to Cauchy-Bunyakovskii unequality. Matrix A is invertible for any n because of $\det(A)$ is a Wronskian for the linear independent system of Hermit functions $h_0(x)$, $h_1(x)$, ... $h_n(x)$ and

$$\det(A) = (2\pi)^{\frac{(n+1)}{2}} W(h_0(0), h_1(0), \dots h_n(0)).$$

We can write the system (15) as

$$\vec{x} = \sum_{k=0}^{n} f_k(\vec{x}) A^{-1}\vec{e}_k \equiv F(\vec{x}) \;\text{ or }\; \vec{x} = A^{-1}(\mathcal{N}(\vec{x})) \equiv F(\vec{x}). \tag{18}$$

Let us describe the function $F : \mathbf{R}^{n+1} \longmapsto \mathbf{R}^{n+1}$. It is continuous except $\vec{x} = 0$ and bounded. Indeed,

$$\|F(\vec{x})\| \leq \|A^{-1}\| \sum_{k=0}^{n} \|N(k)\|, \quad r_n = \|A^{-1}\| \sum_{k=0}^{n} \|N(k)\|. \tag{19}$$

Let us consider function $\mathcal{N}(\vec{x})$. It is continuous function everywhere in $\mathbf{R}^{n+1}$ except $\vec{x} = 0$ and, moreover, $\mathcal{N}(\mathbf{R}^{n+1}\backslash\{0\}) \subset \Pi_1$ where $\Pi_1 = \{\vec{z} \in \mathbf{R}^{n+1} : z_0 = 1\}$ is a plane. Further $A^{-1}(\Pi_1) = \Pi_2$ where $\Pi_2 = \{\vec{y} \in \mathbf{R}^{n+1} : \sum_{k=0}^{n} a_{0j} y_j = 1\}$ is another plane.

$$A_{0j} = \int_{-\infty}^{+\infty} h_j(x) dx = (-\mathbf{i})^j \sqrt{2\pi} h_j(0), \quad \mathbf{i} = \sqrt{-1}, \quad j = 0,1,2,\dots n$$

Thus, we can consider the function $F(\vec{x})$ which is defined on the convex compact set $C_n = \Pi_2 \bigcap B[0, r_n]$ such that $F : C_n \longmapsto C_n$, where $B[0, r_n]$ is a closed ball with radius r_n. Function F is continuous on the C_n because of $\vec{0} \notin C_n$. Now we can use J.Schauder theorem.

Theorem 4.2 (Schauder fixed-point theorem [24]) *Let C be a compact convex subset of a normed space E. Then each continuous map $F : C \longmapsto C$ has at least one fixed point.*

Hence, we can conclude that our system (18) and therefore system (15) has at least one solution. Thus, there is a function $\varphi(x)$ which satisfy to conditions (12) proposed lemma.

Remark 4.1 *Let us a function $\varphi(x)$ satisfies lemma condition. If $\beta \in \mathbf{R}$ then the function $\varphi(x+\beta)$ also satisfies lemma condition. Moreover, if $\int_{-\infty}^{+\infty} \varphi^2(x)\, dx = \alpha$ then $\varphi(\alpha x)$ satisfies lemma condition.*

Thus, we can formulate the following result.

Theorem 4.3 *For any integer p there is a infinitely narrow soliton type solution of the Hopf equation (in the sense of the definition 3.3) up to* e^{-p} *with respect to the norm* $|\cdot|_\nu$*, i.e.*

$$v(t,x,\varepsilon) = l_0 + \Delta l \varphi\left(\frac{x-ct}{\varepsilon}\right), \tag{20}$$

$l_0, \Delta l, c$ *are real numbers,* $\Delta l \neq 0$ *and* $\varphi \in \mathcal{S}(\mathbf{R})$, $\int\limits_{-\infty}^{+\infty} \varphi(y)dy = 1$. *Moreover,*

$$\frac{c-l_0}{\Delta l} = \frac{1}{2}\int\limits_{-\infty}^{+\infty} \varphi^2(x)dx. \tag{21}$$

For example, calculations in case $p = 7$ give the "profile" $\varphi(x)$ (see Fig. 1) for the infinitely narrow soliton $v(t,x,\varepsilon) = \varphi\left(\frac{x-ct}{\varepsilon}\right)$:

$$\varphi(x) = \left\{\frac{c_0}{\sqrt[4]{\pi}} + \frac{c_2(4x^2-2)}{\sqrt{2^2 2!}\sqrt[4]{\pi}} + \frac{c_4(16x^4-48x^2+12)}{\sqrt{2^4 4!}\sqrt[4]{\pi}}\right\} e^{-x^2/2}, \tag{22}$$

where $c_0 = 0.66583$, $c_2 = -0.23404$, $c_4 = 0.05028$, $c = 0.25032$ (c is a velocity of the soliton). Numbers c_0, c_2, c_4 and c were found approximately by iteration method using the following sequence.

$$\vec{x}_{m+1} = A^{-1}(\mathcal{N}(\vec{x}_m)), \quad m = 0, 1, 2, \dots . \tag{23}$$

Matrix A and a nonlinear $\mathcal{N}$ were introduced in the lemma proof.

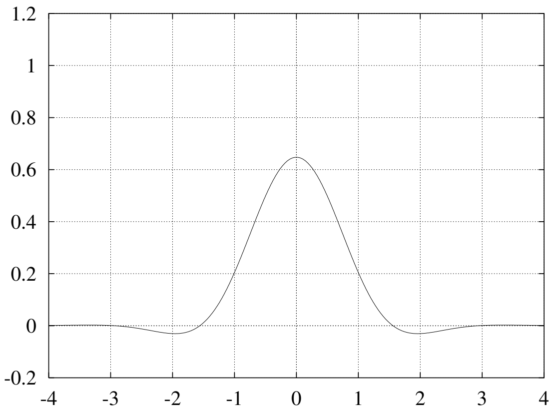

Figure 1: The case $p = 7$, $c = 0.25032$.

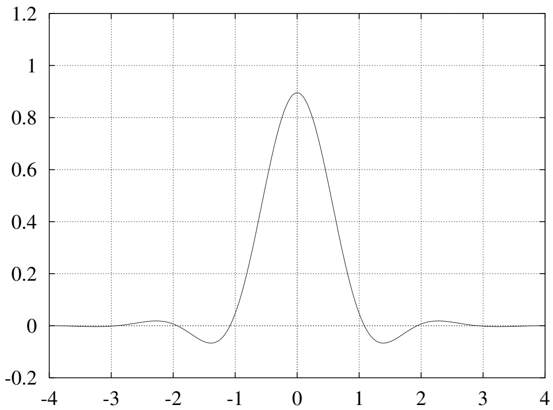

Figure 2: The case $p = 13$, $c = 0.35442$.

Calculations of soliton-like profiles $\varphi(x)$ for the Hopf equation in case $p = 13, 15, 17, 19, 21$ give us pictures (Fig. 2, 3, 4, 5, 6).

For the p greater than 21 matrix A is close to singular and calculations can be inaccurate.

