ON A NONLINEAR DEGENERATE PARABOLIC TRANSPORT-DIFFUSION EQUATION WITH A DISCONTINUOUS COEFFICIENT

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Abstract. We study the Cauchy problem for the nonlinear (possibly strongly) degenerate parabolic transport-diffusion equation

$$\partial_t u + \partial_x\big(\gamma(x) f(u)\big) = \partial_x^2 A(u), \qquad A'(\cdot) \geq 0, \tag{P}$$

where the coefficient $\gamma(x)$ is possibly discontinuous and $f(u)$ is genuinely nonlinear, but not necessarily convex or concave. Existence of a weak solution is proved by passing to the limit as $\varepsilon \downarrow 0$ of a suitable sequence $\{u_\varepsilon\}_{\varepsilon>0}$ of smooth approximations solving (P) with the transport flux $\gamma(x)f(\cdot)$ replaced by $\gamma_\varepsilon(x)f(\cdot)$ and the diffusion function $A(\cdot)$ replaced by $A_\varepsilon(\cdot)$, where $\gamma_\varepsilon(\cdot)$ is smooth and $A_\varepsilon'(\cdot) > 0$. The main technical challenge is to deal with the fact that the total variation $|u_\varepsilon|_{BV}$ cannot be bounded uniformly in ε, and hence one cannot derive directly strong convergence of $\{u_\varepsilon\}_{\varepsilon>0}$. In the purely hyperbolic case ($A' \equiv 0$), where existence has already been established by a number of authors, all existence results to date have used a singular mapping to overcome the lack of a variation bound. Here we derive instead strong convergence via a series of a priori (energy) estimates that allow us to deduce convergence of the diffusion function and use the compensated compactness method to deal with the transport term.

1. Introduction

In this paper we prove existence of a weak solution to the Cauchy problem for a one-dimensional scalar degenerate parabolic equation with a nonlinear transport term that depends explicitly on the spatial position through a coefficient $\gamma(x)$ that may be discontinuous. More precisely, the problem that we study takes the form

$$\begin{cases} \partial_t u + \partial_x\big(\gamma(x) f(u)\big) = \partial_x^2 A(u), & (x,t) \in \Pi_T = \mathbb{R} \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $T > 0$ is fixed, $u : \Pi_T \to \mathbb{R}$ is the unknown function that is sought, and γ, f, A, u_0 are given functions. Regarding $\gamma(\cdot)$, we make the assumption that

$$\underline{\gamma} \leq \gamma(x) \leq \overline{\gamma} \text{ for some constants } \underline{\gamma},\ \overline{\gamma};\ |\gamma(x)| > 0 \text{ a.e. on } \mathbb{R};\ \gamma \in BV(\mathbb{R}). \tag{1.2}$$

In other words, the "transport part" of (1.1) depends explicitly on the spatial location and this dependency may be discontinuous. Regarding the function $f(\cdot)$, we assume that

$$f \in C^2[0,1] \text{ with } f(0) = f(1) = 0;\ f \text{ genuinely nonlinear}, \tag{1.3}$$

but no convexity condition is assumed. As usual, "f genuinely nonlinear" means that there is no subinterval of $[0,1]$ on which f is linear. We require that the diffusion function $A(\cdot)$ satisfies

$$A(\cdot) \in C^1([0,1]);\ A(\cdot) \text{ nondecreasing with } A(0) = 0. \tag{1.4}$$

Finally, we assume that the initial function $u_0(\cdot)$ satisfies

$$u_0 \in L^\infty(\mathbb{R}) \bigcap L^1(\mathbb{R});\ u_0(x) \in [0,1] \text{ for a.e. } x \in \mathbb{R}. \tag{1.5}$$

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The assumption that $u_0 \in [0,1]$ is motivated by considerations like the following. Consider the initial value problem

$$\partial_t u - \partial_x \left(\frac{\mathrm{sign}(x)}{1+|u|} \right) = 0, \quad u(x,0) = 0.$$

The entropy solution to this problem is

$$u(x,t) = \max\left\{ 0, \sqrt{\left|\frac{t}{x}\right|} - 1 \right\}.$$

Clearly $u(x,t)$ is unbounded, and the reason for this is that $f(u) = 1/(1+|u|) \neq 0$ for any u. Furthermore, $u(x,t)$ is also an entropy solution if we modify f to read

$$f(u) = \begin{cases} 0, & u < -1, \\ -3u^3 - 5u^2 - u + 1, & -1 \le u < 0, \\ 1/(1+u), & u \ge 0. \end{cases}$$

In this case f is continuously differentiable, and $f(-1) = 0$, however $f(u) > 0$ for all $u > -1$. Hence, to bound weak solutions, we shall need the assumption that there exist numbers $\alpha < \beta$ such that $f(\alpha) = f(\beta) = 0$ and that $u_0(x) \in [\alpha, \beta]$ for all x. To make the presentation simple, we normalize such that $\alpha = 0$, $\beta = 1$.

As we have just seen, the "degenerate parabolicity" condition (1.4) is general enough to include as a special case of (1.1) the hyperbolic conservation law with discontinuous coefficient:

$$\partial_t u + \partial_x \big(\gamma(x) f(u)\big) = 0. \tag{1.6}$$

This equation is used to model a variety of phenomena, among which are traffic flow [32] and flow of hydrocarbons in porous media. In addition, such equations occur when solving Hamilton-Jacobi equations numerically by dimensional splitting [12].

Independently of the smoothness of $\gamma(\cdot)$, if (1.1) is allowed to degenerate at certain points, that is, $A'(s) = 0$ for some values of s, solutions are not necessarily smooth and weak solutions must be sought. A *weak solution* is defined as follows:

Definition 1.1. *A weak solution of* (1.1) *is a measurable function* $u = u(x,t)$ *satisfying:*

(**D**.1) $u \in L^1(\Pi_T) \bigcap L^\infty(\Pi_T)$ *and* $A(u) \in L^2(0,T; H^1(\mathbb{R}))$.

(**D**.2) *For all* $\varphi \in \mathcal{D}(\Pi_T)$ *with* $\varphi|_{t=T} = 0$,

$$\iint_{\Pi_T} \Big(u \partial_t \varphi + \big[\gamma(x) f(u) - \partial_x A(u)\big] \partial_x \varphi \Big)\, dt\, dx + \int_{\mathbb{R}} u_0(x) \phi(x,0) = 0. \tag{1.7}$$

On the other hand, if $A'(s)$ is zero on an interval $[\alpha, \beta]$, (weak) solutions may be discontinuous and they are not uniquely determined by their initial data. Consequently, an entropy condition must be imposed to single out the physically correct solution. If $\gamma(\cdot)$ is sufficiently "smooth", a weak solution u satisfies the *entropy condition* if for all convex C^2 functions $\eta : \mathbb{R} \to \mathbb{R}$,

$$\partial_t \eta(u) + \partial_x \big(\gamma(x) q(u)\big) + \partial_x^2 r(u) + \gamma'(x)\big(\eta'(u) f(u) - q(u)\big) \le 0 \text{ in } \mathcal{D}'(\Pi_T), \tag{1.8}$$

where $q, r : \mathbb{R} \to \mathbb{R}$ are defined by $q'(u) = \eta'(u) f'(u)$ and $r'(u) = \eta'(u) A'(u)$. By standard limiting argument, (1.8) implies that the Kružkov-type entropy condition

$$\begin{aligned} \partial_t |u - c| + \partial_x \Big(\gamma(x) \mathrm{sign}(u-c)\big(f(u) - f(c)\big)\Big) & \\ + \partial_x^2 \left|A(u) - A(c)\right| + \gamma'(x) \mathrm{sign}(u-c) f(c) &\le 0 \end{aligned} \tag{1.9}$$

holds in $\mathcal{D}'(\Pi_T)$ for all $c \in \mathbb{R}$. For pure hyperbolic equations, the entropy condition (1.9) was introduced by Kružkov [17] and Vol'pert [30]. For degenerate parabolic equations, it must be attributed to Vol'pert and Hudjaev [31]. The main reference on the uniqueness and stability of entropy solutions of degenerate parabolic equations is the recent paper by Carrillo [1] (see also Chen and DiBenedetto [3]), which in turn is a generalization of Kružkov's work on hyperbolic equations. Following [1], it was proved in by Karlsen and Risebro [11] that the entropy solution of (1.1) (as well as a more general equation in multi-dimensions) is unique when $\gamma(\cdot)$ is "smooth". Moreover, in the $L^\infty(0,T;BV(\mathbb{R}^d))$ class of entropy solutions, an L^1 contraction principle as well as "continuous dependence" estimates were proved. Recently there seems to be renewed interest in applying "hyperbolic" techniques to degenerate parabolic equations. For a partial overview of mathematical and numerical theory for degenerate parabolic equations based on "hyperbolic" techniques, see the lecture notes [5].

In this paper, we are interested in taking a first step towards developing a well-posedness theory for degenerate parabolic equations with discontinuous coefficients. To be a bit more precise, we aim at proving existence of a weak solution to (1.1) when the coefficient $\gamma(x)$ may depend discontinuously on x. We will also prove uniqueness of the constructed weak solution.

Let u_ε be the unique classical solution of uniformly parabolic problem

$$\begin{cases} \partial_t u_\varepsilon + \partial_x\big(\gamma_\varepsilon(x) f(u_\varepsilon)\big) = \partial_x^2 A_\varepsilon(u_\varepsilon), & (x,t)\in\Pi_T,\\ u_\varepsilon(x,0) = \omega_\varepsilon * u_0, & x\in\mathbb{R}, \end{cases}$$

where γ_ε is a smooth coefficient, $A_\varepsilon'(\cdot) > 0$, and $\omega_\varepsilon * u_0$ is a smooth initial function (see Section 3 for precise statements). We prove existence of a weak solution of (1.1) by establishing strong convergence of the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ of smooth approximations. Roughly speaking, our main theorem can be stated as follows:

Main Theorem. *The sequence of $\{u_\varepsilon\}_{\varepsilon>0}$ converges strongly in L^1 to a weak solution u of* (1.1). *Finally, a subsequence of $\{A(u_\varepsilon)\}_{\varepsilon>0}$ converges uniformly on compact sets to a Hölder continuous function that coincides with $A(u)$ a.e.*

Since $\gamma(\cdot)$ may be discontinuous , the total variation $|u_\varepsilon|_{BV}$ cannot be bounded uniformly with respect to $\varepsilon > 0$. This point will be discussed below when we put this work in perspective by reviewing the available literature on the subject (which exclusively deals with the hyperbolic case). The lack of a variation bound prevents an application of the standard BV compactness argument to $\{u_\varepsilon\}_{\varepsilon>0}$. To circumvent this analytical difficulty, we establish instead strong compactness of the diffusion function $\{A_\varepsilon(u_\varepsilon)\}_{\varepsilon>0}$ as well as the "total flux" $\{\gamma_\varepsilon(x)f(u_\varepsilon) - \partial_x A_\varepsilon(u_\varepsilon)\}_{\varepsilon>0}$. Using the compactness of these two sequences along with the compensated compactness method of Murat and Tartar [20, 21, 22, 25] to handle the nonlinear transport term, we get strong convergence along a subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$ to a weak solution of (1.1). The constructed weak solution is unique thanks to a stability result in [11]. The detailed proofs are found in Section 3, while the compensated compactness method is recalled in Section 2.

