ON THE PIECEWISELY SMOOTH SOLUTIONS TO NON-HOMOGENEOUS SCALAR CONSERVATION LAWS*

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Abstract. We study the structure and smoothness of non-homogeneous convex conservation laws. The question regarding the number of smoothness pieces is addressed. It is shown that under certain conditions on the initial data the entropy solution has only finite number of discontinuous curves. We also obtain some global estimates on derivatives of the piecewisely smooth entropy solution along the generalized characteristics. These estimates play important roles in obtaining the optimal rate of convergence for various approximation methods to conservation laws.

1. Introduction

We consider the initial value problem for the non-homogeneous scalar conservation law

$$\partial_t u + \partial_x f(u) + g(u) = 0 \quad (x,t) \in \mathbf{R} \times \mathbf{R}^+ \tag{1.1}$$

subject to the piecewise smooth initial condition

$$u(x,0) = u_0(x) \quad x \in \mathbf{R}, \tag{1.2}$$

where f and g are smooth, the flux f is strictly convex

$$f'' \geq \gamma > 0 \tag{1.3}$$

and g satisfies $g(0) = 0$ and a Lipschitz condition with a Lipschitz constant L ($L > 0$), i.e.,

$$|g(u) - g(v)| \leq L|u - v|. \tag{1.4}$$

In general, the problem (1.1)-(1.2) does not possess a global smooth solution even if the initial value is C^∞ smooth. The structure of entropy solutions has been studied by many authors, e.g. Dafermos [2], Lax [7], Oleinik [10] and Schaeffer [13]. The entropy solutions consisting of finite number of shock or rarefaction discontinuities form an important solution class. They are the only solutions that can be computed numerically. Therefore, it is useful to study the solution structure, and in particular to identify the initial conditions for which

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a finite number of shock curves is generated. The first result concerning the size of the shock set was due to Oleinik. In the special case $g \equiv 0$, there is a connection between (1.1) and the Hamilton-Jacobi equation which induces an explicit representation of solutions. Using this representation, Oleinik [10, 11] shows that solutions of (1.1) with $g \equiv 0$ are continuous except on the union of an at most countable shock set. Analogous results were subsequently established by DiPerna [3] for solutions of homogeneous, genuinely nonlinear, systems of hyperbolic conservation laws constructed by Glimm's scheme [4]. These results, however, still allows a very complicated structure such as an everywhere dense shock set. Dafermos [2] has shown that in case that both the (convex) flux and the initial condition are infinitely smooth the solution is C^∞ almost everywhere apart from the shock set which must be closed. Thus, the shock set cannot be everywhere dense but shocks may still accumulate. Schaeffer [13] proved that if $f \in C^\infty$ satisfies (1.3) and u_0 satisfies certain conditions, then the shock set is finite. However, the conditions for the initial data are so abstract that in practice we are unable to verify them.

For homogeneous scalar conservation laws, Tadmor and Tassa [14] and Li and Wang [8] addressed the matter regarding the number of shock curves. It was shown in [14] that if the initial speed has a finite number of decreasing inflection points then it bounds the number of future shock discontinuities. This gives a simple method to bound the shock set. In this work, we will address the same question for the non-homogeneous conservation laws. We first study the mechanism of shock generation for non-homogeneous conservation laws. Then we obtain a formulation to determine the size of the shock set. The proof of our results is based on two ingredients: (i) the generalized characteristics and (ii) the envelope of the characteristic curves.

The second objective of this work is to study the global properties of the piecewisely smooth solutions to the non-homogeneous conservation laws. These properties play important role in obtaining the optimal rate of L^1-convergence for several approximation methods, such as monotone schemes, relaxation approximations [9, 5] and viscosity methods. In the homogeneous case, Tang and Teng [15] obtained similar global properties which enable them to obtain the first-order L^1-convergence rate for the viscosity method. Generally speaking, the L^1-convergence rate of order $O(\sqrt{\epsilon})$ for the several approximation methods is the best possible for $u_0 \in BV$, see e.g. [6, 16, 12], but for convex conservation laws whose entropy solution consists of finitely many discontinuities the L^1-error is bounded by $O(\epsilon)$ or $O(\epsilon|\ln \epsilon|)$, see e.g. [15, 17, 18]. In obtaining the latter result, some global estimates on *derivatives* of the piecewise smooth solutions are essential. Teng [17] extended the main ideas of [15] to treat the relaxation approximations [5] and established similar results on the optimal convergence rate. In this work, we will establish corresponding global properties for non-homogeneous conservation law (1.1). The first-order L^1-convergence rate for various approximations to the non-homogeneous conservation laws will be reported elsewhere. We noticed that the extension is not trivial due to the non-constant property of u along classical characteristics. Roughly speaking, the results we will prove are that along the generalized characteristic $x = \mathbf{X}(t)$

$$\int_{[0,T]\setminus I(\delta)} |u_x(\mathbf{X}(t) \pm 0, t)| dt \le C|\ln\delta|$$

$$\int_{[0,T]\setminus I(\delta)} \|u_{xx}(\cdot,\tau)\|_{L^1(\mathbf{R})} d\tau \le C|\ln\delta|$$

$$\int_{[0,T]\setminus I(\delta)} \|u_x(\cdot,\tau)\|^2_{L^1(\mathbf{R})} d\tau \le C|\ln\delta|$$

provided $\delta > 0$ is sufficiently small, where $I(\delta)$ consists of finitely many subintervals in $[0,T]$ and satisfies $\mathrm{meas}(I(\delta)) = O(\delta)$. The above estimates, together with the standard L^1-contraction lemma and the traveling wave results, may lead to the first-order L^1-convergence rate for various approximations to conservation laws.

2. Generation of shock wave

We discuss the generation of shock waves for non-homogeneous scalar conservation laws

$$\partial_t u + \partial_x f(u) + g(u) = 0 \qquad (x,t) \in \mathbf{R}\times\mathbf{R}^+ \tag{2.1}$$

$$u(x,0) = u_0(x) \qquad x \in \mathbf{R}\,, \tag{2.2}$$

with convex flux f and Lipschitz continuous source term g. Let $X(t;\zeta)$ be (classical) characteristic, i.e. the solution of following equation:

$$\begin{cases} \dfrac{dX(t;\zeta)}{dt} & = a(U(t;u_0(\zeta))) \\ X(0;\zeta) & = \zeta, \end{cases} \tag{2.3}$$

where $a(u) = f'(u)$ and $U(t;\xi)$ is the solution of the following equation

$$\begin{cases} \dfrac{dU(t;\xi)}{dt} & = -g(U(t;\xi)) \\ U(0;\xi) & = \xi. \end{cases} \tag{2.4}$$

Since $g(u)$ is a Lipschitz continuous function, $U(\cdot;\xi)$ exists on $[0,\infty)$ for any $\xi \in (-\infty,\infty)$. Therefore, it follows from (2.3) that $X(\cdot;\zeta)$ also exists on $[0,\infty)$. It is known that $X(t;\zeta)$ is the characteristic drawn from $(\zeta,0)$, and along this characteristic the solution to the non-homogeneous problem (2.1)-(2.2) satisfies

$$u(X(t;\zeta),t) = U(t;u_0(\zeta))$$

as long as $u(x,t)$ is continuous on $x = X(t;\zeta)$.