5 Calculations of the microscopic profiles of the shock wave solutions of the Hopf equation in the sense of $\mathbf{R}\langle\varepsilon\rangle$–distributions.

A solution of the Hopf equation in this case we will seek in the set J. Namely,

$$w(t,x,\varepsilon) = h_0 + \Delta h K\left(\frac{x-at}{\varepsilon}\right),$$

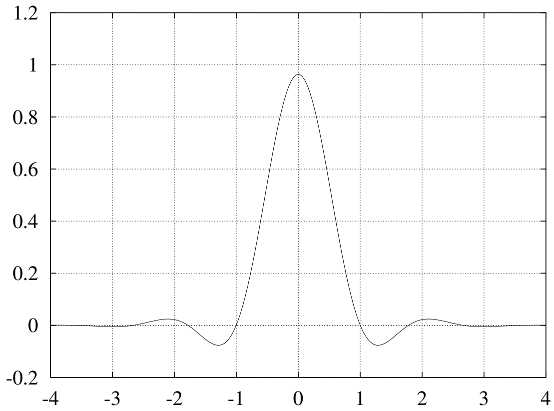

Figure 3: The case $p = 15$, $c = 0.38267$.

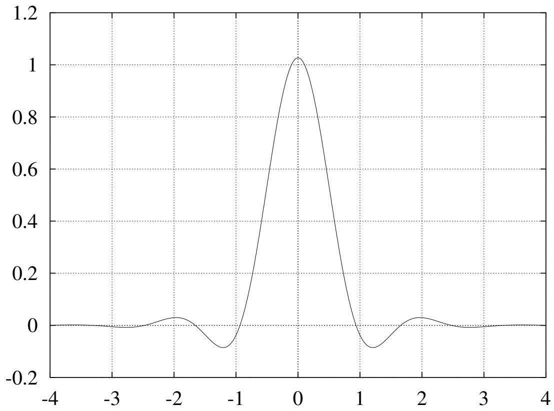

Figure 4: The case $p = 17$, $c = 0.40892$.

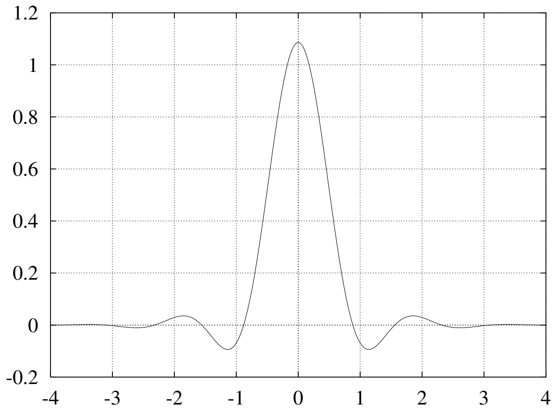

Figure 5: The case $p = 19$, $c = 0.43357$.

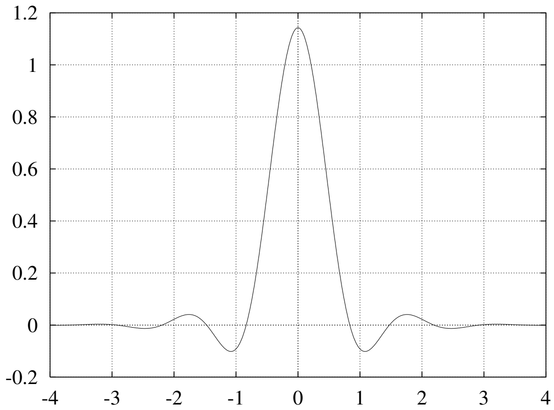

Figure 6: The case $p = 21$, $c = 0.45678$.

$h_0, \Delta h, a$ are real numbers, $\Delta h \neq 0$ and

$$K(x) = \int_{-\infty}^{x} \theta(y)dy, \int_{-\infty}^{+\infty} \theta(y)dy = 1, \quad \theta \in \mathcal{S}(\mathbf{R}).$$

Substitute $w(t, x, \varepsilon)$ into the integral expression (4) using the following formulas

$$\int_{-\infty}^{+\infty} \frac{\partial}{\partial t} \left\{ K\left(\frac{x - at}{\varepsilon}\right) \right\} \psi(x)dx = \sum_{k=0}^{+\infty} (-a)\varepsilon^k m_k \frac{\psi^{(k)}(at)}{k!}, \tag{24}$$

$$\int_{-\infty}^{+\infty} K\left(\frac{x - at}{\varepsilon}\right) \frac{\partial}{\partial x} \left\{ K\left(\frac{x - at}{\varepsilon}\right) \right\} \psi(x)dx = \sum_{k=0}^{+\infty} \varepsilon^k r_k \frac{\psi^{(k)}(at)}{k!}. \tag{25}$$

We denote by

$$m_k(\theta) = \int_{-\infty}^{+\infty} y^k \theta(y)dy, \; r_k(\theta) = \int_{-\infty}^{+\infty} x^k \theta(x) \left(\int_{-\infty}^{x} \theta(y)dy \right) dx, \; k = 0, 1, 2, \dots . \tag{26}$$

Thus, we get

$$\int_{-\infty}^{+\infty} \{w_t + ww_x\}\psi dx = \sum_{k=0}^{+\infty} \left\{(\Delta h)^2 r_k - \Delta h(a-h_0)m_k\right\}\varepsilon^k \frac{\psi^{(k)}(at)}{k!}. \tag{27}$$

From the last expression we have conditions for the function $\theta(x)$

$$r_k(\theta) - \frac{a-h_0}{\Delta h} m_k(\theta) = 0, \ k = 0, 1, 2 \ldots . \tag{28}$$

From the first $(k=0)$ we have

$$\frac{a-h_0}{\Delta h} = \int_{-\infty}^{+\infty} \theta(x) \left(\int_{-\infty}^{x} \theta(y)dy \right) dx = \frac{1}{2}. \tag{29}$$

Therefore, we can rewrite (28) as

$$\frac{1}{2}\int_{-\infty}^{+\infty} x^k\theta(x)dx = \int_{-\infty}^{+\infty} x^k\theta(x) \left(\int_{-\infty}^{x} \theta(y)dy \right) dx \ k = 0, 1, 2 \ldots . \tag{30}$$

The same method one can prove that there is such function $\theta(x) \in \mathcal{S}(\mathbf{R})$ which satisfies the following conditions

$$\frac{1}{2}\int_{-\infty}^{+\infty} x^k\theta(x)dx = \int_{-\infty}^{+\infty} x^k\theta(x) \left(\int_{-\infty}^{x} \theta(y)dy \right) dx \ k = 0, 1, 2 \ldots n. \tag{31}$$

Thus, we can formulate next result.