When $A'(\cdot)\equiv 0$, the classical Kružkov theory applies to the hyperbolic problem (1.6) only if the coefficient γ is continuously differentiable. In the case of a discontinuous coefficient, the notion of entropy solution (1.9) as well as the accompanying existence and uniqueness theory breaks down. When $\gamma(\cdot)$ is discontinuous, the hyperbolic equation (1.6) has often been written as a 2×2 system of equations to facilitate the analysis:

$$\partial_t\gamma = 0, \qquad \partial_t u + \partial_x\big(\gamma f(u)\big) = 0. \tag{1.10}$$

If $f'(\cdot)$ changes sign, then this system is non-strictly hyperbolic, a situation described as resonance. An important consequence of resonance is that no a priori bound on the spatial variation of the conserved quantity is available, in marked contrast to the smooth γ situation where the Kružkov theory applies. For example, when the initial data is approximated by a sequence of piecewise constant functions, this can cause the spatial variation to blow up as the discretization parameter tends to zero [26, 29].

With no spatial variation bound available for the conserved quantity, an alternative method of establishing compactness is required. To date, all existence results for the case of a discontinuous coefficient have employed some form of singular mapping, that is a nonlinear transformation of the conserved quantity. Indeed, the present work is the first to prove strong convergence of approximate solutions without appealing to the singular mapping technique, which was introduced by Temple [26] in order to establish convergence of the Glimm scheme for a 2×2 resonant system of conservation laws modeling the displacement of oil in a reservoir by water and polymer. For the equation (1.6), the singular mapping takes the form

$$\Psi(u,\gamma) = \gamma \int_0^u |f'(\xi)|\,d\xi, \tag{1.11}$$

from which it is clear that Ψ assigns vanishingly small weight to variations in u in the resonant regions (where $f' = 0$). This makes it possible to establish a uniform variation bound for the transformed version of the conserved quantity, thus establishing compactness for the approximating sequence in the transformed variable. The singular mapping Ψ is continuous and strictly monotone as a function of the conserved quantity, which allows the conserved quantity to be recovered after passing to the limit in the transformed variable.

In addition to the Glimm scheme, convergence has been established for the 2×2 Godunov method by Lin, Temple, and Wang [18, 19]. Specifically, they applied the 2×2 Godunov method to the system

$$\partial_t \gamma = 0, \qquad \partial_t u + \partial_x f(\gamma(x), u) = 0, \tag{1.12}$$

and used a version of the singular mapping to establish compactness (see also Hong [8] for an "improved" singular mapping). They also observed that a uniform variation bound (measured via the singular mapping) had not been proven for any scalar schemes that apply to (1.12), nor for the 2×2 Lax-Friedrichs method. Such bounds have since been established for the scalar Engquist-Osher and Godunov schemes [27, 28]. Furthermore, with the present work, we add to that list convergence of the vanishing viscosity/smoothing method. However, no bound has yet been established for either the scalar or 2×2 version of the Lax-Friedrichs scheme, and thus convergence is yet to be proven. Numerical evidence indicates that the Lax-Friedrichs scheme is well-behaved on these problems; it is the theory that is deficient at this point. Our investigation of the compensated compactness approach, which represents a departure from the singular mapping technique, is partially motivated by our desire to find a method that will provide a proof of convergence for the Lax-Friedrichs scheme.

The front tracking method, which is based on the work of Dafermos [4] and Holden, Holden, and Høegh-Krohn [7], has been applied to a number of hyperbolic problems with discontinuous coefficients. Gimse and Risebro [6] used the front tracking method to study the two phase flow equation (s denotes the saturation of one of the phases)

$$\partial_t s + \partial_x \big(f_0(s)(1 - g(x)k(s))\big) = 0, \tag{1.13}$$

where f_0 is the so-called fractional flow function, $g(x)$ models the gravitational pull multiplied by the absolute permeability of the porous medium, and $k(s)$ is the relative permeability of the relevant phase. In (1.13), the spatially varying coefficient $g(x)$ may be discontinuous. Gimse and Risebro proved compactness of the sequence of approximations via a bound on the spatial variation, measured with respect to the singular mapping. For the scalar conservation law with a concave flux, Klingenberg and Risebro [15] used the front tracking technique to establish existence, uniqueness, and asymptotic behavior for the Cauchy problem (1.10). In [15] also, the singular mapping was the method used to establish compactness of the approximating sequence. Concerning uniqueness and stability with respect to perturbations of the initial data, the discontinuity of the flux parameter complicates the analysis. Specifically, the Kružkov entropy condition (1.9) no longer makes sense, thus requiring an alternative approach. To overcome this difficulty, Klingenberg and Risebro used a so-called wave entropy condition, which allowed them to prove uniqueness for the limit of the approximate solutions. For this same concave flux problem, Klausen and Risebro [13] proved continuous dependence on the coefficient γ and on the initial data. The approach in [13] was to prove that the solution constructed via the front tracking approach is the limit of the solutions that result when the coefficient γ is smoothed. The classical Kružkov L^1 stability theory applies when γ is smoothed, and the limit solution inherits this stability. The front tracking method has also been applied to the situation where the flux f is neither concave nor convex. Klingenberg and Risebro [14] established existence and uniqueness for the nonconvex flux $f(u) = \sin(u)$ for $u \in [-\pi, \pi]$. A version of the singular mapping was used here also, and uniqueness was established by passing to the limit in a sequence of solutions corresponding to a smoothed version of γ.

Convergence of scalar difference schemes for the case of a smooth spatially varying flux has been known for many years. For $\gamma \in C^2(\mathbb{R}^d)$, convergence of the Lax-Friedrichs scheme and the upwind scheme was proved in [23]. Under weaker conditions on γ, e.g., $\gamma' \in BV$, and for f convex in u, convergence of the one-dimensional Godunov method for (1.10) (not for (1.1)) was shown by Isaacson and Temple in [9], see Karlsen and Risebro [10] for the multi-dimensional degenerate parabolic case. For the case of a discontinuous coefficient, Towers [27] proved convergence of the scalar Godunov and Engquist-Osher methods for essentially the same concave problem studied by Klingenberg and Risebro [15], and using the same version of the singular mapping as those authors. For piecewise smooth solutions, uniqueness was establish via an L^1 stability proof similar to the classical proof of Quinn [24] for the constant γ conservation law. For the Engquist-Osher scheme, Towers [28] extended the convergence proof to the case of a flux f having any finite number of extrema. The question of uniqueness of limits of the difference scheme was not addressed for the nonconvex problem. We plan to address this question in a later work, which will also discuss uniqueness for the more general problem (1.1).

The singular mapping approach to convergence for these scalar difference schemes appears to depend strongly on the close functional relationship between the viscosity of the Engquist-Osher flux, the Kružkov entropy flux, and the singular mapping. This is true even for the Godunov scheme, where the proof depends on the fact that the Engquist-Osher flux is nearly identical to the Godunov flux when f is concave. This reinforces our impression that the singular mapping approach is not readily applicable to the Lax-Friedrichs scheme, and further motivates our interest in the compensated compactness approach.

The case where the flux f is nonconvex has received less attention in the literature than the convex/concave case, presumably due to additional analytical complexity. An attractive feature of the vanishing viscosity/smoothing approach presented herein is that the absence or presence of inflection points does not enter the analysis, and so no convexity condition is required for the flux f. The (small) price to pay for this is that we must assume that there is no interval where f is linear. Also, sign changes of γ are handled without any special considerations. Sign changes in γ are commonly ruled out [15, 14, 13, 27, 28], again due to added analytical technicalities.

2. Compensated Compactness

In this section we recapitulate the results we shall use from the compensated compactness method due to Murat and Tartar [20, 21, 22, 25]. For a nice overview of applications of the compensated compactness method to hyperbolic conservation laws, we refer to Chen [2].

Let $\mathcal{M}(\mathbb{R}^n)$ denote the space of bounded Radon measures on $\mathbb{R}^n$ and

$$C_0(\mathbb{R}^n) = \left\{ \Psi \in C(\mathbb{R}^n) : \lim_{|x|\to\infty} \Psi(x) = 0 \right\}.$$

If $\mu \in \mathcal{M}(\mathbb{R}^n)$, then

$$\langle \mu, \Psi \rangle = \int_{\mathbb{R}^n} \Psi \, d\mu, \qquad \text{for all} \quad \Psi \in C_0(\mathbb{R}^n).$$

Recall that $\mu \in \mathcal{M}(\mathbb{R}^n)$ if and only if $\left|\langle \mu, \Psi \rangle\right| \le C \left\|\Psi\right\|_{L^\infty(\mathbb{R}^n)}$ for all $\Psi \in C_0(\mathbb{R}^n)$. We define

$$\left\|\mu\right\|_{\mathcal{M}(\mathbb{R}^n)} = \sup \left\{ \left|\langle \mu, \Psi \rangle\right| : \Psi \in \mathcal{D}(\mathbb{R}^n), \left\|\Psi\right\|_{L^\infty(\mathbb{R}^n)} \le 1 \right\}.$$

The space $\left(\mathcal{M}(\mathbb{R}^n), \left\|\cdot\right\|_{\mathcal{M}(\mathbb{R}^n)}\right)$ is a Banach space and it is isometrically isomorphic to the dual space of $\left(C_0(\mathbb{R}^n), \left\|\cdot\right\|_{L^\infty(\mathbb{R}^n)}\right)$, while we define the space of probability measures $\mathrm{Prob}(\mathbb{R}^n)$ as

$$\mathrm{Prob}(\mathbb{R}^n) = \left\{ \mu \in \mathcal{M}(\mathbb{R}^n) : \mu \text{ is nonnegative and } \left\|\mu\right\|_{\mathcal{M}(\mathbb{R}^n)} = 1 \right\}.$$

Then we can state the fundamental theorem in the theory of compensated compactness.

Theorem 2.1. *Let $K \subset \mathbb{R}^n$ be a bounded open set and $u_\varepsilon : \Pi_T \to K$. Then there exists a family of probability measures $\{\nu_{(x,t)}(\lambda) \in \mathrm{Prob}(\mathbb{R}^n)\}_{(x,t)\in\Pi_T}$ (depending weak-$*$ measurably on (x,t)) such that*

$$\operatorname{supp} \nu_{(x,t)} \subset \overline{K} \text{ for a.e. } (x,t) \in \Pi_T.$$

Furthermore, for any continuous function $\Phi : K \to \mathbb{R}$, *we have along a subsequence*

$$\Phi(u_\varepsilon) \overset{*}{\rightharpoonup} \overline{\Phi} \text{ in } L^\infty(\Pi_T) \text{ as } \varepsilon \downarrow 0,$$

where (the exceptional set depends possibly on Φ*)*

$$\overline{\Phi}(x,t) := \left\langle \nu_{(x,t)}, \Phi \right\rangle = \int_{\mathbb{R}^n} \Phi(\lambda)\, d\nu_{(x,t)}(\lambda) \text{ for a.e. } (x,t) \in \Pi_T.$$

In the literature, $\nu_{(x,t)}$ is often referred to as a Young measure. Theorem 2.1 provides us with a representation formula for weak limits in terms of nonlinear functions and Young measures. A uniformly bounded sequence $\{u_\varepsilon\}_{\varepsilon>0}$ converges to u a.e. on Π_T if and only if the corresponding Young measure $\nu_{(x,t)}$ reduces to a Dirac measure located at $u(x,t)$, i.e., $\nu_{(x,t)} = \delta_{u(x,t)}$.

