FRONT MOTION IN MULTI-DIMENSIONAL VISCOUS CONSERVATION LAWS WITH STIFF SOURCE TERMS DRIVEN BY MEAN CURVATURE AND VARIATION OF FRONT THICKNESS

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Abstract

The bistable reaction-diffusion-convection equation

$$\partial_t u + \nabla \cdot \mathbf{f}(u) = -\frac{1}{\epsilon} g(u) + \epsilon \Delta u, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R} \tag{1}$$

is considered. Stationary traveling waves of above equation are proved to exist when $f(u)$ is symmetric and $g(u)$ is antisymmetric about $u = 0$. Solutions of initial value problems tends to almost piecewise constant functions within $O(1)\epsilon$ time. The almost constant pieces are separated by sharp interior layers, called fronts. The motion of these fronts are studied by asymptotic expansion. The equation for the motion of the front is obtained. In the case of $\mathbf{f} = \mathbf{b}u^2$ and $g(u) = au(1 - u^2)$, where $\mathbf{b} \in \mathbb{R}^n$ and $0 < a \in \mathbb{R}$ are constants, the front motion equation takes more explicit form, showing that the front's speed is

$$\epsilon \left(\kappa + \frac{\nabla \mu}{\mu} \cdot \mathbf{T} \right)$$

where κ is the mean curvature of the front, μ the width of the planar traveling of (1) in the normal direction $\mathbf{n}$ of the front, $\mathbf{T}$ a vector tangential to the front. Both κ and $\nabla \mu / \mu \cdot \mathbf{T}$ are elliptic operators, contributing to the shrinkage of closed curves. An ellipse in $\mathbb{R}^2$ is found to preserve its shape while shrinking.

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§1. Introduction

Typical reactive flows are governed by Navier-Stokes equations with reaction source terms. These equations have the form

$$u_t + \nabla \cdot \mathbf{f}(u) = \frac{1}{\epsilon} g(u) + \gamma \Delta u, \quad t > 0, \ \mathbf{x} \in \mathbb{R}^n. \tag{1.1}$$

Here $\epsilon > 0$ is the reaction time, μ the viscosity. We shall consider the scaling $\gamma = \epsilon$ in this paper. We restrict our attention to the bistable type of source term $g(u)$. We assume for convenience that

$$g(\pm 1) = 0, \quad g'(\pm 1) < 0, \quad g(0) = 0, \quad g'(0) > 0 \tag{1.2}$$

with no more zeroes. Under this assumption, ± 1 are the two stable equilibrium points of (1.1). When $\epsilon > 0$ is small, we see that the solution of the initial value problem of (1.1) will quickly become almost ± 1 over subdomians separated by sharp fronts across which u changes from -1 to 1. We shall study the motion of these fronts in this paper.

The one dimensional version of (1.1) and its inviscid version

$$u_t + f(u)_x = \frac{1}{\epsilon} g(u)$$

are studied by [FH1, FH2, FJT, FJ, Lyb, Mas, Sin1, Sin2]. They considered the large time behavior and $\epsilon \to 0+$ limits. No study has been done for multidimensional case of (1.1) yet.

When $\mathbf{f} = 0$, the equation (1.1) is a typical reaction-diffusion equation

$$u_t + \nabla \cdot \mathbf{f}(u) = \frac{1}{\epsilon} g(u) + \gamma \Delta u, \quad t > 0, \ \mathbf{x} \in \mathbb{R}^n.$$

which has been extensively studied. When $\epsilon > 0$ is small, the solution of the reaction-diffusion equation tends to a piecewise constant function within $O(1)\epsilon$ time. The constant pieces are separated by layers called fronts. If the two wells of the potential $\int g(u)du$ are equal, the front will move at the normal speed $\epsilon\kappa$ where κ is the mean curvature of the front. This phenomenon was conjectured by Allen and Cahn, [AC] and justified formally by [Fife, Ca, RSK]. Rigorous results were obtained by [BES, BK, Br, Ch, D, DS, ESS, Il, Son] etc. Among above papers, the one that is closely related to this paper is [RSK]. Many of its techniques are used in this paper. For more information about motion of fronts in reaction diffusion equations, the reader is refereed to the lecture notes [E, Sou] and references cited therein.

The motion by mean curvature when $\mathbf{f} = 0$ is due to the interaction of fast reaction and slow diffusion in the reaction-diffusion equation. What is the effect of convection term $\mathbf{f}$ on the front motion? To answer this question, in this paper, we shall investigate the motion of fronts in (1.1) when $\mathbf{f} \neq 0$. From our earlier results, [FJT], we see that in the $\epsilon \to 0+$ limit, equation (1.1) has two types of planar waves: The first kind is the ordinary Lax shock propagating in the direction of the unit vector $\mathbf{n}$, pointing from 1 to -1 side, at the speed $\mathbf{n} \cdot (\mathbf{f}(1) - \mathbf{f}(-1))/2$ as determined by Rankin-Hugoniot condition. The second type of wave, called rarefaction layer, has the speed $-\mathbf{n} \cdot \mathbf{f}'(0)$. These wave speeds are the $O(1)$ order approximation of the front propagation speeds in (1.1). They are speeds of planar waves. To get the $O(\epsilon)$ order of the wave speeds and to observe the effect of curvature of the front and variance of thickness of front in different directions on the front propagation, we set

$$\mathbf{f}(1) - \mathbf{f}(-1) = \mathbf{f}'(0) = 0. \tag{1.3}$$

In this paper, we assume that

$$\mathbf{f}(-u) = \mathbf{f}(u), \quad g(-u) = -g(u). \tag{1.4}$$

Under this assumption, the condition (1.3) is satisfied.

The main results and the organization of this paper is as follows: In section §2, we shall prove that, under the assumption (1.4), solutions of (1.1) with some initial value will converge as $t \to \infty$ to a planar stationary wave. As a consequence, the existence of stationary planar traveling wave in any direction $\mathbf{n}$ is established. Across these planar waves, u changes from ± 1 to ∓ 1. The speed of these planar waves are necessarily 0. In section §3, we formally derive, via asymptotic expansion, that after $t = O(1)\epsilon$, the domain $\mathbb{R}^n$ will be divided into subdomains over which $u \approx \pm 1$. These subdomains are separated by fronts across which u changes from ± 1 to ∓ 1. This leads to the investigation of the front propagation in the next section §4. Suppose we have a front separating the regions $\{\mathbf{x} \in \mathbb{R}^n \;:\; u(\mathbf{x}, t) > 0\}$ and $\{\mathbf{x} \in \mathbb{R}^n \;:\; u(\mathbf{x}, t) < 0\}$, over which $u \approx \pm 1$. The location of the front can be denoted as the level curve $\Phi(\mathbf{x}, t, \eta, \epsilon) = 0$, where $\eta = \epsilon t$. In section §4, we shall show, through asymptotic expansions, that $\Phi_t = 0$, i.e., the front does not move on the $O(1)$ time scale, as expected since the planar waves are stationary. The front will move in $O(\epsilon^{-1})$ time scale. We obtained the partial differential equation relating Φ_η and partial derivatives of Φ up to second order. This equation contains the planar stationary waves $\psi(\mathbf{n} \cdot \mathbf{x}/\epsilon, \mathbf{n})$ of (1.1) in $\mathbf{n} := \nabla\Phi/|\nabla\Phi|$ direction, determined by

$$\begin{aligned} &\mathbf{n} \cdot \mathbf{f}(u)\zeta = g(u) + u_{\zeta\zeta}, \\ &u(\pm\infty) = \pm 1 \;\text{ or }\; \mp 1 \end{aligned} \tag{1.5}$$

To further understand the behavior of this equation for front motion, we want to study some special cases of $\mathbf{f}$ and g for which explicit solutions of (1.5) can be obtained. For this purpose, we considered in §5 the case where $\mathbf{f}(u) = \mathbf{b}u^2$ and $g(u) = au(1-u^2)$, where $\mathbf{b}$ is a constant vector in $\mathbb{R}^n$ and $a > 0$ is a constant, For such $\mathbf{f}$ and g, the planar stationary waves $\psi(\mathbf{n} \cdot \mathbf{x}/\epsilon, \mathbf{n})$ of (1.1) in $\mathbf{n}$ direction is

$$\psi = \tanh(\zeta/\mu) \tag{1.6}$$

with

$$\mu^{-1} = (-\mathbf{n} \cdot \mathbf{b} \pm \sqrt{(\mathbf{n} \cdot \mathbf{b})^2 + 2a})/2. \tag{1.7}$$