Theorem 5.1 *For any integer p there is a shock wave type solution of the Hopf equation (in the sense of the definition 3.3) up to e^{-p} with respect to the norm $|\cdot|_\nu$.*

$$w(t,x,\varepsilon) = h_0 + \Delta h K\left(\frac{x-at}{\varepsilon}\right), \tag{32}$$

$h_0, \Delta h, a$ are real numbers, $\Delta h \neq 0$ and $K(x) = \int_{-\infty}^{x} \theta(y)dy$, $\int_{-\infty}^{+\infty} \theta(y)dy = 1$ and $\theta \in \mathcal{S}(\mathbf{R})$. Moreover,

$$\frac{a-h_0}{\Delta h} = \frac{1}{2}. \tag{33}$$

Note that the condition (33) is *Rankine — Hugoniot condition* for the velocity of a shock wave.
As in previous section we seek function $\theta(x)$ in the following type:

$$\varphi(x) = a_0h_0(x) + a_1h_1(x) + \ldots + a_nh_n(x), \tag{34}$$

where $h_k(x)$ are Hermite functions. Calculations in case $p=7$ give the following "profile" $(K(x))$ for the shock wave $w(t,x,\varepsilon) = K\left(\frac{x-at}{\varepsilon}\right)$ (where $h_0 = 0, \Delta h = 1$).

$$K(x) = \int_{-\infty}^{x} \left\{ \frac{c_0}{\sqrt[4]{\pi}} + \frac{c_2(4\tau^2-2)}{\sqrt{2^2 2!}\sqrt[4]{\pi}} + \frac{c_4(16\tau^4 - 48\tau^2 + 12)}{\sqrt{2^4 4!}\sqrt[4]{\pi}} \right\} e^{-\tau^2/2} d\tau \tag{35}$$

where $c_0 = 0.79617$, $c_2 = -0.53004$, $c_4 = 0.17923$, $c = 1/2$ is a velocity of the shock wave (see Fig. 7). Numbers c_0, c_2, c_4 were found approximately.

Note that the function $K(x)$ is not unique. There is a different function $K_1(x)$ which satisfies mentioned above conditions. It has the following type

$$\begin{gathered} K_1(x) = \int\limits_{-\infty}^{x} \left\{ \frac{c_0}{\sqrt[4]{\pi}} + \frac{c_1 2\tau}{\sqrt{2^1 1!}\sqrt[4]{\pi}} + \frac{c_2(4\tau^2-2)}{\sqrt{2^2 2!}\sqrt[4]{\pi}} \right\} e^{-\tau^2/2} d\tau + \\ + \int\limits_{-\infty}^{x} \left\{ \frac{c_3(8\tau^3-12\tau)}{\sqrt{2^3 3!}\sqrt[4]{\pi}} + \frac{c_4(16\tau^4-48\tau^2+12)}{\sqrt{2^4 4!}\sqrt[4]{\pi}} \right\} e^{-\tau^2/2} d\tau \end{gathered} \tag{36}$$

where $c_0 = 0.18357$, $c_1 = -0.73567$, $c_2 = 0.74733$, $c_3 = 0.15327$ $c_4 = -0.29539$, $c = 1/2$ is a velocity of the shock wave (see Fig. 8). Coefficients c_0, c_1, c_2, c_3, c_4 were found approximately by the Newton iteration method.

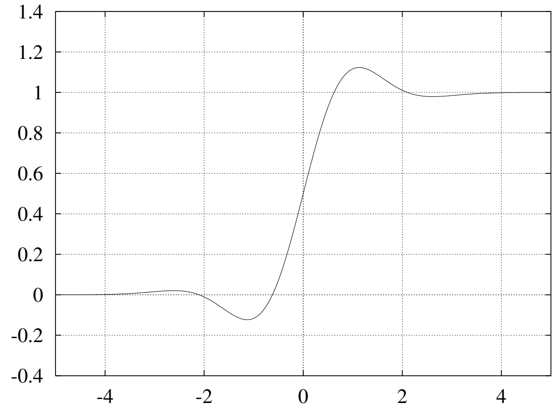

Figure 7: Graph of the function $K(x)$.

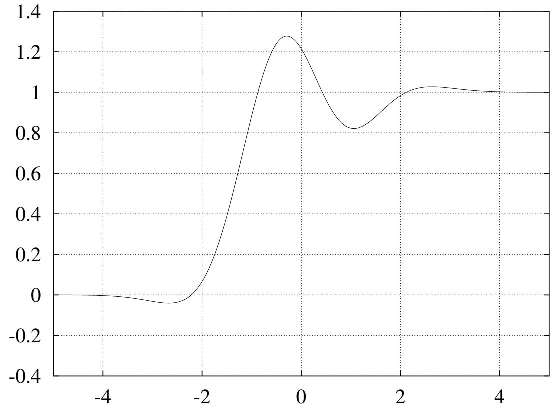

Figure 8: Graph of the function $K_1(x)$.

Taking in account the *Rankine — Hugoniot condition* (33) we also have graphs (Fig. 9, 10) as a shock profiles.

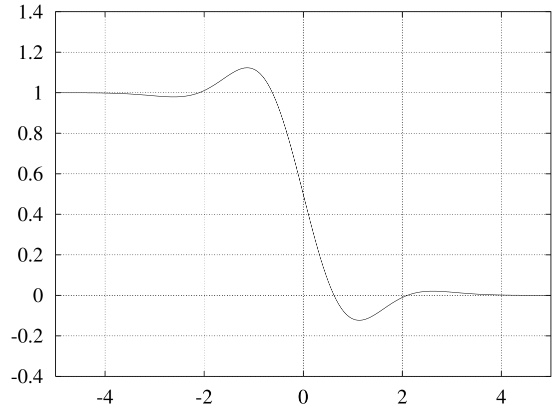

Figure 9: First shock profile $1 - K(x)$.

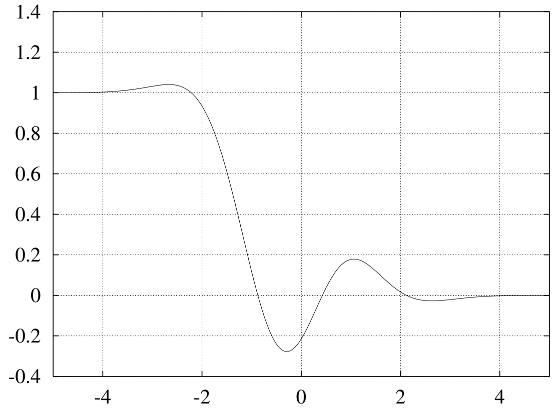

Figure 10: Second shock profile $1 - K_1(x)$.