We have the following "reduction" result:

Lemma 2.1. *Let* $K \subset \mathbb{R}$ *be a bounded open set and* $u_\varepsilon : \Pi_T \to K$. *Suppose that* $u_\varepsilon \overset{*}{\rightharpoonup} u$ *in* $L^\infty(\Pi_T)$. *Suppose also that for any pair of (not necessarily convex)* C^2 *functions* $\eta_1, \eta_2 : \mathbb{R} \to \mathbb{R}$, *we have along a subsequence*

$$\gamma(x) q_1(u_\varepsilon)\eta_2(u_\varepsilon) - \eta_1(u_\varepsilon)\gamma(x)q_2(u_\varepsilon) \overset{*}{\rightharpoonup} \gamma(x)\,\overline{q_1}\,\overline{\eta_2} - \overline{\eta_1}\,\gamma(x)\,\overline{q_2} \text{ in } L^\infty(\Pi_T) \text{ as } \varepsilon \downarrow 0, \tag{2.1}$$

where $q_i : \mathbb{R} \to \mathbb{R}$ *is defined by* $q_i'(u) = \eta_i'(u) f'(u)$, $i = 1, 2$. *Then along a subsequence*

$$\gamma(x) f(u_\varepsilon) \overset{*}{\rightharpoonup} \gamma(x) f(u) \text{ in } L^\infty(\Pi_T) \text{ as } \varepsilon \downarrow 0.$$

Furthermore, if $\gamma(x) \neq 0$ *for a.e.* $x \in \mathbb{R}$ *and there is no interval on which* $f(\cdot)$ *is linear, then a subsequence of* $\{u_\varepsilon\}_{\varepsilon>0}$ *converges to* u *a.e. on* Π_T.

Proof. Applying Theorem 2.1 for the sequence $\{u_\varepsilon\}$ with $\Phi(\lambda) = q_1(\lambda)\eta_2(\lambda) - \eta_1(\lambda)q_2(\lambda)$, we get that

$$\gamma(x) q_1(u_\varepsilon)\eta_2(u_\varepsilon) - \eta_1(u_\varepsilon)\gamma(x)q_2(u_\varepsilon) \overset{*}{\rightharpoonup} \overline{\gamma(x) q_1\eta_2 - \eta_1\gamma(x) q_2} \text{ in } L^\infty(\Pi_T) \text{ as } \varepsilon \downarrow 0.$$

From this and assumption (2.1), we get the following Murat-Tartar commutation relation:

$$\gamma(x)\left[\overline{q_1}\,\overline{\eta_2} - \overline{\eta_1}\,\overline{q_2} - \overline{q_1\eta_2 - \eta_1 q_2}\right] = 0 \text{ for a.e. } (x,t) \in \Pi_T. \tag{2.2}$$

Following Chen [2], we choose

$$\eta_1(\lambda) = \lambda - u(x,t), \qquad q_1(\lambda) = f(\lambda) - f(u(x,t)),$$
$$\eta_2(\lambda) = q_1(\lambda), \qquad q_2(\lambda) = \int_{u(x,t)}^{\lambda} \left(f'(\xi)\right)^2 d\xi,$$

and note that $\overline{\eta_1} \equiv 0$. Inserting this choice into the commutation relation (2.2) yields

$$\begin{aligned} \gamma(x)\Bigg[&\left(\int_{\mathbb{R}} \left(f(\lambda) - f(u(x,t))\right) d\nu_{(x,t)}(\lambda)\right)^2 \\ &+ \int_{\mathbb{R}} \left(\left(\lambda - u(x,t)\right)\int_{u(x,t)}^{\lambda} \left(f'(\xi)\right)^2 d\xi - \left(f(\lambda) - f(u(x,t))\right)^2\right) d\nu_{(x,t)}(\lambda)\Bigg] = 0. \end{aligned} \tag{2.3}$$

By the Cauchy-Schwartz inequality

$$\left(f(\lambda) - f(u(x,t))\right)^2 = \left(\int_{u(x,t)}^{\lambda} f'(\xi)\, d\xi\right)^2 \le \left(\lambda - u(x,t)\right)\int_{u(x,t)}^{\lambda} (f'(\xi))^2\, d\xi,$$

with equality if and only if $f''(\xi) = 0$ for all ξ between $u(x,t)$ and λ. Hence, if $\gamma(x) \neq 0$, both terms in (2.3) must be zero. The first term being zero implies that $\overline{f}(x,t) = f(u(x,t))$. Hence, by the boundedness of γ, we can conclude that $\gamma(x)\overline{f} = \gamma(x)f(u)$ a.e. on Π_T. In view of Theorem 2.1, this proves the first part of the proposition.

The second part of the theorem follows by observing that if $\gamma, f'' \neq 0$ a.e., then the fact that the second term in (2.3) is zero implies $\nu_{(x,t)} = \delta_{u(x,t)}$ a.e. on Π_T (since f is assumed to be genuinely nonlinear). $\square$

Remark 2.1. If $\gamma(\cdot)=0$ on a set of non-zero measure, then it is not possible to conclude that (a subsequence of) u_ε converges strongly to u nor that $f(u_\varepsilon)\overset{*}{\rightharpoonup} f(u)$ in $L^\infty(\Pi_T)$. Nevertheless, Proposition 2.1 can be used to prove that the $L^\infty(\Pi_T)$ weak-$*$ limit u is a weak solution of (1.1). Moreover, if this was our only goal, then we could have replaced the C^2 assumption on f by merely C^1, or even Lipschitz. To see this, we do as Tartar did and insert the functions

$$\begin{aligned}
&\eta_1(\lambda)=\lambda, && q_1(\lambda)=f(\lambda),\\
&\eta_2(\lambda)=\left|\lambda-u(x,t)\right|, && q_2(\lambda)=\operatorname{sign}(\lambda-u(x,t))\big(f(\lambda)-f(u(x,t))\big)
\end{aligned}$$

into the Murat-Tartar commutation relation (2.2). Of course, now we suppose that (2.1) holds for $\eta_1(\lambda)=\lambda$ and any convex (Lipschitz continuous) function $\eta_2:\mathbb{R}\to\mathbb{R}$. The result is

$$\begin{aligned}
\gamma(x)\Bigg[\overline{f}\int_{\mathbb{R}}\left|\lambda-u(x,t)\right|\,d\nu_{(x,t)}(\lambda)-u\int_{\mathbb{R}}\operatorname{sign}(\lambda-u(x,t))\big(f(\lambda)-f(u(x,t))\big)\,d\nu_{(x,t)}(\lambda)\\
-\int_{\mathbb{R}}\Big(f(\lambda)\left|\lambda-u(x,t)\right|-\lambda\operatorname{sign}(\lambda-u(x,t))\big(f(\lambda)-f(u(x,t))\big)\Big)\,d\nu_{(x,t)}(\lambda)\Bigg]=0,
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
\gamma(x)\int_{\mathbb{R}}\Big(\big[\overline{f}(x,t)-f(\lambda)\big]\left|\lambda-u(x,t)\right|\\
+\big[\lambda-u(x,t)\big]\operatorname{sign}(\lambda-u(x,t))\big(f(\lambda)-f(u(x,t))\big)\Big)\,d\nu_{(x,t)}(\lambda)=0,
\end{aligned}\tag{2.4}$$

or

$$\gamma(x)\big[f(u(x,t))-\overline{f}(x,t)\big]\int_{\mathbb{R}}\left|\lambda-u(x,t)\right|\,d\nu_{(x,t)}(\lambda)=0.$$

Consequently, we have either $\gamma(x)\overline{f}(x,t)=\gamma(x)f(u(x,t))$ or, if $\gamma(x)\neq 0$, $\overline{f}=f(u(x,t))$ or $\nu_{(x,t)}=\delta_{u(x,t)}$, which also implies $\gamma(x)\overline{f}(x,t)=\gamma(x)f(u(x,t))$. This proves our claim.

Before we continue, we need to recall the celebrated Div-Curl lemma.

Lemma 2.2 (Div-Curl). *Let $Q\subset\mathbb{R}^2$ be a bounded domain. Suppose*

$$\begin{aligned}
&v_\varepsilon^1\rightharpoonup\overline{v}^1, && v_\varepsilon^2\rightharpoonup\overline{v}^2,\\
&w_\varepsilon^1\rightharpoonup\overline{w}^1, && w_\varepsilon^2\rightharpoonup\overline{w}^2,
\end{aligned}$$

in $L^2(Q)$ as $\varepsilon\downarrow 0$. Suppose also that the two sequences

$$\left\{\operatorname{div}\left(v_\varepsilon^1,v_\varepsilon^2\right)\right\}_{\varepsilon>0}\ \textit{and}\ \left\{\operatorname{curl}\left(w_\varepsilon^1,w_\varepsilon^2\right)\right\}_{\varepsilon>0}\ \textit{lie in a (common) compact subset of}\ H^{-1}_{\mathrm{loc}}(Q),$$

where $\operatorname{div}\left(v_\varepsilon^1,v_\varepsilon^2\right)=\partial_{x_1}v_\varepsilon^1+\partial_{x_2}v_\varepsilon^2$ and $\operatorname{curl}\left(w_\varepsilon^1,w_\varepsilon^2\right)=\partial_{x_1}w_\varepsilon^2-\partial_{x_2}w_\varepsilon^1$. Then along a subsequence

$$\left(v_\varepsilon^1,v_\varepsilon^2\right)\cdot\left(w_\varepsilon^1,w_\varepsilon^2\right)\to\left(\overline{v}^1,\overline{v}^2\right)\cdot\left(\overline{w}^1,\overline{w}^2\right)\ \textit{in}\ \mathcal{D}'(Q)\ \textit{as}\ \varepsilon\downarrow 0.$$

Theorem 2.2. *Suppose that $\{u_\varepsilon\}_{\varepsilon>0}\subset L^\infty(\Pi_T)$ uniformly in ε. Suppose also that for any C^2 function $\eta:\mathbb{R}\to\mathbb{R}$, the sequence of distributions*

$$\left\{\partial_t\eta(u_\varepsilon)+\partial_x\big(\gamma(x)q(u_\varepsilon)\big)\right\}_{\varepsilon>0}\ \textit{lies in a compact subset of}\ H^{-1}_{\mathrm{loc}}(\Pi_T),\tag{2.5}$$

where $q:\mathbb{R}\to\mathbb{R}$ is defined by $q'(u)=\eta'(u)f'(u)$. Then along a subsequence

$$u_\varepsilon\overset{*}{\rightharpoonup}u\ \textit{in}\ L^\infty(\Pi_T)\ \textit{as}\ \varepsilon\downarrow 0,\qquad \gamma(x)f(u_\varepsilon)\overset{*}{\rightharpoonup}\gamma(x)f(u)\ \textit{in}\ L^\infty(\Pi_T)\ \textit{as}\ \varepsilon\downarrow 0.\tag{2.6}$$

Furthermore, if $\gamma(x)\neq 0$ for a.e. $x\in\mathbb{R}$ and there is no interval on which $f(\cdot)$ is linear, then a subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$ converges to u a.e. on Π_T.