The function $\mu(\mathbf{n})$ is the width of the traveling wave of (1.1) in $\mathbf{n}$ direction. With the help of (1.6) and (1.7), we obtained the explicit expression for the front equation:

$$\frac{\Phi_\eta}{|\nabla \Phi|} = \kappa + \frac{\nabla \mu}{\mu} \cdot \mathbf{T}, \tag{1.8}$$

where κ is the mean curvature of the level curve $\Phi(\mathbf{x}, t, \eta, \epsilon)$ =constant, the vector $\mathbf{T} := (\mathbf{b} - (\mathbf{b} \cdot \mathbf{n})\mathbf{n})/(2\mu^{-1} + \mathbf{b} \cdot \mathbf{n})$ is in a tangential direction of the level curve. Both κ and $\frac{\nabla \mu}{\mu} \cdot \mathbf{T}$ are elliptic operators on Φ. Equation (1.8) states that the front motion of (1.1) is driven by mean curvature of the front and relative rate of change in a tangential direction of the thickness of planar traveling waves of (1.1). In two dimensional case, the contributions of both κ and $\frac{\nabla \mu}{\mu} \cdot \mathbf{T}$ are to make a circular front to shrink. An example in $\mathbb{R}^2$ is given to show that equation (1.8) have an explicit solution representing an elliptic front. This front maintains its shape while shrinking. The time it takes for this elliptic front to shrink to a point and then to disappear is given. We guess that simple closed fronts of other shape, for example a circle will evolve, as η increases, towards the shape of this ellipse before it shrinks to a point. In section §6. we solve a differential equation to provide its solutions needed in §4 and §5.

§2. Planar Traveling Waves of (1.1)

In this section, we consider planar traveling waves of (1.1) when $\gamma = A\epsilon$. A planar traveling wave of (1.1) in the direction of the unit vector $\mathbf{n}$ is a solution of (1.1) of the form $u(\epsilon^{-1}(\mathbf{x} \cdot \mathbf{n} - ct))$. We see that a planar traveling wave of (1.1) connecting $u_\pm = \pm 1$ or ∓ 1 must satisfy the following connecting orbit problem of ODE:

$$\begin{aligned} &-cu' + \mathbf{n} \cdot \mathbf{f}(u)' = g(u) + Au'', \\ &u(\pm\infty) = u_\pm. \end{aligned} \tag{2.1}$$

For simplicity, we use the following notation

$$f(u) := \mathbf{n} \cdot \mathbf{f}(u). \tag{2.2}$$

Then the system (2.1) become

$$\begin{aligned} & -cu' + f(u)' = g(u) + Au'', \\ & u(\pm\infty) = u_\pm. \end{aligned} \tag{2.3}$$

The traveling wave equation of

$$u_t + f(u)_x = g(u) + Au_{xx} \tag{2.4}$$

is the same as (2.3). To prove the existence of solutions of (2.3), we shall show that the solution of (2.4) with some initial data $u(x,0)$ will converge to a stationary solution under condition (1.4). Some of the lemmas in this section are proved in our earlier paper [FJ]. However, the proof in [FJ] is for the special case $g(u) = u(1-u^2)$. Here we present a proof holds for general $g \in C^1(\mathbb{R}, \mathbb{R})$. We shall further prove that the speed of traveling waves of (2.4) and hence that of (2.1) is 0.

Lemma 2.1. *Let $u(x,t)$ be the solution of (1.1) with initial data $u(x,0)$. If $u_x(x,0) < 0$ (> 0), then $u_x(x,t) < 0$ (> 0).*

Proof. Let $v = u_x$, then v satisfies

$$\begin{aligned} & v_t + f'(u)v_x + f''(u)v^2 = v_{xx} + g'(u)v, \\ & v(x,0) = u_x(x,0) > 0 \quad (< 0). \end{aligned} \tag{2.5}$$

The maximum principle type of argument applies to (2.5) to yield that $v(x,t) > 0$ (< 0) if $v(x,0) > 0$ (< 0). ∎

In the rest of this section, we assume the initial data satisfies

$$u_x(x,0) < 0, \text{ and } -1 \le u(x,0) \le 1. \tag{2.6}$$

In this case the solution $u(x,t)$ is decreasing. Then the transformation from (x,t) to

$$\begin{aligned} & w = u(x,t), \\ & s = t \end{aligned} \tag{2.7}$$

is one-to one. Then for any smooth function $h(w,s)$, the chain rule reads

$$
\begin{aligned}
h_t &= h_w u_t + h_s, \\
h_x &= h_w u_x.
\end{aligned} \tag{2.8}
$$

Let $v := u_x$. After changing variables according to (2.7) and (2.8), the equation (2.5) and (2.6) become

$$
\begin{aligned}
& v_s = v^2 \left[v_{ww} + \left(\frac{g(w)}{v} \right)_w - f''(w) \right], \quad w \in (-1, 1), \\
& v(w, 0) < 0.
\end{aligned} \tag{2.9}
$$

From Lemma 2.1, we know that the solution of (2.9) satisfies $v(w, s) < 0$ for $s > 0$.

Lemma 2.2. *If $v_s(w, 0) > 0 \ \ (< 0)$, then $v_s(w, s) > 0 \ \ (< 0)$ for all $s > 0$.*

Proof. Taking $\partial/\partial s$ on (2.9), we obtain

$$
(v_s)_s = v^2 (v_s)_{ww} + 2v[v_{ww} - f''(w)]v_s - g(w)(v_s)_w + g'(w)(v_s). \tag{2.10}
$$

Again, the maximum principle type of argument applies to (2.10) to yield that if $v_s(w, 0) > 0 \ \ (< 0)$, then $v_s(w, s) > 0 \ \ (< 0)$ for all $s > 0$. ∎

Lemma 2.3.

(i) Let

$$
u(x, 0) = -\tanh(x/\delta), \quad (\tanh(x/\delta)) \tag{2.10}
$$

where $\delta > 0$ is a constant and $v(w, 0)$ be the function $u_x(x, 0)$ with variables (w, s) given in (2.8). Then when $\delta > 0$ is small enough, the solution v of (2.9) satisfies $v_s(w, s) > 0$ (< 0) for all $s > 0$.

(ii) Let $u(x, 0)$ be the solution of

$$
u_x = \frac{1}{\delta_1} \frac{g(u)}{u}, \quad u(0, 0) = 0. \tag{2.11}
$$

If $\delta_1 > 0$ are large enough, then the solution of (2.9) with initial data $v(w, 0) = u_x(x(w, 0), 0)$ satisfies $v_s(w, s) > 0$ for all $s > 0$.

(iii) Let $u(x, 0)$ be the solution of

$$
u_x = -\frac{1}{\delta_2} \frac{g(u)}{u}, \quad u(0, 0) = 0. \tag{2.12}
$$

If $\delta_2 > 0$ are large enough, then the solution of (2.9) with initial data $v(w,0) = u_x(x(w,0),0)$ satisfies $v_s(w,s) < 0$ for all $s > 0$.

Proof. (i) We only prove the case where $u(x,0) = \tanh(x/\delta)$. The other case can be similarly proved by multiply minus sign to v since the equation is linear in v.

From Lemma 2.2, it suffices to prove that the initial data given by (2.10) satisfies $(2.9)_2$ and $v_s(w,0) > 0$ for $w \in (-1,1)$. It is easy to see that $v(w,0) = u_x(x,0) > 0$ for $u(x,0) = \tanh(x/\delta)$. Also, the limiting process $w \to \pm 1$ correspond to $x \to \mp\infty$ and hence $v(w \to \pm 1, 0) = 0$. We compute straightforwardly to get

$$\begin{aligned}
&v_s(w,0) = v_t + v_x \frac{\partial x}{\partial s} = v_t - v_x \frac{u_t}{u_x} \\
&= u_{xxx} - u_{xx}^2/u_x - f''(u)u_x^2 + g'(u)u_x - g(u)u_{xx}/u_x \\
&= u_x \left(\frac{u_{xx}}{u_x}\right)_x - f''(u)u_x^2 + u_x \left(\frac{g(u)}{u_x}\right)_x \\
&= \operatorname{sech}^4\left(\frac{x}{\delta}\right)\left[\frac{2}{\delta^3} - \frac{f''(u)}{\delta^2}\right] + \delta u_x \left(\frac{g(u)}{1-u^2}\right)_x \\
&= \operatorname{sech}^4\left(\frac{x}{\delta}\right)\left[\frac{2}{\delta^3} - \frac{f''(u)}{\delta^2}\right] + \delta u_x^2 \left(\frac{g(u)}{1-u^2}\right)_u \\
&= \operatorname{sech}^4\left(\frac{x}{\delta}\right)\left[\frac{2}{\delta^3} - \frac{f''(u)}{\delta^2} + \frac{O(1)}{\delta}\right]
\end{aligned}$$

Here we used conditions (1.2) and (1.4) and $u_x = (1-u^2)/\delta$. The conclusion follows from Lemma 2.2 immediately.