Here we describe how it is possible to find coefficients $c_0, c_1, \dots, c_n$ in this case by the Newton iteration method for the following system of nonlinear equations.

$$P(\vec{c}) = A\vec{c} - 2\sum_{k=0}^{n} (S(k)\vec{c}, \vec{c})\vec{e}_k = 0, \;\; \vec{c} = (c_0, c_1, \dots, c_n) \tag{37}$$

Vector $\vec{e}_k = (e_0, e_1, \dots, e_n)$ such that $e_k = 1$ and $e_j = 0$ for all $j \neq k$. $S(k)$ are matrices with elements

$$S_{ij}(k) = \int_{-\infty}^{+\infty} x^k h_i(x) \cdot \int_{-\infty}^{x} h_j(y)\, dy\, dx, \;\; i,j,k = 0,1,2\dots n \tag{38}$$

Matrix A have elements

$$A_{ij} = \int_{-\infty}^{+\infty} x^i h_j(x)\, dx, \;\; i,j = 0,1,2\dots n \tag{39}$$

We can write the formula for the Newton iteration method [11].

$$\vec{x}_{m+1} = \vec{x}_m - \left[P'(\vec{x}_m)\right]^{-1} \left[P(\vec{x}_m)\right], \tag{40}$$

where $[P'(\vec{x})]$ is a linear map depending on the vector $\vec{x}$.

$$\left[P'(\vec{x})\right][\vec{h}] = A\vec{h} - 2\left\{\sum_{k=0}^{n}(S(k)\vec{x},\vec{h})\vec{e}_k + \sum_{k=0}^{n}(S^T(k)\vec{x},\vec{h})\vec{e}_k\right\} \tag{41}$$

Calculations of shock profiles $K(x)$ for the Hopf equation in case $p = 8$, 9, 10, 11, 12, 13 give us the following pictures (Fig. 11, 12, 13, 14, 15, 16). Here, we show only two different types of the shock type solutions of the Hopf equation. We can find more solutions if we take a different initial data for the Newton iteration method.

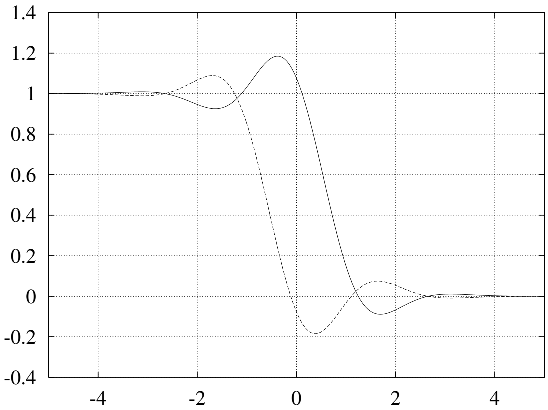

Figure 11: Shock profiles when $p = 8$.

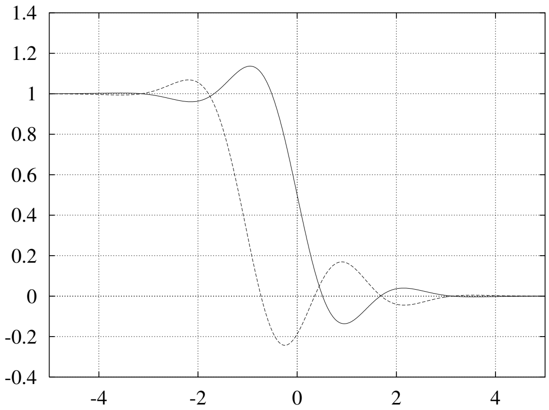

Figure 12: Shock profiles when $p = 9$.

Remark 5.1 *It is not easy to see that there is exist function*

$$\theta(x) = \sum_{n=1}^{\infty} a_n h_n(x), \;\; \vec{a} = (a_0, a_1, \dots, a_n, \dots) \in l_2, \tag{42}$$

such that

$$\frac{1}{2}\int_{-\infty}^{+\infty} x^k \theta(x) dx = \int_{-\infty}^{+\infty} x^k \theta(x) \left(\int_{-\infty}^{x} \theta(y) dy \right) dx, \;\; k = 0,1,2\dots. \tag{43}$$

We think that it is true.

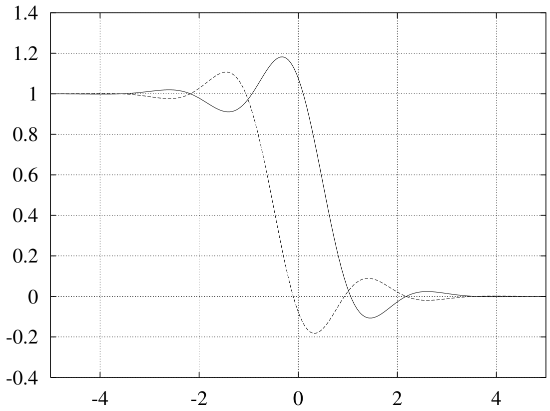

Figure 13: Shock profiles when $p = 10$.

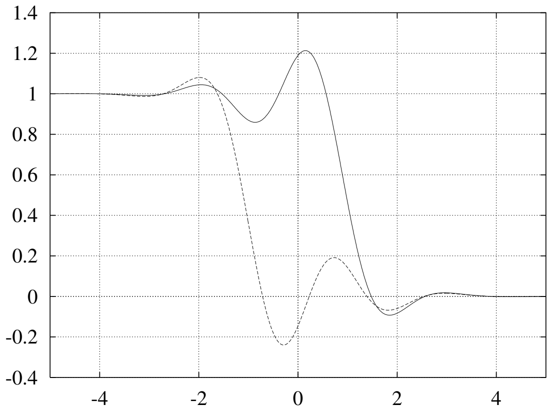

Figure 14: Shock profiles when $p = 11$.

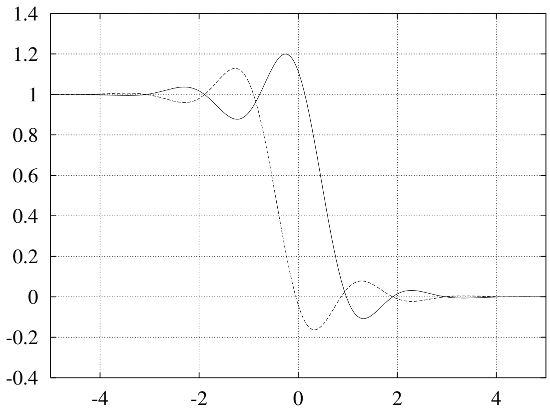

Figure 15: Shock profiles when $p = 12$.

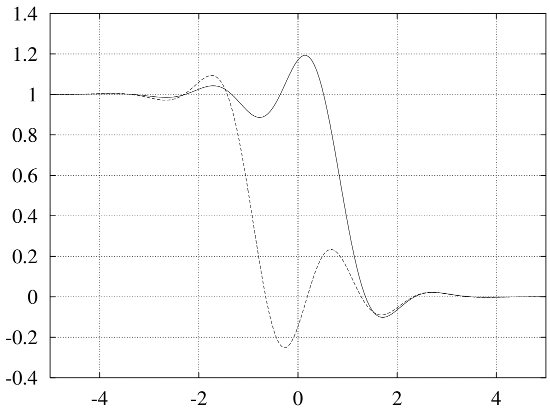

Figure 16: Shock profiles when $p = 13$.

6 Calculations of the microscopic profiles of the shock wave solutions of equations of elasticity theory in the sense of $\mathbf{R}\langle\varepsilon\rangle-$distributions.

Let us consider the following system.

$$\begin{aligned} &u_t + (u^2)_x = \sigma_x \ \text{ (the conservation law for momentum)} \\ &\sigma_t + u\sigma_x = k^2 u_x \ \text{ (the Hooke law)} \end{aligned} \tag{44}$$

Here, u is the velocity of a medium and σ is the stress. We suppose that density of a medium is equal to the constant 1 and k^2 some constant.