Proof. Let $\eta_1,\eta_2:\mathbb{R}\to\mathbb{R}$ be a pair of C^2 functions and define q_i by $q_i'(u)=\eta_i'(u)f'(u)$, $i=1,2$. Consider then the vector fields

$$v_\varepsilon=\left(\eta_1(u_\varepsilon),\gamma(x)q_1(u_\varepsilon)\right),\qquad w_\varepsilon=\left(-\gamma(x)q_2(u_\varepsilon),\eta_2(u_\varepsilon)\right).$$

In view of Theorem 2.1, the L^∞ bounds on u_ε and $\gamma(x)$ imply that along subsequences

$$v_\varepsilon \overset{*}{\rightharpoonup} \overline{v} := (\overline{\eta_1}, \gamma(x)\overline{q_1}) \text{ in } L^\infty(\Pi_T), \qquad w_\varepsilon \overset{*}{\rightharpoonup} \overline{w} := (-\gamma(x)\overline{q_2}, \overline{\eta_2}) \text{ in } L^\infty(\Pi_T).$$

By assumption (2.5), the sequences

$$\begin{aligned} \{\operatorname{div}(v_\varepsilon)\}_{\varepsilon>0} &= \left\{\left(\partial_t \eta_1(u_\varepsilon) + \partial_x\left(\gamma(x)q_1(u_\varepsilon)\right)\right\}_{\varepsilon>0}, \\ \{\operatorname{curl}(w_\varepsilon)\}_{\varepsilon>0} &= \left\{\partial_t \eta_2(u_\varepsilon) + \partial_x\left(\gamma(x)q_2(u_\varepsilon)\right)\right\}_{\varepsilon>0} \end{aligned}$$

lie in a (common) compact subset of $H^{-1}_{\mathrm{loc}}(\Pi_T)$. Also, we have $\{v_\varepsilon\}_{\varepsilon>0}, \{w_\varepsilon\}_{\varepsilon>0} \subset L^\infty(\Pi_T)$ and therefore $\{v_\varepsilon\}_{\varepsilon>0}, \{w_\varepsilon\}_{\varepsilon>0} \subset L^2_{\mathrm{loc}}(\Pi_T)$ uniformly in ε. The Div-Curl lemma then gives (up to the extraction of a subsequence)

$$v_\varepsilon \cdot w_\varepsilon \to \overline{v} \cdot \overline{w} \text{ in } \mathcal{D}'(\Pi_T).$$

Since we work with bounded functions, we have that $\{v_\varepsilon \cdot w_\varepsilon\}_{\varepsilon>0}$ converges weakly-$*$ in $L^\infty(\Pi_T)$ along a subsequence to (necessarily) $\overline{v} \cdot \overline{w}$. Therefore along a subsequence

$$\gamma(x)q_1(u_\varepsilon)\eta_2(u_\varepsilon) - \eta_1(u_\varepsilon)\gamma(x)q_2(u_\varepsilon) \overset{*}{\rightharpoonup} \gamma(x)\,\overline{q_1}\,\overline{\eta_2} - \overline{\eta_1}\,\gamma(x)\,\overline{q_2} \text{ in } L^\infty(\Pi_T).$$

In view of Lemma 2.1, this concludes the proof. □

The following compactness interpolation result (known as Murat's lemma [22]) is useful in obtaining the H^{-1}_{loc} compactness needed in Theorem 2.2.

Lemma 2.3. *Suppose that $\{\mathcal{L}_\varepsilon\}_{\varepsilon>0}$ is bounded in $W^{-1,\infty}_{\mathrm{loc}}(\Pi_T)$. Suppose also that $\mathcal{L}_\varepsilon = \mathcal{L}^1_\varepsilon + \mathcal{L}^2_\varepsilon$, where $\{\mathcal{L}^1_\varepsilon\}_{\varepsilon>0}$ lies in a compact subset of $H^{-1}_{\mathrm{loc}}(\Pi_T)$ and $\{\mathcal{L}^2_\varepsilon\}_{\varepsilon>0}$ lies in a bounded subset of $\mathcal{M}_{\mathrm{loc}}(\Pi_T)$. Then $\{\mathcal{L}_\varepsilon\}_{\varepsilon>0}$ lies in a compact subset of $H^{-1}_{\mathrm{loc}}(\Pi_T)$.*

3. Existence of Weak Solution

Existence of a weak solution will be proved by establishing convergence of a suitable sequence of smooth functions solving regularized problems. Let $\omega \in C_0^\infty(\mathbb{R})$ be a nonnegative function satisfying $\omega(x) = \omega(-x)$, $\omega(x) \equiv 0$ for $|z| \geq 1$, and $\int_{\mathbb{R}} \omega(z)\,dz = 1$. For $\varepsilon > 0$, let $\omega_\varepsilon(x) = \frac{1}{\varepsilon}\omega\left(\frac{x}{\varepsilon}\right)$ and introduce the "smoothed" coefficient $\gamma_\varepsilon = \omega_\varepsilon * \gamma$. We then denote by u_ε the unique classical solution of uniformly parabolic problem

$$\begin{cases} \partial_t u_\varepsilon + \partial_x\left(\gamma_\varepsilon(x) f(u_\varepsilon)\right) = \partial_x^2 A_\varepsilon(u_\varepsilon), & (x,t) \in \Pi_T, \\ u_\varepsilon(x,0) = \omega_\varepsilon * u_0, & x \in \mathbb{R}, \end{cases} \tag{3.1}$$

where $A_\varepsilon(u) = A(u) + \varepsilon u$. Our goal is to pass to the limit in u_ε as $\varepsilon \downarrow 0$. As was already mentioned in the introduction, our main problem is the lack of a BV estimate on u_ε (which is uniform in ε) and hence strong convergence of $\{u_\varepsilon\}_{\varepsilon>0}$. Instead, we shall derive a series of a priori estimates which will imply strong compactness of $\{A(u_\varepsilon)\}_{\varepsilon>0}$. This strong compactness together with some a priori estimates on the "total flux" $\gamma_\varepsilon(x)f(u_\varepsilon) - \partial_x A_\varepsilon(u_\varepsilon)$ will make it possible for us to use the compensated compactness method to obtain the desired strong convergence. Finally, we will prove (this is the easy part) that any limit point of a convergent subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$ is a weak solution of (1.1). Uniqueness of the constructed weak solution is a direct consequence of a stability result in [11]. Before continuing, let us also mention that the compensated compactness method has been applied before to certain degenerate parabolic equations (with smooth coefficients) by Zhao [34] and Yin [33].

Our first lemma gives uniform L^1 and L^∞ estimates on u_ε (the proof of the latter exploits assumption (1.3)).

Lemma 3.1. *There exists a constant $C > 0$, independent of ε, such that*

$$\|u_\varepsilon(\cdot,t)\|_{L^1(\mathbb{R})}, \|u_\varepsilon(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq C, \qquad \textit{for all} \quad t \in (0,T).$$

Proof. From the L^1 contraction property proved in [11] it follows that

$$\|u_\varepsilon(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|u_\varepsilon(\cdot,0)\|_{L^1(\mathbb{R})}, \qquad \text{for all} \quad t \in (0,T).$$

Regarding the L^∞ estimate, we will prove that if $u_0(x) \in [0,1]$ then $u_\varepsilon(x,t) \in [0,1]$ for all (x,t). For $\delta > 0$, we let v solve the auxiliary initial value problem

$$\partial_t v + \partial_x \big(\gamma_\varepsilon(x) f(v)\big) = \partial_x^2 A_\varepsilon(v) - \delta, \quad v(x,0) = u_\varepsilon(x,0). \tag{3.2}$$

It is straightforward to show that for each $\varepsilon > 0$, $v \to u_\varepsilon$ strongly in $L^\infty(\Pi_T)$ as $\delta \downarrow 0$. Set

$$K = \Big\{(x,t) \mid v(x,t) > 1 \text{ and } t < T\Big\}.$$

If K is nonempty, set $\bar t = \inf\{t \in K\}$. By continuity there must be a point $\bar x$ such that $v(\cdot,\bar t)$ has a local maximum at $\bar x$ and $v(\bar x,\bar t) = 1$. Furthermore

$$\partial_x v(\bar x,\bar t) = 0, \quad \partial_x^2 v(\bar x,\bar t) \le 0, \quad \text{and} \quad \partial_t v(\bar x,\bar t) \ge 0.$$

Using (3.2) at $(\bar x,\bar t)$, and the assumption that $0 = f(1) = f(v(\bar x,\bar t))$, we find that

$$\begin{aligned} 0 \le \partial_t v(\bar x,\bar t) &+ \partial_x \gamma_\varepsilon(\bar x) f(v(\bar x,\bar t)) + \gamma_\varepsilon(\bar x) f'(v(\bar x,\bar t))\, \partial_x v(\bar x,\bar t) \\ &= A_\varepsilon''(v(\bar x,\bar t))(\partial_x v(\bar x,\bar t))^2 + A_\varepsilon'(v(\bar x,\bar t))\, \partial_x^2 v(\bar x,\bar t) - \delta \le -\delta < 0. \end{aligned}$$

This contradiction implies that K is empty, and $v(x,t) \le 1$ in Π_T, hence also $u_\varepsilon(x,t) \le 1$ in Π_T. Similarly one shows that $u_\varepsilon(x,t) \ge 0$. □

Our next lemma provides us with a uniform $L^2(\Pi_T)$ space and time translation estimate on $A(u_\varepsilon)$, and hence strong L^2_{loc} compactness of $\{A(u_\varepsilon)\}_{\varepsilon>0}$. Later we will use this lemma to pass to the limit in the nonlinear diffusion term.

Lemma 3.2. *There exists a constant $C > 0$ which depends on T but not ε such that*

$$\Big\| A(u_\varepsilon(\cdot + y, \cdot + \tau)) - A(u_\varepsilon(\cdot,\cdot)) \Big\|_{L^2(\Pi_{T-\tau})} \le C\left(|y| + \sqrt{\tau}\right), \qquad \forall y \in \mathbb{R} \text{ and } \forall \tau \ge 0. \tag{3.3}$$

In particular, we have that $\{A(u_\varepsilon)\}_{\varepsilon>0}$ is strongly compact in $L^2_{\mathrm{loc}}(\Pi_T)$.