(ii) Now, the initial data $u(x,0)$ is determined by

$$u_x = \frac{g(u)}{\delta_1 u}, \quad u(0,0) = 0.$$

By the conditions (1.2) on $g(u)$, we see that range of $u(x,0)$ is $(-1,1)$ and $u_x(x,0) > 0$. The computation similar to that in the proof of (i) yields that

$$\begin{aligned}
v_s(w,0) &= u_x \left(\frac{u_{xx}}{u_x}\right)_x - f''(u)u_x^2 + u_x \left(\frac{g(u)}{u_x}\right)_x \\
&= \left(\frac{O(1)}{\delta_1} + O(1) + \delta_1\right) u_x^2.
\end{aligned}$$

When $\delta_1 > 0$ is sufficiently large, $v_s(w,0) > 0$ and hence $v_s(w,s) > 0$.

(iii) The proof is almost the same as that of (ii). ∎

Corollary 2.4. *Let $v(w,s)$ be the solution of (2.9) with initial data $-\tanh(x/\delta)$ (or $\tanh(x/\delta)$) where $\delta > 0$ sufficiently small. Then $v(w,s) \to \theta(w) < 0$ (> 0) for $w \in (-1,1)$ as $s \to \infty$. Furthermore, the function $\theta(w)$ satisfies*

$$-\infty = \int_0^1 \frac{1}{\theta(w)} dw < \int_0^w \frac{1}{v(w,0)} dw < \int_0^w \frac{1}{v(w,0)} dw < \int_0^{-1} \frac{1}{\theta(w)} = \infty \tag{2.13}$$

$$\left(\infty = \int_0^1 \frac{1}{\theta(w)} dw > \int_0^w \frac{1}{v(w,0)} dw > \int_0^w \frac{1}{v(w,0)} dw > \int_0^{-1} \frac{1}{\theta(w)} = -\infty \right). \tag{2.13$'$}$$

Proof. we only prove for the case where $v(w,0) = -\tanh(x/\delta)$. The other case can be prove in the same way. Lemma 2.3 states that the solution of (2.9) with initial data (2.10) with $\delta > 0$ sufficiently small satisfies $v_s(w,s) > 0$ and hence $v(w,s)$ is increasing as s increases. On the other hand, v is also bounded from above by 0. Therefore, the limit $\lim_{s\to\infty} v(w,s) =: \theta(w)$ exists for all $w \in (-1,1)$. To prove that $\theta(w) < 0$ for $w \in (-1,1)$, we consider the solution v_2 of (2.9) given in Lemma 2.3(iii) with $\delta_2 > 0$ large enough. This solution satisfies $0 > v_2(w,0) > v_2(w,s)$ for all $s > 0$ and $w \in (-1,1)$. We note that the comparison principle holds for equation (2.9). To compare the initial datum $v(w,0)$ and $v_2(w,0)$, we consider the equation defining $v_2(w,0)$:

$$\begin{aligned} &u_{2x} = -\frac{g(u_2)}{\delta_2 u_2} = -(1-u_2^2)\frac{g(u_2)}{\delta_2 u_2 (1-u_2^2)}, \\ &u_2(0) = 0, \\ &u_2(x(w)) = w, \\ &v_2(w,0) = u_{2x}(x(w)). \end{aligned} \tag{2.14}$$

By condition (1.2), the factor

$$\left| \frac{g(u)}{\delta_2 u(1-u^2)} \right| \le C$$

for $u \in [-1,1]$. Comparing to the equation that the initial data $v(w,0)$ satisfies:

$$\begin{aligned} &u_x = -\frac{1-u^2}{\delta}, \\ &u(0) = 0, \\ &u(x(w)) = w, \\ &v(w,0) = u_x(x(w)). \end{aligned}$$

Choosing $\delta_2 > 0$ large enough, we see that when $w = u(x) = u(x_2)$, $0 > v_2(w,0) > v(w,0)$. Thus, we have

$$0 > v_2(w,0) > v_2(w,s) > v(w,s) > v(w,0) \tag{2.15}$$

and hence $0 > v_2(w, 0) \geq \theta(w)$. Furthermore, the equality

$$\frac{\partial x}{\partial w} = \frac{1}{u_x}$$

and inequalities (2.14) and the property of (2.10) and hence that of $v(w, 0)$ implies that

$$\int_0^1 \frac{1}{\theta(w)} dw < \int_0^w \frac{1}{v(w,0)} dw = -\infty.$$

The other half of (2.13) can be proved similarly. ∎.

Theorem 2.5. *If $f(u) = f(-u)$, then the solution $u(x, t)$ of (2.4) with initial data*

$$u(x, 0) = -\tanh(x/\delta) \quad (\text{ or } \tanh(x/\delta)) \tag{2.16}$$

with constant $\delta > 0$ small enough, converges to a stationary solution of (2.4).

Proof. again, we present the proof got the case where the initial data is

$$u(x, 0) = -\tanh(x/\delta).$$

The other case can be proved similarly.

From the definition (2.7) of the transformation $(x, t) \mapsto (u, s)$, we see that

$$\frac{\partial x}{\partial u} = \frac{1}{u_x}.$$

For the solution given in Corollary 3.2.4, we have

$$\frac{\partial x}{\partial u} = \frac{1}{\theta(u)} \tag{2.17}$$

or equivalently

$$\lim_{s\to\infty} x(u, s) = \lim_{s\to\infty} (x(u, s) - x(0, s)) = \int_0^u \frac{1}{\theta(u)} du. \tag{2.18}$$

The integral in (2.18) is regular for all $w \in (-1, 1)$ in view of Corollary 2.4. When $f(u) = f(-u)$, the solution of (2.4) with initial data (2.16) is antisymmetric about the point $x = 0$ and hence $x(0, s) \equiv 0$. Then (2.18) yields

$$x = \lim_{s\to\infty} x(u, s) = G(u). \tag{2.19}$$

Estimate (2.13) and $\theta(u) < 0$ guarantee that for each $x \in \mathbb{R}$, there is a $u(x) \in (-1, 1)$ satisfies (2.19). By definition (2.7), we have

$$u(x) = u(x(u(x), s), s) = u(x, s) + u_x(\eta, s)(x(u(x), s) - x)$$

for some η between x and $x(u(x), s)$. Since u_x is bounded as indicated by (2.14), we can take $s \to \infty$ to obtain

$$\lim_{s \to \infty} u(x, s) = u(x)$$

for all $x \in \mathbb{R}$. It is easy to see that this limit $u(x)$ must be a stationary solution of (2.4). ∎

Above theorem states that there is a stationary traveling wave of (2.4) if $f(u)$ is symmetric and g is antisymmetric about $u = 0$. Next, we shall prove that all traveling waves of (2.4) are stationary under this condition.

By straightforward calculation, we obtain the eigenvalues of the linearized equation of (2.4) at $u = \pm 1$: At $u = 1$, we have

$$\lambda_{\pm}(1) = \frac{1}{2A}[f'(1) - c \pm \sqrt{(f'(1) - c)^2 + 8A}], \tag{2.20}$$

while at $u = -1$,

$$\lambda_{\pm}(-1) = \frac{1}{2A}[f'(-1) - c \pm \sqrt{(f'(-1) - c)^2 + 8A}]. \tag{2.21}$$

We rewrite (2.3) as

$$\begin{aligned} &u' = p, \\ &Ap\frac{dp}{du} + g(u) = (-c + f'(u))p, \\ &p(u = 1) = p(u = -1) = 0. \end{aligned} \tag{2.22}$$

We can compute the value of dp/du at $u = \pm 1$ to obtain

$$\frac{dp}{du}\Big|_{u=1} = \lambda_{\pm}(1) \tag{2.23}$$

and

$$\frac{dp}{du}\Big|_{u=-1} = \lambda_{\pm}(-1) \tag{2.24.}$$

Theorem 2.6. *If $f(u) = f(-u)$ for $u \in \mathbb{R}$, then the speed of traveling waves of (2.4) is necessarily* 0.

Proof. For definiteness, we consider the case where $u_- = 1$ and $u_+ = -1$ in (2.3). The other case can be handled similarly.