We will seek for a solution of this system in the following form

$$u(t,x,\varepsilon) = u_0 + \Delta u U\left(\frac{x - vt}{\varepsilon}\right), \tag{45}$$

$u_0, \Delta u, v$ are real numbers, $\Delta u \neq 0$ and $U(x) = \int\limits_{-\infty}^{x} \widetilde{U}(y)dy$, $\int\limits_{-\infty}^{+\infty} \widetilde{U}(y)dy = 1$ and $\widetilde{U} \in \mathcal{S}(\mathbf{R})$.

$$\sigma(t,x,\varepsilon) = \sigma_0 + \Delta\sigma\Sigma\left(\frac{x - vt}{\varepsilon}\right), \tag{46}$$

$\sigma_0, \Delta\sigma, v$ are real numbers, $\Delta\sigma \neq 0$ and $\Sigma(x) = \int\limits_{-\infty}^{x} \widetilde{\Sigma}(y)dy$, $\int\limits_{-\infty}^{+\infty} \widetilde{\Sigma}(y)dy = 1$ and $\widetilde{\Sigma} \in \mathcal{S}(\mathbf{R})$. Note that v is a velocity of the shock waves.

In the other hand, we suppose

$$\widetilde{U}(x) = a_0 h_0(x) + a_1 h_1(x) + \ldots + a_n h_n(x), \;\; \vec{a} = (a_0, a_1, \ldots, a_n), \tag{47}$$

$$\widetilde{\Sigma}(x) = c_0 h_0(x) + c_1 h_1(x) + \ldots + c_n h_n(x), \;\; \vec{c} = (c_0, c_1, \ldots, c_n) \tag{48}$$

where $h_k(x)$ are Hermite functions.

We understand the solution of the system in sense of $\mathbf{R}\langle\varepsilon\rangle$–distributions.

Definition 6.1 *Functions $u \in J$ and $\sigma \in J$ is a solution of the system (44) up to e^{-p}, $p \in \mathbf{N}_0$ in the sense of $\mathbf{R}\langle\varepsilon\rangle$–distributions if for any $t \in [0, T]$*

$$\int\limits_{-\infty}^{+\infty} \{u_t(t,x,\varepsilon) + 2u(t,x,\varepsilon)u_x(t,x,\varepsilon) - \sigma_x(t,x,\varepsilon)\} \psi(x)dx = \sum_{k=p}^{+\infty} \xi_k \varepsilon^k \in \mathbf{R}\langle\varepsilon\rangle, \tag{49}$$

$$\int\limits_{-\infty}^{+\infty} \left\{\sigma_t(t,x,\varepsilon) + u(t,x,\varepsilon)\sigma_x(t,x,\varepsilon) - k^2 u_x(t,x,\varepsilon)\right\} \psi(x)dx = \sum_{k=p}^{+\infty} \eta_k \varepsilon^k \in \mathbf{R}\langle\varepsilon\rangle \tag{50}$$

for every $\psi \in \mathcal{S}(\mathbf{R})$.

In case when p is equal to $+\infty$ functions $u(t,x,\varepsilon)$ and $\sigma(t,x,\varepsilon))$ exactly satisfies the system (44) in the sense of $\mathbf{R}\langle\varepsilon\rangle$–distributions.

Substituting u and σ into (49), (50) we get the following relations for the moments.

$$\{2u_0\Delta u - v\Delta u\} m_k(\widetilde{U}) + 2(\Delta u)^2 m_k(\widetilde{U}U) - \Delta\sigma m_k(\widetilde{\Sigma}) = 0, \;\; k = 0, 1, 2, \ldots n \tag{51}$$

$$\{u_0\Delta\sigma - v\Delta\sigma\} m_k(\widetilde{\Sigma}) + \Delta u \Delta\sigma m_k(\widetilde{\Sigma}U) - k^2 \Delta u m_k(\widetilde{U}) = 0, \;\; k = 0, 1, 2, \ldots n \tag{52}$$

We denote as usual by

$$m_k(\widetilde{U}) = \int\limits_{-\infty}^{+\infty} x^k \widetilde{U}(x)dx, \tag{53}$$

$$m_k(\widetilde{U}U) = \int\limits_{-\infty}^{+\infty} x^k \widetilde{U}(x) \left(\int\limits_{-\infty}^{x} \widetilde{U}(y)dy \right) dx, \;\; k = 0, 1, 2, \ldots, \tag{54}$$

$$m_k(\widetilde{\Sigma}U) = \int\limits_{-\infty}^{+\infty} x^k \widetilde{\Sigma}(x) \left(\int\limits_{-\infty}^{x} \widetilde{U}(y)dy \right) dx, \;\; k = 0, 1, 2, \ldots \tag{55}$$

It is easy to find v from (51) when k=0. Indeed,

$$\{2u_0\Delta u - v\Delta u\} + (\Delta u)^2 - \Delta\sigma = 0.$$

Therefore,

$$v = 2u_0 + \Delta u - \frac{\Delta\sigma}{\Delta u}. \tag{56}$$

Substitute v into the (51). We have

$$\{\Delta\sigma - (\Delta u)^2\}m_k(\widetilde{U}) + 2(\Delta u)^2 m_k(\widetilde{U}U) - \Delta\sigma m_k(\widetilde{\Sigma}) = 0, \;\; k = 0,1,2,\dots n \tag{57}$$

Because of ΔU and $\Delta\sigma$ some real numbers, therefore, all three vectors with coordinates $m_k(\widetilde{U})$, $m_k(\widetilde{U}U)$ and $m_k(\widetilde{\Sigma})$, $k = 0,1,2,\dots n$, respectively should be collinear. However,

$$m_0(\widetilde{U}) = m_0(\widetilde{\Sigma}) = 1.$$

Hence,

$$m_k(\widetilde{U}) = m_k(\widetilde{\Sigma}), \;\; k = 0,1,2,\dots n.$$

Thus, $\vec{a} = \vec{c}$ and from (57) follows that

$$m_k(\widetilde{U}) = 2m_k(\widetilde{U}U), \;\; k = 0,1,2,\dots n.$$

This system we already know how to solve by the Newton iteration method. See conditions (31) and solution in this case.