It follows from (2.4) that

$$U(t;u_0(\zeta)) = u_0(\zeta) + \int_0^t -g(U(\tau;u_0(\zeta)))d\tau\,.$$

Differentiating the above equation with respect to ζ gives

$$\frac{dU(t;u_0(\zeta))}{d\zeta} = u_0'(\zeta) + \int_0^t -g'(U(\tau;u_0(\zeta)))\frac{dU(\tau;u_0(\zeta))}{d\zeta}d\tau$$

which leads to

$$\frac{dU(t;u_0(\zeta))}{d\zeta} = u_0'(\zeta)\exp\left\{\int_0^t -g'(U(\tau;u_0(\zeta)))d\tau\right\}. \tag{2.5}$$

On the other hand, it follows from (2.3) that

$$X(t;\zeta) = \zeta + \int_0^t a(U(\tau;u_0(\zeta)))d\tau \tag{2.6}$$

which yields

$$\frac{\partial X(t;\zeta)}{\partial\zeta} = 1 + \int_0^t a'(U(\tau;u_0(\zeta)))\frac{dU(\tau;u_0(\zeta))}{d\zeta}d\tau. \tag{2.7}$$

Remark 2.1. Throughout this paper the derivatives of $u_0(\zeta)$ are refereed to smooth points of u_0.

The following definitions are due to Dafermos in [2].

Definition 2.1. *A Lipschitz continuous curve $x = \xi(t)$, defined on an interval of $[0,\infty)$, is called a generalized characteristic if for almost all t in the interval*

$$\xi'(t) \in [a(u(\xi(t)+,t)), a(u(\xi(t)-,t))]. \tag{2.8}$$

Definition 2.2. *A Lipschitz continuous curve $x = \eta(t)$ defined on $[\bar{t},\infty)$ is called a shock if*

$$a(u(\eta(t)-,t)) > a(u(\eta(t)+,t)) \tag{2.9}$$

$$\eta'(t) = \frac{f(u(\eta(t)+,t)) - f(u(\eta(t)-,t))}{u(\eta(t)+,t) - u(\eta(t)-,t)}. \tag{2.10}$$

It is shown in [2] that the generalized characteristics must propagate either at classical characteristic speed or at shock speed. More precisely, let $x = \xi(t)$ be a characteristic. Then for $t > 0$

$$\dot{\xi}(t) = \begin{cases} a(u(\xi(t)\pm,t)) & \text{if} \quad u(\xi(t)+,t) = u(\xi(t)-,t) \\ \dfrac{f(u(\xi(t)+,t)) - f(u(\xi(t)-,t))}{u(\xi(t)+,t) - u(\xi(t)-,t)} & \text{if} \quad u(\xi(t)+,t) > u(\xi(t)-,t). \end{cases} \tag{2.11}$$

Definition 2.3. *A point $(\bar{x},\bar{t}) \in (-\infty,\infty)\times(0,\infty)$ with $\bar{x} = \eta(\bar{t})$ is called a shock generation point if the (unique) forward generalized characteristic through $(\bar{x},\bar{t})$ is a shock, while every backward generalized characteristic through $(\bar{x},\bar{t})$ is classical characteristic.*

A typical case of the shock generation point is shown in Fig. 1.

Let $(x(\zeta),t(\zeta))$ be the *envelope* of the characteristic curves $x = X(t;\zeta)$. From the theory of geometry we know that $(x(\zeta),t(\zeta))$ must satisfy

$$\begin{cases} x = X(t;\zeta) \\ \dfrac{\partial X(t;\zeta)}{\partial\zeta} = 0. \end{cases} \tag{2.12}$$

Therefore, $t(\zeta)$ satisfies $X_\zeta(t;\zeta) = 0$. Using (2.7) gives

$$1 + \int_0^{t(\zeta)} a'(U(\tau; u_0(\zeta))) \frac{dU(\tau; u_0(\zeta))}{d\zeta} d\tau = 0. \tag{2.13}$$

We further use (2.5) to obtain

$$1 + u_0'(\zeta) \int_0^{t(\zeta)} a'(U(\tau; u_0(\zeta))) \exp\left\{ \int_0^{\tau} -g'(U(s; u_0(\zeta)))ds \right\} d\tau = 0. \tag{2.14}$$

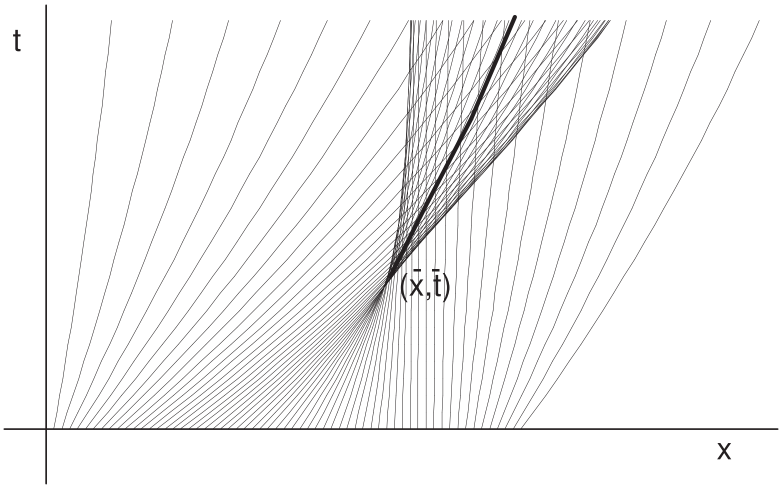

FIGURE 1. Illustration of shock generation: Thin lines are classical characteristics $x = X(t, \zeta)$, thick solid line is shock curve and $(\bar{x}, \bar{t})$ is shock generation point.

Notice that a' is always positive and hence the integrand above is always positive. Therefore a necessary condition for (2.14) is $u_0'(\zeta) < 0$.

2.1. **Size of the shock set.** We now state a theorem concerning the number of shock curves in the entropy solution of (1.1) and (1.2); its proof mainly follows the work of Dafermos [2].

Theorem 2.1. *Let u be the entropy solution of the non-homogeneous convex conservation laws, subject to the bounded and piecewise C^1 initial data u_0. Let $(x(\zeta), t(\zeta))$ be the envelope of the characteristic curves $x = X(t;\zeta)$ defined by (2.3)-(2.4). Then the number of disjoint shock curves is equal to or less than the number of critical points of $t(\zeta)$ plus the number of negative jumps of $u_0(\zeta)$.*

Proof. It easy to show that if the initial $u_0(x)$ has a negative jump at a point $\bar{x}$ (i.e. $u_0(\bar{x}-0) > u_0(\bar{x}+0)$), then a shock curve starts from $(\bar{x}, 0)$. Therefore the number of initial shock equals to the number of negative jumps of u_0.