For a solution of (2.3) to exist, it is necessary and sufficient that an unstable manifold of (2.3) issued from $u_- = 1$ intersect a stable manifold of (2.3) entering $u_+ = -1$. The slopes of these manifolds at $u = \pm 1$ are

$$\frac{dp}{du}\Big|_{u=1} = \lambda_+(1) = \frac{1}{2A}[f'(1) - c + \sqrt{(f'(1)-c)^2 + 8A}], \tag{2.25}$$

$$\frac{dp}{du}\Big|_{u=-1} = \lambda_-(-1) = \frac{1}{2A}[f'(-1) - c - \sqrt{(f'(-1)-c)^2 + 8A}], \tag{2.26}$$

respectively.

When $c = 0$, we know from Theorem 2.5 that an unstable manifold of issued from $u_- = 1$ intersects and hence coincides with a stable manifold entering $u_+ = -1$. Since the traveling wave given in Theorem 2.5 is monotone, the manifold is in the $p < 0$ half plane. We denote this manifold in $(u, p)-$plane as $\Gamma(0)$.

We see from (2.25) that the slope of unstable manifold from $(u = 1, u' = 0)$, denoted as $\Gamma_+(1, c)$, decreases as c increases. The slope of stable manifold, denoted as $\Gamma_-(-1, c)$, at $(u = -1, u' = 0)$ is also decreasing as c increases. If $c > 0$, then the stable manifold at $u = -1$ is below $\Gamma(0)$ near $u = -1$, while the unstable manifold at $u = 1$ is above $\Gamma(0)$ near $u = 1$. The opposite occurs when $c < 0$. Thus, for a connection between $u = -1$ and $u = 1$ with $c \neq 0$ to be possible, it is necessary that one of the following four cases holds, see Figure 2.1:

$$Figure 2.1$$

Case 1. $\Gamma_+(1, c)$ intersects $\Gamma(0)$.

Case 2. $\Gamma_+(1, c)$ crosses $u' = 0$ from above at $u > 1$.

Case 2 is impossible since at $(u > 1, u' = 0)$,

$$Au'' = -g(u) > 0$$

which prevents $\Gamma_+(1, c)$ to go below $u' = 0$.

Now, to prove this theorem, we only have to prove that Case 1 cannot happen. To this end, we assume the contrary, i.e. $\Gamma(0)$ and $\Gamma_+(1, c)$ intersect at some point (u^*, p^*). We denote $\Gamma(0)$ by $p_0(u)$ and $\Gamma_+(1, c)$ by $p_1(u)$. Then, we have the following equations from (2.22):

$$Ap_0 \frac{dp_0}{du} + g(p_0) = (-0 + f(u))p_0, \tag{2.27}$$

$$Ap_1 \frac{dp_1}{du} + g(p_1) = (-c + f(u))p_1, \tag{2.28}$$

we further use (u^*, p^*) to denote the point of intersection with the maximum u^*. Then at the point (u^*, p^*) and when $c > 0$, we have

$$p_0(u^*) = p_1(u^*) = p^* < 0 \tag{2.29}$$

and

$$\frac{dp_0}{du} \mid_{u=u^*} \leq \frac{dp_1}{du} \mid_{u=u^*} . \tag{2.30}$$

The difference of (2.27) and (2.28) reads

$$0 \leq Ap^* \left[\frac{dp_0}{du} \mid_{u=u^*} - \frac{dp_1}{du} \mid_{u=u^*} \right] = cp_1 < 0 \tag{2.31}$$

which is a contradiction. Similarly, we will also get a contradiction when $c < 0$. These contradictions completes the proof. ∎

Corollary 2.7. *If $f(u) = f(-u)$ and $g(u) = -g(-u)$, then solutions of (2.3) are anti-symmetric about a point $\xi = \xi_0$, i.e. $u(\xi - \xi_0) = -u(-(\xi - \xi_0))$.*

Proof. Assume the contrary, that is, there is a solution of (2.3) that is not anti-symmetric, and hence u' is not symmetric about any point ξ. Then there is a non-symmetric solution of (2.22), that is

$$p(u_0) \neq p(-u_0) \tag{2.32}$$

for some point $u_0 \in (-1, 1)$. We recall from last theorem that $c = 0$ in (2.22). Then we see that $p(-u)$ is also a solution of (2.22) under the assumptions on f and g. If (2.32) held, then $p(u)$ and $p(-u)$ would intersect at some point $u_1 \in (-1, 1)$. However, in our proof of last theorem, we see that any two manifolds of (2.22) of $(u = \pm 1, p = 0)$ either coincide or do not intersect on $(-1, 1)$ at all. This contradiction completes our proof. ∎

§3. Asymptotic Behavior of u

Here we consider the behavior of u away from a front when $\epsilon > 0$ is small through asymptotic expansion. To do this, we have to assume functions $f(u)$ and $g(u)$ are smooth enough. We observe that in the region where $u(\mathbf{x}, t, \epsilon) > \delta > 0$, the estimate $u(\mathbf{x}, t, \epsilon) =$

$1 - O(1)\exp(O(1)t/\epsilon)$ holds. This hints us to use the "fast variable" $\tau := t/\epsilon$ and start with the ansatz

$$u(\mathbf{x}, t, \epsilon) = v_0(\mathbf{x}, \tau) + \epsilon v_1(\mathbf{x}, \tau) + \epsilon^2 v_2(\mathbf{x}, \tau) + O(\epsilon^3). \tag{3.1}$$

Plugging (3.1) into (1.1) and equating the coefficients of ϵ^{-1}, 1 and ϵ to zero, we obtain

$$\partial_\tau v_0 = g(v_0), \tag{3.2a}$$

$$\partial_\tau v_1 + \nabla \cdot \mathbf{f}(v_0) = g'(v_0)v_1, \tag{3.3a}$$

$$\partial_\tau v_2 + \nabla \cdot (\mathbf{f}'(v_0)v_1) = g'(v_0)v_2 + \frac{1}{2}g''(v_0)v_1^2 + \Delta v_0. \tag{3.4a}$$

The corresponding initial datum for above equations are

$$v_0(\mathbf{x}, 0) = u(\mathbf{x}, 0), \tag{3.2b}$$

$$v_1(\mathbf{x}, 0) = 0, \tag{3.3b}$$

$$v_2(\mathbf{x}, 0) = 0, \tag{3.4b}$$

respectively. It is clear that (3.2), (3.3) and (3.4) are ordinary differential equations of v_0, v_1 and v_2 respectively and $\mathbf{x}$ acts only as a parameter.

Now, we study the behavior of $u(\mathbf{x}, t)$ as $\tau \to \infty$ for $\mathbf{x}$ away from fronts. If we divide $\mathbb{R}^n$ into connected components of $\{u(\mathbf{x}, 0) > 0\}$ and $\{u(\mathbf{x}, 0) \le 0\}$, then inside each such components, $v_0(\mathbf{x}, \tau) \to 1$ or -1, due to the assumption on $g(u)$, (1.2). Similarly, we can prove that v_1 and v_2 converge to zero exponentially as $\tau \to \infty$. From this analysis, we can imagine the picture of $u(\mathbf{x}, t, \epsilon)$ when $\epsilon > 0$ is small and $t > 0$: At any time $t > 0$, $\mathbb{R}^n$ is divided into subdomains inside which $u(\mathbf{x}, t, \epsilon)$ is close to 1 or -1. Near the boundaries of these subdomains, there are sharp layers, call them fronts, across which u changes from 1 to -1. Then, the characterization of the behavior of $u(\mathbf{x}, t, \epsilon)$ is reduced to that of these sharp layers.

In next section, we shall investigate the behavior of these fronts.

§4. Derivation of the Equation for the Front Motion

In this section, we shall formally derive the motion equation for fronts of (1.1) through asymptotic expansions.

It is clear that the ansatz (3.1) is no longer useful near the fronts since it requires some smoothness uniform in $\epsilon > 0$ to be valid. Thus, we shall introduce new ansatz near a front.