Substitute v into the (52)and take into account previous equalities we have

$$\left\{\frac{(\Delta\sigma)^2}{\Delta u} - u_0\Delta\sigma - \Delta u\Delta\sigma\right\} m_k(\widetilde{\Sigma}) + \Delta u\Delta\sigma m_k(\widetilde{\Sigma}\Sigma) - k^2\Delta u m_k(\widetilde{\Sigma}) = 0 \tag{58}$$

where $k = 0,1,2,\dots n$. The last expression gives us relation for constants $\Delta\sigma$, Δu, u_0, k^2. Namely,

$$\left\{\frac{(\Delta\sigma)^2}{\Delta u} - u_0\Delta\sigma - \Delta u\Delta\sigma\right\} + \frac{1}{2}\Delta u\Delta\sigma - k^2\Delta u = 0 \tag{59}$$

or

$$(\Delta\sigma)^2 - \left(u_0 + \frac{1}{2}\Delta u\right)\Delta u\Delta\sigma - k^2(\Delta u)^2 = 0 \tag{60}$$

If Δu, u_0, k^2 are known then from the last equation one can find $\Delta\sigma$

$$\Delta\sigma_{1,2} = \frac{1}{2}\left(u_0 + \frac{1}{2}\Delta u\right) \pm \frac{1}{2}|\Delta u|\sqrt{(u_0 + \frac{1}{2}\Delta u)^2 + 4k^2} \tag{61}$$

In particular, if $\Delta u = -1$, $u_0 = 1$ then

$$\Delta\sigma_{1,2} = -\frac{1}{4} \pm \frac{1}{4}\sqrt{1 + 16k^2}.$$

Shock profiles of the considered system (44) one can find on pictures (Fig. 17, 18, 19, 20). We considered case when $p = 13$ and $\Delta u = -1$, $u_0 = 1$, $k^2 = 0.1$. We can also take any real σ_0 but here we took $\sigma_0 = 0.5$ and then calculated $\Delta\sigma_1$ (Fig. 17, 19, the velocity of shocks is $v = 1.1531$) and $\Delta\sigma_2$ (Fig. 18, 20, the velocity of shocks is $v = 0.34689$). We consider two different types of shock profiles. The first is on the Fig. 17, 18. The second is on the Fig. 19, 20.

Theorem 6.1 *For any integer p there is a solution of the system of equations (44) in the sense of the definition 6.1. Moreover,*

$$v = 2u_0 + \Delta u - \frac{\Delta\sigma}{\Delta u}$$

and

$$(\Delta\sigma)^2 - \left(u_0 + \frac{1}{2}\Delta u\right)\Delta u\Delta\sigma - k^2(\Delta u)^2 = 0.$$

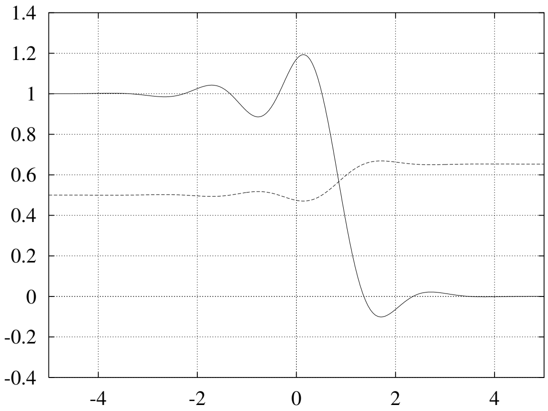

Figure 17: Shock profiles of velocity and stress.

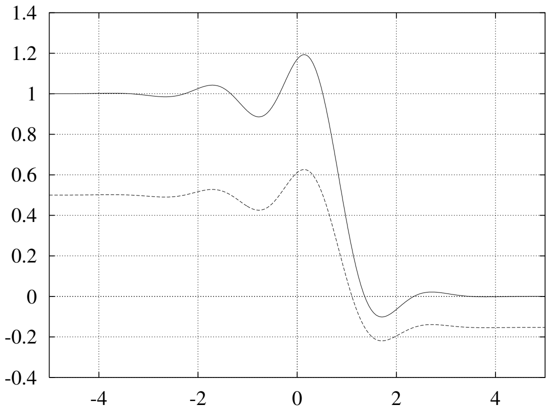

Figure 18: Shock profiles of velocity and stress.

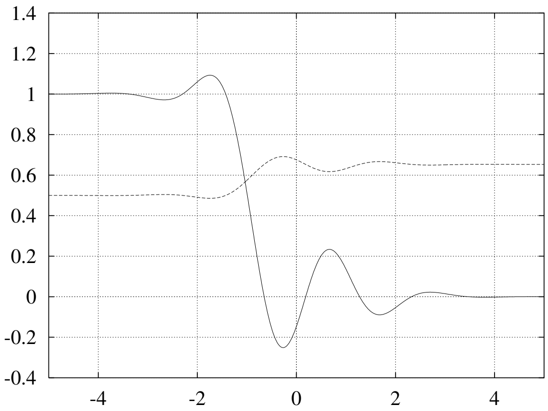

Figure 19: Shock profiles of velocity and stress.

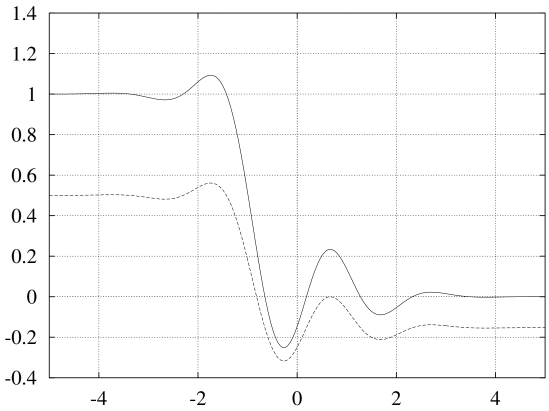

Figure 20: Shock profiles of velocity and stress.

Let us consider the following system.

$$\begin{array}{l} \rho_t + (\rho u)_x = 0 \ \text{(the conservation law for mass)} \\ (\rho u)_t + (\rho u^2)_x = \sigma_x \ \text{(the conservation law for momentum)} \\ \sigma_t + u\sigma_x = k^2 u_x \ \text{(the Hooke law)} \end{array} \tag{62}$$

Here, u is the velocity of a medium and σ is the stress. We suppose that k^2 is some constant.

Definition 6.2 *Functions $u \in J$, $\rho \in J$ and $\sigma \in J$ is a solution of the system (62) up to e^{-p}, $p \in \mathbf{N}_0$ in the sense of $\mathbf{R}\langle\varepsilon\rangle$–distributions if for any $t \in [0,T]$*

$$\int_{-\infty}^{+\infty} \{\rho_t(t,x,\varepsilon) + \rho_x u + \rho u_x\} \psi(x) dx = \sum_{k=p}^{+\infty} \xi_k \varepsilon^k \in \mathbf{R}\langle\varepsilon\rangle, \tag{63}$$

$$\int_{-\infty}^{+\infty} \{\rho_t u + \rho u_t + \rho_x u^2 + 2\rho u u_x - \sigma_x\} \psi(x) dx = \sum_{k=p}^{+\infty} \zeta_k \varepsilon^k \in \mathbf{R}\langle\varepsilon\rangle, \tag{64}$$

$$\int_{-\infty}^{+\infty} \{\sigma_t + u\sigma_x - k^2 u_x\} \psi(x) dx = \sum_{k=p}^{+\infty} \eta_k \varepsilon^k \in \mathbf{R}\langle\varepsilon\rangle. \tag{65}$$

for every $\psi \in \mathcal{S}(\mathbf{R})$.