We next consider the number of shock generation points. Assume $(\bar{x}, \bar{t})$ be a shock generation point as indicated in Fig. 1. It follows from Lemmas 5.2, 5.6 and 5.7 of Dafermos [2] that $(\bar{x}, \bar{t})$ is located on an envelope $(x(\zeta), t(\zeta))$ of the (classical) characteristics $x = X(t;\zeta)$

and the shock generation point $\bar{t}$ is a critical point of $t(\zeta)$. Therefore the number of newly generated shock curves is less or equal to the number of critical points of $t(\zeta)$. This completes the proof of this theorem. □

It is known that a necessary condition for a critical point of $t(\zeta)$ is

$$\frac{dt(\zeta)}{d\zeta} = 0. \tag{2.15}$$

Also, it follows from (2.13) that

$$\frac{dt(\zeta)}{d\zeta}\frac{da(U(\tau;u_0(\zeta)))}{d\zeta} + \int_0^{t(\zeta)} \frac{d^2a(U(\tau;u_0(\zeta)))}{d\zeta^2}d\tau = 0. \tag{2.16}$$

Combining (2.15), (2.16) and Theorem 2.1 gives the following result.

Corollary 2.1. *Let $t(\zeta)$ be defined implicitly by*

$$1 + \int_0^{t(\zeta)} \frac{da(U(\tau;u_0(\zeta)))}{d\zeta}d\tau = 0\,. \tag{2.17}$$

If the following equation

$$\int_0^{t(\zeta)} \frac{d^2a(U(\tau;u_0(\zeta)))}{d\zeta^2}d\tau = 0 \tag{2.18}$$

has finite number of zero points for ζ, then the entropy solution u has only a finite number of disjoint shock curves.

2.2. **An application of Theorem 2.1.** We consider a special case where the source term is linear, namely $g(u) = Lu$. In this case, (2.5) yields

$$U(t;u_0(\zeta)) = u_0(\zeta)\exp\{-Lt\}.$$

Using this result, we obtain from (2.17) that

$$1 - \frac{u_0'(\zeta)}{Lu_0(\zeta)}\left[a\left(u_0(\zeta)e^{-Lt(\zeta)}\right) - a(u_0(\zeta))\right] = 0. \tag{2.19}$$

Similarly, we obtain from (2.18) that

$$\frac{1}{L}u_0'(\zeta)\left[e^{-Lt(\zeta)}a'\left(u_0(\zeta)e^{-Lt(\zeta)}\right) - a'(u_0(\zeta))\right] + \frac{u_0(\zeta)u_0''(\zeta)}{u_0'(\zeta)^2} - 1 = 0. \tag{2.20}$$

Corollary 2.2. *Consider the nonhomogeneous conservation law (2.1)-(2.2) with* linear *source, i.e. $g(u) = Lu$. If (2.20) has finite number of zero points for ζ, then the entropy solution of (2.1)-(2.2) has only a finite number of disjoint shock curves.*

Taking limit of (2.20) as $L \to 0$ yields

$$\frac{u_0(\zeta)}{a'(u_0(\zeta))u_0'(\zeta)^2}\frac{d^2}{d\zeta^2}a(u_0(\zeta)) = 0.$$

Thus for the case of $L = 0$, i.e., homogeneous conservation laws, we have derived the results similar to those obtained by Tadmor and Tassa [14] and by Li and Wang [8]. Namely, if the initial speed has a finite number of decreasing inflection points then it bounds the number of future shock discontinuities.

3. Solution structure

As indicated in Theorem 2.1 if $t(\zeta)$ has finitely many critical points then the entropy solution of (2.1)-(2.2) consists of a finite number of smooth pieces and the number of disjoint shock curves is less than or equals to the number of critical points of $t(\zeta)$. More precisely, we assume that there exists $0 = t_0 < t_1 < \cdots < t_P < \infty$ such that in each time interval (t_{p-1}, t_p), there are finite number of smooth curves $x = X_m^{(p-1)}(t)$, $m = 1, 2, \ldots, M_{p-1}$ satisfying

(P_1). For $t \in (t_{p-1}, t_p)$,

$$X_m^{(p-1)}(t) < X_{m+1}^{(p-1)}(t), \quad m = 1, \ldots, M_{p-1} - 1;$$

In other words, these curves do not cross each other in the interval $t \in (t_{p-1}, t_p)$.

(P_2). At time $t = t_p$, there are two possibilities. The first one is that at least two curves will meet each other. Namely, there exists an m such that

$$X_m^{(p-1)}(t_p + 0) = X_{m+1}^{(p-1)}(t_p - 0)$$

see Figure 2c. The second possibility is that a *new shock* $X_l^{(p)}(t)$ is generated starting from $t = t_p$, see Figure 2d.

(P_3). The solution $u(x,t)$ is smooth everywhere except on the finitely many smooth curves $x = X_m^{(p-1)}(t)$.

(P_4). Let $u_{p,m}^{\pm} := u(X_m^{(p-1)}(t) \pm 0, t)$ be the right-hand and left-hand limits along $x = X_m^{(p-1)}(t)$. Then $u_{p,m}^{\pm}$ satisfy either shock wave conditions:

$$u_{p,m}^{-} > u_{p,m}^{+}$$
$$\dot{X}_m^{(p-1)}(t) = \frac{f(u_{p,m}^{+}) - f(u_{p,m}^{-})}{u_{p,m}^{+} - u_{p,m}^{-}}$$

or weak discontinuous conditions:

$$u_{p,m}^{-} = u_{p,m}^{+} =: u_{p,m}, \quad \dot{X}_m^{(p-1)}(t) = f'(u_{p,m})$$
$$u_x(X_m^{(p-1)}(t) - 0, t) \neq u_x(X_m^{(p-1)}(t) + 0, t)$$

where $\dot{X}(t) = dX(t)/dt$. In the first case, $x = X_m^{(p-1)}(t)$ is called a *shock curve* (Figure 2b), and in the second case it is called a *weak discontinuous curve* (Figure 2a);

(P_5). Each shock $X_m^{(p-1)}(t)$ continues to the next time interval (t_p, t_{p+1}), with a possibility that it collides with another shock curve (Figure 2c), and at the time interval (t_P, ∞) there exists at most one shock curve $X_1^{(P)}(t)$.

The piecewise smooth entropy solution is a finite combination of the cases discussed above. In obtaining global estimates, we only need to consider the cases (a)-(d) plotted in Figure 2.

4. Global estimates

In this section, we will establish some global estimate for the derivatives of the piecewise-smooth entropy solution along generalized characteristics, especially along shock waves. Theses estimates play important role in theoretical and numerical analysis for conservation laws with source terms. For homogeneous conservation laws the corresponding estimates were