Proof. Multiply $\partial_t u_\varepsilon + \partial_x(\gamma_\varepsilon(x) f(u_\varepsilon)) = \partial_x(A_\varepsilon'(u_\varepsilon)\partial_x u_\varepsilon)$ by u_ε and then do integration by parts in x to obtain

$$\iint_{\Pi_T} \Big(\frac{1}{2}\partial_t (u_\varepsilon)^2 - \gamma_\varepsilon(x) f(u_\varepsilon)\partial_x u_\varepsilon + A_\varepsilon'(u_\varepsilon)(\partial_x u_\varepsilon)^2\Big)\, dt\, dx = 0.$$

From this equality it follows that

$$\iint_{\Pi_T} A_\varepsilon'(u_\varepsilon)(\partial_x u_\varepsilon)^2\, dt\, dx = \frac{1}{2}\|u_\varepsilon(\cdot,0)\|^2_{L^2(\mathbb{R})} - \frac{1}{2}\|u_\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R})} + \iint_{\Pi_T} \gamma_\varepsilon(x)\partial_x \mathcal{F}(u_\varepsilon)\, dt\, dx,$$

where $\mathcal{F}(u_\varepsilon) = \int_0^{u_\varepsilon} f(\xi)\, d\xi$. Integration by parts gives

$$\left| \iint_{\Pi_T} \gamma_\varepsilon(x)\partial_x \mathcal{F}(u_\varepsilon)\, dt\, dx \right| = \left| \iint_{\Pi_T} \partial_x \gamma_\varepsilon(x) \mathcal{F}(u_\varepsilon)\, dt\, dx \right| \le C T\, |\gamma|_{BV(\mathbb{R})},$$

so that we end up with

$$\iint_{\Pi_T} A_\varepsilon'(u_\varepsilon)(\partial_x u_\varepsilon)^2\, dt\, dx \le \frac{1}{2}\|u_\varepsilon(\cdot,0)\|^2_{L^2(\mathbb{R})} + CT\, |\gamma|_{BV(\mathbb{R})} \le C,$$

where the constant does not depend ε. From this and Lemma 3.1, we conclude that

$$\iint_{\Pi_T} (\partial_x A(u_\varepsilon))^2\, dt\, dx \le \max_u A'(u) \iint_{\Pi_T} A'(u_\varepsilon)(\partial_x u_\varepsilon)^2\, dt\, dx \le C, \tag{3.4}$$

where the constant C does not depend on ε. From this it immediately follows that (3.3) holds when $\tau = 0$. To show that (3.3) holds when $y = 0$ we calculate as follows

$$
\begin{aligned}
&\iint_{\Pi_{T-\tau}} \Big(A(u_\varepsilon(x,t+\tau)) - A(u_\varepsilon(x,t))\Big)^2 \,dt\,dx \\
&\quad\le \|A\|_{\mathrm{Lip}} \iint_{\Pi_{T-\tau}} \left(\int_t^{t+\tau} \partial_t u_\varepsilon(x,\xi)\,d\xi\right) \left(A(u_\varepsilon(x,t+\tau)) - A(u_\varepsilon(x,t))\right) dt\,dx \\
&\quad\le \|A\|_{\mathrm{Lip}} \iint_{\Pi_{T-\tau}} \left(\int_t^{t+\tau} \Big(-\partial_x\left(\gamma_\varepsilon(x,\xi) f(u_\varepsilon(x,\xi))\right) + \partial_x^2 A_\varepsilon(u_\varepsilon(x,\xi))\Big)\,d\xi\right) \\
&\qquad\qquad\qquad \times \Big(A(u_\varepsilon(x,t+\tau)) - A(u_\varepsilon(x,t))\Big)\,dt\,dx \\
&\quad= \|A\|_{\mathrm{Lip}} \int_0^\tau \Bigg\{ \iint_{\Pi_{T-\tau}} \Big(-\partial_x\left(\gamma_\varepsilon(x,t+s) f(u_\varepsilon(x,t+s))\right) + \partial_x^2 A_\varepsilon(u_\varepsilon(x,t+s))\Big) \\
&\qquad\qquad\qquad \times \Big(A(u_\varepsilon(x,t+\tau)) - A(u_\varepsilon(x,t))\Big)\,dt\,dx \Bigg\}\,ds \\
&\quad\le \|A\|_{\mathrm{Lip}} \int_0^\tau \Bigg\{ \iint_{\Pi_{T-\tau}} \gamma_\varepsilon(x,t+s) f(u_\varepsilon(x,t+s)) \Big(\partial_x A(u_\varepsilon(x,t+\tau)) - \partial_x A(u_\varepsilon(x,t))\Big)\,dt\,dx \\
&\qquad\qquad + \iint_{\Pi_{T-\tau}} -\partial_x A_\varepsilon(u_\varepsilon(x,t+s)) \Big(\partial_x A(u_\varepsilon(x,t+\tau)) - \partial_x A(u_\varepsilon(x,t))\Big)\,dt\,dx \Bigg\}\,ds \\
&\quad\le 2\,\|A\|_{\mathrm{Lip}}\,\tau \Big\{ \|\gamma_\varepsilon f(u_\varepsilon)\|_{L^2(\Pi_T)} \|\partial_x A(u_\varepsilon)\|_{L^2(\Pi_T)} + \|\partial_x A_\varepsilon(u_\varepsilon)\|_{L^2(\Pi_T)} \|\partial_x A(u_\varepsilon)\|_{L^2(\Pi_T)} \Big\} \\
&\quad\le C\,\tau,
\end{aligned}
$$

where we have used the equation for u_ε and Hölder's inequality.

Equipped with the uniform space and time translation estimate (3.3), it is an easy exercise to use Kolmogorov's compactness criterion to conclude the proof of the lemma. $\square$

From Lemma 3.1 we know that $M := \|u_\varepsilon\|_{L^\infty(\Pi_T)} \le 1$ (uniformly in ε). Let

$$K = \max_{\lambda \in [0,1]} |A(\lambda)| = A(1).$$

For any function $\Phi \in C([0,K])$, we then have

$$\|\Phi(A(u_\varepsilon))\|_{L^\infty(\Pi_T)} \le C,$$

so that along a subsequence

$$\Phi(A(u_\varepsilon)) \overset{*}{\rightharpoonup} \overline{\Phi} \ \text{ in } L^\infty(\Pi_T), \tag{3.5}$$

and, from Theorem 2.1,

$$\overline{\Phi}(x,t) = \int_{\mathbb{R}} \Phi(A(\lambda))\,d\nu_{(x,t)}(\lambda), \qquad \forall (x,t) \in \Pi_T \setminus N_\Phi, \tag{3.6}$$

for some exceptional set N_Φ that depends possibly on Φ and $|N_\Phi| = 0$. One can choose a sequence $\{\Phi_j\}_{j=1}^\infty \subset C([0,K])$ (e.g., the polynomials with rational coefficients) that is dense in $C([0,K])$ and set

$$N = \bigcup_{j=1}^\infty N_{\Phi_j}. \tag{3.7}$$

Then $|N| = 0$ and

$$\text{(3.6) holds at any point } (x,t) \in \Pi_T \setminus N \text{ for each } \Phi \in C([0,K]). \tag{3.8}$$

From Lemmas 3.1 and 3.2, we know that $A(u_\varepsilon)$ converges along a subsequence to some function $\overline{A}$ a.e. on Π_T. In view of (3.8), we may assume without loss of generality that

$$\Psi\left(\overline{A}(x,t)\right) = \lim_{\varepsilon\downarrow 0} \Psi\left(A(u_\varepsilon(x,t))\right) = \int_{\mathbb{R}} \Psi\left(A(\lambda)\right)\, d\nu_{(x,t)}(\lambda) \text{ for all } \quad (x,t) \in \Pi_T \setminus N, \tag{3.9}$$

for any $\Phi \in C([0,K])$. Since $A(u_\varepsilon(x,t)) \in [0,K]$ for all $\varepsilon > 0$, we have from (3.9) (with $\Psi(\xi) = \xi$) that

$$\overline{A}(x,t) \in [0,K] \text{ for all } (x,t) \in \Pi_T \setminus N.$$

Let u denote the $L^\infty(\Pi_T)$ weak-$*$ limit of $\{u_\varepsilon\}_{\varepsilon>0}$. We can assume without loss of generality that

$$u(x,t) = \int_{\mathbb{R}} \lambda\, d\nu_{(x,t)}(\lambda) \text{ for all } (x,t) \in \Pi_T \setminus N. \tag{3.10}$$

For $\xi \in [0,K]$, define the functions

$$l(\xi) = \min\Big\{\lambda \in [0,1] : A(\lambda) = \xi\Big\}, \qquad L(\xi) = \max\Big\{\lambda \in [0,1] : A(\lambda) = \xi\Big\}. \tag{3.11}$$

In the special case where $A(\cdot)$ is strictly increasing (so that the inverse function $A^{-1}(\cdot)$ exists), $l(\xi) = L(\xi) = A^{-1}(\xi)$ for all ξ. The function $l(\xi)$ is left-continuous and hence lower semicontinuous, while the function $L(\cdot)$ is right-continuous and hence upper semicontinuous. Furthermore,

$$l(A(\lambda)) \le \lambda \le L(A(\lambda)) \text{ for all } \lambda \in [0,1],$$
$$l(A(\lambda)) = \lambda = L(A(\lambda)) \text{ for a.e. } \lambda \in [0,1].$$

Observe that

$$l\left(\overline{A}(x,t)\right)) \le L\left(\overline{A}(x,t)\right) \text{ for all } (x,t) \in \Pi_T \setminus N. \tag{3.12}$$

For any $(x,t) \in \Pi_T \setminus N$, introduce

$$I(x,t) := \left[l\left(\overline{A}(x,t)\right), L\left(\overline{A}(x,t)\right)\right]$$

and, in view of (3.12), observe that $I(x,t)$ is a single point or a closed interval. We shall also need the (measurable) sets

$$\begin{aligned} H &:= \Big\{(x,t) \in \Pi_T \setminus N : l\left(A(u(x,t))\right) < L\left(A(u(x,t))\right)\Big\}, \\ P &:= \Big\{(x,t) \in \Pi_T \setminus N : l\left(A(u(x,t))\right) = L\left(A(u(x,t))\right)\Big\}. \end{aligned} \tag{3.13}$$

We now have the following lemma:

Lemma 3.3. *We have*

(i) $\operatorname{supp} \nu_{(x,t)} \subseteq I(x,t)$ *for all* $(x,t) \in \Pi_T \setminus N$,
(ii) $\nu_{(x,t)} = \delta_{u(x,t)}$ *for all* $(x,t) \in P$, *and*
(iii) $\overline{A}(x,t) = A(u(x,t))$ *for all* $(x,t) \in \Pi_T \setminus N$.

Proof. Suppose that there exists a point $(x_0,t_0) \in \Pi_T \setminus N$ such

$$\operatorname{supp} \nu_{(x_0,t_0)} \not\subset I(x_0,t_0),$$

which implies that

$$\nu_{(x_0,t_0)}\Big([0,1] \setminus I(x_0,t_0)\Big) > 0.$$

Observing that

$$A(\lambda) \ne \overline{A}(x_0,t_0), \qquad \forall \lambda \in [0,1] \setminus I(x_0,t_0),$$

we get, from (3.6) (with $\Psi(\xi) = \left|\xi - \overline{A}(x_0,t_0)\right|$)

$$\begin{aligned} 0 \equiv \left|\overline{A}(x_0,t_0) - \overline{A}(x_0,t_0)\right| &= \int_{\mathbb{R}} \left|A(\lambda) - \overline{A}(x_0,t_0)\right|\, d\nu_{(x_0,t_0)}(\lambda) \\ &\ge \int_{[0,1]\setminus I(x_0,t_0)} \left|A(\lambda) - \overline{A}(x_0,t_0)\right|\, d\nu_{(x_0,t_0)}(\lambda) > 0, \end{aligned}$$

which is a contradiction. This proves (**i**).