We consider the front Γ_0 in $\mathbb{R}^n$ that separates the subdomains $\{u(\mathbf{x}, 0) > 0\}$ and $\{u(\mathbf{x}, 0) < 0\}$. The equation of this front is $\phi_0(\mathbf{x}) = 0$. The front at time t evolved from Γ_0 is denoted as $\Gamma(t)$. Suppose that the front $\Gamma(t)$ can be represented by a function $\phi(\mathbf{x}, t, \eta)$ with

$$\eta = \epsilon t \tag{4.1}$$

and

$$\phi(\mathbf{x}, 0, \eta) = \phi_0(\mathbf{x}). \tag{4.2}$$

The layer around $\Gamma(t)$ is expected to be of width ϵ. Thus we introduce the stretched spatial variable

$$y := \epsilon^{-1}\phi(\mathbf{x}, t, \eta). \tag{4.3}$$

We introduce the following ansatz for $u(\mathbf{x}, t, \epsilon)$ around the front $\Gamma(t)$:

$$u(\mathbf{x}, t, \epsilon) = u_0(y, \mathbf{x}, \tau, t, \eta, \epsilon) + \epsilon u_1(y, \mathbf{x}, \tau, t, \eta, \epsilon) + O(\epsilon^2). \tag{4.4}$$

Here $\tau = t/\epsilon$ is defined in §3. Putting (4.4) into (1.1) and equating the coefficients of ϵ^{-1} and ϵ^0, we obtain

$$\partial_\tau u_0 + \phi_t \partial_y u_0 + (\nabla\phi \cdot \mathbf{f}'(u_0))\partial_y u_0 = g(u_0) + (\nabla\phi)^2 \partial_y^2 u_0, \tag{4.5}$$

$$\begin{aligned} &\partial_\tau u_1 + (\phi_t + \nabla\phi \cdot \mathbf{f}'(u_0))\partial_y u_1 \\ &\quad + (\nabla\phi \cdot \mathbf{f}''(u_0)\partial_y u_0 - g'(u_0))u_1 - (\nabla\phi)^2 \partial_y^2 u_1 \\ &\quad = -\partial_t u_0 - \nabla \cdot \mathbf{f}(u_0) + 2\nabla(\partial_y u_0) \cdot \nabla\phi + \nabla^2\phi \partial_y u_0 - \phi_\eta \partial_y u_0. \end{aligned} \tag{4.6}$$

Equation (4.5) is a parabolic partial differential equations with variables $\tau > 0$ and $y \in \mathbb{R}^n$ with parameters $\mathbf{x}, t, \eta$. To uniquely determine u_0, we need the initial data $u_0(y, \mathbf{x}, \tau =$

$0, t, \eta, \epsilon)$. Consider the coordinate system $\mathbf{x} = (\xi, z)$ where ξ is the signed distance from the point $\mathbf{x}$ to the $n-1$ dimensional surface Γ_0 and z is the $n-1$ dimensional coordinate system on Γ_0. According to the definition of y, (4.3), we have

$$\epsilon y = \phi_0(\mathbf{x}) = \phi_0(\xi, z) = \phi(0, z) + \xi \partial_\xi \phi_0(0, z) + O(\xi^2) = \xi \partial_\xi \phi_0(0, z) + O(\xi^2) \tag{4.7}$$

where in the last step, we used that along Γ_0, $\phi_0(\mathbf{x}) = \phi_0(0, z) = 0$. Let initial data of (1.1) be $u(\mathbf{x}, 0, \epsilon) = a(\mathbf{x})$. Then from (4.4) and (4.7) we have the initial data for u_0:

$$u_0(y, 0, z, \tau = 0, t, \eta, \epsilon) = a(0, z) + \epsilon y \frac{\partial_\xi a(0, z)}{\partial_\xi \phi_0(0, z)}. \tag{4.8}$$

Similar statements hold for u_1.

When $\tau \to \infty$, it is expected that the solution u_0 of (4.5) approach to a traveling wave. So, we assume that

$$u_0(y, \mathbf{x}, \tau, t, \eta, \epsilon) \sim \psi(y - c\tau, \mathbf{x}, t, \eta) \tag{4.9}$$

where ψ is a traveling wave of (4.5) with speed c. The function ψ satisfies the traveling wave equation of (4.5):

$$\begin{aligned} &(\phi_t - c + \nabla\phi \cdot f'(\psi))\psi' = g(\psi) + (\nabla\phi)^2 \psi'', \\ &\psi(\pm\infty) = u_\pm \end{aligned} \tag{4.10}$$

where $u_\pm$ are the two stable equilibria of (1.1).

In Section 2, we proved that the traveling wave equation

$$\begin{aligned} &(-s + F'(\psi))\psi' = g(\psi) + A\psi'', \\ &\psi(\pm\infty) = u_\pm \end{aligned} \tag{4.11}$$

has a solution when $F(u)$ is symmetric and $g(u)$ is anti-symmetric about $u = 0$. The speed s must be zero and the solution is anti-symmetric around a point $\xi = \xi_0$. Apply this result to (4.10), we see that

$$c - \phi_t = 0 \tag{4.12}$$

and hence

$$\begin{aligned} &(\nabla\phi \cdot f'(\psi))\psi' = g(\psi) + (\nabla\phi)^2 \psi'', \\ &\psi(\pm\infty) = u_\pm. \end{aligned} \tag{4.13}$$

From the definition of y, (4.3), we can see that a fixed point in $\mathbf{x}$-coordinate system travels at a speed ϕ_t/ϵ relative to the y-coordinate system. The traveling wave of (4.5) travels at a speed c/ϵ relative to the y-coordinate system. Then the speed of this traveling wave relative to the $\mathbf{x}$-coordinate system is $(c-\phi_t)/\epsilon$ which is 0 according to (4.12). In other words, the front $\Gamma(t)$ is the level curve

$$\Phi := \phi(\mathbf{x}, t, \epsilon) - ct = 0,$$

which does not move on the t time scale. Furthermore, (4.12) shows that $\nabla\phi$ is independent of t and therefore the traveling wave equation (4.10) and hence (4.13) are independent of t: $\psi = \psi(\xi, \mathbf{x})$.

Now, we shall find the motion of the front on the η time scale: To this end, we consider the equation (4.6) for the next order of approximation u_1. We assume that as $\tau \to \infty$, u_1 approaches a traveling wave of (4.6):

$$u_1 \to \omega(y - c\tau, \mathbf{x}, t, \eta). \tag{4.14}$$

The traveling wave equation of (4.6) is

$$\begin{aligned}&(-c + \phi_t + \nabla\phi \cdot \mathbf{f}'(\psi) - g'(\psi))\omega' + (\nabla\phi \cdot \mathbf{f}''(\psi))\psi'\omega - (\nabla\phi)^2\omega'' \\ &= -\nabla \cdot \mathbf{f}(\psi) + 2\nabla\psi' \cdot \nabla\phi + \psi'\nabla^2\phi - \psi'\phi_\eta.\end{aligned} \tag{4.15}$$

Now, we use $\phi_t - c = 0$ and (4.10) to obtain

$$\begin{aligned}&[(\nabla\phi \cdot \mathbf{f}')\omega]' - g'(\psi)\omega - (\nabla\phi)^2\omega'' \\ &= -\nabla \cdot \mathbf{f}(\psi) + 2\nabla\psi' \cdot \nabla\phi + \psi'\nabla^2\phi - \psi'\phi_\eta.\end{aligned} \tag{4.16}$$

We consider the adjoint equation for the left hand side of (4.16):

$$(\nabla\phi)^2\psi_1'' + (\nabla\phi \cdot \mathbf{f}'(\psi)\psi_1' + g'(\psi)\psi_1 = 0. \tag{4.17}$$

It is closely related to the equation for ψ, (4.13). Taking derivative on (4.13), we get

$$(\nabla\phi)^2(\psi_2)'' - ((\nabla\phi \cdot \mathbf{f}'(\psi)\psi_2)' + g'(\psi)\psi_2 = 0 \tag{4.18}$$

with $\psi_2 = \psi'$. It is the adjoint equation of (4.17). Since ψ is monotone, the solution $\psi_2 = \psi'$ of (4.18) satisfies $\psi_2 \neq 0$. We shall see from Theorem 6.1 in §6 later that the general solution of (4.17) is

$$\psi_1 = \psi' \left[C_2 \exp\left(\int_0^\xi \frac{-\nabla\phi \cdot f'(\psi)}{|\nabla\phi|^2} d\theta \right) + \int_0^\xi \frac{C_1}{|\nabla\phi|^2\psi'^2} \exp\left(\int_{\theta_1}^\xi \frac{-\nabla\phi \cdot f'(\psi)}{|\nabla\phi|^2} d\theta_2 \right) d\theta_1 \right]$$

where C_1 and C_2 are any constants. We choose

$$\psi_1 = \psi' \exp\left(\int_0^\xi \frac{-\nabla\phi\cdot f'(\psi)}{|\nabla\phi|^2}d\theta\right) =: \psi'(\xi)W(\xi). \tag{4.19}$$