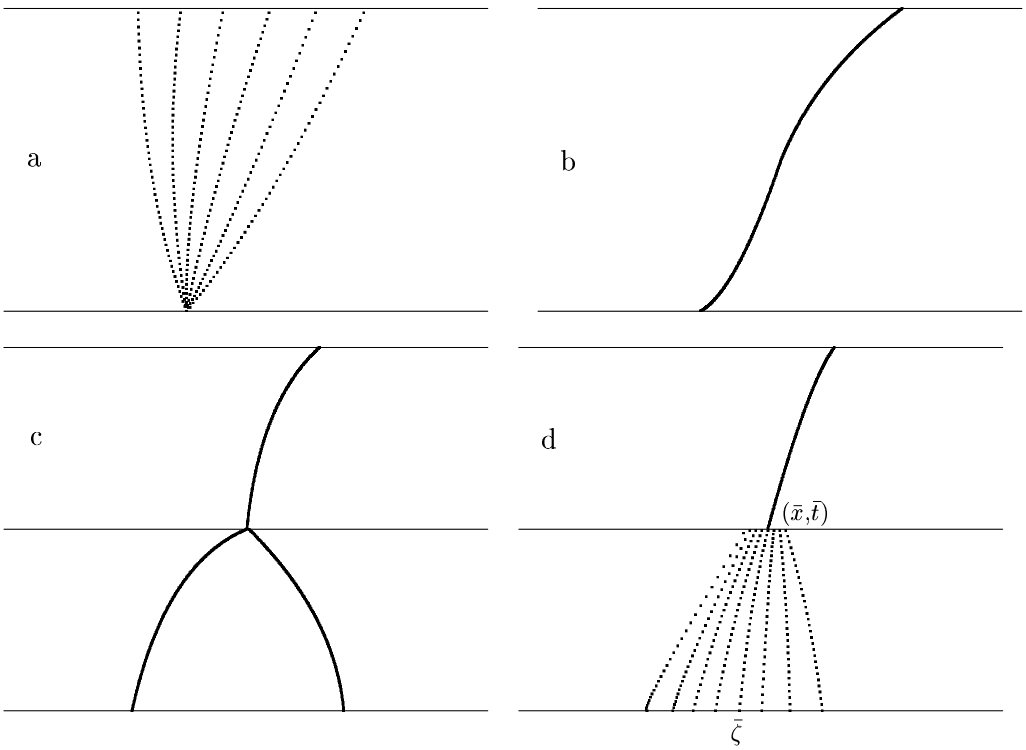

FIGURE 2. Structures of piecewise smooth solution: (a)–central rarefaction wave; (b)–shock curve; (c)–interacting shock curves; (d)–new formed shock curve.

obtained in [15] and [17]. In this section, they will be extended to the non-homogeneous case. The applications of these global estimates will be reported elsewhere.

In order to obtain the main results in this section, we need the following lemma which gives some estimates on $U(t;\zeta)$ defined by (2.4).

Lemma 4.1. *Let $U(t;\xi)$ be the solution of* (2.4). *Then the following estimates hold:*

$$|\xi|\exp\{-Lt\} \le |U(t;\xi)| \le |\xi|\exp\{Lt\}; \tag{4.1}$$

$$\exp\{-Lt\} \le \frac{\partial U(t;\xi)}{\partial \xi} \le \exp\{Lt\}; \tag{4.2}$$

$$\left|\frac{\partial^2 U(t;\xi)}{\partial \xi^2}\right| \le G_2 t\exp\{2Lt\}; \tag{4.3}$$

$$\exp\{-L(t-\tau)\} \le \left|\frac{U(\tau;\xi_1)-U(\tau;\xi_2)}{U(t;\xi_1)-U(t;\xi_2)}\right| \le \exp\{L(t-\tau)\}, \quad \tau\in[0,t] \tag{4.4}$$

where L is the Lipschitz constant for g.

The lemma follows easily from (2.4) and the Lipschitz continuity assumption on g.

Theorem 4.1. *Let u be the entropy solution of the non-homogeneous convex conservation laws, subject to the bounded and piecewise C^2 initial data u_0, and $x = \mathbf{X}(t)$ be a generalized characteristic. If $t(\zeta)$ defined by* (2.17) *has finite number of critical points, then for sufficiently small $\delta > 0$,*

1. *If $(\mathbf{X}(\bar{t}), \bar{t})$ is a shock generation point with $\bar{t} > 0$, then for any $T > \bar{t}$*

$$\int_{[0,T]\setminus[\bar{t}-\delta,\bar{t}+\delta]} |u_x(\mathbf{X}(t) \pm 0, t)|\, dt \le C(T)|\ln\delta| + C(T)\,; \tag{4.5}$$

2. *If $\mathbf{X}(0)$ is a discontinuous point of $u_0(x)$ with positive jump, then for any $T > 0$*

$$\int_{[\delta,T]} |u_x(\mathbf{X}(t) \pm 0, t)|\, dt \le C(T)|\ln\delta| + C(T); \tag{4.6}$$

3. *Otherwise*

$$\int_{[0,T]} |u_x(\mathbf{X}(t) \pm 0, t)|\, dt \le C(T). \tag{4.7}$$

Proof. It follows from the assumptions above and Theorem 2.1 that the entropy solution u is piecewise smooth with finitely many discontinuities. In the first case, if $(\mathbf{X}(\bar{t}), \bar{t})$ is a shock generation point, then the generalized characteristic $x = \mathbf{X}(t)$ defined on $(\bar{t}, \infty)$ is a shock curve. Thus we can trace two classical characteristic lines from each side of $\mathbf{X}(t)$ for $t \in (\bar{t}, T)$ backward to $t = 0$. The intersection points of two characteristic lines with $t = 0$ are $\zeta^-(\mathbf{X}(t), t)$ and $\zeta^+(\mathbf{X}(t), t)$, which satisfy: For $\bar{t} < t \le T$

$$\zeta^+(\mathbf{X}(t), t) > \zeta^-(\mathbf{X}(t), t)\,, \quad \mathbf{X}(t) = X(t; \zeta^\pm(\mathbf{X}(t), t))\,. \tag{4.8}$$

The characteristic relation gives

$$u(\mathbf{X}(t) \pm 0, t) = u(X(t; \zeta^\pm) \pm 0, t) = U(t; u_0(\zeta^\pm)), \tag{4.9}$$

where (X, U) are the solution of (2.3) and (2.4). Differentiating both side of (4.8) with respect to t gives

$$\dot{\zeta}^\pm = \frac{\mathbf{X}'(t) - a(U(t; u_0(\zeta^\pm)))}{X_\zeta(t; \zeta^\pm)}, \quad \dot{\zeta}^\pm := \frac{d\zeta^\pm(\mathbf{X}(t), t)}{dt},$$

or equivalently

$$0 < \frac{1}{X_\zeta(t; \zeta^\pm)} = \frac{\dot{\zeta}^\pm}{\mathbf{X}'(t) - a(u(\mathbf{X}(t) \pm 0, t))}, \tag{4.10}$$

where the first inequality is due to the facts that (i) $\zeta^-(t)$ is decreasing and $\zeta^+(t)$ is increasing and (ii) the entropy condition $a(u(\mathbf{X} - 0, t)) > \mathbf{X}'(t) > a(u(\mathbf{X} + 0, t))$. Observe that

$$\begin{aligned} \mathbf{X}'(t) - a(u(\mathbf{X}(t) \pm 0, t)) &= \int_0^1 a(\theta u^+ + (1-\theta)u^-)\, d\theta - a(u^\pm) \\ &= \pm \int_0^1 f''(u^*_\pm)\theta\, d\theta (u^- - u^+), \end{aligned} \tag{4.11}$$

where $u^\pm = u(\mathbf{X}(t) \pm 0, t)$ and $u^*_\pm$ denote some intermediate value. Therefore, we have

$$|\mathbf{X}'(t) - a(u(\mathbf{X}(t) \pm 0, t))| \ge \frac{\gamma}{2}|u^+ - u^-|. \tag{4.12}$$

where γ given by (1.3) is the constant related to the convexity of the flux function. On the other hand, it follows from (2.3) that

$$X(t; \zeta_1) - X(t; \zeta_2) - (\zeta_1 - \zeta_2) = \int_0^t f''(u^*)\big(U(\tau; u_0(\zeta_1)) - U(\tau; u_0(\zeta_2))\big)\, d\tau.$$