Statement (**ii**) follows immediately from (**i**) since by (3.10), if $(x,t) \in P$ and $I(x,t)$ is a single point,

$$l\left(\overline{A}(x,t)\right) = L\left(\overline{A}(x,t)\right) = \int_{\mathbb{R}} \lambda \, d\nu_{(x,t)}(\lambda) = u(x,t).$$

From (**ii**) and (3.9) (with $\Psi(\xi) = \xi$), we know already that (**iii**) holds for all $(x,t) \in P$. Let $(x,t) \in H$ and keep in mind that $I(x,t)$ is now an interval on which $A(\cdot)$ is constant. Hence, from (3.9) and (3.10) we get

$$\overline{A}(x,t) = \int_{I(x,t)} A(\lambda) \, d\nu_{(x,t)}(\lambda) = A\left(\int_{I(x,t)} \lambda \, d\nu_{(x,t)}(\lambda)\right) = A(u(x,t)).$$

Thus, we have shown that (**iii**) holds for all $(x,t) \in (H \bigcup P) = \Pi_T \setminus N$. □

Remark 3.1. Note statement (**ii**) of Lemma 3.3 implies that $\{u_\varepsilon\}_{\varepsilon>0}$ converges to u a.e. on P. The proof of this claim is classical. Let $K := P \bigcap [a,b]$ for any $a, b \in \mathbb{R}$ (this is a measurable set), and note that $u_\varepsilon^2 \overset{*}{\rightharpoonup} u^2$ in $L^\infty(K)$. Then we have

$$\iint_K (u_\varepsilon - u)^2 \, dt \, dx = \iint_K \left(u_\varepsilon^2 - 2u_\varepsilon u + u^2\right) dt \, dx \to 0 \text{ as } \varepsilon \downarrow 0,$$

from which the claim follows.

In the next lemma we sum up the compactness properties of the “diffusion part” of (3.1).

Lemma 3.4. *A subsequence of* $\{A(u_\varepsilon)\}_{\varepsilon>0}$ *converges strongly to* $A(u)$ *in* $L^2_{\text{loc}}(\Pi_T)$, *where* u *is the* $L^\infty(\Pi_T)$ *weak-∗ limit of* $\{u_\varepsilon\}_{\varepsilon>0}$. *Furthermore,*

$$A(u) \in L^\infty(\Pi_T) \bigcap L^2(0,T;H^1(\mathbb{R})).$$

Proof. The proof is an immediate consequence of Lemmas 3.2 and 3.3. □

Before we continue, we shall need the following interpolation lemma due to Kružkov [16]:

Lemma 3.5 (Kružkov [16]). *Let* $u(x,t)$ *be a bounded measurable function defined on* Π_T. *Assume that there exists a nondecreasing continuous function (where we indicate the dependence on* u *by writing “*$;u$*”)* $\nu(\cdot;u) : [0,\infty) \to [0,\infty)$ *such that* $\nu(0;u) = 0$ *and*

$$\int_{\mathbb{R}} |u(x+y,t) - u(x,t)| \, dx \le \nu(|y|;u), \qquad \forall y \in \mathbb{R}, \forall t \in (0,T). \tag{3.14}$$

Suppose that for any $\phi \in C_0^\infty(\mathbb{R})$ *and any* $t_1,t_2 \in (0,T)$,

$$\left|\int_{\mathbb{R}} \Big(u(x,t_2) - u(x,t_1)\Big)\phi(x)\,dx\right| \le C\Big(\|\phi\|_{L^\infty(R)} + \|\partial_x\phi\|_{L^\infty(R)}\Big)\,|t_2 - t_1|, \tag{3.15}$$

where the constant does not depend on ϕ *or* t. *Then for any* $t_1, t_2 \in (0,T)$ *and all* $\varepsilon > 0$

$$\int_{\mathbb{R}} |u(x,t_2) - u(x,t_1)| \, dx \le C\left(\frac{|t_2 - t_1|}{\varepsilon} + \nu(\varepsilon;u)\right). \tag{3.16}$$

Our next lemma provides us with a series of priori estimates that imply strong compactness of the “total flux” sequence $\{\gamma_\varepsilon(x)f(u_\varepsilon) - \partial_x A_\varepsilon(u_\varepsilon)\}_{\varepsilon>0}$. However, these a priori estimates only hold if the initial function u_0 satisfies, in addition to (1.5), the stronger regularity condition

$$|\gamma(x)f(u_0) - \partial_x A(u_0)|_{BV(\mathbb{R})} < \infty. \tag{3.17}$$

In the proof of the next lemma, we shall need the approximate sign function

$$\operatorname{sign}_\eta \xi := \begin{cases} \operatorname{sign}(\xi) & \text{if } |\xi| > \eta, \\ \xi/\eta & \text{if } |\xi| \le \eta, \end{cases} \qquad \eta > 0. \tag{3.18}$$

Lemma 3.6. *Suppose that* (3.17) *holds and introduce the function*

$$v_\varepsilon(x,t) = \gamma_\varepsilon(x)f(u_\varepsilon) - \partial_x A_\varepsilon(u_\varepsilon).$$

There exists a constant $C > 0$, *independent of* ε, *such that for all* $t \in (0,T)$

(i) $\|v_\varepsilon(\cdot,t)\|_{L^\infty(\mathbb{R})} \le C$,
(ii) $|v_\varepsilon(\cdot,t)|_{BV(\mathbb{R})} \le C$,
(iii) $\|v_\varepsilon(\cdot,t+\tau) - v_\varepsilon(\cdot,t)\|_{L^1(\mathbb{R})} \le C\sqrt{\tau},\ \forall \tau \ge 0$.

In particular, we have that $\{v_\varepsilon\}_{\varepsilon>0}$ *is strongly compact in* $L^1_{\mathrm{loc}}(\Pi_T)$.

Proof. We rewrite v_ε as

$$v_\varepsilon(x,t) = \int^x \partial_t u_\varepsilon(\xi,t)\,d\xi,$$

and observe that v_ε satisfies the linear uniformly parabolic equation

$$\partial_t v_\varepsilon + \gamma_\varepsilon(x) f'(u_\varepsilon)\partial_x v_\varepsilon = \partial_x\big(A'_\varepsilon(u_\varepsilon)\partial_x v_\varepsilon\big). \tag{3.19}$$

Then the maximum principle for (3.19) gives

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\mathbb{R})} \le \|v_\varepsilon(\cdot,0)\|_{L^\infty(\mathbb{R})}.$$

We shall derive a BV estimate for v_ε. Differentiate (3.19) with respect to x, set $w_\varepsilon = \partial_x v_\varepsilon$, multiply with $\mathrm{sign}_\eta\, w_\varepsilon$, and integrate over $(x,s)\in \Pi_t := \mathbb{R}\times[0,t]$. The final result reads

$$\iint_{\Pi_t} \Big(\partial_t w_\varepsilon \mathrm{sign}_\eta w_\varepsilon + \partial_x\big(\gamma_\varepsilon(x)f'(u_\varepsilon)w_\varepsilon\big)\mathrm{sign}_\eta w_\varepsilon - \partial_x^2\big(A'_\varepsilon(u_\varepsilon)w_\varepsilon\big)\mathrm{sign}_\eta w_\varepsilon\Big)\,ds\,dx = 0.$$

Since for each fixed $\varepsilon>0$, $\partial_x w_\varepsilon$ is summable,

$$\iint_{\Pi_t} \partial_x\big(\gamma_\varepsilon(x)f'(u_\varepsilon)w_\varepsilon\big)\mathrm{sign}_\eta w_\varepsilon\,ds\,dx = -\iint_{\Pi_t} \gamma_\varepsilon(x)f'(u_\varepsilon)w_\varepsilon\,\mathrm{sign}'_\eta(w_\varepsilon)\,\partial_x w_\varepsilon\,ds\,dx \to 0 \text{ as } \eta\downarrow 0.$$

Similarly, for each fixed $\varepsilon>0$ we have

$$\begin{aligned}
\iint_{\Pi_t} \partial_x^2\big(A'_\varepsilon(u_\varepsilon)w_\varepsilon\big)\mathrm{sign}_\eta w_\varepsilon\,ds\,dx &= -\iint_{\Pi_t} \partial_x\big(A'_\varepsilon(u_\varepsilon)w_\varepsilon\big)\,\mathrm{sign}'_\eta(w_\varepsilon)\,\partial_x w_\varepsilon\,ds\,dx\\
&= -\iint_{\Pi_t} \partial_x A'_\varepsilon(u_\varepsilon)w_\varepsilon\,\mathrm{sign}'_\eta(w_\varepsilon)\,\partial_x w_\varepsilon\,ds\,dx\\
&\qquad -\iint_{\Pi_t} A'_\varepsilon(u_\varepsilon)\,\mathrm{sign}'_\eta(w_\varepsilon)\,(\partial_x w_\varepsilon)^2\,ds\,dx\\
&\le -\iint_{\Pi_t} \partial_x A'_\varepsilon(u_\varepsilon)w_\varepsilon\,\mathrm{sign}'_\eta(w_\varepsilon)\,\partial_x w_\varepsilon\,ds\,dx \to 0 \text{ as } \eta\downarrow 0,
\end{aligned}$$

since $\partial_x A'(u_\varepsilon)$ is bounded. Finally,

$$\begin{aligned}
\iint_{\Pi_t} \partial_t w_\varepsilon \mathrm{sign}_\eta w_\varepsilon\,ds\,dx &= \iint_{\Pi_t} \partial_t\left(\int_0^{w_\varepsilon(x,t)} \mathrm{sign}_\eta(\xi)\,d\xi\right)ds\,dx\\
&= \int_{\mathbb{R}}\left(\int_0^{w_\varepsilon(x,T)} \mathrm{sign}_\eta(\xi)\,d\xi\right)dx - \int_{\mathbb{R}}\left(\int_0^{w_\varepsilon(x,0)} \mathrm{sign}_\eta(\xi)\,d\xi\right)dx.\\
&\to \int_{\mathbb{R}} |w_\varepsilon(x,T)|\,dx - \int_{\mathbb{R}} |w_\varepsilon(x,0)|\,dx \text{ as } \eta\downarrow 0.
\end{aligned}$$

Summing up, $\int_{\mathbb{R}} |w_\varepsilon(x,t)|\,dx = \int_{\mathbb{R}} |w_\varepsilon(x,0)|\,dx$. From this we conclude that

$$|v_\varepsilon(\cdot,t)|_{BV(\mathbb{R})} \le |v_\varepsilon(\cdot,0)|_{BV(\mathbb{R})}.$$

We next prove that v_ε is L^1 Hölder continuous in time with exponent $1/2$. Multiplying (3.19) by a test function $\varphi \in C_0^\infty$ and then do integration by parts, we get

$$\begin{aligned}\int_{\mathbb{R}} \partial_t v_\varepsilon \varphi(x) &= -\int_{\mathbb{R}} \gamma_\varepsilon(x) f'(u_\varepsilon)\partial_x v_\varepsilon \varphi(x)\,dx + \int_{\mathbb{R}} A'_\varepsilon(u_\varepsilon)\partial_x v_\varepsilon \partial_x \varphi(x)\,dx \\ &\le C\Big(\|\varphi\|_{L^\infty(\mathbb{R})} + \|\partial_x\varphi\|_{L^\infty(\mathbb{R})}\Big),\end{aligned}$$

since v_ε is of bounded variation. Consequently,

$$\int_{\mathbb{R}} \Big(v_\varepsilon(x,t+\tau) - v_\varepsilon(x,t)\Big)\varphi(x)\,dx \le C\Big(\|\varphi\|_{L^\infty(\mathbb{R})} + \|\partial_x\varphi\|_{L^\infty(\mathbb{R})}\Big)\tau.$$

Using Kružkov's interpolation lemma (Lemma 3.5), we can conclude that

$$\int_{\mathbb{R}} \big|v_\varepsilon(x,t+\tau) - v_\varepsilon(x,t)\big|\,dx \le C\sqrt{\tau}.$$

The estimates **(i)** – **(iii)** and an application of Kolmogorov's compactness criterion concludes the proof of the lemma. □

To be able to use the compensated compactness method to treat the "nonlinear transport part" of (3.1), we need the next lemma.