A straightforward calculation on the decay rate of ψ, the solution of (4.13), and that of (4.19) at $\xi = \pm\infty$ shows that $\psi'\psi_1$ decays at $\xi = \pm\infty$ exponentially. We multiply (4.16) by ψ_1 given in (4.19) and integrate by parts on the left hand side, we obtain

$$\begin{aligned} \phi_\eta &= \nabla^2\phi + \frac{\int_{-\infty}^{\infty}[-\nabla\cdot\mathbf{f}(\psi) + 2\nabla\phi\cdot\nabla\psi']\psi_1 d\xi}{\int_{-\infty}^{\infty}\psi'\psi_1 d\xi} \\ &= \nabla^2\phi + \frac{\int_{-\infty}^{\infty}[-\nabla\cdot\mathbf{f}(\psi)\psi' W - \nabla\phi\cdot\psi'^2\nabla W]d\xi}{\int_{-\infty}^{\infty}\psi'^2 W d\xi} + \nabla\phi\cdot\frac{\nabla\int_{-\infty}^{\infty}\psi'^2 W d\xi}{\int_{-\infty}^{\infty}\psi'^2 W d\xi}. \end{aligned} \tag{4.20}$$

To further clarify the dependence of ψ and ψ_1 on $\mathbf{x}$, we perform the transformation

$$\zeta := \frac{\xi}{|\nabla\phi|} \tag{4.21}$$

in (4.13) to yield

$$\mathbf{n}\cdot\mathbf{f}(\psi)_\zeta = g(\psi) + \psi_{\zeta\zeta} \tag{4.22}$$

where

$$\mathbf{n} := \frac{\nabla\phi}{|\nabla\phi|}. \tag{4.23}$$

is the unit normal vector of level curves of ϕ. We do the same on (4.17) to get

$$\mathbf{n}\cdot\mathbf{f}'(\psi))\psi_{1\zeta} + g'(\psi)\psi_1 + \psi_{1\zeta\zeta} = 0. \tag{4.24}$$

From (4.22-24), we can see that ψ and ψ_1 only depend on ζ and $\mathbf{n}(\mathbf{x})$. In fact, the function $\psi(\zeta, \mathbf{n}(\mathbf{x}))$ is the traveling wave of (1.1) in the direction $\mathbf{n}$. In this new variable, ψ_1 in (4.19) has the form

$$\psi_1 = \frac{1}{|\nabla\phi|}\psi_\zeta(\zeta,\mathbf{n})W(\zeta,\mathbf{n}) \tag{4.25a}$$

where

$$W = \exp\left(-\mathbf{n}\cdot\int_0^\zeta \mathbf{f}'(\psi(\theta,\mathbf{n}))d\theta\right). \tag{4.25b}$$

We note that the gradient ∇ in (4.18) is $\nabla_{\mathbf{x}}$ with ξ fixed, denoted more precisely by $\nabla\big|_{\xi}$. If we want to use the new variable $(\zeta, \mathbf{x}) = (\xi/|\nabla\phi|, \mathbf{x})$ in (4.18), we have to use the chain rule

$$
\begin{aligned}
\nabla\big|_{\xi} v &= \nabla\big|_{\zeta} v + \nabla\big|_{\xi} (\zeta) \partial_\zeta v \\
&= \nabla\big|_{\zeta} v + \zeta|\nabla\phi|\nabla_{\mathbf{x}} \left(\frac{1}{|\nabla\phi|}\right) \partial_\zeta v
\end{aligned} \tag{4.26}
$$

in (4.20). This leads us to

$$
\begin{aligned}
\phi_\eta &= \nabla^2\phi - \nabla\phi \cdot \frac{\nabla|\nabla\phi|}{|\nabla\phi|} + \nabla\phi \cdot \frac{\nabla \int_{-\infty}^{\infty} \psi_\zeta^2 W d\zeta}{\int_{-\infty}^{\infty} \psi_\zeta^2 W d\zeta} \\
&- \frac{\nabla\phi}{\int_{-\infty}^{\infty} \psi_\zeta^2 W d\zeta} \cdot \int_{-\infty}^{\infty} \psi_\zeta^2 \left[\nabla W + |\nabla\phi|\nabla\left(\frac{1}{|\nabla\phi|}\right) W_\zeta \zeta\right] d\zeta \\
&- \frac{|\nabla\phi|}{\int_{-\infty}^{\infty} \psi_\zeta^2 W d\zeta} \int_{-\infty}^{\infty} \psi_\zeta W \left[\nabla\cdot\mathbf{f} + |\nabla\phi|\nabla\left(\frac{1}{|\nabla\phi|}\right)\cdot \mathbf{f}(\psi)_\zeta \zeta\right] d\zeta.
\end{aligned} \tag{4.27}
$$

We note that the first two terms on the right hand side of (4.27) is

$$
|\nabla\phi|\nabla\cdot\frac{\nabla\phi}{|\nabla\phi|} = |\nabla\phi|\kappa_\phi
$$

where κ_ϕ is the mean curvature of the level curve of ϕ. Therefore, the equation of motion for ϕ, (4.27), takes the form

$$
\begin{aligned}
\phi_\eta &= |\nabla\phi|\kappa + \nabla\phi \cdot \frac{\int_{-\infty}^{\infty} (\nabla\psi_\zeta^2) W d\zeta}{\int_{-\infty}^{\infty} \psi_\zeta^2 W d\zeta} + \mathbf{n}\cdot\nabla(|\nabla\phi|) \frac{\int_{-\infty}^{\infty} \psi_\zeta^2 W_\zeta \zeta d\zeta}{\int_{-\infty}^{\infty} \psi_\zeta^2 W d\zeta} \\
&- \frac{|\nabla\phi|}{\int_{-\infty}^{\infty} \psi_\zeta^2 W d\zeta} \int_{-\infty}^{\infty} \psi_\zeta W \left[\nabla\cdot\mathbf{f} + |\nabla\phi|\nabla\left(\frac{1}{|\nabla\phi|}\right)\cdot \mathbf{f}(\psi)_\zeta \zeta\right] d\zeta.
\end{aligned} \tag{4.28}
$$

When $\mathbf{f} = 0$, the function ψ determined by (4.22) is independent of $\mathbf{x}$, and $W = 1$. Then the equation (4.28) reduces to the well known motion by mean curvature equation

$$
\phi_\eta = |\nabla\phi|\kappa_\phi \tag{4.29}
$$

when $\mathbf{f} = 0$.

§5. Some Special Cases Where (4.22) and (4.24) Have Exact Solutions

To investigate the behavior of the front propagation equation (4.28) further, it is better that we have exact solutions of (4.22) and (4.24) for some special $\mathbf{f}(u)$ and $g(u)$. Here, we assume

$$g(u) = au(1-u^2) \tag{5.1}$$

and

$$\mathbf{f}(u) = \mathbf{b}u^2 \tag{5.2}$$

where a is a constant and $\mathbf{b} \in \mathbb{R}^n$ is a constant vector. For definiteness, we shall assume

$$a \geq 0. \tag{5.3}$$

Then, equations (4.22) takes the form

$$\begin{aligned} &2b\psi\psi' = a\psi(1-\psi^2) + \psi'', \\ &\psi(\pm\infty) = \pm 1. \text{ (or } \quad \mp 1) \end{aligned} \tag{5.4a}$$

where

$$b := \mathbf{n} \cdot \mathbf{b}. \tag{5.4b}$$

It is easy to check that

$$\psi(\zeta) = \tanh(\delta\zeta) \tag{5.5a}$$

with

$$\delta := \frac{1}{2}(-b \pm \sqrt{b^2 + 2a}) \tag{5.5b}$$

are solutions of (5.4).

Under the choice of (5.1) and (5.2) for $\mathbf{f}$ and g, equation (4.24) becomes

$$\psi_1'' + 2b\psi\psi_1 + g'(\psi)\psi_1 = 0. \tag{5.6}$$

The solution of (5.6) chosen for deriving (4.28) is

$$\psi_1 = \psi' W \tag{5.7}$$

with

$$W(\zeta) = \exp\left(-2b\int_0^\zeta \psi(\theta)d\zeta\right). \tag{5.8}$$

From (5.5), we find that

$$W(\zeta) = \text{sech}^{2b/\delta}(\delta\zeta). \tag{5.9}$$

Now, we plugging the expressions (5.9) and (5.5) into (4.28) to see what (4.28) becomes under assumptions (5.1) and (5.2). We write (4.28) as

$$\phi_\eta = |\nabla\phi|\kappa + II + III + IV. \tag{5.10}$$

We compute the term II

$$II := \nabla\phi \cdot \frac{\int_{-\infty}^{\infty}(\nabla\psi_\zeta^2)Wd\zeta}{\int_{-\infty}^{\infty}\psi_\zeta^2 Wd\zeta} \tag{5.11}$$

as follows:

$$\begin{aligned}\int_{-\infty}^{\infty}\psi_\zeta^2 Wd\zeta &= \delta^2\int_{-\infty}^{\infty}\text{sech}^{4+2b/\delta}(\delta\zeta)d\zeta\\ &= \delta\int_{-\infty}^{\infty}\text{sech}^{4+2b/\delta}(\theta)d\theta,\end{aligned} \tag{5.12}$$

where $\theta = \delta\zeta$,

$$\begin{aligned}&\int_{-\infty}^{\infty}(\nabla\psi_\zeta^2)Wd\zeta\\ &= 2\nabla\delta\int_{-\infty}^{\infty}(\text{sech}^{4+2b/\delta}\theta - 2\theta\text{sech}^{4+2b/\delta}\theta\tanh\theta)d\theta\\ &= 2\nabla\delta\int_{-\infty}^{\infty}\text{sech}^{4+2b/\delta}\theta d\theta + 4\nabla\delta\int_{-\infty}^{\infty}\theta\text{sech}^{3+2b/\delta}\theta d(\text{sech}\theta)\\ &= \frac{2\delta+2b}{2\delta+b}\int_{-\infty}^{\infty}\text{sech}^{4+2b/\delta}\theta d\theta.\end{aligned} \tag{5.13}$$