The above result, with the aid of (4.4), leads to

$$\big|X(t;\zeta_1) - X(t;\zeta_2) - \big(\zeta_1 - \zeta_2\big)\big| \le F_2 \big|U(t;u_0(\zeta_1)) - U(t;u_0(\zeta_2))\big| \frac{e^{Lt}-1}{L},$$

or equivalently

$$\big|U(t;u_0(\zeta_1)) - U(t;u_0(\zeta_2))\big| \ge \big|X(t;\zeta_1) - X(t;\zeta_2) - \big(\zeta_1 - \zeta_2\big)\big| \frac{L}{(e^{Lt}-1)F_2},$$

where

$$F_2 = \max_{|u| \le \|u_0\|_\infty e^{LT}} f''(u) > 0.$$

Substituting ζ^+ and ζ^- into the above inequality, on account of (4.8) and (4.9), gives

$$|u^+ - u^-| \ge |\zeta^+ - \zeta^-| \frac{L}{(e^{Lt}-1)F_2}. \tag{4.13}$$

Combining (4.10), (4.12) and (4.13) gives

$$0 \le \frac{1}{X_\zeta(t;\zeta^\pm)} \le C(T) \frac{|\dot\zeta^\pm|}{|\zeta^+ - \zeta^-|}.$$

Since ζ^- is decreasing and ζ^+ is increasing, we have

$$|\dot\zeta^\pm| = \pm\dot\zeta^\pm \le \dot\zeta^+ - \dot\zeta^-$$

and therefore

$$0 \le \frac{1}{X_\zeta(t;\zeta^\pm)} \le C(T) \frac{\dot\zeta^+ - \dot\zeta^-}{\zeta^+ - \zeta^-} \quad \text{for} \quad \bar t \le t \le T. \tag{4.14}$$

Differentiating both side of the second identity in (4.9) with respect to $\zeta^\pm$ gives

$$u_x(\mathbf{X}(t) \pm 0, t) = \frac{U_\xi(t;u_0(\zeta^\pm))u_0'(\zeta^\pm)}{X_\zeta(t;\zeta^\pm)} \quad \text{for} \quad \bar t \le t \le T.$$

It follows from (4.2), an estimate for U_ξ, that

$$|u_x(\mathbf{X}(t) \pm 0, t)| \le \frac{\|u_0'\|_\infty \exp\{Lt\}}{X_\zeta(t;\zeta^\pm)} \quad \text{for} \quad \bar t \le t \le T, \tag{4.15}$$

where

$$\|u_0'\|_\infty := \max_{l=0,1,\dots,L} \Big\{ \sup_{x \in (\gamma_l, \gamma_{l+1})} |u_0'(x)| \Big\}.$$

Here γ_l $(l = 1, \dots, L)$ are discontinuous points of u_0, $\gamma_0 = -\infty$ and $\gamma_{L+1} = \infty$. It follows from (4.14) and (4.15) that

$$\int_{\bar t+\delta}^{T} |u_x(\mathbf{X}(t) \pm 0, t)|\, dt \le C(T) \ln\left(\frac{\zeta^+(T) - \zeta^-(T)}{\zeta^+(\bar t+\delta) - \zeta^-(\bar t+\delta)} \right), \tag{4.16}$$

where $\zeta^\pm(t) := \zeta^\pm(\mathbf{X}(t), t)$. Now let us estimate $\zeta^+(t) - \zeta^-(t)$. We need to show that

$$\zeta^+(t) - \zeta^-(t) \ge c(T)(t - \bar t) \quad \text{for} \quad t \in [\bar t, T], \tag{4.17}$$

where $c(T)$ is a positive constant. Since $(\mathbf{X}(\bar t), \bar t)$ is a shock generation point, as indicated by Theorem 2.1 that

$$\frac{\partial X(\bar t;\bar\zeta)}{\partial \zeta} = 0 \quad \text{for} \quad \bar\zeta \in [\zeta^-(\bar t+0), \zeta^+(\bar t+0)]. \tag{4.18}$$

Here, we only consider the case when $\bar{\zeta} \equiv \zeta^-(\bar{t}+0) = \zeta^+(\bar{t}+0)$. For the case of $\zeta^-(\bar{t}+0) < \zeta^+(\bar{t}+0)$, a centered compression wave, the proof is similar to the center rarefaction wave, cf. the proof of (4.6). Using the Mean value theorem and the equality (4.18) gives

$$
\begin{aligned}
0 < \frac{\partial X(t;\zeta^{\pm})}{\partial \zeta} &= \frac{\partial X(t;\zeta^{\pm})}{\partial \zeta} - \frac{\partial X(\bar{t};\bar{\zeta})}{\partial \zeta} \\
&= \frac{\partial X(t;\zeta^{\pm})}{\partial \zeta} - \frac{\partial X(t;\bar{\zeta})}{\partial \zeta} + \frac{\partial X(t;\bar{\zeta})}{\partial \zeta} - \frac{\partial X(\bar{t};\bar{\zeta})}{\partial \zeta} \\
&= \frac{\partial^2 X(\bar{t};\bar{\zeta}^*_{\pm})}{\partial \zeta^2}(\zeta^{\pm} - \bar{\zeta}) + \frac{\partial^2 X(\bar{t}^*;\bar{\zeta})}{\partial \zeta \partial t}(t-\bar{t}),
\end{aligned} \tag{4.19}
$$

where $\bar{t}^*$ and $\bar{\zeta}^*_{\pm}$ denote some intermediate values. It follows from (2.13) that

$$
0 = \frac{\partial X(\bar{t},\bar{\zeta})}{\partial \zeta} = 1 + \int_0^{\bar{t}} a'(U(\tau,u_0(\bar{\zeta})))\frac{dU(\tau,u_0(\bar{\zeta}))}{d\zeta}d\tau. \tag{4.20}
$$

Since $a'(u) = f''(u) > 0$ and

$$
\frac{dU(\tau,u_0(\bar{\zeta}))}{d\zeta} = u_0'(\bar{\zeta})\exp\left\{-\int_0^{\tau} g'(U(s,u_0(\bar{\zeta})))ds\right\} \quad \text{for} \quad \tau \in [0,\bar{t}],
$$

we conclude from (4.20) that

$$
u_0'(\bar{\zeta}) < 0. \tag{4.21}
$$

It follows from (2.7) , an expression for X_ζ, and the above inequality that

$$
\begin{aligned}
\frac{\partial^2 X(\bar{t}^*;\bar{\zeta})}{\partial \zeta \partial t} &= a'(U(\bar{t}^*,u_0(\bar{\zeta})))u_0'(\bar{\zeta})\exp\Big\{-\int_0^{\bar{t}^*} g'(U(s,u_0(\bar{\zeta})))ds\Big\} \\
&\leq \gamma u_0'(\bar{\zeta})\exp\{-LT\}
\end{aligned} \tag{4.22}
$$

$$
\begin{aligned}
\frac{\partial^2 X(\bar{t};\bar{\zeta}^*)}{\partial \zeta^2} &= \int_0^{\bar{t}}\Big\{a''(U(\tau;u_0(\bar{\zeta}^*)))\left[\frac{dU(\tau,u_0(\bar{\zeta}^*))}{d\zeta}\right]^2 \\
&\quad + a'(U(\tau,u_0(\bar{\zeta}^*)))\frac{d^2U(\tau,u_0(\bar{\zeta}^*))}{d\zeta^2}\Big\}d\tau.
\end{aligned} \tag{4.23}
$$