Lemma 3.7. *Suppose that* (3.17) *holds. Then for any C^2 function $\eta : \mathbb{R} \to \mathbb{R}$, the sequence of distributions*

$$\big\{\partial_t \eta(u_\varepsilon) + \partial_x\big(\gamma(x) q(u_\varepsilon)\big)\big\}_{\varepsilon>0} \text{ lies in a compact subset of } H^{-1}_{\mathrm{loc}}(\Pi_T),$$

where $q : \mathbb{R} \to \mathbb{R}$ is defined by $q'(u) = \eta'(u) f'(u)$.

Proof. Let us define the distribution $\mathcal{L}_\varepsilon$ by

$$\langle \mathcal{L}_\varepsilon, \varphi\rangle = \iint_{\Pi_T} \Big(\eta(u_\varepsilon)\partial_t\varphi + \gamma(x) q(u_\varepsilon)\partial_x\varphi\Big)\,dt\,dx, \qquad \varphi \in \mathcal{D}(\Pi_T).$$

Using the equation for u_ε and the definition of q, in the sense of distributions we have

$$\begin{aligned}&\partial_t \eta(u_\varepsilon) + \partial_x\big(\gamma(x) q(u_\varepsilon)\big) \\ &\quad = \eta'(u_\varepsilon)\partial_x^2 A_\varepsilon(u_\varepsilon) + \partial_x\Big(\left[\gamma(x) - \gamma_\varepsilon(x)\right] q(u_\varepsilon)\Big) + \gamma'_\varepsilon(x)\Big(q(u_\varepsilon) - \eta'(u_\varepsilon) f(u_\varepsilon)\Big) \\ &\quad = \partial_x\Big(\eta'(u_\varepsilon)\partial_x A_\varepsilon(u_\varepsilon)\Big) - \eta''(u_\varepsilon) A'_\varepsilon(u_\varepsilon)\big(\partial_x u_\varepsilon\big)^2 \\ &\qquad + \partial_x\Big([\gamma(x) - \gamma_\varepsilon(x)] q(u_\varepsilon)\Big) + \gamma'_\varepsilon(x)\Big(q(u_\varepsilon) - \eta'(u_\varepsilon) f(u_\varepsilon)\Big).\end{aligned} \tag{3.20}$$

In view of (3.20), we therefore have

$$\langle \mathcal{L}_\varepsilon, \varphi\rangle = \big\langle \mathcal{L}^1_\varepsilon, \varphi\big\rangle + \big\langle \mathcal{L}^2_\varepsilon, \varphi\big\rangle + \big\langle \mathcal{L}^3_\varepsilon, \varphi\big\rangle + \big\langle \mathcal{L}^4_\varepsilon, \varphi\big\rangle,$$

where

$$\begin{aligned}\big\langle \mathcal{L}^1_\varepsilon, \varphi\big\rangle &= \iint_{\Pi_T} \eta'(u_\varepsilon)\partial_x A_\varepsilon(u_\varepsilon)\partial_x\varphi\,dt\,dx, \\ \big\langle \mathcal{L}^2_\varepsilon, \varphi\big\rangle &= \iint_{\Pi_T} \eta''(u_\varepsilon) A'_\varepsilon(u_\varepsilon)\big(\partial_x u_\varepsilon\big)^2\varphi\,dt\,dx, \\ \big\langle \mathcal{L}^3_\varepsilon, \varphi\big\rangle &= \iint_{\Pi_T} \left[\gamma(x) - \gamma_\varepsilon(x)\right] q(u_\varepsilon)\partial_x\varphi\,dt\,dx, \\ \big\langle \mathcal{L}^4_\varepsilon, \varphi\big\rangle &= -\iint_{\Pi_T} \gamma'_\varepsilon(x)\Big(q(u_\varepsilon) - \eta'(u_\varepsilon) f(u_\varepsilon)\Big)\varphi\,dt\,dx.\end{aligned}$$

Using Lemma 3.1 and (3.4), we get

$$\left| \iint_{\Pi_T} \eta''(u_\varepsilon) A_\varepsilon'(u_\varepsilon) \left(\partial_x u_\varepsilon\right)^2 dt\, dx \right| \le C,$$

and hence $\left|\left\langle \mathcal{L}_\varepsilon^2, \varphi \right\rangle\right| \le C \left\|\varphi\right\|_{L^\infty(\Pi_T)}$. Again thanks to to Lemma 3.1 and the fact that $|\gamma_\varepsilon|_{BV(\mathbb{R})}$ is bounded uniformly with respect to ε, we also have

$$\left|\left\langle \mathcal{L}_\varepsilon^4, \varphi \right\rangle\right| \le C \left\|\varphi\right\|_{L^\infty(\Pi_T)}.$$

Therefore $\left\|\mathcal{L}_\varepsilon^2 + \mathcal{L}_\varepsilon^4\right\|_{\mathcal{M}(\Pi_T)} \le C$, i.e., $\left\{\mathcal{L}_\varepsilon^2 + \mathcal{L}_\varepsilon^4\right\}_{\varepsilon>0}$ is bounded in $\mathcal{M}(\Pi_T)$

Next, we have

$$\left|\left\langle \mathcal{L}_\varepsilon^3, \varphi \right\rangle\right| \le \left\|\gamma - \gamma_\varepsilon\right\|_{L^2(\Pi_T)} \left\|\partial_x \varphi\right\|_{L^2(\Pi_T)} \to 0 \text{ as } \varepsilon \downarrow 0,$$

so that $\left\{\mathcal{L}_\varepsilon^3\right\}_{\varepsilon>0}$ is compact in $H^{-1}_{\mathrm{loc}}(\Pi_T)$. Finally, let us consider $\mathcal{L}_\varepsilon^1$. We write

$$\left\langle \mathcal{L}_\varepsilon^1, \varphi \right\rangle = \left\langle \mathcal{L}_\varepsilon^{1,1}, \varphi \right\rangle + \left\langle \mathcal{L}_\varepsilon^{1,2}, \varphi \right\rangle,$$

where

$$\left\langle \mathcal{L}_\varepsilon^{1,1}, \varphi \right\rangle = \iint_{\Pi_T} \eta'(u_\varepsilon) \partial_x A(u_\varepsilon) \partial_x \varphi \, dt\, dx, \qquad \left\langle \mathcal{L}_\varepsilon^{1,2}, \varphi \right\rangle = \iint_{\Pi_T} \eta'(u_\varepsilon) \varepsilon \partial_x u_\varepsilon \partial_x \varphi \, dt\, dx.$$

Using (3.4) once more, we get

$$\left|\left\langle \mathcal{L}_\varepsilon^{1,2}, \varphi \right\rangle\right| \le C\sqrt{\varepsilon} \left\|\partial_x \varphi\right\|_{L^2(\Pi_T)} \to 0 \text{ as } \varepsilon \downarrow 0,$$

so that also $\left\{\mathcal{L}_\varepsilon^{1,2}\right\}_{\varepsilon>0}$ is compact in $H^{-1}_{\mathrm{loc}}(\Pi_T)$.

In what follows, we use the term "converges" as shorthand for "converges along a subsequence". The semicontinuity of $l(\cdot)$ and $L(\cdot)$ implies that

$$l(\xi) \le \liminf_{\eta \to \xi} l(\eta), \qquad L(\xi) \ge \limsup_{\eta \to \xi} L(\eta).$$

In addition, we have

$$l\left(A(u_\varepsilon(x,t))\right) \le u_\varepsilon(x,t) \le L\left(A(u_\varepsilon(x,t))\right) \text{ for all } (x,t) \in \Pi_T \text{ and } \varepsilon > 0.$$

Consequently,

$$\begin{aligned} &\liminf_{\varepsilon \downarrow 0} u_\varepsilon(x,t) \ge l\left(A(u(x,t))\right) \text{ for a.e. } (x,t) \in \Pi_T, \\ &\limsup_{\varepsilon \downarrow 0} u_\varepsilon(x,t) \le L\left(A(u(x,t))\right) \text{ for a.e. } (x,t) \in \Pi_T. \end{aligned} \tag{3.21}$$

By (3.21), and since

$$A'(\lambda) = 0 \text{ for all } \lambda \in [0,1] \text{ such that } l(A(\lambda)) < L(A(\lambda)),$$

it follows that

$$A'(u_\varepsilon) \to 0 \text{ a.e. on } H \text{ as } \varepsilon \downarrow 0.$$

Therefore, in view of Lemma 3.1 and (3.4),

$$\begin{aligned} &\iint_{H \bigcap [a,b]} \left|\eta'(u_\varepsilon) A'(u_\varepsilon) \partial_x u_\varepsilon\right| \, dt\, dx \\ &\quad \le C \left(\iint_{H \bigcap [a,b]} A'(u_\varepsilon) \, dt\, dx \right)^{1/2} \left(\iint_{H \bigcap [a,b]} A'(u_\varepsilon) \left(\partial_x u_\varepsilon\right)^2 dt\, dx \right)^{1/2} \to 0 \text{ as } \varepsilon \downarrow 0, \end{aligned}$$

for any interval $[a,b] \subset \mathbb{R}$. Hence

$$\eta'(u_\varepsilon) \partial_x A(u_\varepsilon) \to 0 \text{ a.e. on } H \text{ as } \varepsilon \downarrow 0.$$

On the other hand, Lemma 3.3 shows that $\{u_\varepsilon\}_{\varepsilon>0}$ converges a.e. on P. From Lemma 3.6, we have that

$$\{\gamma_\varepsilon(x)f(u_\varepsilon) - \partial_x A_\varepsilon(u_\varepsilon)\}_{\varepsilon>0} \text{ converges a.e. on } \Pi_T.$$

Since $\{u_\varepsilon\}_{\varepsilon>0}$ converges a.e. on P, we conclude that

$$\{\eta'(u_\varepsilon)\partial_x A_\varepsilon(u_\varepsilon)\}_{\varepsilon>0} \text{ converges a.e. on } P.$$

Since $\partial_x A_\varepsilon(u_\varepsilon) = \partial_x A(u_\varepsilon) + \varepsilon\partial_x u_\varepsilon$ and $\eta'(u_\varepsilon)\varepsilon\partial_x u_\varepsilon \to 0$ a.e. on Π_T, we conclude that also

$$\{\eta'(u_\varepsilon)\partial_x A(u_\varepsilon)\}_{\varepsilon>0} \text{ converges a.e. on } P.$$

Hence we have shown that

$$\{\eta'(u_\varepsilon)\partial_x A(u_\varepsilon)\}_{\varepsilon>0} \text{ converges a.e. on } H\bigcup P = \Pi_T \setminus N \text{ (and hence a.e. on } \Pi_T).$$

Moreover, from Lemma 3.1 and (3.4), $\eta'(u_\varepsilon)\partial_x A(u_\varepsilon) \in L^2(\Pi_T)$. Consequently,

$$\{\eta'(u_\varepsilon)\partial_x A(u_\varepsilon)\}_{\varepsilon>0} \text{ converges strongly in } L^2(\Pi_T),$$

and $\{\mathcal{L}_\varepsilon^{1,1}\}_{\varepsilon>0}$ belongs to a compact subset of $H^{-1}_{\mathrm{loc}}(\Pi_T)$.