In case when p *is equal to* $+\infty$ *functions* $u(t,x,\varepsilon)$, $\rho(t,x,\varepsilon)$ *and* $\sigma(t,x,\varepsilon)$ *exactly satisfies the system (62) in the sense of* $\mathbf{R}\langle\varepsilon\rangle$*–distributions.*

We will seek for a solution of this system in the form (45), (46), (47), (48),

$$\rho(t,x,\varepsilon) = \rho_0 + \Delta\rho R\left(\frac{x-vt}{\varepsilon}\right), \tag{66}$$

$\rho_0, \Delta\rho, v$ are real numbers, $\Delta\rho \neq 0$ and $R(x) = \int\limits_{-\infty}^{x} \widetilde{R}(y)dy$, $\int\limits_{-\infty}^{+\infty} \widetilde{R}(y)dy = 1$ and $\widetilde{R} \in \mathcal{S}(\mathbf{R})$. We suppose

$$\widetilde{R}(x) = b_0h_0(x) + b_1h_1(x) + \ldots + b_nh_n(x),\ \ \vec{b} = (b_0, b_1, \ldots, b_n), \tag{67}$$

Substituting (45) and (66) into (63) we get the following relations for the moments.

$$\{-v\Delta\rho + \Delta\rho u_0\}m_k(\widetilde{R}) + \Delta\rho\Delta u m_k(\widetilde{R}U) + \rho_0\Delta u m_k(\widetilde{U}) + \Delta\rho\Delta u m_k(\widetilde{U}R) = 0 \tag{68}$$

where $k = 0, 1, \ldots n.$. Note that $m_0(\widetilde{R}U) = 1 - m_0(\widetilde{U}R)$. Suppose $k = 0$ and we get

$$-v\Delta\rho + \Delta\rho u_0 + \Delta\rho\Delta u - \rho_0\Delta u = 0$$

or

$$v = u_0 + \Delta u + \rho_0\frac{\Delta u}{\Delta\rho} \tag{69}$$

Last expression gives us the following

$$-\Delta u(\rho_0 + \Delta\rho)m_k(\widetilde{R}) + \Delta\rho\Delta u m_k(\widetilde{R}U) + \rho_0\Delta u m_k(\widetilde{U}) + \Delta\rho\Delta u m_k(\widetilde{U}R) = 0 \tag{70}$$

where $k = 0, 1, \ldots n.$.

All four vectors with coordinates $m_k(\widetilde{R})$, $m_k(\widetilde{R}U)$, $m_k(\widetilde{U})$, $m_k(\widetilde{U}R)$ should be collinear. Consider $m_k(\widetilde{R})$ and $m_k(\widetilde{U})$. Because of $m_0(\widetilde{R}) = m_0(\widetilde{U}) = 1$ then $m_k(\widetilde{R}) = m_k(\widetilde{U})$ for k from 0 to n and therefore $\vec{a} = \vec{b}$. From last equality we have

$$m_k(\widetilde{R}U) = m_k(\widetilde{U}R) = m_k(\widetilde{U}U)$$

where $k = 0, 1, \ldots n.$. Thus

$$-\Delta u(\rho_0 + \Delta\rho)m_k(\widetilde{U}) + 2\Delta\rho\Delta u m_k(\widetilde{U}U) + \rho_0\Delta u m_k(\widetilde{U}) = 0 \tag{71}$$

where $k = 0, 1, \ldots n.$. It means

$$m_k(\widetilde{U}) = 2m_k(\widetilde{U}U),\ \ k = 0, 1, \ldots n. \tag{72}$$

Substituting (45), (46) and (66) into (64) we get the following relations for the moments.

$$\begin{gathered} u_0\Delta\rho(u_0 - v)m_k(\widetilde{R}) + \rho_0\Delta u(2u_0 - v)m_k(\widetilde{U}) + \\ +\Delta u\Delta\rho(2u_0 - v)\{m_k(\widetilde{U}R) + m_k(\widetilde{R}U)\} + \\ +2\rho_0(\Delta u)^2 m_k(\widetilde{U}U) + \rho(\Delta u)^2 m_k(\widetilde{R}U^2) + \\ +2\Delta\rho(\Delta u)^2 m_k(\widetilde{U}RU) - \Delta\sigma m_k(\Sigma) = 0 \end{gathered} \tag{73}$$

where $k = 0, 1, \dots n$.. Suppose $k = 0$ and we get

$$\rho_0(\Delta u)^2\left(\frac{\rho_0}{\Delta\rho} + 1\right) + \Delta\sigma = 0. \tag{74}$$

Moreover $m_k(\widetilde{R}) = m_k(\widetilde{\Sigma}),\ \ k = 0, 1, \dots n$ and then $\vec{b} = \vec{c}$. Finally,

$$\vec{a} = \vec{b} = \vec{c} \text{ and } m_k(\widetilde{R}) = 3m_k(\widetilde{U}U^2),\ \ k = 0, 1, \dots n.$$

Substituting (45) and (46) into (65) we get the following relations for the moments.

$$\{u_0\Delta\sigma - v\Delta\sigma\}m_k(\widetilde{\Sigma}) + \Delta u\Delta\sigma m_k(\widetilde{\Sigma}U) - k^2\Delta u m_k(\widetilde{U}) = 0 \tag{75}$$

where $k = 0, 1, \dots n$.. Suppose $k = 0$ and using expression for the velocity (69) we get

$$\Delta\sigma\Delta u\left(\frac{\rho_0}{\Delta\rho} + \frac{1}{2}\right) + k^2\Delta u = 0. \tag{76}$$

From the (75) we can also find the following equality for the velocity

$$v = u_0 + \Delta u + \frac{1}{2}u_0 - k^2\frac{\Delta u}{\Delta\sigma} \tag{77}$$

It is well known result in the elasticity theory.

Thus, if ρ_0, Δu and k^2 are known then the rest constants we can find from the system

$$\begin{cases} \rho_0(\Delta u)^2\left(\rho_0 + \Delta\rho\right) + \Delta\sigma\Delta\rho = 0, \\ \Delta\sigma\left(\rho_0 + \frac{1}{2}\Delta\rho\right) + k^2\Delta\rho = 0. \end{cases} \tag{78}$$

Hence,

$$\Delta\sigma = -\frac{2k^2\Delta\rho}{2\rho_0 + \Delta\rho} \tag{79}$$

and

$$\Delta\rho_{1,2} = \frac{-\frac{3}{2}\rho_0^2(\Delta u)^2 \pm \sqrt{\frac{\rho_0^4(\Delta u)^4}{4} + 4k^2(\Delta u)^2\rho_0^3}}{\rho_0(\Delta u)^2 - 2k^2}. \tag{80}$$

Theorem 6.2 *For any integer p there is a solution of the system of equations (62) in the sense of the definition 6. Moreover,*

$$\begin{cases} \rho_0(\Delta u)^2\left(\rho_0 + \Delta\rho\right) + \Delta\sigma\Delta\rho = 0, \\ \Delta\sigma\left(\rho_0 + \frac{1}{2}\Delta\rho\right) + k^2\Delta\rho = 0, \\ v = u_0 + \Delta u + \rho_0\frac{\Delta u}{\Delta\rho}. \end{cases}$$

Shock profiles of the considered system (62) can be found on pictures (Fig. 21, 22, 23, 24). We considered case when $p = 13$ and $\Delta u = -1$, $u_0 = 1$, $k^2 = 0.1$. We can also take any real ρ_0, σ_0 (see conditions of the theorem 6.2) but here we took $\rho_0 = 1.1$, $\sigma_0 = 0.5$ and then we should calculate $\Delta\rho$ and $\Delta\sigma$. We consider two different types of shock profiles. The first is on the Fig. 21, 22. The second is on the Fig. 23, 24.