Substituting

$$
\begin{aligned}
\frac{dU(\tau,u_0(\zeta))}{d\zeta} &= \frac{\partial U(\tau,u_0(\zeta))}{\partial \xi}u_0'(\zeta) \\
\frac{d^2U(\tau,u_0(\bar{\zeta}))}{d\zeta^2} &= \frac{\partial^2 U(\tau,u_0(\zeta))}{\partial \zeta^2}u_0'(\zeta)^2 + \frac{\partial U(\tau,u_0(\zeta))}{\partial \zeta}u_0''(\zeta)
\end{aligned}
$$

into the equation (4.23) and using estimates (4.2) and (4.3) give

$$
\left|\frac{\partial^2 X(\bar{t};\bar{\zeta}^*)}{\partial \zeta^2}\right| \leq C(T). \tag{4.24}
$$

This, together with (4.19) and (4.22), yield

$$
C(T)|\zeta^{\pm}(t) - \bar{\zeta}| \geq -\gamma u_0'(\bar{\zeta})(t-\bar{t})\exp\{-LT\} > 0 \quad \text{for} \quad t \in (\bar{t},T].
$$

The above estimate is (4.17), with $c(T) == -\gamma u_0'(\bar{\zeta})\exp\{-LT\}/C(T) > 0$. Substituting (4.17) into (4.16) yields

$$\int_{\bar{t}+\delta}^{T} |u_x(\mathbf{X}(t) \pm 0, t)|\, dt \le C(T) \ln\left(\frac{T-\bar{t}}{\delta}\right) \quad \text{for} \quad 0 < \delta \ll 1. \tag{4.25}$$

The proof for $\int_0^{\bar{t}-\delta} |u_x(\mathbf{X}(t) \pm 0, t)|\, dt$ is similar to that of $\int_{\bar{t}+\delta}^{T} |u_x(\mathbf{X}(t) \pm 0, t)|\, dt$. The only difference is that $\mathbf{X}(t)$ for $t \in [0, \bar{t})$ is a classical characteristic and $u(x,t)$ is a classical solution at the neighborhood of $\mathbf{X}(t)$. The estimate takes the following form

$$\int_{0}^{\bar{t}-\delta} |u_x(\mathbf{X}(t) \pm 0, t)|\, dt \le C(T) \ln\left(\frac{\bar{t}}{\delta}\right) \quad \text{for} \quad 0 < \delta \ll 1. \tag{4.26}$$

We omit the detail proof here. Combining (4.25) and (4.26) leads to the conclusion (4.5).

We now consider the second case, i.e. $\mathbf{X}(0)$ is a discontinuous point of $u_0(x)$, with $u_0(\mathbf{X}(0)-0) < u_0(\mathbf{X}(0)+0)$. In this case there is no shock generation point along $\mathbf{X}(t)$ for $t \in [0, \infty)$ and thus $u_x(\mathbf{X}(t) \pm 0, t)$ is a continuous function of t for $t \in (0, \infty)$. Therefore we only need to prove (4.6) for some small $T > 0$. Since the entropy solution $u(x,t)$ is piecewise smooth with only finitely many discontinuous curves, the following classical characteristic relationship holds for $0 < t < T$, with some $T > 0$:

$$\begin{aligned} \mathbf{X}(t) &= X(t; u(\mathbf{X}(+0), +0)) \\ u(\mathbf{X}(t), t) &= U(t; u(\mathbf{X}(+0), +0)), \end{aligned}$$

where $u(\mathbf{X}(+0), +0) \in [u_0(\mathbf{X}(0)-0), u_0(\mathbf{X}(0)+0)]$ and (X, U) satisfy

$$\begin{cases} \dfrac{dX(t;\xi)}{dt} = a(U(t;\xi)) \\ X(0;\xi) = \mathbf{X}(0) \end{cases} \tag{4.27}$$

and

$$\begin{cases} \dfrac{dU(t;\xi)}{dt} = -g(U(t;\xi)) \\ U(0;\xi) = \xi, \quad \xi \in [u_0(\mathbf{X}(0)-0), u_0(\mathbf{X}(0)+0)]. \end{cases} \tag{4.28}$$

Therefore, we have

$$u_x(\mathbf{X}(t), t) = \left.\frac{U_\xi(t;\xi)}{X_\xi(t;\xi)}\right|_{\xi = u(\mathbf{X}(+0), +0)} \quad \text{for} \quad 0 < t < T. \tag{4.29}$$

It is known from (4.27) and (4.28) that

$$\begin{aligned} U_\xi(t;\xi) &= \exp\left\{-\int_0^t g'(U(\tau,\xi))d\tau\right\}, \\ X_\xi(t;\xi) &= \int_0^t a'(U(\tau,\xi))U_\xi(\tau;\xi)d\tau. \end{aligned}$$

Therefore we obtain

$$|U_\xi(t;\xi)| \le \exp\{LT\}, \quad |X_\xi(t;\xi)| \ge \gamma t \exp\{-LT\}.$$

The above estimates, together with (4.29), yield

$$|u_x(\mathbf{X}(t),t)| \le \frac{\exp\{2LT\}}{\gamma t} \quad \text{for} \quad 0 < t \le T.$$

The desired result (4.6) follows from the above inequality.

In the rest cases, $u_x(\mathbf{X}(t) \pm 0, t)$ are continuous with respect to t on $[0, \infty)$. Therefore (4.7) is established directly. Hence the proof of Theorem 4.1 is complete. □

In what follows we will use L^1-norm for the derivatives of u, for example $\|u_{xx}(\cdot,t)\|_{L^1(\mathbf{R})}$, $\|u_x(\cdot,t)^2\|_{L^1(\mathbf{R})}$ and so on, where the integration in the norm is to be understood as piecewise integration excluding the discontinuous points of the derivatives.

Theorem 4.2. *Let $\{(\bar{x}_j, \bar{t}_j)\}_1^J$ with $\bar{t}_j < \bar{t}_{j+1}$ be the shock generation points and assume that $u_0(x)$ has compact support.*

1. *If at $t = 0$ there are some starting center rarefaction points for the entropy solution $u(x,t)$, then for any $T > \bar{t}_J$*

$$\int_{[\delta,T]\setminus S(\delta)} \|u_{xx}(\cdot,\tau)\|_{L^1(\mathbf{R})} d\tau \le C|\ln\delta| \tag{4.30}$$

$$\int_{[\delta,T]\setminus S(\delta)} \|u_x(\cdot,\tau)^2\|_{L^1(\mathbf{R})} d\tau \le C|\ln\delta| \tag{4.31}$$

provided that δ is sufficiently small, where

$$S(\delta) := \bigcup_{j=1}^{J} [\bar{t}_j - \delta, \bar{t}_j + \delta].$$

If an interval $[\kappa, T]$ does not include any $\bar{t}_j$ $(j = 1, \ldots, J)$ and 0, then

$$\int_{[\kappa,T]} \|u_{xx}(\cdot,\tau)\|_{L^1(\mathbf{R})} d\tau \le C \tag{4.32}$$

$$\int_{[\kappa,T]} \|u_x(\cdot,\tau)^2\|_{L^1(\mathbf{R})} d\tau \le C. \tag{4.33}$$