Summing up, we have proved that the sequence of distributions $\{\mathcal{L}_\varepsilon\}_{\varepsilon>0}$ is the sum of two terms, one which is compact in $H^{-1}_{\mathrm{loc}}(\Pi_T)$ and one which is bounded in $\mathcal{M}(\Pi_T)$. In addition, Lemma 3.1 implies that $\{\mathcal{L}_\varepsilon\}_{\varepsilon>0}$ belongs to a bounded subset of $W^{-1,\infty}_{\mathrm{loc}}(\Pi_T)$. Hence, the proof of the lemma is now finished by appealing to Lemma 2.3. □

Our main result is the following theorem:

Theorem 3.1. *Suppose that conditions* (1.2)–(1.5) *hold. Then there exists a weak solution (in the sense of Definition 1.1) of the Cauchy problem* (1.1). *Furthermore, u can be constructed as the strong limit of the sequence $\{u_\varepsilon\}_{\varepsilon>0}$, where u_ε solves the regularized problem* (3.1).

Let v be another weak solution constructed as the strong limit of the sequence $\{v_\varepsilon\}_{\varepsilon>0}$, where v_ε solves the regularized problem (3.1) *corresponding to initial data v_0. Then*

$$\int_{\mathbb{R}} |u(x,t) - v(x,t)|\, dx \le \int_{\mathbb{R}} |u_0(x) - v_0(x)|\, dx. \tag{3.22}$$

Consequently, the constructed weak solution u of (1.1) *is unique.*

Suppose that the initial function u_0 satisfies the additional regularity condition stated in (3.17). *Then the constructed weak solution u has the following regularity properties:*

(i) $\left|\big(\gamma(x)f(u) - \partial_x A(u)\big)(\cdot,t)\right|_{BV(\mathbb{R})} \le C,\ \forall t \in (0,T)$.

(ii) $\|u(\cdot,t+\tau) - u(\cdot,t)\|_{L^1(\mathbb{R})} \le C\tau,\ \forall \tau \ge 0$.

Proof. First, let us assume that the additional regularity condition in (3.17) holds. By Lemma 3.7 and Theorem 2.2, we have that

$$u_\varepsilon \to u \text{ along a subsequence a.e. in } \Pi_T \text{ as } \varepsilon \downarrow 0.$$

Lemma 3.1 states that the limit u belongs to $L^1(\Pi_T)\bigcap L^\infty(\Pi_T)$, so that the convergence holds true in $L^p(\Pi_T)$ for any $p \in [1,\infty)$. From Lemma 3.4, it follows that $A(u) \in L^2(0,T;H^1(\mathbb{R}))$. Hence, the limit u satisfies (**D**.1). Multiplying the equation for u_ε by a test function $\varphi \in \mathcal{D}(\Pi_T)$ (with $\varphi|_{t=T} = 0$) and then do integration by parts in x and t, we get

$$\iint_{\Pi_T} \Big(u\partial_t\varphi + \gamma_\varepsilon(x)f(u_\varepsilon)\partial_x\varphi + (A(u_\varepsilon) + \varepsilon u_\varepsilon)\,\partial_x^2\varphi\Big)\, dt\, dx + \int_{\mathbb{R}} \omega_\varepsilon * u_0\, \phi(x,0) = 0.$$

Sending $\varepsilon \downarrow 0$, it follows (after an integration by parts) that the limit u satisfies (**D**.2). In addition, (**i**) and (**ii**) are direct consequences of Lemma 3.6. This concludes the proof when (3.17) holds.

We will now remove the extra assumption (3.17) by using a stability result for "smooth" $\gamma(\cdot)$ found in [11], which tells us that $\int_{\mathbb{R}} |u_\varepsilon(x,t) - v_\varepsilon(x,t)|\, dx \le \int_{\mathbb{R}} |u_\varepsilon(x,0) - v_\varepsilon(x,0)|\, dx$, where v_ε solves (3.1) corresponding to initial data v_0 satisfying (3.17). Sending $\varepsilon \downarrow 0$ yields (3.22) whenever u_0, v_0 satisfy (3.17). If u_0 satisfies (1.5), we can certainly find a sequence $\{u_0^m\}_{m=1}^\infty$ such that each

u_0^m satisfy (3.17) and $u_0^m \to u_0$ in $L^1(\mathbb{R})$ as $m \uparrow \infty$. Let u^ε be a weak solution of (1.1) with initial data u_0^m. Using (3.22), we get

$$\int_{\mathbb{R}} |u^m(x,t) - u^n(x,t)|\, dx \le \int_{\mathbb{R}} |u_0^m(x) - u_0^n(x)|\, dx \to 0 \text{ as } m, n \uparrow \infty.$$

Hence $\{u^m\}_{m=1}^\infty$ is a Cauchy sequence in $L^1(\Pi_T)$. It is not difficult to check that the limit u of this sequence satisfies (**D**.1) and (**D**.2). This concludes the proof of the theorem. $\square$

Remark 3.2. In the pure hyperbolic case, Theorem 3.1 (**i**) implies that the total variation of $f(u)$ is finite if $u_0 \in BV(\mathbb{R})$ (recall that $\gamma \neq 0$ a.e.), although the total variation of u need not be finite. This fact has already been established by Klausen and Risebro [13]. However, their proof is much more complicated than the elementary proof given here (see the proof of Lemma 3.6).

Remark 3.3. It is worth mentioning that if $A(\cdot)$ is strictly increasing we do not need the compensated compactness method to get strong convergence of $\{u_\varepsilon\}_{\varepsilon>0}$. This is the typical situation that one has to deal with in models for two-phase flow in porous media (see, e.g., [5]). In this case, we have strong L^2_{loc} convergence of $\{u_\varepsilon\}_{\varepsilon>0}$ directly from Lemma 3.3.

Provided the initial function u_0 is sufficiently smooth, it is possible to upgrade the strong L^2 compactness of $\{A(u_\varepsilon)\}_{\varepsilon>0}$ to strong compactness in the Hölder space $C^{1,\frac{1}{2}}$. This is the content of the following proposition, which also shows that $A(u)$ is Hölder continuous, i.e., significantly more regular than anticipated by Definition 1.1.

Proposition 3.1. *Suppose that conditions* (1.2)-(1.5) *hold. In addition, suppose that* (3.17) *holds. Then there exists a constant C, independent of ε, such that*

$$\Big|A(u_\varepsilon(x+y, t+\tau)) - A(u_\varepsilon(x,t))\Big| \le C\Big(|y| + \sqrt{\tau} + \varepsilon\Big),$$

for all $y \in \mathbb{R}$ and $\tau \ge 0$ with $t + \tau < T$. In particular, $\{A(u_\varepsilon)\}_{\varepsilon>0}$ converges along a subsequence to some function $\overline{A}$ uniformly on compact subsets of Π_T as $\varepsilon \downarrow 0$ and

$$\overline{A} \in C^{1,\frac{1}{2}}(\Pi_T).$$

If u denotes the weak solution in Theorem 3.1, then $\overline{A} = A(u)$ a.e. on Π_T.

Proof. Since $u_\varepsilon, \gamma_\varepsilon(x) f(u_\varepsilon)$ are uniformly bounded, we get from (**i**)

$$\|\partial_x A_\varepsilon(u_\varepsilon(\cdot,t))\|_{L^\infty(\mathbb{R})} \le C, \qquad \forall t \in (0,T).$$

From this estimate and the L^∞ bound on u_ε, we get

$$A(u_\varepsilon(x+y,t)) - A(u_\varepsilon(x,t)) \le C\big(|y| + \varepsilon\big). \tag{3.23}$$

Following Zhao [34], we show next that $A(u_\varepsilon)$ is Hölder continuous in time. To this end, let $\tau > 0$ and note that

$$\begin{aligned}
\int_x^{x+\sqrt{\tau}} \Big(u_\varepsilon(x,t+\tau) - u_\varepsilon(x,t)\Big)\, dx &= \int_x^{x+\sqrt{\tau}} \int_t^{t+\tau} \partial_t u_\varepsilon(x,\xi)\, d\xi\, dx \\
&= \int_x^{x+\sqrt{\tau}} \int_t^{t+\tau} \Big(-\partial_x\big(\gamma_\varepsilon(x) f(u_\varepsilon(x,\xi)) + \partial_x^2 A_\varepsilon(u_\varepsilon(x,\xi))\big)\Big)\, d\xi\, dx \\
&= \int_t^{t+\tau} \left(\Big[-\gamma_\varepsilon(x) f(u_\varepsilon(x,\xi))\Big]_x^{x+\sqrt{\tau}} + \Big[\partial_x A_\varepsilon(u_\varepsilon(x,\xi))\Big]_x^{x+\sqrt{\tau}}\right) d\xi\, dx \\
&\le C\big(t+\tau-t\big) = C\tau.
\end{aligned}$$

By the mean value theorem there exists an x^* between x and $x + \sqrt{\tau}$ such that

$$\Big(u_\varepsilon(x^*, t+\tau) - u_\varepsilon(x^*,t)\Big)\sqrt{\tau} \le C\tau \Longrightarrow \Big|u_\varepsilon(x^*, t+\tau) - u_\varepsilon(x^*,t)\Big| \le C\sqrt{\tau}.$$

Consequently, we can calculate as follows

$$
\begin{aligned}
&\Big|A(u_\varepsilon(x,t+\tau)) - A(u_\varepsilon(x,t))\Big| \\
&\quad\le \Big|A(u_\varepsilon(x,t+\tau)) - A(u_\varepsilon(x^*,t+\tau))\Big| + \Big|A(u_\varepsilon(x^*,t+\tau)) - A(u_\varepsilon(x^*,t))\Big| \\
&\qquad + \Big|A(u_\varepsilon(x^*,t)) - A(u_\varepsilon(x,t))\Big| \\
&\quad\le C\Big(|x-x^*| + \tau^{1/2} + |x-x^*|\Big) \le C\sqrt{\tau}.
\end{aligned}
\tag{3.24}
$$

In view of (3.23) and (3.24), an application of the Ascoli-Arzela compactness criterion concludes the proof of the proposition. □

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