Hence we have

$$II = \frac{2\delta+2b}{2\delta+b}\nabla\phi\cdot\frac{\nabla\delta}{\delta}. \tag{5.14}$$

Similarly, we can obtain

$$III := \frac{\nabla\phi}{|\nabla\phi|}\cdot\nabla(|\nabla\phi|)\frac{\int_{-\infty}^{\infty}\psi_\zeta^2 W_\zeta \zeta d\zeta}{\int_{-\infty}^{\infty}\psi_\zeta^2 W d\zeta} = -\frac{b}{2\delta+b}\frac{\nabla\phi}{|\nabla\phi|}\cdot\nabla(|\nabla\phi|), \tag{5.15}$$

and

$$\begin{aligned} IV &:= -\frac{|\nabla\phi|}{\int_{-\infty}^{\infty}\psi_\zeta^2 W d\zeta}\int_{-\infty}^{\infty}\psi_\zeta W\left[\nabla\cdot\mathbf{f} + |\nabla\phi|\nabla\left(\frac{1}{|\nabla\phi|}\right)\cdot\mathbf{f}(\psi)_\zeta\zeta\right]d\zeta \\ &= -\frac{|\nabla\phi|}{\int_{-\infty}^{\infty}\psi_\zeta^2 W d\zeta}\int_{-\infty}^{\infty}\zeta\psi_\zeta^2 W f'(\psi)\cdot\left(\frac{\nabla\delta}{\delta} - \frac{\nabla|\nabla\phi|}{|\nabla\phi|}\right)d\zeta \\ &= -\frac{|\nabla\phi|}{2\delta+b}\left(\frac{\nabla\delta}{\delta} - \frac{\nabla|\nabla\phi|}{|\nabla\phi|}\right)\cdot\mathbf{b}. \end{aligned} \tag{5.16}$$

Plugging (5.14-16) into (5.10), we obtain

$$\frac{\phi_\eta}{|\nabla\phi|} = \kappa + \frac{\nabla\delta}{\delta}\cdot\left(\frac{(2\delta+2b)\mathbf{n}-\mathbf{b}}{2\delta+b}\right) - \frac{\nabla|\nabla\phi|}{|\nabla\phi|}\cdot\left(\frac{b\mathbf{n}-\mathbf{b}}{2\delta+b}\right) \tag{5.17}$$

With the notation

$$\mathbf{T} := \frac{\mathbf{b}-b\mathbf{n}}{2\delta+b} \tag{5.18}$$

equation (5.17) becomes

$$\frac{\phi_\eta}{|\nabla\phi|} = \kappa + \frac{\nabla\delta}{\delta}\cdot(\mathbf{n}-\mathbf{T}) + \frac{\nabla|\nabla\phi|}{|\nabla\phi|}\cdot\mathbf{T} \tag{5.19}$$

Recalling that $b = \mathbf{n}\cdot\mathbf{b}$ and $||\mathbf{n}|| = 1$, we see that $\mathbf{n}\cdot\mathbf{T} = 0$ We recall that $\mathbf{n} = \nabla\phi/|\nabla\phi|$ is the unit normal of the level curve $\phi = C$, Thus, $\mathbf{T}$ is in the tangent direction of the level curve. We claim that

$$\frac{\nabla\delta}{\delta}\cdot\mathbf{n} + \frac{\nabla|\nabla\phi|}{|\nabla\phi|}\cdot\mathbf{T} = 0 \tag{5.20}$$

Indeed, we further compute $\nabla\delta$ defined in (5.5b) to get

$$\frac{-1}{2\delta+b}\nabla b.$$

In the rest of this section, we use the convention that summation is taken for repeated indeces. Using (4.23), (5.4b) and (5.18) in (5.20) we obtain

$$\begin{aligned} &= \frac{-1}{2\delta+b}\nabla b\cdot\mathbf{n} - \frac{\nabla|\nabla\phi|}{|\nabla\phi|}\cdot\frac{\mathbf{b}-b\mathbf{n}}{2\delta+b} \\ &= \frac{-|\nabla\phi|^{-1}}{2\delta+b}[n_k b_j(\phi_{x_jx_k} - n_j n_l\phi_{x_kx_l}) + n_j\phi_{x_jx_k}(b_l n_l n_k - b_k)] \\ &= \frac{-|\nabla\phi|^{-1}}{2\delta+b}[n_k b_j\phi_{x_jx_k} - b_k n_j\phi_{x_jx_k} - n_k b_j n_j n_l\phi_{x_kx_l} + n_j b_l n_l n_k\phi_{x_jx_k}] \\ &= 0 \end{aligned} \tag{5.21}$$

as desired. Thus, we can further simplify the equation (5.19) as

$$\frac{\phi_\eta}{|\nabla\phi|} = \kappa + \frac{\nabla\delta^{-1}}{\delta^{-1}} \cdot \mathbf{T} \tag{5.22}$$

Recalling (5.5), we see that δ^{-1} is the width of the traveling wave of (1.1) in the $\mathbf{n}$ direction. Then the laster term in the equation (5.22) represents the relative rate of change of the thickness of the front in the tangential direction $\mathbf{T}$ of the level curve ϕ =constant.

Since the original equation (1.1) is of parabolic type, we expect the equation (5.19) for its front is also of parabolic type. We prove this in the following theorem:

Theorem 5.1.
(i) The operator

$$\kappa = \nabla \cdot \frac{\nabla\phi}{|\nabla\phi|}$$

in (5.22) is elliptic.
(ii) The operator

$$\frac{\nabla\delta^{-1}}{\delta^{-1}} \cdot \mathbf{T}$$

in (5.22) is elliptic.

Proof. (i) Although κ is well known to be an elliptic operator, we still present the proof since we will use it later. We compute κ to get

$$\begin{aligned}\kappa &= \frac{|\nabla\phi|\nabla^2\phi - \nabla\phi \cdot \nabla|\nabla\phi|}{|\nabla\phi|^2} \\ &= \frac{|\nabla\phi|\nabla^2\phi - \phi_{x_j}\phi_{x_k}\phi_{x_k x_j}/|\nabla\phi|}{|\nabla\phi|^2}.\end{aligned}$$

We replace $\phi_{x_j x_k}$ in the above by $\lambda_j\lambda_k$, use $\mathbf{n} = \nabla\phi/|\nabla\phi|$ and apply Schwaltz inequality, then the operator κ becomes

$$\frac{\lambda_j\lambda_j - n_j n_k \lambda_j \lambda_k}{|\nabla\phi|} = |\nabla\phi|^{-1}((\Lambda,\Lambda) - (\Lambda,\mathbf{n})^2) \geq 0 \tag{5.23}$$

where Λ is the vector $(\lambda_1, \lambda_2, ..., \lambda_n)$. The equality in (5.23) holds if and only if $\Lambda = ||\Lambda||\mathbf{n}$. This shows that κ is an elliptic operator.