2. *If at $t = 0$ there is no center rarefaction point for the entropy solution $u(x,t)$, then for any $T > \bar{t}_J$*

$$\int_{[0,T]\setminus S(\delta)} \|u_{xx}(\cdot,\tau)\|_{L^1(\mathbf{R})} d\tau \le C|\ln\delta| \tag{4.34}$$

$$\int_{[0,T]\setminus S(\delta)} \|u_x(\cdot,\tau)^2\|_{L^1(\mathbf{R})} d\tau \le C|\ln\delta| \tag{4.35}$$

provided that δ is sufficiently small. If an interval $[0, T]$ does not include any $\bar{t}_j$ $(j = 1, \ldots, J)$, then

$$\int_{[0,T]} \|u_{xx}(\cdot,\tau)\|_{L^1(\mathbf{R})} d\tau \le C \tag{4.36}$$

$$\int_{[0,T]} \|u_x(\cdot,\tau)^2\|_{L^1(\mathbf{R})} d\tau \le C. \tag{4.37}$$

Proof. Here we only prove (4.30)-(4.33); the estimates (4.34)-(4.37) can be proved in a similar way. Let $x = \mathbf{X}_j(t)$ for $t \in [0, \infty)$ be generalized characteristic passing through $(\bar{x}_j, \bar{t}_j)$ for $j = 1, \ldots, J$, respectively. From each point (x, t) we can draw a characteristic line backward to $t = 0$ and the intersection point is $\zeta(x, t)$, which satisfies

$$x = X(t, \zeta) \quad \text{and} \quad u(x, t) = U(t, u_0(\zeta)).$$

Since $\zeta(x, t)$ is an increasing function of x, we have from the above equations

$$\begin{aligned}
\zeta_x(x,t) &= \frac{1}{X_\zeta(t;\zeta)} \geq 0, \quad \zeta_{xx} = -\frac{X_{\zeta\zeta}\zeta_x}{X_\zeta{}^2}, && (4.38)\\
u_x(x,t) &= U_\xi(t; u_0(\zeta)) u_0'(\zeta) \frac{1}{X_\zeta(t;\zeta)} \\
u_{xx}(x,t) &= U_{\xi\xi}(t; u_0(\zeta)) \left(u_0'(\zeta)\zeta_x\right)^2 + U_\xi(t; u_0(\zeta)) \left(u_0''(\zeta)\zeta_x{}^2 + u_0'(\zeta)\zeta_{xx}\right) \\
&= \left[U_{\xi\xi} u_0'{}^2 + U_\xi \left(u_0'' - u_0' \frac{X_{\zeta\zeta}}{X_\zeta}\right)\right] \frac{1}{X_\zeta{}^2}. && (4.39)
\end{aligned}$$

Since $\{(\bar{x}_j, \bar{t}_j)\}_1^J$ are the shock generation points, $X_\zeta(t; \zeta(x, t)) = 0$ at these points. From (4.39) we know that $u_{xx}(x, t)$ becomes infinity at $\{(\bar{x}_j, \bar{t}_j)\}_1^J$. Therefore, $u_{xx}(x, t)$ will not change its signs at some of their neighborhood, which are denoted by

$$\omega_j := \{(x,t) \,|\, \mathbf{X}_k(\tau) - h \leq x \leq \mathbf{X}_k(\tau) + h, \, \bar{t}_j - h \leq \tau \leq \bar{t}_j + h\}$$

with some $h > 0$, and then

$$|u_{xx}(x,t)| \leq C(T) \quad \text{for} \quad (x,t) \in \Omega(T) \backslash \bigcup_{j=1}^{J} \omega_j, \tag{4.40}$$

where

$$\Omega(T) := (-\infty, \infty) \times [0, T].$$

We first divide the integration into two parts

$$\begin{aligned}
&\int_{[\delta,T]\backslash S(\delta)} \|u_{xx}(\cdot,\tau)\|_{L^1(\mathbf{R})} d\tau \\
&\leq \iint_{\Omega(T)\backslash \bigcup_{j=1}^J \omega_j} |u_{xx}| dx dt + \iint_{\bigcup_{j=1}^J \{\omega_j \backslash \Delta_j(\delta)\}} |u_{xx}| dx dt \\
&\leq \iint_{\Omega(T)\backslash \bigcup_{j=1}^J \omega_j} |u_{xx}| dx dt + \sum_{j=1}^{J} \iint_{\omega_j \backslash \Delta_j(\delta)} |u_{xx}| dx dt \\
&\equiv I + II,
\end{aligned}$$

where

$$\Delta_j(\delta) := \{(x,t) \,|\, \mathbf{X}_k(\tau) - h \leq x \leq \mathbf{X}_k(\tau) + h, \, \bar{t}_j - \delta \leq \tau \leq \bar{t}_j + \delta\}$$

and $0 < \delta < h$. From (4.40) we know that

$$I \leq C(T). \tag{4.41}$$

Since u_{xx} does not change its signs in ω_j, we have

$$\begin{aligned}\iint_{\omega_j\setminus\Delta_j(\delta)}|u_{xx}|dxdt &= \left|\iint_{\omega_j\setminus\Delta_j(\delta)}u_{xx}dxdt\right|\\ &\le \int_{[\bar t_j-h,\bar t_j+h]\setminus[\bar t_j-\delta,\bar t_j+\delta]}\{|u_x(\mathbf{X}_j(\tau)+0,\tau)|+|u_x(\mathbf{X}_j(\tau)-0,\tau)|\}\,d\tau+C(T).\end{aligned}$$

Substituting the estimate (4.5) into above inequality gives

$$\iint_{\omega_j\setminus\Delta_j(\delta)}|u_{xx}|dxdt\le C(T)\delta+C(T),$$

and thus

$$II\le C(T)\delta+C(T). \tag{4.42}$$

Summing up (4.41) and (4.42) yields (4.30). The estimates (4.32) is an easy conclusion from the fact that u_{xx} is bounded on the region of $(-\infty,\infty)\times[\kappa,T]$.

It remains to prove (4.31) and (4.33). The proof is based on the integration by parts:

$$\begin{aligned}\int\int_{\mathbf{R}}u_x^2dxdt &= \sum_j\int uu_x\Big|_{\mathbf{X}_j(t)+0}^{\mathbf{X}_{j+1}(t)-0}dt-\int\int_{\mathbf{R}}uu_{xx}dxdt\\ &\le C\sum_j\int|u_x(\mathbf{X}_j(t)\pm 0,t)|dt+C\int\|u_{xx}\|_{L^1(\mathbf{R})}dt\,.\end{aligned}$$

The above result, together with Theorem 4.1, (4.30) and (4.32), lead to the desired results (4.31) and (4.33). ☐

5. The sharpness of the theoretical estimates

In this section we present an exact entropy solution to show that the global estimates given in the previous section are sharp. The example is of the form

$$\begin{cases}\partial_t u+\partial_x\left(u^2/2\right)=Lu\\ u(x,0)=\begin{cases}-C_L(x-x^3), & |x|\le 1\\ 0, & |x|>1.\end{cases}\end{cases} \tag{5.1}$$

where L is a constant and

$$C_L:=\begin{cases}1, & L=0\\ \dfrac{L}{e^L-1}>0, & L\ne 0.\end{cases}$$

It follows from (2.3) and (2.4) that

$$\begin{cases}X(t;\zeta)=\begin{cases}\zeta-(\zeta-\zeta^3)(e^{Lt}-1)/(e^L-1), & |\zeta|\le 1\\ \zeta, & |\zeta|>1\end{cases}\\ u(X(t;\zeta),t)=\begin{cases}-C_Le^{Lt}(\zeta-\zeta^3), & |\zeta|\le 1\\ 0, & |\zeta|>1,\end{cases}\end{cases} \tag{5.2}$$

where the characteristics $x=X(t;\zeta)$ with $L=1$ are drawn on Fig. 3 by using Mathematica.