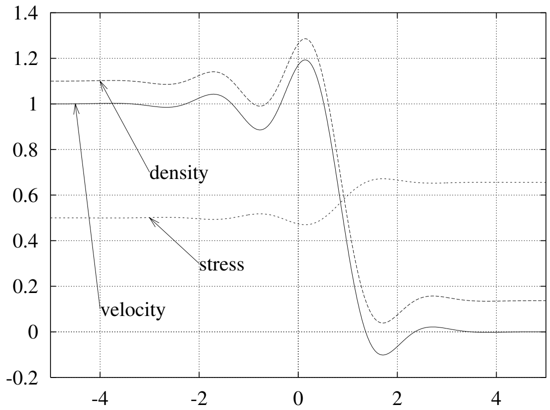

Figure 21: Shock profiles, $v = 1.1417$.

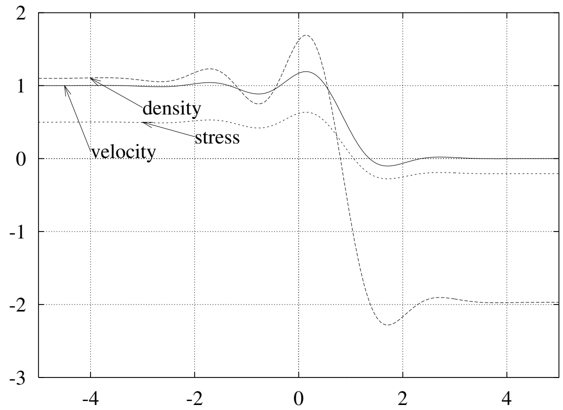

Figure 22: Shock profiles, $v = 0.35833$.

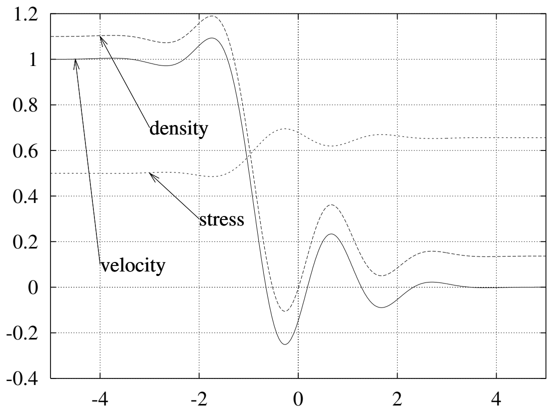

Figure 23: Shock profiles, $v = 1.1417$.

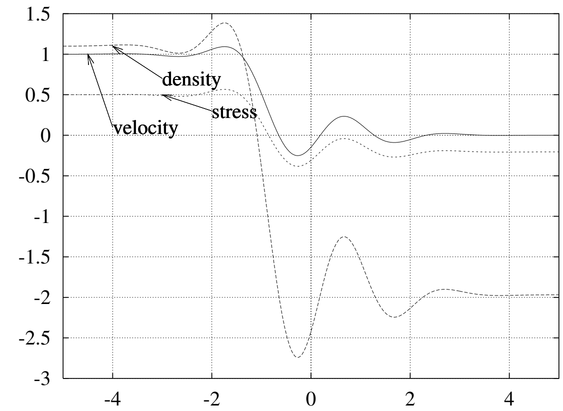

Figure 24: Shock profiles, $v = 0.35833$.

7 Conclusions and remarks. The algebra of generalized functions.

In fact, we considered only special kind of solutions from the sets I and J. Moreover, mentioned solutions are "approximate" solutions. It is open question about existence of the solution of the Hopf equation in sense of the definition 3.3 when $p = \infty$. It is evident that to extend our approach for more general cases one should consider a whole algebra instead of sets I and J. For example we can use the following construction. Denote by

$$\mathcal{K}_P = \{f(x,\varepsilon) \in C^\infty(\mathbf{R} \times (0,1)) : \forall k \in \mathbf{N}_0 \, \exists p, q \in \mathbf{Z} \text{ s.t. } \\ \frac{\partial^k}{\partial x^k} f(x,\varepsilon) = O\left(|x|^p \frac{1}{\varepsilon^q}\right), |x| \to \infty, \varepsilon \to 0\}$$

and

$$\mathcal{I}_P = \{f(x,\varepsilon) \in C^\infty(\mathbf{R} \times (0,1)) : \forall k \in \mathbf{N}_0 \, \forall n, m \in \mathbf{Z} \text{ s.t. } \\ \frac{\partial^k}{\partial x^k} f(x,\varepsilon) = O\left(\frac{1}{|x|^n} \varepsilon^m\right), |x| \to \infty, \varepsilon \to 0\}.$$

It is not hard to see that $\mathcal{K}_P$ is an algebra and $\mathcal{I}_P$ is an ideal in $\mathcal{K}_P$. Therefore, we can construct a quotient algebra $\mathcal{P} = \mathcal{K}_P/\mathcal{I}_P$.

An element f of the algebra $\mathcal{P}$ call *regular* if it have a representative $f(x,\varepsilon)$ from $\mathcal{K}_P$ such that

$$\int\limits_{-\infty}^{+\infty} f(x,\varepsilon)\psi(x)dx = a_{f,\varepsilon}(\psi) \in \mathbf{R}\langle\varepsilon\rangle \text{ for every function } \psi(x) \in \mathcal{S}(\mathbf{R}).$$

Set of all regular elements of the algebra $\mathcal{P}$ denote by $\mathcal{P}_R$. Clearly that elements of the algebra $\mathcal{P}$ and $\mathbf{R}\langle\varepsilon\rangle$-distributions are more general objects than Sobolev-Schwartz distributions.

From our point of view proposed algebra is a similar to algebras which are constructed in the paper [6].

In this paper we concentrate on the solutions of a special type (we considered sets I, J). However one may consider some analog of the initial problem for the Hopf equation in our terms. Namely, we can write the following. The smooth functions $u_0(x,\varepsilon)$ and $v_0(x,\varepsilon)$ are initial data i.e.

$$u(0,x,\varepsilon) = u_0(x,\varepsilon), \quad \partial_t u(0,x,\varepsilon) = v_0(x,\varepsilon)$$

for the Hopf equation if

$$\int\limits_{-\infty}^{+\infty} \{v_0(x,\varepsilon) + u_0(x,\varepsilon)\partial_x u_0(x,\varepsilon)\}\, \psi(x)dx = \sum_{k=p}^{+\infty} \xi_k \varepsilon^k \in \mathbf{R}\langle\varepsilon\rangle \tag{81}$$

for every $\psi \in \mathcal{S}(\mathbf{R})$.

We should notice that there is also a Non-Archimedean approach which is developed by V.Vladimirov, I.Volovich, E.Zelenov [28], A.Khrennikov [12]. This approach based on p-adic valued distributions and used for the construction of some models in Mathematical Physics.

Our method one can apply to the problems of hydrodynamics and non-linear optics.

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