(ii) We further compute $\nabla\delta$ defined in (5.5b) to get

$$\begin{aligned}\frac{\nabla\delta}{\delta} \cdot \mathbf{T} &= \frac{-1}{2\delta + b}\nabla b \cdot \mathbf{T} \\ &= \frac{-1}{2\delta + b} T_k b_j \partial_{x_k} n_j \\ &= \frac{-1}{2\delta + b} T_k b_j (\phi_{x_j x_k} - n_j n_l \phi_{x_l x_k})|\nabla\phi|^{-1}.\end{aligned} \tag{5.24}$$

Replacing $\phi_{x_j x_k}$ by $\lambda_j \lambda_k$ and using (5.18) in (5.24), we see that (5.24) becomes

$$\begin{aligned}
&\left(\frac{-1}{2\delta+b}\right)^2 |\nabla\phi|^{-1} b_j(b_k - b_i n_i n_k)((\lambda_j\lambda_k - n_j n_l \lambda_l \lambda_k) \\
&= \left(\frac{-1}{2\delta+b}\right)^2 |\nabla\phi|^{-1} (b_k\lambda_k - b_i n_i n_k \lambda_k)((\lambda_j b_j - n_j b_j n_l \lambda_l) \\
&= \left(\frac{-1}{2\delta+b}\right)^2 |\nabla\phi|^{-1} ((\mathbf{b}\cdot\Lambda) - (\mathbf{b}\cdot\mathbf{n})(\mathbf{n}\cdot\Lambda))^2 \geq 0
\end{aligned} \tag{5.25}$$

with equality holds if and only if parts of $\mathbf{b}$ and Λ perpendicular to $\mathbf{n}$ are perpendicular to each other: $(\mathbf{b}-(\mathbf{b}\cdot\mathbf{n})\mathbf{n})\cdot(\Lambda-(\Lambda\cdot\mathbf{n})\mathbf{n}) = 0$. The inequality (5.25) completes the proof. ∎

Remark: Equation (5.22) is degenerate since when $\mathbf{n} = \Lambda/||\Lambda||$, both (5.25) and (5.23) are 0. xxx

Remark: From above theorem, we see that the convection term of (1.1), $\nabla\cdot\mathbf{f}(u)$, also contributes to the parabolicity of the front motion equation.

Now, we consider the evolution of a front that is a circle in $\mathbb{R}^2$ at initial time under the equation (5.22). Take $\phi(x,y,0) = x^2/2 + y^2/2 - r^2$ as the initial data where $r > 0$ is a constant. We can calculate to get that

$$\kappa = \frac{1}{|\nabla\phi|} > 0, \tag{5.26}$$

and

$$-\frac{\nabla\delta}{\delta}\cdot\mathbf{T} = \frac{1}{(2\delta+b)^2}|\nabla\phi|^{-1}(\mathbf{b}\cdot\mathbf{b} - (\mathbf{b}\cdot\mathbf{n})^2) \geq 0. \tag{5.27}$$

We see from (5.22) that $\phi_\eta(x,y,0) > 0$. We then expect that the front represented by $\phi = 0$ shrinks as η increases. We also see that the circular front shrinks fastest when the normal direction of the front $\mathbf{n}$ is perpendicular to the vector $\mathbf{b}$ in the function $\mathbf{f}$, (5.2), and slowest when $\mathbf{n}$ is paralell to $\mathbf{b}$. This suggests that some elliptic front may preserve its shape as it shrinks. The following example shows that it is indeed the case.

Example: Assume $\mathbf{b} = (b_1, 0)$. Let the initial data for (5.22) be

$$\phi(x,y,0) = \frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - r_0^2. \tag{5.28}$$

If

$$a_1 = a_2\sqrt{\frac{2a+b_1^2}{2a}}, \tag{5.29}$$

then the solution of (5.22) is

$$\phi(x,y,\eta) = \frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - (r(\eta))^2, \tag{5.30}$$

where

$$r(\eta) = \sqrt{-a_1^2 a_2^{-2}\eta + r_0^2}. \tag{5.31}$$

Thus, the front which is at $\phi = 0$ will maintain the shape of the ellipse of (5.30) while shrinking as indicated by (5.31). At the rescaled time $\eta = r_0^2 a_1^{-2} a_2^2$, the ellips will shrink to a point and then disappear.

Proof. By straightforward calculation, using the ansatz (5.30), we see that

$$\kappa = \frac{1}{a_1^2 a_2^2 |\nabla\phi|^3}\left(\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2}\right), \tag{5.32}$$

$$\begin{aligned} -\frac{\nabla\delta}{\delta}\cdot\mathbf{T} &= \frac{1}{(2\delta+b)^2|\nabla\phi|}(b_k - (\mathbf{b}\cdot\mathbf{n})n_k)b_j(\phi_{x_j x_k} - n_j n_l \phi x_l x_k) \\ &= \frac{1}{(2\delta+b)^2|\nabla\phi|}(b_k - (\mathbf{b}\cdot\mathbf{n})n_k)b_j(\delta_{jk}a_j^{-2} - n_j n_l \delta_{lk} a_k^{-2}) \\ &= \frac{1}{(2\delta+b)^2|\nabla\phi|}a_k^{-2}(b_k - (\mathbf{b}\cdot\mathbf{n})n_k)^2 \\ &= \frac{b_1^2 y^2}{(2\delta+b)^2 a_1^2 a_2^6 |\nabla\phi|^5}\left(\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2}\right), \end{aligned} \tag{5.33}$$

To determine $r(\eta)$, we plug (5.33), (5.32) into (5.22) and use (5.5b) to obtain

$$\begin{aligned} \phi_\eta &= \frac{1}{a_1^2 a_2^2 |\nabla\phi|^2}\left[1 + \frac{b_1^2 y^2}{|\nabla\phi|^2(b^2 n_1 + 2a)a_2^4}\right]\left(\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2}\right) \\ &= \frac{b^2+2a}{a_1^2 a_2^2}\frac{1}{\frac{(2a+b^2)x^2}{a_1^4} + \frac{2ay^2}{a_2^4}}\left(\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2}\right). \end{aligned} \tag{5.34}$$

When (5.29) holds, above equality yields that

$$\phi_\eta = -(r^2)_\eta = a_1^2 a_2^{-2},$$

and hence

$$r(\eta) = \sqrt{-a_1^2 a_2^{-2}\eta + r_0^2}$$

as desired. ∎

§6. Solutions of (4.24)

In this section, we shall find solutions of (4.24). This can be described in a broader context of the relation between solutions of the boundary value problem

$$(p_0(\xi)v')' + (p_1(\xi)v)' + p_2(\xi)v = 0, \tag{6.1}$$

and that of

$$(p_0(\xi)w')' - p_1(\xi)w' + p_2(\xi)w = 0. \tag{6.2}$$

We see that the equation (4.17) and (4.18), (4.24) and derivative of (4.22) are special cases of (6.2) and (6.1).

Suppose we already know a special solution of (6.1), denoted as $v_0(\xi)$. Assume this solution satisfies

$$v_0(\xi) \neq 0. \tag{6.3}$$

We let

$$w(\xi) = v_0(\xi)W(\xi). \tag{6.4}$$

This way, we can eliminate p_2 in (6.2):

$$\begin{aligned} 0 &= (p_0(\xi)w')' - p_1(\xi)w' + p_2(\xi)w \\ &= [(p_0v_0')' + p_2v_0 - p_1v_0']W + [2p_0v_0' + p_0'v_0 - p_1v_0]W' + p_0v_0W'' \\ &= [-(p_1v_0)' - p_1v_0']W + [2p_0v_0' + p_0'v_0 - p_1v_0]W' + p_0v_0W'' \\ &= v_0(p_0W' - p_1W)' + 2v_0'(p_0W' - p_1W) \end{aligned} \tag{6.5}$$

or simply

$$v_0(p_0W' - p_1W)' + 2v_0'(p_0W' - p_1W) = 0. \tag{6.6}$$

Integrating the equation (6.6) once to get

$$p_0W' - p_1W = C_1v_0^{-2}$$

and hence

$$W(\xi) = C_2 \exp\left(\int_0^\xi \frac{p_1}{p_0}d\eta\right) + C_1\int_0^\xi p_0^{-1}(\eta)v_0^{-2}(\eta)\exp\left(\int_\eta^\xi \frac{p_1}{p_0}d\zeta\right). \tag{6.7}$$

We summarize above result in the following theorem:

Theorem 6.1. *Assume that (6.1) has a solution v_0 satisfying $v_0' \neq 0$. Then the general solution of (6.2) is*

$$w(\xi) = v_0 W(\xi) \tag{6.8}$$

where $W(\xi)$ is given by (6.7).

Proof. We already see that the function given by (6.7) are solutions of (6.2) for any constants C_1 and C_2. It remains to prove that any solution of (6.2) can be expressed by (6.7) with suitable constants C_1 and C_2. Since solutions of (6.2) are uniquely determined by initial data $w(0)$ and $w'(0)$, it suffices to prove that for any given initial data $w(0)$ and $w'(0)$, there are constants C_1 and C_2 such that the $w(\xi)$ given by (6.8) satisfies the initial data. It is clear from (6.7) and (6.8) that $w(0) = v_0(0)C_2$ or $C_2 = w(0)/v_0(0)$. Similarly, we see that C_1 can also be determined by $w(0)$ and $w'(0)$. This completes the proof. ∎

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