Solving $X_\zeta(t;\zeta)=0$ for t gives

$$t(\zeta)=\frac{1}{L}\ln\left(\frac{-3\zeta^2+e^L}{-3\zeta^2+1}\right)\quad \text{for}\quad |\zeta|\le \min\left\{\frac{e^{L/2}}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right\}$$

and

$$t'(\zeta)=\frac{6\zeta}{C_L(1-3\zeta^2)(e^L-3\zeta^2)}\quad \text{for}\quad |\zeta|\le \min\left\{\frac{e^{L/2}}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right\}.$$

Therefore, $\bar{\zeta}=0$ is a critical point (more precisely a local minimum) of $t(\zeta)$ and its corresponding shock *generation* point is

$$(\bar{x},\bar{t})=\Big(X\big(t(\bar{\zeta});\bar{\zeta}\big),t(\bar{\zeta})\Big)=(0,1).$$

¿From the anti-symmetry of the initial date $u_0(x)$ with respect to $x=0$ one knows that the shock curve is

$$x=0\quad \text{for}\quad 1\le t<\infty.$$

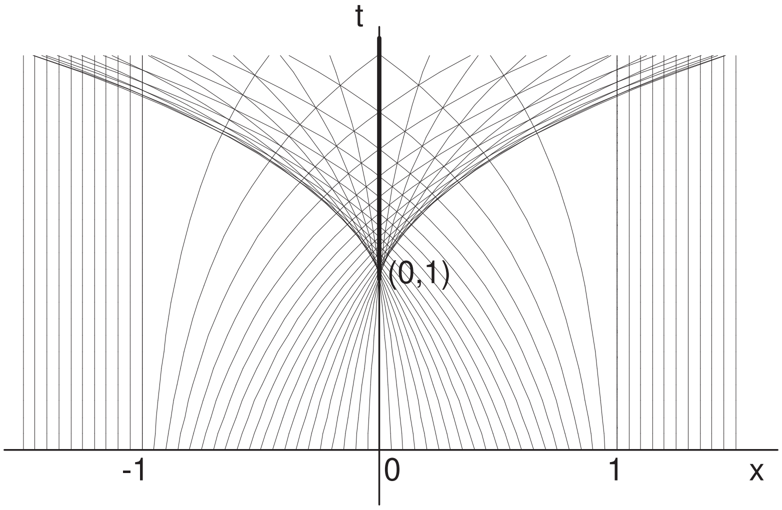

FIGURE 3. Illustration of the characteristics: Thin lines are classical characteristics $x=X(t,\zeta)$, thick solid line is shock curve and $(0,1)$ is shock generation point.

Now let us derive an explict expression for the entropy solution of (5.1) for $t>1$. Since the entropy solution $u(x,t)$ is anti-symmetry with respect to $x=0$, we only consider the case of $x>0$. Solving the equation $X(t;\zeta)=x$ or equivalently

$$x=\begin{cases}\zeta-(\zeta-\zeta^3)\left(\dfrac{e^{Lt}-1}{e^L-1}\right), & 0<\zeta\le 1,\ t>1\\ \zeta, & \zeta>1,\ t>1\end{cases}$$

for positive ζ by using trigonometric solution of cubic [1] gives

$$\zeta_+(x,t)=\begin{cases}2\sqrt[3]{r}\cos\theta, & 0<x\le 1,\ t>1\\ x, & x>1,\ t>1\end{cases}\tag{5.3}$$

where

$$r = \sqrt{-\left(\frac{p}{3}\right)^3}, \quad \theta = \frac{1}{3}\arccos\left(-\frac{q}{2r}\right)$$

with

$$p = \frac{e^L - e^{Lt}}{e^{Lt}-1}, \qquad q = -x\left(\frac{e^L-1}{e^{Lt}-1}\right).$$

Since $t > 1$ and $x > 0$, we have $p < 0$, $q < 0$, $r > 0$ and $0 \le \theta < \pi/6$. This shows that $\zeta_+(x,t)$ defined in (5.3) is positive. Therefore the explicit expression of $u(x,t)$ for $0 < x \le 1$ and $t > 1$ is

$$u(x,t) = -C_L e^{Lt}\left(\zeta_+(x,t) - \zeta_+(x,t)^3\right).$$

As a consequence we obtain

$$u_x(x,t) = -C_L e^{Lt}\Big(1 - 3(\zeta_+)^2\Big)(\zeta_+)_x.$$

It follows from (5.3) that for $t > 1$

$$\zeta_+(0+,t) = \left(\frac{e^{Lt}-e^L}{e^{Lt}-1}\right)^{1/2}$$

and

$$(\zeta_+)_x(0+,t) = \frac{1}{2}\frac{e^L-1}{e^{Lt}-e^L}.$$

Hence for $t > 1$ we have

$$u_x(0+,t) = -\frac{1}{2}\frac{3e^L - 2e^{Lt}-1}{e^{Lt}-1}\frac{Le^{Lt}}{e^{Lt}-e^L}.$$

Straightforward calculation gives

$$\begin{aligned}
\int_{1+\delta}^{T} |u_x(0+,t)|dt &\ge \left|\int_{1+\delta}^{T} u_x(0+,t)dt\right| \\
&= \frac{1}{2}\left|\ln\left[\left(\frac{1-e^{LT}}{1-e^{(1+\delta)L}}\right)^3\left(\frac{e^L-e^{(1+\delta)L}}{e^L-e^{LT}}\right)\right]\right| \\
&= \frac{1}{2}\left|\ln\left(\frac{e^{-3\theta_1 LT}(-LT)^3 e^{-\theta_2\delta L}(-\delta L)}{e^{-3\theta_3(1+\delta)L}(-(1+\delta)L)^3 e^{\theta_4(1-T)L}(1-T)L}\right)\right| \quad \text{(Mean-Value Theorem is used)} \\
&= \frac{1}{2}\left|\ln\left(\frac{e^{-3\theta_1 LT}T^3 e^{-\theta_2\delta L}\delta}{e^{-3\theta_3(1+\delta)L}(1+\delta)^3 e^{\theta_4(1-T)L}(T-1)}\right)\right| \\
&= \frac{1}{2}\Big|\,O(T) + \ln\delta\Big|,
\end{aligned}$$

where $0 < \theta_1, \theta_2, \theta_3, \theta_4 < 1$, $\delta > 0$ and $T > 1+\delta$. The last equality indicates that the estimate in (4.5) is sharp. The sharpness of the other estimates obtained in the last section can be verified in a similar way.

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